BHSEC MATHEMATICS

BOOK – I

FOR CLASS XI

For Class XI Students of Bhutan

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PREFACE

We feel happy in presenting the revised version of our immensely popular book **ISC Mathematics** strictly in accordance with the *Mathematics syllabus for class 11 for 2010 and after* released by the Curriculum and Professional Support Division (CAPSD) of the Ministry of Education, Royal Government of Bhutan, as part of its Bhutan Higher Secondary Education Certificate (BHSEC) course.

The special features of this book are:

1. **It follows strictly the prescribed syllabus** and incorporates the latest trends in the teaching of Mathematics.
2. For the convenience of the teachers and students the detailed syllabus has been given right in the beginning.
3. The chapters are in the same sequence as given in the syllabus to facilitate teaching in the class and coverage of the syllabus.
4. The development is logical, and the preparation of each new idea is based on the preceding material.
5. Great pains have been taken to present the subject matter in a very easy to understand manner. To achieve this, the authors had sometimes to sacrifice brevity and give detailed explanation to bring home to the students the finer points of every topic. A sincere effort has been made to explain the '**How and Why**' of every concept to make the fundamentals clear. The authors are of the view that a textbook is not just a collection of formulae and questions but much more than this. A textbook should make an in-depth study of the subject and lay solid and sound foundation for further study.
6. The clearly development textual explanations are followed by appropriate solved **examples** which are **large in number** and include almost all types of questions possible on a particular topic or concept.
7. Effort has been made to include **quality questions** in exercises for practice, keeping in mind the latest trend and style of questions. The questions are ample in number and well-graded and would cater to the needs of all types of students—***average***, ***above average*** and ***brilliant***. **Hints have been provided to difficult and tricky questions** so that the student does not get stuck up and is able to maintain his pace.
8. Revision exercises containing **multi-choice questions** will, we hope expose the students to a variety of problems, requiring intelligent approach and help them in acquainting themselves with the latest trends and getting firm grasp of the fundamentals and thorough knowledge of different topics.
9. A sincere effort has been made to maintain **Mathematical accuracy and rigour.**
10. Historical notes have been interspersed throughout the text.

For proper feedback as per the requirement of the BHSEC syllabus, the authors are thankful to Mr. Karma Yeshey and Mr. Geewanath Sharma, Curriculum Officers, CAPSD, Ministry of Education, Bhutan.

Feedback and suggestions for further improvement would be most welcome.

October, 2009 **AUTHORS**

SYLLABUS FOR CLASS XI (2010 Onwards)

The syllabus for the Pure Mathematics course and the Business Mathematics course are as given below. It will be noticed that while the students taking the Business Mathematics will study comparatively less content under certain units like Trigonometry, Calculus and Coordinate Geometry, they will study an additional unit called Commercial Mathematics. The commonalties and the differences of the contents between the Pure Mathematics and Business Mathematics are clearly indicated below. A good estimate of the expected times that should be spent in the formal teaching of each topic is given in hours with the topics.

	UNIT 1 – ALGEBRA	**Pure Mathematics**	**Business Mathematics**
1	**Sequence and Series** **(10 hrs)** ▪ AP, GP: Their meanings and finding the nth term (T_n) and the sum of the series (S_n); Insertion of arithmetic and geometric means between two numbers; sum to infinity of GP ($\lvert r \rvert < 1$); ▪ Special sums, *i.e.*, Σn, Σn^2, n^3, $n \in N$; Explain the meaning and use of Σ (summation notation) ▪ Problems involving the above sequences	All	All
2	**Binomial Theorem** **(7 hrs)** ▪ Binomial expansion for positive integral indices; use of Pascal's triangle; and the binomial theorem, *i.e.*, $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \dots {}^nC_ny^n$ ▪ Meaning of ${}^nC_r$ ▪ Binomial theorem for the expansion of binomial expressions having negative or fractional indices ▪ Finding the general term of the expansions ▪ Application of the theorem for approximation, *e.g.*, $(0.99)^8 = (1 - 0.01)^8$	All	All
3	**Logarithms** **(5 hrs)** ▪ Revise the laws of Exponents taught in class IX ▪ Relationship between Logarithmic and Exponential expressions ▪ Laws of Logarithm and their properties including the change of base	All	All
4	**Remainder and Factor Theorem** **(5 hrs)** ▪ Meaning of Rational Integral Function ▪ Remainder Theorem ▪ Factor Theorem ▪ Factorization of cubic and quadratic polynomials	All	All
5	**Quadratic Equations and Functions** **(15 hrs)** ▪ Solution of Quadratic equations by factorization and use of their graphs/sketches	All	All

	■ Solution of Quadratic equations by the Formula method ■ Nature of roots - Real roots, Complex roots, Equal roots ■ Introduction to the concept of imaginary and complex numbers through the square root of –1 ■ Sum and Product of roots ■ Forming quadratic equations with given roots and related data ■ Graph of quadratic function $ax^2 + bx + c$; Sign of quadratics function; Maximum and minimum value of quadratic functions ■ Quadratic inequalities: Solution of quadratic inequalities; Use of graphs and number lines should be employed		
6	**Partial Fractions** **(10 hrs)** ■ Rational functions of the form $f(x) / g(x)$, where $f(x)$ and $g(x)$ are polynomial functions in x Case I – degree of numerator < degree of denominator Type 1 – Non repeated linear factors Type 2 – Repeated linear factor Type 3 – Quadratic factors (may not be factorizable) Case II – degree of numerator ≥ degree of denominator Type 1 – Non repeated linear factor Type 2 – Repeated linear factor	All	All
	UNIT 2 – TRIGONOMETRY	**Pure Mathematics**	**Business Mathematics**
1	**Angles and Arc lengths** **(3 hrs)** ■ Angles: Convention of signs of angles; Magnitude of an angle; Measures of angles; Circular measures ■ The relation $S = r\theta$, where θ is in radians; Relation between radians and degrees ■ Arc length and area of a sector of a circle	All	All
2	**Trigonometric functions** **(7 hrs)** ■ Trigonometric ratios; Relationship between trigonometric ratios ■ Proving simple trigonometric identities ■ Signs of trigonometric ratios ■ Limits of trigonometric ratios ■ Trigonometric ratios of standard angles ■ Trigonometric ratios of allied angles ■ Periods of trigonometric functions ■ Graphs of simple trigonometric functions (only sketches); Students should be exposed to Computer Generated Graphs through the use of computers	All	All

3	**Compound and Multiple Angles** **(7 hrs)** ▪ Addition and Subtraction formulas: sin $(A \pm B)$; cos $(A \pm B)$; tan $(A \pm B)$; tan $(A + B + C)$; etc., Double angle, triple angle, half angle and one third angle formula as special cases ▪ Sums and differences as products: *e.g.*, sin C + sin D = 2 sin$\{(C + D)/2\}$ cos$\{(C - D)/2\}$ ▪ Product to sums or differences: *e.g.*, 2 sin A cos B = sin $(A + B)$ + sin $(A - B)$ etc. ▪ Conditional identities (involving angles of triangles)	All	All
4	**Trigonometric Equations** **(4 hrs)** ▪ Solutions of trigonometric equations (General solution and solution in specified range) ▪ Type 1: Equations in which only one function of a single angle is involved, *e.g.*, sin $5\theta = 0$ ▪ Type 2: Equations expressible in terms of one trigonometric ratio of the unknown angle ▪ Type 3: Equations involving multiple and sub-multiple angles ▪ Equations involving compound angles ▪ Linear equations of the form $a\cos\theta + b\sin\theta = c$, where $\lvert c\rvert \leq (a^2 + b^2)^{1/2}$ and $a, b \neq 0$	All	This chapter is NOT for Business Mathematics
5	**Properties of Triangles** **(4 hrs)** ▪ Sine Rule (including ambiguous case for triangles) ▪ Cosine Rule ▪ Projection formula ▪ Napier's Formula for the area of a triangle (Proof and use)	All	This chapter is NOT for Business Mathematics
6	**Heights and Distances** **(3 hrs)** ▪ Practical problems based on angle of elevation and depression (in 2-D)	All	This chapter is NOT for Business Mathematics
	UNIT 3 – CALCULUS	**Pure Mathematics**	**Business Mathematics**
1	**Functions** **(5 hrs)** Concept of real valued functions; Domain and Range; Inverse functions; Classification of functions; Sketch of graphs of exponential functions, logarithmic functions, step functions, and simple trigonometric function like sin x cos x and tan x	All	All
2	**Limits** **(5 hrs)** Notion and meaning of limits; Fundamental theorems on limits; Limits of algebraic and trigonometric functions	All	All

3	**Continuity** **(5 hrs)** Continuity of a function at a point $x = a$; Continuity of a function in a range	All	All
4	**Differentiation** **(15 hrs)** ▪ Meaning and geometrical interpretation of derivatives; Differentiation from first principle; Derivative of simple algebraic and trigonometric functions and their formulae; Derivative of sums, differences, products and quotients of functions; ▪ Application of derivatives: Equation of tangent and normal; Approximation; Rate measure	All	All EXCEPT the portion of the Application of derivatives concerning *Approximation and Rate Measure*
5	**Integration** **(15 hrs)** ▪ Indefinite integral: integration as the inverse of differentiation; Anti-derivatives of polynomials and functions like $(ax + b)^n$, $\sin (x)$, $\cos (x)$, $\sec^2 (x)$, $\text{cosec}^2(x)$ ▪ Integration by simple substitution for simple polynomial functions and simple trigonometric functions	All	All
	UNIT 4 – COORDINATE GEOMETRY	**Pure Business**	**Mathematics Mathematics**
1	**Points and their Coordinates in 2-Dimensions(7 hrs)** ▪ Cartesian system of coordinates ▪ Distance formula ▪ Section formula ▪ Centroid of a triangle ▪ Incentre of a triangle ▪ Area of a triangle using its three vertices ▪ Area of a quadrilateral ▪ Slope or gradient of a line ▪ Angle between two lines ▪ Conditions of perpendicularity and parallelism of two lines	All	All
2	**The Straight Line** **(7 hrs)** ▪ Various forms of equation f lines: point slope form; two points form; intercept form; perpendicular/normal form; general equation of a line; slope/gradient; distance of a point from a line; distance between parallel lines; Angles between two lines; equations of lines bisecting the angle between the lines; Identical lines ▪ Family of lines: Lines parallel to $ax + by + c = 0$ are of the form $ay + bx + k = 0$; Lines perpendicular to $ax + by + c = 0$ are of the form $ay – bx + k = 0$; any	All	All EXCEPT the portion on the Family of lines *In other words, the portion on the Family of lines is EXCLUDED for B/Maths*

	line through the intersection of two lines L_1 and L_2 is of the form $L_1 + KL_2 = 0$, where $K \in R$		
3	**Locus and its Equation** **(4 hrs)** ▪ Definition of a locus and methods to find the equation of a locus; problems should be limited to fairly simple ones	All	This chapter is NOT for Business Mathematics
4	**Equations of Circles** **(6 hrs)** ▪ Equation of a circle in: Standard form; diameter form; general form; parametric form ▪ Given the equation of a circle, to find the centre and the radius ▪ Finding the equation of a circle, given 3 non-collinear points; and given other sufficient data	All	All
5	**Theorems on Circles** **(8 hrs)** ▪ Theorems on chords of a circle ▪ Theorems on arcs and angles ▪ Theorems on angles in alternate segment ▪ Theorems on congruent arc and chords ▪ Theorems on tangent lines and circles	All	All
	UNIT 5 – STATISTICS	**Pure Mathematics**	**Business Mathematics**
1	**Measures of Central Tendency** **(4 hrs)** ▪ Mean, Median, Mode; finding by direct methods, formulas, and graphs	All	All
2	**Dispersion** **(4 hrs)** ▪ Range: Quartiles, inter quartiles ▪ Standard deviation – by direct method, short cut method and step deviation method; the meaning of Standard deviation should be emphasized	All	All
	UNIT 6 – COMMERCIAL MATHEMATICS	**Pure Mathematics**	**Business Mathematics**
1	**Simple Interest and Compound Interest** **(5 hrs)** ▪ Meanings and methods of the interest calculations ▪ Problems involving the two types of interests	This is NOT for P/Maths	All
2	**Discount** **(5 hrs)** Trade discount; problems based on it, Present value, True discount, Bill of Exchange; banker's gain; days of grace; problems based on these.	This is NOT for P/Maths	All

MODE OF ASSESSMENT

There are two types of assessment, depending on what you do with them: Formative Assessment and Summative Assessment. Formative Assessment is observation to guide further instruction; and the observation is normally not measured, or its measurement is not recorded to grade the students. Summative Assessment is used to determine a mark or a grade. There are various ways provided to accomplish formative and summative assessment *(Please see the "Mathematics Curriculum Guide for Teachers, Class XI" produced by CAPSD.* The mode of assessment given here is for summative assessment of students in class XI. However, observations and analysis made on students' performance in these summative assessments could very well be used for further instructions. The Summative assessment in class XI will be done as per the following break-downs:

Mode of summative assessment
Class XI

Year beginning – Mid year 40%

- **Class Participation 5%**
- **Assignments 5%**
- **Units Tests 5%**
- **Mid-term Exams 25%**

Mid year – Year end 60%

- **Class Participation 5%**
- **Assignments 5%**
- **Units Tests 5%**
- **Annual Exams 45%**

A brief rationale on each of the components of the assessment above follows:

Year beginning to Mid year

Class Participation: Student's active involvement in the class is important for his/her learning. Class participation would consist of student's positive attitude and behaviors towards earning: his/her ability to follow instructions, cooperation displayed in doing group works, confidence in asking questions and answering the questions asked, etc to mention a few. Teacher should develop criteria to assess students for the class participation. A better alternative would be to work out the criteria with the students in the beginning of the year. It is important that the students know the criteria and are reminded of them from time to time. This would force the students to be active, cooperative, critical thinkers and confident communicators in the class. This would also force the teachers to drive students towards these qualities. These are desirable and healthy disposition we would want in our children. Whatever reasonable assessment tools and marking scheme the teacher has chosen to use for the class participation up to the mid term should be worked out to be worth 5% of the whole year assessment, for entering into the student progress report form.

Assignment: Reasonable amounts of assignment, which we normally call home works, should be assigned quite regularly. More importantly, they should be checked, and prompt feedback provided to the students on their works. The teacher will check at least two times each student's home works during the first half term of the year; they can devise their own marking scheme. The average mark from the total should be worked out to be worth 5% for entering onto the students' Progress Report Card.

Unit Tests: A unit test should be conducted at the end of teaching a unit. It should be carried out during one of the class periods. The teacher should keep proper record of the students' achievement in the series of unit tests. A minimum of two unit tests should be conducted before the mid term examination. The total marks obtained in the unit tests should be worked out to be worth 5% for entering into the student's Progress Report Card.

Mid-term Examination: The mid-term examination format may be based on the specifications provided for the annual examinations below. The mark obtained in it should be brought down to 25% for entering into the Progress Report Card.

Mid year to Year end

Class Participation: To be done similarly as during the first term of the year.

Assignments : To be done similarly as during the first term of the year.

Unit Tests: To be done similarly as during the first half term of the year, but with the units covered after the mid term examination.

Year End Examination: The annual examination paper will be set for 100 marks, with writing time of **Three hours**. The paper will consist of two sections:

- **Section A** will be composed of 15 multiple choice questions, covering the entire syllabus. Each MCQ should carry one Key/Correct Answer and three distracters. Each MCQ is worth 2 marks, making the section worth 30 marks in total.
- **Section B** will be made up of about 13 open answer type questions set from the entire syllabus, out of which students will attempt 10 questions. Each question will be set to carry 7 marks, making the section worth 70 marks in total.

NOTE:

1. For Pure Mathematics, the weighting accorded for each of the units for the annual examination is as given below:

	UNITS	MARKS %
1.	Algebra	30 %
2.	Trigonometry	20%
3.	Calculus	25%
4.	Coordinate Geometry	15%
5.	Statistics	10%
	Total	**100%**

2. For the Business Mathematics the weighting accorded for each of the units for the annual examination is as given below:

	UNITS	MARKS %
1.	Algebra	25%
2.	Trigonometry	15%
3.	Calculus	25%
4.	Coordinate Geometry	15%
5.	Statistics	10%
6.	Commercial Mathematics	10%
	Total	**100%**

3. Care should also be taken in the preparation of questions having a balance of them requiring conceptual understanding, problem solving, communication, reasoning, and applications of procedural knowledge and skills. Some questions should cross strands or units. Along with these, test blue print based on Blooms Taxonomy would also be needed to be used in the preparation of the paper.
4. The marks obtained out of 100 in this examination should be worked out to be worth 45% for entering into the student's progress report card.

CONTENTS

UNIT 1
ALGEBRA

- Sequence and Series
- Binomial Theorem
- Logarithms
- Remainder and Factor Theorem
- Quadratic Equations
- Partial Fractions

1

Sequences and Series

1.01. Sequence

A sequence may be thought of as a set of numbers specified in a definite order by some assigned rule or law.

Thus, the set of numbers 5, 9, 13, 17, 21...... in which each succeeding number is obtained on adding 4 to the preceding number forms a sequence.

A sequence such as 2, 4, 8, 16, 32, 64 which has a last term is called a **finite sequence**. A sequence, such as 1, 4, 9, 16, 25, 36.... which has no last term is an **infinite sequence**.

1.02. *n*th term of a sequence

To refer to the terms of any sequence, we often choose some letter as a, or T and then attach to that letter a small numeral, or subscript, to denote a particular term of the sequence. For instance, we call the first term a_1 or T_1, the second term a_2 or T_2 and so on. In general a_n or T_n is the ***n*th term** of the sequence (also called the general term).

1.03. Series

An expression consisting of the term of a sequence, alternating with the symbol '+' is called a **series**.

For example, associated with the sequence $\frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \ldots\ldots$

We have the series $\frac{3}{5} + \frac{5}{7} + \frac{7}{9} + \ldots\ldots$

1.04. Progressions

A sequence is also called a progression. We now study two important types of sequences :

(1) The Arithmetic Progression, (2) The Geometric Progression,

1.05. Arithmetic Progression. Def. (A.P.)

A sequence is called an arithmetic progression (Abbrev. A.P.) *if its terms continually increase or decrease by the same number.* The fixed number by which the terms increase or decrease is called the **common difference**. The following are examples of sequences in **A.P** :

Sequences	*Common difference*
2,6,10,14,.....	4
10, 5,0,–5,–10,......	–5
$a, a + d, a + 2d, a + 3d, \ldots,$	d

The last sequence in the above examples is a standard A.P.

[**Test.** Subtract from each term, the preceding term. If the differences are equal, the progression is arithmetic.]

Thus three quantities, a, b, c will be in A.P. if $b - a = c - b$, *i.e.*, $\mathbf{2b = a + c}$.

Aid to memory : *Three quantities are in A.P. if twice the middle = sum of the extremes.*

1.06. *n*th term of an A.P.

Let us consider the arithmetic progression

$$a, a + d, a + 2d, a + 3d, ...$$

What is the coefficient of d in the 10th term, the 25th term, the 83rd term ?

It is obvious that the coefficient of d in any term is one less than the number indicating the position of the term, and so the nth term of an A.P. is $a + (n - 1) d$.

$$\boxed{T_n = a + (n - 1)\, d}$$

Aid to memory : First Term + (number of the terms – 1) common difference.

Note. 1. The formula $l = a + (n - 1)\, d$ contains four quantities l, a, n, d. Three quantities being given, the fourth can be found out. If only two quantities are given, two conditions of the problem should be given.

Note. 2. $d = T_2 - T_1 = T_3 - T_2 = ... = T_n - T_{n-1}$.

Ex. 1. ***The nth term of an A.P. is 4n – 1. Write down the first 4 terms and the 18th term of the A.P.***

Sol. $T_n = 4n - 1$.

Putting $n = 1, 2, 3, 4$ and 18, we get

$T_1 = 4 \times 1 - 1 = 3,\ T_2 = 4 \times 2 - 1 = 7,\ T_3 = 4 \times 3 - 1 = 11.$

$T_4 = 4 \times 4 - 1 = 15,\ T_{18} = 4 \times 18 - 1 = 71.$

Hence, the first four terms of the A.P. are 3, 7, 11, 15 and the 18th term is 71.

Ex. 2. ***Find the 9th and pth terms of the A.P. 2, 5, 8.....***

Sol. Here $a = 2$ $\therefore$ 9th term $= 2 + (9 - 1) \times 3 = 26$

$d = 3$ pth term $= 2 + (p - 1) \times 3 = 3p - 1$.

Ex. 3. ***Which term of the series 31 + 29 + 27 + ... is 3 ?***

Sol. Here l is given and n is to be determined ; $a = 31, d = -2$

If 3 be the nth term, then $3 = 31 + (n - 1)(-2) = 31 - 2n + 2$

$\therefore\ 2n = 30$ $\therefore\ n = 15$

Hence 3 is the 15th term.

Ex. 4. ***The 8th term of a series in A.P. is 23 and the 102nd term is 305. Find the series.***

Sol. Let a be the first term and d the common difference.

Then, $T_8 = a + 7d = 23$ ...(1)

$T_{102} = a + 101d = 305$ ....(2)

Solving (1) and (2), $a = 2, d = 3$.

Hence, the required series is 2 + 5 + 8 + 11 + ...

Ex. 5. ***In a certain A.P. the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term.*** **[S.C.]**

Sol. Let a be the first term and d the common difference. Then,

By the given condition, $a + 23d = 2(a + 9d)$ $\therefore\ a = 5d$ ...(1)

Now, 72nd term $= a + 71d = 5d + 71d = 76d$ [From (1)] ...(2)

34th term $= a + 33d = 5d + 33d = 38d$ [From (l)] ....(3)

Obviously, (2) is twice (3). Hence proved.

Ex. 6. ***If p times the pth term of an A.P. is equal to q times the qth term, prove that (p + q)th term is zero.***

Sol. Let a be the first term and d the common difference. It is given that $p.t_p = q.t_q$.

i.e., $p\,[a + (p - 1)\,d] = q\,[a + (q - 1)\,d]$

$\Rightarrow \quad (p - q)\,a = (q^2 - q - p^2 + p)\,d \Rightarrow -(q - p)\,a = (q - p)\,[(q + p) - 1]\,d$

$\Rightarrow \quad -a = [(q + p) - 1]\,d \Rightarrow a + [(q + p) - 1]\,d = 0 \;\Rightarrow\; t_{p+q} = 0.$ $(\because\ q - p \neq 0)$

Ex. 7. ***If a, b and c be respectively the pth, qth and rth terms of an A.P., prove that***

$$\boldsymbol{a\,(q - r) + b\,(r - p) + c\,(p - q) = 0.}$$

Sol. Let A be the first term and D the common difference.

Since $t_p = a$, therefore, $A + (p - 1)\ D = a$ ...(1)

Since $t_q = b$, therefore, $A + (q - 1)\,D = b$ ...(2)

Since $t_r = c$, therefore, $A + (r - 1)\ D = c$ ...(3)

Multiplying (1) by $(q - r)$, (2) by $(r - p)$ and (3) by $(p - q)$, and adding

$A\,[(q - r) + (r - p) + (p - q)] + [(p - 1)\,(q - r) + (q - 1)\,(r - p) + (r - 1)\,(p - q)]\,D$

$= a\,(q - r) + b\,(r - p) + c\,(p - q)$

$\Rightarrow \quad A\,(0) + (0)\,D = a\,(q - r) + b\,(r - p) + c\,(p - q)$. Hence, $a\,(q - r) + b\,(r - p) + c\,(p - q) = 0.$

EXERCISE 1(*a*)

1. Write the first six terms of an A.P. in which

(*i*) $a = 5, d = 4$; (*ii*) $a = 98, d = -3$; (*iii*) $a = 7\frac{1}{2}, d = 1\frac{1}{2}$; (*iv*) $a = x, d = 3x + 2$.

2. Write the 5th and 8th terms of an A.P. whose 10th term is 43 and the common difference is 4.

3. In each of the following find the terms required

(*a*) The seventh term of 2, 7, 12,..... (*b*) The fifth term of 21, 28, 35, ...

(*c*) The eighteenth term of 9, 5, 1,.....

4. Find the first four terms and the eleventh term of the series whose *n*th term is

(*a*) $4n - 2$, (*b*) $6n + 5$, (*c*) $101 - 3n$.

5. The 5th term of an A.P. is 11 and the 9th term is 7. Find the 16th term.

6. Which term of the series 5, 8, 11,...... is 320 ?

7. The fourth term of an A.P. is ten times the first. Prove that the sixth term is four times as great as the second term. **[S.C.]**

8. The fourth term of an A.P. is equal to 3 times the first term, and the seventh term exceeds twice the third term by 1. Find the first term and the common difference. **[S.C.]**

9. Prove that the product of the 2nd and 3rd terms of an arithmetic progression exceeds the product of the 1st and 4th by twice the square of the difference between the 1st and 2nd.

10. (*i*) Which term of the progression 19, $18\frac{1}{5}$, $17\frac{2}{5}$, ... is the first negative term ?

* (*ii*) What is the smallest number of terms which must be taken for their sum to be negative ? Calculate the sum exactly. **[S.C.]**

11. (*i*) Find the value of k so that $8k + 4$, $6k - 2$, and $2k + 7$ will form an A.P.

(*ii*) Find a, b such that 7.2, a, b, 3 are in A.P. **[S.C.]**

* This part (*ii*) may be done after Art. 1.08.

12. Given that the $(p + l)$th term of an arithmetic progression is twice the $(q + 1)$th term, prove that the $(3p + l)$th term is twice the $(p + q + 1)$th term. **[S.C.]**

13. Determine 2nd term and rth term of an A.P. whose 6th term is 12 and 8th term is 22. **[S.C.]**

14. The 2nd, 31st and last term of an A.P. are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. Find the first term and the number of terms.

15. If 7 times the 7th term of an A.P. is equal to 11 times its 11th term, show that the 18th term of the A.P. is zero.

16. Determine k so that $k + 2$, $4k - 6$ and $3k - 2$ are three consecutive terms of an A.P. **[I.S.C.]**

17. The pth term of an A.P. is q and the qth term is p, show that the mth term is $p + q - m$.

18. Let T_r be the rth term of an A.P. whose first term is a and common difference is d. If for some positive integers m, n, $T_m = \frac{1}{n}$, $T_n = \frac{1}{m}$, then $(a - d)$ equals (1) $1/mn$ (2) 1 (3) 0 (4) $\frac{1}{m} + \frac{1}{n}$.
[AIEEE 2004, U.P. SEE 2007]

ANSWERS

1. (*i*) 5, 9, 13, 17, 21, 25; (*ii*) 98, 95, 92, 89, 86, 83; (*iii*) $7\frac{1}{2}, 9, 10\frac{1}{2}, 12, 13\frac{1}{2}, 15$; (*iv*) $x, 4x+2, 7x+4, 10x+6, 13x+8, 16x+10$ **2.** 23, 35 **3.** (*a*) 32, (*b*) 49, (*c*) –59
4. (*a*) 2, 6, 10, 14, 42, (*b*) 11,17, 23, 29, 71. (*c*) 98, 95, 92, 89; 68
5. 0 **6.** 106 **8.** $a = 3,\ d = 2$ **10.** (*i*) 25th (*ii*) 49, –9.8
11. (*i*) $7\frac{1}{2}$ (*ii*) $a = 5.8$, $b = 4.4$ **13.** $-8,\ 5r - 18$ **14.** 8, 59 **16.** 3. **18.** (3)

1.07. Sum of a stated number of terms of an Arithmetic Series (A.S.)

More than 200 years ago, a class of German school children was asked to find the sum of all the integers from 1 to 100 inclusive. One boy in the class, an eight-year-old named *Carl Fredrick Gauss* (1777–1855), who later established his reputation as one of the greatest mathematicians, announced the answer almost at once. The teacher overawed at this asked Gauss to explain how he got this answer. Gauss explained that he had added these numbers in pairs as follows:

$$1 + 100 = 101,\ 2 + 99 = 101,\ 3 + 98 = 101, \text{ and so forth.}$$

These are 50 equal pairs. The answer can be obtained by multiplying 101 by 50 to get 5050.

We hope you have understood the method applied by Gauss. You may write the integers in two ways as under:

$$S = 1 + 2 + 3 + \ldots\ldots + 98 + 99 + 100$$

and $$S = 100 + 99 + 98 + \ldots\ldots + 3 + 2 + 1$$

$\therefore$ $$2S = 101 + 101 + 101 + \ldots\ldots. + 101 + 101 + 101 \text{ (Adding)}$$

$\Rightarrow$ $$2S = 101 \times 100 \Rightarrow S = \frac{101 \times 100}{2} \Rightarrow S = 101 \times 50 = 5050.$$

The argument depends on the fact that in the series

$1 + 2 + 3 + \ldots.. + 98 + 99 + 100$, the terms can be paired so that the terms equidistant from the ends of the series have the same sum.

1.08. Sum of *n* terms of an A.P.

Let the A.P. be $a, a + d, a + 2d, \ldots\ldots$ Let l be the last term, and S the required sum.

Then, $$S = a + (a + d) + (a + 2d) + \ldots + (l - d) + l$$

Reversing the right hand terms

$$S = l + (l-d) + (l-2d) + \ldots + (a+d) + a.$$

Adding,

$$2S = (a + l) + (a + l) + \ldots \text{ to } n \text{ terms} = n(a + l)$$

$\therefore$ $$\boxed{S = \frac{n}{2}(a+l)} \quad \ldots(1)$$

If we substitute the value of l *viz.*, $l = a + (n-1)d$, in this formula, we get

$$S = \frac{n}{2}[a + a + (n-1)d], \text{ i.e., } \boxed{S = \frac{n}{2}[2a + (n-1)d]}$$

Note. 1. Each of these formulas contains four quantities. Three being known, the fourth can be found out.

Note 2. If the sum of n terms be a function of n, the sum of $(n-1)$ terms can be found on putting $(n-1)$ for n. Thus if $S_n = 3n^2 + 2n \Rightarrow S_{n-1} = 3(n-1)^2 + 2(n-1)$.

Note 3. Also, from (2) above, we have

$$S_n - S_{n-1} = \frac{n}{2}[2a+(n-1)d] - \frac{n-1}{2}[2a+(n-1-1)d]$$

$$= a[n-(n-1)] + \frac{(n-1)}{2}d[n-(n-2)]$$

$$= a + \frac{(n-1)}{2}d \times 2 = a + (n-1)d = n\text{th term}$$

Hence, $\boxed{n\text{th term} = S_n - S_{n-1}}$

Ex. 8. ***Find the sum of (i) 20 terms, (ii) n terms of the progression 1, 3, 5, 7, 9.....***

Sol. Here, $a = 1, d = 2$

(*i*) $S_{20} = \dfrac{20}{2}\{2 \times 1 + (20-1) \times 2\} = 10 \times 40 = 400$

(*ii*) $S_n = \dfrac{n}{2}\{2a + (n-1)d\} = \dfrac{n}{2}\{2 \times 1 + (n-1)2\} = n^2.$

Ex. 9. ***Find the sum of the series 72 + 70 + 68 + ...+ 40.***

Sol. Here the last term is given. We will first have to find the number of terms.

$a = 72$ $\qquad l = a + (n-1)d$

$d = -2 \quad \Rightarrow \quad 40 = 72 + (n-1)(-2) \qquad \therefore \quad n = 17.$

$l = 40$

$\therefore \quad S_{17} = \dfrac{17}{2}(72 + 40) = 952.$ $\qquad$ [Using $S_n = \dfrac{n}{2}(a+l)$]

Ex. 10. ***The sum of n terms of an A.P. is $4n^2 + 5n$. Find the series.***

Sol. $S_1 = 4 \times 1^2 + 5 \times 1 = 9, S_2 = 4 \times 2^2 + 5 \times 2 = 26, S_3 = 4 \times 3^2 + 5 \times 3 = 51, \ldots$

$\therefore \quad T_1 = S_1 = 9, T_2 = S_2 - S_1 = 26 - 9 = 17, T_3 = S_3 - S_2 = 51 - 26 = 25$

Hence the series is 9 + 17 + 25 + ...

Alternatively, $T_n = S_n - S_{n-1} = (4n^2 + 5n) - [4(n-1)^2 + 5(n-1)] = 8n + 1.$

Now, put $n = 1, 2, 3, \ldots\ldots$

Ex. 11. ***How many terms of the A.P. 24, 20, 16..... must be taken so that the sum may be 72 ? Explain the double answer.***

Sol. Let n be the number of the terms. Then

$$72 = \frac{n}{2}\{2 \times 24 + (n-1)(-4)\} = n(26 - 2n) = 26n - 2n^2$$

$\therefore \quad n^2 - 13n + 36 = 0 \quad \Rightarrow \quad (n-4)(n-9) = 0 \;\; \therefore \;\; n = 4, 9.$

The double answer shows that there are 2 sets of numbers satisfying the condition of the problem. Writing the series up to 9 terms, we have

$$24 + 20 + 16 + 12 + 8 + 4 + 0 - 4 - 8$$

As the last 5 terms cancel out, the sum of 4 terms is the same as that of 9 terms.

Ex. 12. ***On each birthday, John received from his father Rs 5 for each year of his age. Find the total sum John had received by the time he was 21 years old.*** **[S.C]**

Sol. Here $a = 5, d = 5$ and $n = 21$; $S = \frac{21}{2}\{10 + 20 \times 5\} = 55 \times 21 = \text{Rs } 1155.$

Ex. 13. ***The sums of n terms of two arithmetic series are in the ratio of 2n + 1: 2n – 1. Find the ratio of their 10th terms.***

Sol. Let the two A.S. be $a, a + d, a + 2d$....... and $A, A + D, A + 2D$......

It is given that $\dfrac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}.[2A+(n-1)D]} = \dfrac{2n+1}{2n-1} \Rightarrow \dfrac{2a+(n-1)d}{2A+(n-1)D} = \dfrac{2n+1}{2n-1}$ ...(1)

We have to find the ratio $\dfrac{t_{10}}{T_{10}} = \dfrac{a+9d}{A+9D} = \dfrac{2a+18d}{2A+18D}$.

Putting $n = 19$ (1), we get $\dfrac{t_{10}}{T_{10}} = \dfrac{2a+18d}{2A+18D} = \dfrac{2 \times 19 + 1}{2 \times 19 - 1} = \mathbf{\dfrac{39}{37}}$.

Ex. 14. ***The sums of first p, q, r terms of an A.P. are a, b, c respectively. Prove that***

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$$

Sol. Let A be the first term and D the common difference. It is given that $S_p = a$

i.e., $\dfrac{p}{2}[2A + (p-1)D] = a \quad \Rightarrow \quad A + \dfrac{p-1}{2}D = \dfrac{a}{p}$ ...(1)

Similarly, $A + \dfrac{q-1}{2}D = \dfrac{b}{q}$ ...(2) and $A + \dfrac{r-1}{2}D = \dfrac{c}{r}$ ...(3)

Multiplying (1) by $(q - r)$, (2) by $(r - p)$, (3) by $(p - q)$, and adding, we get

$$A\,(q + r + r - p + p - q) + \frac{D}{2}[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$$

$$= \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q)$$

$$\Rightarrow A(0) + \frac{D}{2}(0) = \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q)$$

$$\Rightarrow \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$$

Practical Problems

Ex. 15. ***A man saved Rs 16,500 in 10 years. In each year after the first he saved Rs 100 more than he did in the preceding year. How much did he save in the first year?***

Sol. The yearly savings form an A.P. whose common difference is 100 and the sum of whose 10 terms is Rs 16,500. If a denotes in rupees the saving of the first year, then

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \Rightarrow \quad 16,500 = \frac{10}{2}[2a + 9 \times 100]$$

$$\Rightarrow \quad 16,500 = 10a + 4,500 \quad \Rightarrow \quad 10a = 12,000 \quad \Rightarrow \quad a = 1,200$$

$\therefore$ The first year's saving = **Rs 1,200.**

Ex. 16. ***80 coins are placed in a st. line on the ground. The distance between any two consecutive coins is 10 metres. How far must a person travel to bring them one by one to a basket placed 10 metres behind the first coin?***

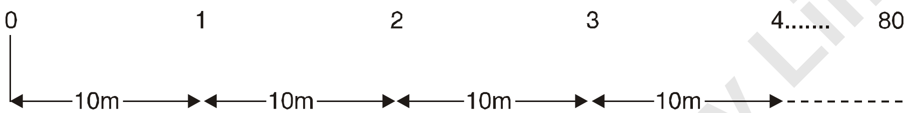


Fig. 1.01

Sol. Let 1, 2, 3,... 80 represent the positions of the coins and 0 that of the basket.

The distance covered in bringing the first coin = 10 + 10 = 20

The distance covered in bringing the second coin = 20 + 20 = 40

The distance covered in bringing the third coin = 30 + 30 = 60 and so on

$$\therefore \text{ The total distance covered } = \frac{n}{2}[2a + (n-1)\,d] = \frac{80}{2}[2 \times 20 + (80-1)\,.\,20]$$

$$= 40\,(40 + 1580) = 40 \times 1620 = \mathbf{64,800 \text{ metres.}}$$

EXERCISE 1 (*b*)

1. Find the sum of :
   (*i*) 10 terms of 5 + 8 + 11 +.....;
   (*ii*) 18 terms of 57 + 49 + 41 + ...;
   (*iii*) n terms of 4 + 7 + 10 + ... ;
   (*iv*) 24 terms and n terms of $2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{6}, 5, ...;$
   (*v*) 101 + 99 + 97 + . . . + 47.
2. Find the sum of all the numbers between 100 and 200 which are divisible by 7. **[S.C.]**
3. The sum of a series of terms in A.P. is 128. If the first term is 2 and the last term is 14, find the common difference.
4. The sum of 30 terms of a series in A.P., whose last term is 98, is 1635. Find the first term and the common difference.
5. If the sums of the first 8 and 19 terms of an A.P. are 64 and 361 respectively, find (*i*) the common difference and (*ii*) the sum of n terms of the series.
6. The sum of terms of an A.P. is 136, the common difference 4, and the last term 31, find n.
7. Find the number of terms of the series 21, 18, 15, 12...which must be taken to give a sum of zero. **[S.C.]**
8. The sum of n terms of a series is $(n^2 + 2n)$ for all values of n. Find the first 3 terms of the series. **[S.C.]**

9. Find the smallest number of terms which may be taken in order that the sum of the arithmetical series 325 + 350 + 375 + ... may exceed 10000. **[S.C.]**

10. The third term of an arithmetical progression is 7, and the seventh term is 2 more than 3 times the third term. Find the first term, the common difference and the sum of the first 20 terms. **[I.S.C.]**

11. The interior angles of a polygon are in arithmetic progression. The smallest angle is 52° and the common difference is 8°. Find the number of sides of the polygon. **[*I.S.C. 1990*]**

[Hint. $\frac{n}{2}\,|\,[2 \times 52^o + (n-1)\,8^o] = (n-2) \times 180^o$

$\Rightarrow$ $n = 30$ or $n = 3$. But when $n = 30$, the last angle is $52^o + (30-1)8^o = 284°$

which is not possible, since no interior angle of a polygon is more than 180°

$\therefore$ Number of sides of the given polygon is 3.]

12. Determine the sum of first 35 terms of an A.P. if $t_2 = 1$ and $t_7 = 22$.

13. Find the sum of all natural numbers between 100 and 1000 which are multiples of 5.

14. How many terms of the A.P. 1,4, 7,.... are needed to give the sum 715? **(I.S.C.)**

15. Find the rth term of an A.P., sum of whose first n terms is $2n + 3n^2$.

16. The first term of an A.P. is a, the second term is b and the last term is c. Show that the sum of A.P. is $\dfrac{(b+c-2a)(c+a)}{2\,(b-a)}$.

17. In an arithmetical progression, the sum of p terms is m and the sum of q terms is also m. Find the sum of $(p + q)$ terms. **[I.S.C.]**

18. The sum of the first fifteen terms of an arithmetical progression is 105 and the sum of the next fifteen terms is 780. Find the first three terms of the arithmetical progression. **[I.S.C.]**

19. The sum of the first six terms of an arithmetic progression is 42. The ratio of the 10th term to the 30th term of the A.P. is $\frac{1}{3}$. Calculate the first term and the 13th term. **[I.S.C.]**

20. A sum of Rs 6240 is paid off in 30 instalments, such that each instalment is Rs 10 more than the preceding instalment. Calculate the value of the first instalment. **[I.S.C.]**

21. The nth term of an A.P. is p and the sum of the first n term is s. Prove that the first term is $\dfrac{2s - pn}{n}$. **[S.C.]**

22. The sum of the first n terms of the arithmetical progression 3, $5\frac{1}{2}$, 8...... is equal to the 2nth term of the arithmetical progression $16\frac{1}{2}$, $28\frac{1}{2}$, $40\frac{1}{2}$. Calculate the value of n. **[I.S.C.]**

23. If the sum of the first 4 terms of an arithmetic progression is p, the sum of the first 8 terms is q and the sum of the first 12 terms is r, express $3p + r$ in terms of q. **[I.S.C.]**

24. The last term of an A.P. 2, 5, 8, 11, ... is x. The sum of the terms of the A.P. is 155. Find the value of x. **[*I.S.C. 1991*]**

25. A gentleman buys every year Banks' certificates of value exceeding the last year's purchase by Rs 25. After 20 years he finds that the total value of the certificates purchased by him is Rs 7,250. Find the value of the certificates purchased by him (i) in the 1st year (ii) in the 13th year.

26. If the sums of the first n terms of two A.P.'s are in the ratio $7n - 5 : 5n + 17$; show that the 6th terms of the two series are equal.

27. (*i*) If the ratio of the sum of m terms and n terms of an A.P. be $m^2 : n^2$, prove that the ratio of its mth and nth terms is $(2m - 1) : (2n - 1)$.

[Hint. $$\frac{S_m}{S_n} = \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2} \Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n}$$

$$\Rightarrow n[2a + (m-1)\, d] = m[2a + (n-1)\, d]$$

Solving, we get $d = 2a$. Substitute in $\dfrac{a+(m-1)d}{a+(n-1)d}$.]

(*ii*) Let $a_1, a_2, a_3, \ldots\ldots$ be terms of an A.P. If $\dfrac{a_1 + a_2 + \ldots.. + a_p}{a_1 + a_2 + \ldots. + a_q} = \dfrac{p^2}{q^2}$, $(p \neq q)$ then find $\dfrac{a_6}{a_{21}}$.

[*AIEEE 2006*]

28. If the sums of n, $2n$, $3n$ terms of an A.S are S_1, S_2, S_3 respectively, prove that $S_3 = 3\,(S_2 - S_1)$.

29. If the sum of p terms of an A.S is q and the sum of q terms is p, show that the sum of $(p + q)$ terms is $-(p + q)$.

30. The ratio between the sum of n terms of two A.P.'s is $(7n + 1) : (4n + 27)$. Find the ratio of their 11th terms.

ANSWERS

1. (*i*) 185, (*ii*) -198 (*iii*) $\dfrac{n}{2}(3n+5)$, (*iv*) 290, $\dfrac{5n(n+5)}{12}$ (*v*) 2072

2. 2107 **3.** $\dfrac{4}{5}$ **4.** 11, 3 **5.** 2, n^2. **6.** 8.

7. 15 **8.** 3, 5, 7 **9.** 19 **10.** -1, 4, 740 **11.** 3

12. 2387 **13.** 98450 **14.** 22 **15.** $6r - 1$ **17.** 0

18. $-14, -11, -8$ **19.** 2, 26 **20.** Rs 63. **22.** $n = 18$

23. $3p + r = 3q$. **24.** $x = 29$ **25.** (*i*) Rs. 125 (*ii*) Rs. 425 **27.** (*ii*) $\dfrac{11}{41}$ **30.** 148 : 111

1.09. Arithmetic Mean

The Arithmetic Mean between two numbers is the number which when placed between them forms an arithmetic progression with them. Thus, if a and b are the given numbers and x their arithmetic mean, then a, x, b *are* in A.P.

$$x - a = b - x \Rightarrow 2x = a + b \Rightarrow x = \frac{a+b}{2}$$

Hence, the **arithmetic mean between two numbers is one-half of their sum.**

If we have to find many arithmetic means between two numbers, we can do so by first finding the common difference.

Ex. 17. ***Find the arithmetic mean between 5 and 9.***

Sol. Reqd. A.M. $= \dfrac{5+9}{2} = 7.$

Special Types of Questions

Ex. 18. ***The sum of three numbers in A.P. is 51 and the product of their extremes is 273. Find the numbers.***

Sol. Let the numbers be $a - d, a, a + d$.

Then, $(a-d) + a + (a+d) = 51 \quad \Rightarrow \quad a = 17$

and $(a-d)(a+d) = 273 \quad \Rightarrow \quad 17^2 - d^2 = 273 \quad \Rightarrow \quad d = \pm 4.$

Hence, the numbers are 13, 17, 21.

Ex. 19. ***The sum of four numbers in A.P. is 28 and the sum of their squares is 216. Find the numbers.***

Sol. Let the numbers be $a - 3d, a - d, a + d, a + 3d$

Then $(a-3d) + (a-d) + (a+d) + (a+3d) = 28 \quad \therefore a = 7$

and $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 216$

$\Rightarrow 2(a^2 + 9d^2) + 2(a^2 + d^2) = 216 \quad \Rightarrow \quad a^2 + 5d^2 = 54 \quad \Rightarrow \quad 5d^2 = 54 - 49 = 5 \quad \therefore d = \pm 1.$

Hence the numbers are 4, 6, 8 and 10.

> **Important Note.** If we have to find an odd number of terms in an A.P., whose sum is given, it is convenient to take the middle term, a and to take d as the common difference. Thus, three terms are taken as $\boldsymbol{a - d, a, a + d,}$ and five terms are taken as $\boldsymbol{a - 2d, a - d, a, a + d, a + 2d}$.
> If we have to find even number of terms, we take $a - d, a + d$ as the middle terms and $2d$ as the common difference.
> Thus, four terms are taken as $a - 3d, a - d, a + d, a + 3d$.

1.10. Useful results

I. Let $a, a + d, a + 2d, \ldots$ be any A.P. ...(1)

(*i*) On adding k to each of its terms, we get the sequence $a + k, a + d + k, a + 2d + k, \ldots$ which is an A.P. with first term $= a + k$, common difference $= d$.

(*ii*) Subtracting k from each term, the new sequence is $a - k, a + d - k, a + 2d - k, \ldots$ which is an A.P. with first term $= a - k$, common difference $= d$.

(*iii*) Multiplying each term by k, the new sequence is $ka, k(a + d), k(a + 2d), \ldots$ which is an A.P. with first term $= ka$, common difference $= kd$.

(*iv*) Dividing each term by k the new sequence is $\dfrac{a}{k}, \dfrac{a+d}{k}, \dfrac{a+2d}{k}, \ldots$ which is an A.P. with first term $= \dfrac{a}{k}$, common difference $= \dfrac{d}{k}$.

From the above, it is clear that

> ***If each term of a given A.P. is increased or decreased or multiplied or divided by the same number, then the resulting series is also an A.P.***

II. If a, b, c are in A.P., then by def.

$$b-a=c-b \Rightarrow \frac{b-a}{c-b}=1 \Rightarrow \frac{a-b}{b-c}=1$$

i.e., if three numbers are in A.P. then $\boxed{\dfrac{\textbf{First term – Second term}}{\textbf{Second term – Third term}} = \mathbf{1.}}$

Ex. 20. ***If a, b, c are in A.P. prove that the following are also in A.P.***

(i) $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ ***(ii)*** $b+c,\ c+a,\ a+b$

(iii) $a^2(b+c),\ b^2(c+a),\ c^2(a+b)$ ***(iv)*** $\dfrac{a(b+c)}{bc}, \dfrac{b(c+a)}{ca}, \dfrac{c(a+b)}{ab}$.

Sol. (*i*) a, b, c are in A.P.

Dividing each term by abc. $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in A.P.

(*ii*) a, b, c are in A.P.

Subtracting $a + b + c$ from each term

$\Rightarrow$ $a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are in A.P.

$\Rightarrow$ $-(b+c), -(c+a), -(a+b)$ are in A.P.

Multiplying each term by -1, $\Rightarrow b+c, c+a, a+b$ are in A.P.

(*iii*) a, b, c are in A.P.

Multiplying each term by $ab + bc + ca$

$\Rightarrow$ $a(ab+bc+ca), b(ab+bc+ca), c(ab+bc+ca)$ are in A.P.

$\Rightarrow$ $a(ab+ca)+abc, b(ab+bc)+abc, c(bc+ca)+abc$ are in A.P.

Subtracting abc from each term

$\Rightarrow$ $a(ab+ca), b(ab+bc), c(bc+ca)$ are in A.P.

i.e. $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

Second method

$a^2(b+c), b^2(c+a), c^2(a+b)$ will be in A.P.

if $a^2(b+c)+abc, b^2(c+a)+abc, c^2(a+b)+abc$ are in A.P. [adding abc to each term]

or if $a(ab+bc+ca), b(ab+bc+ca), c(ab+bc+ca)$ are in A.P.

or if a, b, c are in A.P. [Dividing each term by $ab+bc+ca$]

which is true. Hence, the result.

Third method

$a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

if $\dfrac{a^2(b+c)-b^2(c+a)}{b^2(c+a)-c^2(a+b)}=1$ or $\dfrac{a^2b+a^2c-b^2c-b^2a}{b^2c+b^2a-c^2a-c^2b}=1$

or $\dfrac{ab(a-b)+c(a^2-b^2)}{bc(b-c)+a(b^2-c^2)}=1$ or $\dfrac{(ab+bc+ca)(a-b)}{(ab+bc+ca)(b-c)}=1$

or $\dfrac{a-b}{b-c}=1$, which is true, because a, b, c are given to be in A.P.

$\therefore$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

(*iv*) a, b, c are in A.P.

Dividing each term by abc, $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

Multiplying each term by $ab + bc + ca$,

$\Rightarrow$ $$\frac{ab + bc + ca}{bc}, \frac{ab + bc + ca}{ca}, \frac{ab + bc + ca}{ab} \text{ are in A.P.}$$

$\Rightarrow$ $$\frac{a(b+c)+bc}{bc}, \frac{b(c+a)+ca}{ca}, \frac{c(a+b)+ab}{ab} \text{ are in A.P.}$$

Subtracting 1 from each term, $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are in A.P.

EXERCISE 1 (*c*)

1. Find the A.M. between :
   (*i*) 6 and 12 (*ii*) 5 and 22 (*iii*) $(\cos\theta + \sin\theta)^2$ and $(\cos\theta - \sin\theta)^2$
   (*iv*) $(x + y)^2$ and $(x - y)^2$.
2. Find two numbers whose product is 91 and whose A.M. is 10.
3. The sum of three numbers in A.P. is 33, and the sum of their squares is 461 . Find the numbers.
4. There are four numbers is A. P., the sum of the two extremes is 8, and the product of the middle two is 15. What are the numbers?
5. The sum of the first three terms of an A.P. is 36 while their product is 1620. Find the A.P.
6. The sum of the first four terms of an A.P. is 16 and the sum of their squares is 84. Find the terms.
7. The angles of a triangle are in A.P. If the greatest angle is double the least, find the angles.
8. The sum of the first three consecutive terms of an A.P. is 9 and the sum of their squares is 35. Find S_n.
9. The sum of five consecutive terms of an A.P. is 25 and the sum of their squares is 135. Find the terms.
10. The angles of a quadrilateral are in A.P. and the greatest is double the first. Find the circular measure of the least angle.
11. If a, b, c are in A.P., show that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are also in A.P.
12. If $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are A.P., show that

    (*i*) yz, zx, xy are in A.P. (*ii*) xy, zx, yz are in A.P. (*iii*) $\frac{y+z}{x}, \frac{z+x}{y}, \frac{x+y}{z}$ are in A.P.
13. If $(b + c)^{-1}, (c + a)^{-1}, (a + b)^{-1}$ are in A.P., then show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in A.P.
14. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

15. If x, y, z are in A.P., show that $(xy)^{-1}, (zx)^{-1}, (yz)^{-1}$ are also in A.P.

[Hint. x, y, z are in A.P. Reversing the order of the terms $\Rightarrow z, y, x$ are in A.P.

Dividing each term by xyz, $\frac{1}{xy}, \frac{1}{zx}, \frac{1}{yz}$ are in A. P.]

16. If a^2, b^2, c^2 are in A.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

ANSWERS

1. (*i*) 9	(*ii*) 13.5	(*iii*) 1,	(*iv*) $x^2 + y^2$
2. 7, 13	**3.** 4, 11, 18	**4.** 1, 3, 5, 7	**5.** 9, 12, 15 or 15, 12, 9
6. 1, 3, 5, 7, or 7, 5, 3, 1		**7.** 40°, 60°, 80°	
8. n^2 or $n(6-n)$	**9.** 3, 4, 5, 6, 7 or 7, 6, 5, 4, 3	**10.** $\pi/3$.	

GEOMETRIC SEQUENCES

1.11. What is a Geometric Sequence ?

Consider the following sequences :

(*a*) 2, 4, 8, 16........ (*b*) 2, 6, 18, 54, 162.......

(*c*) 10, 20, 40, 80, 160...... (*d*) 9, 6, 4, $\frac{8}{3}, \frac{16}{9}$ ....

These are not arithmetic sequences. What is the speciality of these sequences? Do you see that in each of them, the ratio of the second term to the first is equal to the ratio of the third to its predecessor, the ratio of the fourth term to its predecessor and so on. Such a sequence is called a **geometric sequence.**

Definition. ***A Geometric Sequence (G.S.) is one in which the ratio of any term to its predecessor is always the same number. This ratio is called the* common ratio.**

A geometric sequence is also called a **geometric progression (G .P.).**

The following are also examples of geometric sequences :

Sequence	***Common ratio***
1, 3, 9, 27, 81........	$r = 3$
16/27, –8/9, 4/3, –2........	$r = -3/2$
x, x^2, x^3, x^4	$r = x$

[Test: Divide each term by the preceding term, if the quotients are equal the sequence is geometric.]

If a denotes the first term and r the common ratio in a G.P., then the standard G.P. is,

$$a, ar, ar^2, ...$$

Ex. 21. ***The first three terms of a geometric sequence are 48, 24, 12. What are the common ratio and fourth term of this sequence?***

Sol. Step 1. *To find the common ratio, divide any term by its predecessor.*

$$24 \div 48 = \frac{1}{2} \quad \text{or} \quad 12 \div 24 = \frac{1}{2} \qquad \therefore \text{ The common ratio is } \frac{1}{2}.$$

Step 2. *To find the fourth term, multiply the third term by the common ratio.* $12 \times \frac{1}{2} = 6$

$\therefore$ *the fourth term is 6.*

Ex. 22. ***Prove that the function f defined by*** $f : n \to \frac{1}{3^n}$ ***where*** $n \in N$***, is a G.S.***

Sol. Putting $n = 1, 2, 3......$ the successive terms of the sequence are obtained as

$$\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ..., \frac{1}{3^n}, ...,$$

Since $\frac{f(n)}{f(n-1)} = \frac{\frac{1}{3^n}}{\frac{1}{3^{n-1}}} = \frac{1}{3}, \forall\, n \in N, n \geq 2$ therefore the given sequence is a G.S.

> $\forall$ means 'for all'

1.12. The *n*th term

The following table suggests an expression for t_n in terms of n, r, and a

1st term	2nd term	3rd term	4th term	...	nth term
t_1 a	t_2 ar	t_3 ar^2	t_4 ar^3	....	t_n ar^{n-1}

Observing that the exponent of r in each term is one less than the number of the term, you can make the following conjecture:

The nth term of a geometric sequence whose first term is a and common ratio is a non-zero number r is

$$\boxed{t_n = ar^{n-1}}$$

Ex. 23. ***The nth term of a geometric sequence is*** $2(1.5)^{n-1}$ ***for all values of n. Write down the value of a (the first term) and the common ratio (r).*** **[I.S.C.]**

Sol. $t_n = 2.\,(1.5)^{n-1}$ $\quad\therefore\ t_1 = 2.\,(1.5)^0 = 2$ (Putting $n = 1$)

$$t_2 = 2.(1.5)^{2-1} = 3 \quad \therefore \text{ Common ratio } r = \frac{t_2}{t_1} = \frac{3}{2} = 1.5.$$

Ex. 24. ***Find the (i) sixth term, (ii) nth term of the sequence 2, 6, 18,...***

Sol. Here, $a = 2, r = \frac{6}{2} = 3.$

$\therefore$ 6th term $T_6 = ar^5 = 2 \times 3^5 = 486$ and nth term $T_n = ar^{n-1} = 2 \times 3^{n-1}$.

Ex. 25. ***Write the G. P. whose 4th term is 54 and the 7th term is 1458.***

Sol. $t_4 = ar^3 = 54$ ...(1) $\quad t_7 = ar^6 = 1458$ ...(2)

Dividing (2) by (1) we get $r^3 = \frac{1458}{54} = 27 \Rightarrow r = 3 \therefore$ From (1), $27a = 54 \Rightarrow a = 2$.

Hence the G.P. is 2, 6, 18, 54,....

Ex. 26. ***Which term of the series 8 + 1.6 + 0.32 + ... is 0.00256 ?***

Sol. Here $a = 8$ and $r = \frac{1.6}{8} = \frac{1}{5}$.

Let 0.00256 be the nth term. Then,

$$0.00256 = ar^{n-1} = 8\left(\frac{1}{5}\right)^{n-1} \quad \therefore \quad \left(\frac{1}{5}\right)^{n-1} = 0.00032 = \frac{32}{100000} = \left(\frac{2}{10}\right)^5 = \left(\frac{1}{5}\right)^5$$

$\therefore$ $n - 1 = 5$ or $n = 6$

$\therefore$ 0.00256 is the 6th term of the series.

Ex. 27. ***If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P., then prove that $x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$.***

Sol. Since a, b, c are in A.P. therefore $2b = a + c$ and since x, y, z are in G.P. therefore $y = xr, z = xr^2$ where r is the common ratio.

$$x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = x^{b-c} \cdot (xr)^{c-a} \cdot (xr^2)^{a-b}$$
$$= x^{b-c+c-a+a-b} \cdot r^{c-a+2a-2b} = x^0 \cdot r^{c+a-2b}$$
$$= x^0 \cdot r^{2b-2b} = x^0 \cdot r^0 = 1.$$

1.13. Geometric Mean

The terms between two given numbers of a G.P. are called the **geometric means** between the given numbers.

For example, in the sequence 5, 15, 45, 135; 15, 45 are the two geometric means between 5 and 135.

When three simple real numbers form a geometric sequence, the middle one is called the geometric mean of the numbers.

For example, 5, 25, 125 form a G.P., therefore 25 is the Geometric mean of 5 and 125.

If G is a geometric mean of two non-zero numbers a and b, then

a, G, b form a geometric sequence, so $\frac{G}{a} = \frac{b}{G} \Rightarrow \boldsymbol{G^2 = ab}$

If a and b are positive numbers, the geometric mean of the numbers is usually defined to be the positive root of the equation $G^2 = ab$, whereas if a and b are negative numbers, their geometric mean is the negative root of the equation.

Thus $G^2 = ab \Rightarrow \boldsymbol{G = \sqrt{ab}}$ *if* a and b are positive numbers.

and $\Rightarrow \boldsymbol{G = -\sqrt{ab}}$ *if* a and b are negative numbers.

Note. If a and b are two numbers of opposite signs, then G.M. between them does not exist. Thus, if –1 and 9 are two numbers, then $G^2 = -9$. This has no real root.

The positive geometric mean between two numbers is the mean proportional between them and is $\boldsymbol{G = \sqrt{ab}}$. This shows that we have a unique G.M. between two numbers.

Note 1. This also shows that if three numbers a, b, c form a G.P., then $\boldsymbol{b^2 = ac}$
i.e., ***(Square of the middle term) = Product of the extremes.***

Note 2. If there are n positive integers $a_1, a_2, a_3 \ldots\ldots a_n$, then their geometric mean is defined to be equal to $\boldsymbol{(a_1 . a_2 . a_3 \ldots\ldots a_n)^{1/n}}$.

Ex. 28. ***Find the geometric mean of 6 and 24.***

Sol. Let G be the geometric mean, then $G^2 = 6 \times 24 = 144 \Rightarrow G = 12$ or $G = -12$

$\therefore$ The geometric mean of 6 and 24 is 12.

Note. The G.M. of –6 and –24 is $-\sqrt{-6 \times -24} = -\sqrt{144} = \mathbf{-12}$.

Just as you may have more than one arithmetic mean, you may also have more than one geometric mean. The terms between any two given terms of a geometric sequence are geometric means. Thus, in the G.P. 2, 4, 8, 16........ , the terms 4 and 8 are the geometric means between 2 and 16. To insert geometric mean between given numbers, we first find the common ratio.

1.14. Theorem

Let A and G be the arithmetic and geometric means of two positive numbers a* and *b, then $A \geq G$. [See Theorem 2 after Art. 4.21]

Proof. **Case 1.** If a and b are two unequal positive numbers, then

$$A = \frac{a+b}{2}, G = \sqrt{ab} \qquad \therefore\ A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}.$$

Since $(\sqrt{a}-\sqrt{b})^2$ is always + ve, therefore, $A - G > 0 \Rightarrow A > G$.

Case 2. If $a = b$, then $A = G$. Hence, $A \geq G$.

EXERCISE 1 (*d*)

1. Find (*a*) the 7th term of 2, 4, 8,..... (*b*) the 9th term of $1, \frac{1}{2}, \frac{1}{2^2}, ...$ (*c*) the *n*th term of $\frac{15}{8}, \frac{3}{8}, \frac{3}{40}, ...$

2. The second term of a G.P. is 18 and the fifth term is 486. Find (*a*) the first term, (*b*) the common ratio. **[S.C.]**

3. The fourth term of a G.P. is greater than the first term, which is positive by 372. The third term is greater than the second by 60. Calculate the common ratio and the first term of the progression. **[S.C.]**

4. Find the value of x for which $x + 9$, $x - 6$, 4 are the first three terms of a geometrical progression and calculate the fourth term of progression in each case. **[S.C.]**

5. If 5, x, y, z, 405 are the first five terms of a geometric progression, find the values of x, y, and z. **[S.C.]**

6. If the A.M. and G.M. between two numbers are respectively 17 and 8, find the numbers.

7. The second, third and sixth terms of an A.P. are consecutive terms of a geometric progression. Find the common ratio of the geometric progression. **[S.C.]**

   [**Hint.** $(a + 2d)^2 = (a + d).(a + 5d) \Rightarrow d = -2a$, common ratio $= \frac{a+2d}{a+d}$.]

8. The first, eighth, and twenty-second terms of an arithmetic progression are three consecutive terms of a geometric progression. Find the common ratio of the geometric progression. Given also that the sum of the first twenty-two terms of the arithmetic progression is 275, find its first term. **[S.C.]**

9. a_1, a_2, a_3, a_4, a_5 are first five terms of an A.P. such that $a_1 + a_3 + a_5 = -12$ and $a_1 a_2 a_3 = 8$. Find the first term and the common difference. **(*I.S.C. 1992*)**

10. The 5th, 8th and 11th terms of a G.P. are P, Q and S respectively. Show that $Q^2 = PS$.

11. The $(p + q)$th term and $(p - q)$th terms of a G.P. are a and b respectively. Find the pth term.

    [**Hint.** $T_{p+q} = xr^{p+q-1} = a, T_{p-q} = xr^{p-q-1} \Rightarrow r^{2q} = \frac{a}{b} \Rightarrow r = \left(\frac{a}{b}\right)^{\frac{1}{2q}}$
    $\Rightarrow \frac{1}{r} = \left(\frac{b}{a}\right)^{\frac{1}{2q}}$. Now, $T_p = xr^{p-1} = xr^{(p+q-1)}.\left(\frac{1}{r}\right)^q = a\left(\frac{b}{a}\right)^{\frac{q}{2q}} = \sqrt{ab}$.]

12. There are seven houses, in each are seven cats. Each cat kills seven mice. Each mouse would have eaten seven ears of wheat. Each ear of wheat would have produced seven measures of grains. How much grain is saved ?

13. If the pth, qth, rth terms of a G.P. are x, y, z respectively, prove that $x^{q-r}.y^{r-p}.z^{p-q} = 1$.

14. In a set of four numbers, the first three are in G.P. and the last three are in A.P. with difference 6. If the first number is the same as the fourth, find the four numbers. ***(I.I.T.)***

[Hint. Let the three numbers be $a-d, a, a+d$.]

15. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and a, b, c, are in G.P., prove that x, y, z are in A.P. ***(I.I.T.)***

16. Construct a quadratic equation in x such that the A.M. of its roots is A and G.M. is G. ***(I.I.T.)***

ANSWERS

1. (*a*) 128 (*b*) $\frac{1}{256}$ (*c*) $\frac{15}{8}\left(\frac{1}{5}\right)^{n-1}$ **2.** 6, 3 **3.** 5, 3

4. $x = 0, 16; -2\frac{2}{3}, 1\frac{3}{5}$ **5.** 15, 45, 135 or –15, 45, – 135.

6. 32, 2 **7.** 3 **8.** 2, 5 **9.** $T_1 = 2$; Common Diff. = – 3

11. $\sqrt{ab}$ **12.** $7^5 = 16807$ **14.** 8, – 4, 2, 8 **16.** $x^2 - 2Ax + G^2 = 0$.

1.15. Sum of *n* terms of a Geometric Series

Consider the series $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$

Let S_n denote the sum of n terms of this series. Write the series and subtract from it, term by term, the product of r and the series as shown below :

$$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}$$
$$r.\,S_n = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$$

Subtracting, $S_n - rS_n = a - ar^n \Rightarrow S_n(1-r) = a(1-r^n)$

$$\Rightarrow \qquad \boldsymbol{S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1).}$$

We can write and use the above formula in the form $S_n = \frac{a(r^n - 1)}{r-1}$; $(r \neq 1)$.

Cor. $\boldsymbol{S_n = \frac{a - lr}{1-r}}$ **or** $\boldsymbol{\frac{lr - a}{r-1}}$ if l is the last term.

The corollary can be deduced by writing $l = t_n = ar^{n-1}$ so that $lr = ar^n$.

Ex. 29. ***Find the sum of each of the following series :***

(i) ***5 – 10 + 20 – .... to 6 terms.*** ***(ii)*** $\boldsymbol{4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots}$ ***to 10 terms.***

(iii) 243 + 324 + 432 + ... to n terms.

Sol. (*i*) $S_n = \frac{a(1-r^n)}{1-r}$ $(r \neq 1)$. Here $a = 5$, $r = -2$, and $n = 6$

$$\therefore \qquad S_6 = \frac{5[1-(-2)^6]}{1-(-2)} = \frac{5 - 5 \times 64}{3} = \frac{-315}{3} = -105.$$

(*ii*) $a = 4$, $r = \frac{1}{2}$ and $n = 10$

$$\therefore \qquad S_{10} = \frac{4\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}} = \frac{4.\left[1 - \frac{1}{1024}\right]}{\frac{1}{2}} = 8 \times \frac{1023}{1024} = 8 \text{ (Approx.).}$$

(*iii*) $a = 243,\ r = \frac{324}{243} = \frac{4}{3}$ (which is > 1)

$$\therefore\ S_n = \frac{a(r^n - 1)}{r - 1} = \frac{243\left[\left(\frac{4}{3}\right)^n - 1\right]}{\frac{4}{3} - 1} = \frac{243.\left[\frac{4^n}{3^n} - 1\right]}{\frac{1}{3}} = 3^6.\left(\frac{4^n - 3^n}{3^n}\right) = 3^{6-n}.(4^n - 3^n).$$

Ex. 30. ***How many terms of the series 2 + 6 + 18 +..... must be taken to make the sum equal to 728?***

Sol. Here $a = 2$, $r = 3$ and $S = 728$.

Let n be the number of terms required, then, $728 = \frac{2(3^n - 1)}{3 - 1}$

$\Rightarrow\ 3^n - 1 = 728 \quad \Rightarrow \quad 3^n = 729 = 3^6 \quad \Rightarrow \quad n = 6.$

Hence, the number of terms required is 6.

Ex. 31. ***Sum to n terms the series :***

(i) 7 + 77 + 777 +......... ***(ii) 0.7 + 0.77 + 0.777 + ...***

Sol. (*i*) $S_n = 7 + 77 + 777 + ...$ to n terms $= 7\,[1 + 11 + 111 + ...$to n terms$]$

$$= \frac{7}{9}[9 + 99 + 999 + ...\text{ to } n \text{ terms}]$$

$$= \frac{7}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + ...\text{ to } n \text{ terms}]$$

$$= \frac{7}{9}[\{10 + 10^2 + 10^3 + ...\text{to } n \text{ terms}\} - (1 + 1 + 1 + ...\text{ to } n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10.(10^n - 1)}{10 - 1} - n\right] = \frac{7(10^{n+1} - 10)}{81} - \frac{7}{9}n.$$

(*ii*) $S_n = 0.7 + 0.77 + 0.777 + ...$ to n terms

$$= 7\,[0.1 + 0.11 + 0.111 + ...\text{to } n \text{ terms}]$$

$$= \frac{7}{9}[0.9 + 0.99 + 0.999 + ...\text{to } n \text{ terms}]$$

$$= \frac{7}{9}[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + ...\text{to } n \text{ terms}]$$

$$= \frac{7}{9}\left[n - \frac{0.1\{1 - (0.1)^n\}}{1 - 0.1}\right]$$

$$= \frac{7}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right] = \frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right].$$

Ex. 32. ***The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the amount of the grains***

for subsequent squares. How many grains would have to be given to the inventor? (There are 64 squares in the chess board).

Sol. The required number of grains $= 1 + 2 + 2^2 + \ldots\ldots$ to 64 terms

$$= \frac{1 \times (2^{64}) - 1}{2 - 1} = 2^{64} - 1.$$

Ex. 33. ***An insect starts from a point and travels in a straight path one mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point ?***

Sol. Let the required time be n seconds.

Then $3 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ to n terms. or $3 = \dfrac{1 \times \left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$

$$\Rightarrow \quad \frac{3}{2} = 1 - \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = \frac{-1}{2} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{-1}{2}.$$

There is no value of n for which $\left(\frac{1}{2}\right)^n = \frac{-1}{2}$ as exponential functions are always positive.

Hence, the insect would never reach a point 3 mm away from its starting point.

Ex. 34. ***A man borrows Rs 32,760 without interest and agrees to pay back in 12 monthly instalments, each instalment being twice the preceding one. Find the second and the last instalments.***

Sol. Since each instalment is twice the preceding one, the instalments are in G..P. with common ratio 2. Let Rs a be the first instalment. Then, we have $r = 2$, $n = 12$, $S_{12} = 32{,}760$, $a = ?$

$$S_n = \frac{a(r^n - 1)}{r - 1} \Rightarrow S_{12} = \frac{a(2^{12} - 1)}{2 - 1} = \frac{a(4096 - 1)}{1} = \text{Rs } 32760 \text{ (given)}$$

$\Rightarrow \quad 4095\, a = 32760 \qquad \Rightarrow \quad a = 8.$

$\therefore$ Second instalment $= ar = $ Rs $(8 \times 2) =$ **Rs 16.**

and last instalment $= ar^{n-1} = $ Rs $(8 \times 2)^{12-1} = $ Rs (8×2^{11})

$=$ Rs $(8 \times 2048) =$ **Rs 16,384.**

1.16. Sum of an Infinite Geometric Progression

We have $S_n = \dfrac{a - ar^n}{1 - r} = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}.$

If $|r| < 1$, $r^n \to 0$ as $n \to \infty$. Therefore, $\boldsymbol{S_\infty = \dfrac{a}{1 - r}}$.

Caution. Sum to infinity exists only when r is numerically less than 1, *i.e.*, $|r| < 1$.

If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

Ex. 35. ***Sum the following series to n terms and to infinity,***

$$\boldsymbol{1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ldots}$$

Sol. $a = 1, \ r = -\frac{1}{2}$

$$\therefore \ S_n = \frac{a(1-r^n)}{1-r} = \frac{1 \times \left[1-\left(-\frac{1}{2}\right)^n\right]}{1-\left(-\frac{1}{2}\right)} = \frac{\left[1-\left(-\frac{1}{2}\right)^n\right]}{\frac{3}{2}} = \frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^n\right]$$

$$\Rightarrow S_\infty = \underset{n\to\infty}{Lt} \ \frac{2}{3}\left[1-\left(\frac{-1}{2}\right)^n\right] = \frac{2}{3}(1-0) = \frac{2}{3}.$$

Note. We could straightway use the result $\left[S_\infty = \frac{a}{1-r} = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}\right]$

Ex. 36. ***The first term of a G.P. is 2 more than the second term and the sum to infinity is 50. Find the G.P.***

Sol. Let the first two terms be a, $a-2$.

Then common ratio $= \frac{a-2}{a}$ $\quad \therefore \ S_\infty = \frac{a}{1-\frac{a-2}{a}} = \frac{a^2}{2}$ $\quad \left[S_\infty = \frac{a}{1-r}, |r| < 1.\right]$

$$\therefore \quad \frac{a^2}{2} = 50 \Rightarrow a^2 = 100 \Rightarrow a = \pm 10.$$

If $a = -10$, $\ r = \frac{-10-2}{-10} = \frac{12}{10} = \frac{6}{5}$, which >1

$\therefore$ The value $a = -10$ is inadmissible $\therefore$ $a = 10$.

Hence, the series is 10, 8, $\frac{32}{5}$, ...

Recurring Decimals.

Ex. 37. ***Find the value of 0.423***

Sol. $0.4\overline{23} = 0.423\,23\,23$ .... .ad inf.

$$= 0.4 + 0.023 + 0.00023 + \ldots\ldots\ldots\ldots\text{ad inf.}$$

$$= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \ldots\ldots\ldots\ldots\ldots\text{ad inf.}$$

$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\text{ad inf.}\right)$$

$$= \frac{4}{10} + \frac{23}{10^3} \times \frac{1}{1-\left(\frac{1}{10^2}\right)} = \frac{4}{10} + \frac{23}{1000} \times \frac{100}{99} = \frac{419}{990}. \qquad \left[S_\infty = \frac{a}{1-r}\right]$$

Ex. 38. ***The side of a given square is equal to a. The mid-points of its sides are joined to form a new square. Again, the mid-points of the sides of this new square are joined to form another square. This process is continued indefinitely. Find the sum of the areas of the squares and the sum of the perimeters of the squares.*** **[I.S.C. 1991]**

Sol. Since the diagonal of a square is $\sqrt{2}$ times the side of a square, we get the following series as an infinite G.P. of the areas of the squares :

$$a^2+\left(\sqrt{2}\times\frac{a}{2}\right)^2+\left(\sqrt{2}\times\sqrt{2}\times\frac{a}{4}\right)^2+\left(\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\frac{a}{8}\right)^2+\text{.....to }\infty$$

$$=a^2+\frac{a^2}{2}+\frac{a^2}{4}+\frac{a^2}{8}+\text{.....to }\infty,\qquad \therefore\ S_\infty=\frac{\text{First term}}{1-\text{Common ratio}}=\frac{a^2}{1-\frac{1}{2}}=2a^2\text{ sq units.}$$

Similarly, since the diagonal of a square is $\sqrt{2}$ times the side of a square, we get the following series as an infinite G.P. of the sum of the perimeters of the squares :

$$4\left[a+\left(\sqrt{2}\times\frac{a}{2}\right)+\left(\sqrt{2}\times\sqrt{2}\times\frac{a}{4}\right)+\text{.....to }\infty\right]=4\left[a+\frac{a}{\sqrt{2}}+\frac{a}{2}+\text{.....to }\infty\right]$$

$$4\times\frac{a}{1-\frac{1}{\sqrt{2}}}=\frac{4\sqrt{2}a}{\sqrt{2}-1}=\frac{4\sqrt{2}(\sqrt{2}+1)a}{(\sqrt{2}-1)(\sqrt{2}+1)}=4\sqrt{2}(\sqrt{2}+1)a\text{ units.}$$

EXERCISE 1(*e*)

1. Find the sum to

(*a*) 8 terms of $3+6+12+$ .....

(*b*) 20 terms of $2+6+18+$....... **[S.C.]**

(*c*) 10 terms of $1+\sqrt{3}+3+$ .....

(*d*) *n* terms of $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$.......

2. Sum the following series to infinity :

(*i*) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$......

(*ii*) 16, –8, 4 ,.....

(*iii*) $\sqrt{2}-\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}-\frac{1}{4\sqrt{2}}+$.....

(*iv*) $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+$......

3. Find the sum of a geometric series in which $a=16, r=\frac{1}{4}, l=\frac{1}{64}$.

4. Find the sum of the series $81-27+9-\ldots\ldots-\frac{1}{27}$.

5. The first three terms of a G.P. are *x*, *x* + 3, *x* + 9. Find the value of *x* and the sum of first eight terms. **[S.C.]**

6. Of how many terms is $\frac{55}{72}$, the sum of the series $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-$ .....?

7. The second term of a G.P. is 2 and the sum of infinite terms is 8. Find the first term.

8. (*a*) Find the value of $0.2\dot{3}\dot{4}$ regarding it as a geometric series.

(*b*) Evaluate : (*i*) $0.9\overline{7}$ (*ii*) $0.\overline{45}$ (*iii*) $0.23\dot{4}\dot{5}$

(*c*) Find a rational number which when expressed as a decimal will have $1.2\overline{56}$ as its expansion.

[Hint. $1.2\overline{56}=1.2+0.056+0.00056+0.0000056+\ldots\ldots=1.2+\frac{0.056}{1-0.01}$.]

9. If $a + b + ... + l$ is a G.P., prove that its sum is $\dfrac{bl - a^2}{b - a}$.

10. The nth term of a geometrical progression is $\dfrac{2^{2n-1}}{3}$ for all values of n. Write down the numerical values of the first three terms and calculate the sum of the first 10 terms, correct to 3 significant figures. **[I.S.C.]**

11. A geometrical progression of positive terms and an arithmetical progression have the same first term. The sum of their first terms is 1, the sum of their second terms is $\frac{1}{2}$ and the sum of their third terms is 2. Calculate the sum of their fourth terms. **[I.S.C.]**

12. In a geometric progression, the third term exceeds the second by 6 and the second exceeds the first by 9. Find (*i*) the first term, (*ii*) the common ratio and (*iii*) the sum of the first ten terms. Simplify the answer. **[I.S.C.]**

13. In an infinite geometric progression, the sum of first two terms is 6 and every term is four times the sum of all the terms that follow it. Find.

(*i*) the geometric progression and (*ii*) its sum to infinity. **[I.S.C.]**

14. Three numbers are in A.P. and their sum is 15. If 1, 4 and 19 be added to these numbers respectively, the numbers are in G.P. Find the numbers. **[I.S.C.]**

15. Calculate the least number of terms of the geometric progression 5 + 10 + 20 + ... whose sum would exceed 1000000. **[I.S.C.]**

[Hint. $S_n = \dfrac{5.(2^n - 1)}{2 - 1} = 5.\,(2^n - 1)$.

We have to find the least value of n such that

$$5(2^n - 1) > 10{,}00{,}000 \;\Rightarrow\; 2^n - 1 > 2{,}00{,}000$$

$$\Rightarrow 2^n > 2{,}00{,}001 \qquad ...(1)$$

If $n = 17$, then $2^n = 2^{17} = 131072$, which is $< 2{,}00{,}001$

If $n = 18$, then $2^n = 2^{18} = 262144$, which is $> 2{,}00{,}001$.

Hence, the least value of $n = 18$.

Alternatively. From (1), we have $n \log 2 > \log 200001 \Rightarrow n > \dfrac{\log 200001}{\log 2}$

$$\Rightarrow n > \frac{5.3010}{0.3010} \Rightarrow n > 17.61$$

Hence, $n = 18$.]

16. If S be the sum, P the product and R the sum of the reciprocals of n terms in G.P., prove that $p^2 = \left(\dfrac{S}{R}\right)^n$.

17. If you save Re 1 today, Rs 2 the next day, Rs 4 the succeeding day and so on, what will be your total saving in 12 days?

18. A man borrows Rs 8190 without interest and repays the loan in 12 monthly instalments, each instalment being twice the preceding one. Find the first and the last instalments.

19. A man borrows Rs 9,841 without interest and repays the loan in 9 instalments, each instalment (beginning with the second) being three times the preceding one. Find the amount of the last instalment.

20. An air pump used to extract air from a vessel removes one-tenth of the air at each stroke. Find what fraction of the original volume of air is left after the 5th stroke.
[Hint. If Vcc is the original volume of air, then air left after.

$$\text{first stroke } = V - \frac{1}{10}V = \frac{9}{10}Vcc$$

$$\text{second stroke } = \frac{9V}{10} - \frac{1}{10}\left(\frac{9V}{10}\right) = \left(\frac{9}{10}\right)Vcc$$

and so on. Thus the quantities of air left after different strokes are is G.P. whose first term is $\frac{9V}{10}$ and common ratio is $\frac{9}{10}\cdot$ Find the 5th term.]

21. A bouncing ball rebounds each time to a height equal to one-half the height of the previous bounce. If it is dropped from a height of 16 metres, find the total distance it has travelled when it hits the ground for the 10th time **[W.B.J.E.E.]**

22. If $\frac{2}{3} = \left(x - \frac{1}{y}\right) + \left(x^2 - \frac{1}{y^2}\right) + \ldots\ldots \text{to } \infty$

and $xy = 2$, then calculate the values of x and y with the condition that $x < 1$. **[I.S.C.]**

23. $S_1, S_2, S_3, \ldots\ldots, S_n$ are sums of n infinite geometric progressions. The first terms of these progressions are $1, 2^2 - 1, 2^3 - 1, \ldots\ldots, 2^n - 1$ and the common ratios are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots, \frac{1}{2^n}$.
Calculate the value of $S_1 + S_2 + \ldots\ldots + S_n$. **[I.S.C.]**

24. Find three numbers a, b, c between 2 and 18 such that :

(*a*) their sum is 25, and

(*b*) the numbers 2, a, b are consecutive terms of an arithmetic progression, and

(*c*) the numbers b, c, 18 are consecutive terms of a geometric progression. **[I.S.C.]**

[Hint. $a + b + c = 25$, $2a - b = 2$, $18b = c^2$.]

25. Three numbers, whose sum is 21, are in A.P. If 2, 2, 14 are added to them respectively, the resulting numbers are in G.P. Find the numbers. **[I.S.C 1990.]**

26. If $x = 1 + a + a^2 + \ldots\ldots \infty$, $a < 1$ and $y = 1 + b + b^2 + \ldots\ldots\ldots \infty$, $b < 1$,

then prove that $1 + ab + a^2b^2 + \ldots\ldots \infty = \dfrac{xy}{(x + y - 1)}$.

27. If the sums of n, $2n$ and $3n$ terms of a G.P. are S_1, S_2 and S_3 respectively, then prove that

$$S_1 (S_3 - S_2) = (S_2 - S_1)^2.$$

28. If $S_1, S_2, S_3, \ldots\ldots, S_p$ are the sums of infinite geometric series whose first terms are 1, 2, 3,...., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots, \frac{1}{p+1}$ respectively, prove that

$$S_1 + S_2 + S_3 + \ldots\ldots + S_p = \frac{1}{2}\, p\,(p + 3).$$ **(*T.S. Rajendra*)**

ANSWERS

1. (*a*) 765, (*b*) $3^{20}-1$, (*c*) $121(\sqrt{3}+1)$, (*d*) $\frac{81}{8}\left\{1-\left(\frac{2}{3}\right)^n\right\}$

2. (*i*) 2, (*ii*) $\frac{32}{3}$, (*iii*) $\frac{2\sqrt{2}}{3}$, (*iv*) $\frac{3\sqrt{3}}{2}$

3. $21\frac{21}{64}$ **4.** $60\frac{20}{27}$ **5.** 3; 765 **6.** 5 **7.** 4

8. (*a*) $\frac{116}{495}$, (*b*) (*i*) $\frac{44}{45}$, (*ii*) $\frac{5}{11}$, (*iii*) $\frac{129}{550}$, (*c*) $\frac{622}{495}$

10. $\frac{2}{3}, \frac{8}{3}, \frac{32}{3}$; 233000. **11.** $9\frac{1}{2}$ **12.** (*i*) -27 (*ii*) $\frac{2}{3}$ (*iii*) $-\frac{58025}{729}$.

13. (*i*) $5, 1, \frac{1}{5}.....$ (*ii*) $6\frac{1}{4}$ **14.** 2, 5, 8 or 26, 5, –16 **15.** 18

17. Rs 40.95 **18.** Rs 2; Rs 4096 **19.** Rs 6561 **20.** $\left(\frac{9}{10}\right)^5 Vcc$, where V is the original volume

21. $47\frac{15}{16}$ m **22.** $x=\frac{1}{2}, y=4$ **23.** $2(2^n-1)$ **24.** $a=5, b=8, c=12$

25. 1, 7, 13 or 25, 7, –11

1.17. Special types of problems

Ex. 39. ***Find three real numbers in G.P. whose sum is 30 and product is 216.***

Sol. Let the three real numbers in G.P. be $\frac{a}{r}$, a, ar.

$\because$ Their product = 216 $\quad\therefore\ \frac{a}{r}.a.ar = 216 \quad\Rightarrow\ a^3 = 6^3 \quad\Rightarrow a = 6$

Their sum = 30 $\quad\therefore\ \frac{a}{r}+a+ar=30 \ \Rightarrow\ \frac{6}{r}+6+6r=30 \quad\Rightarrow\ 6r^2-24r+6=0$

$\Rightarrow\ r^2-4r+1=0$, which is a quadratic equation in r.

(Disct.) $= b^2-4ac = (-4)^2-4(1) = 12 \qquad \therefore\ r = \frac{4\pm\sqrt{12}}{2} = \frac{4\pm 2\sqrt{3}}{2} = 2\pm\sqrt{3}$

when $a=6$, $r = 2+\sqrt{3}$, the three numbers are $\frac{6}{2+\sqrt{3}}$, 6, $6(2+\sqrt{3})$

i.e., $\frac{6}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$, 6, $6(2+\sqrt{3})$ *i.e.,* $6(2-\sqrt{3})$, 6, $6(2+\sqrt{3})$

when $a=6$, $r=2-\sqrt{3}$, the three numbers are $\frac{6}{2-\sqrt{3}}$, 6, $6(2-\sqrt{3})$

i.e., $\frac{6}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}$, 6, $6(2-\sqrt{3})$, *i.e.,* $6(2+\sqrt{3})$, 6, $6(2-\sqrt{3})$

Note. When product is known, suppose the numbers in G.P. are as under:

(i) If the odd numbers of terms are to be considered, suppose them as follows by taking a as the mid-term and common ratio r,

$$\frac{a}{r}, a, ar \;; \frac{a}{r^2}, \frac{a}{r} a, ar, ar^2.$$

(ii) If even number of terms are to be considered, suppose them as under by taking as $\frac{a}{r}$, ar as the two mid-terms and r^2 as the common ratio.

$$\frac{a}{r^3}, \frac{a}{r} ar\ ar^3 \quad \text{[4 terms]} \quad \frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r} ar, ar^3, ar^5. \quad \text{[6 terms]}$$

Ex. 40. ***The sum of the first three terms of G.P. is 7 and the sum of their squares is 21. Determine the first five terms of the G.P.***

Sol. Here the product of the three terms in G.P. is not given, so it is not necessary to suppose them in a special manner.

Let the first three terms of the G.P. be a, ar, ar^2.

$\because$ Their sum = 7 $\quad \therefore\ a + ar + ar^2 = 7 \quad \Rightarrow a(1 + r + r^2) = 7$ ...(1)

$\because$ Sum of the squares of the terms = 21 $\quad \therefore\ a^2 + a^2r^2 + a^2r^4 = 21$

$\Rightarrow a^2(1 + r^2 + r^4) = 21 \quad \Rightarrow \quad a^2(1 + r^2 + r)(1 + r^2 - r) = 21$ **[Note this step]** ...(2)

Squaring (1), $a^2(1 + r + r^2)^2 = 49.$ ...(3)

Dividing (2) by (3), $\dfrac{1 - r + r^2}{1 + r + r^2} = \dfrac{3}{7} \quad \Rightarrow\ 7 - 7r + 7r^2 = 3 + 3r + 3r^2$

$\Rightarrow\ 2r^2 - 5r + 2 = 0 \quad \Rightarrow\ (2r - 1)(r - 2) = 0 \quad \Rightarrow\ r = \dfrac{1}{2}$ or 2.

when $r = \dfrac{1}{2}$ from (1), $a\left(1 + \dfrac{1}{2} + \dfrac{1}{4}\right) = 7 \Rightarrow a = 4$

$\therefore$ The first five terms of the G. P. are 4, 2, 1, $\dfrac{1}{2}, \dfrac{1}{4}$.

when $r = 2$ from (1), $a(1 + 2 + 4) = 7 \Rightarrow a = 1$

$\therefore$ The first five terms of the G.P. are 1, 2, 4, 8, 16.

Ex. 41. ***If a, b, c, d are in G.P. prove that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are also in G.P.***

Sol. Let r be the common ratio of the G. P. Then $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = r$

$\Rightarrow\ b = ar, c = br = ar^2, d = cr = ar^3$

$a^2 - b^2, b^2 - c^2, c^2 - d^2$ will be in G.P. if $(b^2 - c^2)^2 = (a^2 - b^2)(c^2 - d^2)$

i.e., if $(a^2r^2 - a^2r^4)^2 = (a^2 - a^2r^2)(a^2r^4 - a^2r^6)$

i.e., if $a^4r^4(1 - r^2)^2 = a^2(1 - r^2)\, a^2r^4(1 - r^2) = a^4r^4(1 - r^2)^2$, which is true.

Hence, $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

Method. To solve questions of this type

(1) *Let the common ratio of the G.P. be r.*

(2) *Write the values of the given terms in terms of r.*

(3) *Apply the condition that the other terms are in G.P. Establish its truth by substituting known values from (2).*

EXERCISE 1 (*f*)

1. Find three numbers in G.P. whose sum is 19 and product is 216.

2. The sum of the first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the G.P.

3. The product of first three terms of a G.P. is 1000. If we add 6 to its second term and 7 to its third term, the resulting three terms form an A.P. Find the terms of the G.P.

4. If a, b, c are in G.P. show that the following are also in G.P.

(*i*) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ (*ii*) $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ (*iii*) a^2, b^2, c^2

(*iv*) b^2c^2, c^2a^2, a^2b^2 (*v*) $a^2 + b^2, ab + bc, b^2 + c^2$.

5. If a, b, c, d are in G. P, show that the following are also in G. P.

(*i*) $a + b, b + c, c + d$ (*ii*) $a^2 + b^2, b^2 + c^2, c^2 + d^2$

6. If a, b, c, d are in G.P. prove that

(*i*) $(b + c)(b + d) = (c + a)(c + d)$ (*ii*) $(a – d)^2 = (b – c)^2 + (c – a)^2 + (d – b)^2$

7. If the mth, nth and pth terms of an A.P. and G.P. are equal and are x, y, z respectively, prove that

$$x^{y-z} . y^{z-x} . z^{x-y} = 1.$$

8. If the pth, qth and rth terms of an A. P. are in G.P., prove that the common ratio of the G.P. is $\frac{q-r}{p-q}$.

[**Hint.** Common ratio $= \frac{a+(q-1)\,d}{a+(p-1)\,d} = \frac{a+(r-1)\,d}{a+(q-1)\,d} = \frac{q-r}{p-q}\left(\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then each } = \frac{a-c}{b-d}\right)$.]

9. If $\frac{1}{x+y}, \frac{1}{2y}, \frac{1}{y+z}$ are the three consecutive terms of an A.P. prove that x, y, z are the three consecutive terms of a G.P.

[Hint. $\frac{1}{x+y}, \frac{1}{2y}, \frac{1}{y+z}$ are in A.P. $\Rightarrow \frac{1}{2y} - \frac{1}{x+y} = \frac{1}{y+z} - \frac{1}{2y}$

$\Rightarrow \frac{x-y}{x+y} = \frac{y-z}{y+z} \Rightarrow \frac{(x-y)+(x+y)}{(x-y)-(x+y)} = \frac{(y-z)+(y+z)}{(y-z)-(y+z)}$

[If $\frac{a}{b} = \frac{c}{d}$, then by componendo and dividendo $\frac{a+b}{a-b} = \frac{c+d}{c-d}$]

$\Rightarrow \frac{2x}{-2y} = \frac{2y}{-2z} \Rightarrow \frac{y}{x} = \frac{z}{y} \Rightarrow x, y, z$ are in G.P.]

10. Find three numbers in G.P. whose sum is 19 and the sum of whose squares is 133.

ANSWERS

1. 4, 6, 9	**2.** $\frac{4}{3}, -1, \frac{3}{4}, ...;$ or $\frac{3}{4} -1, \frac{4}{3}, ...$
3. 20, 10, 5 or 5, 10, 20	**10.** 4, 6, 9 or 9, 6, 4.

1.18. Summation notation (Use of Σ)

Suppose we have n data values. The symbol x_i denotes the ith value in the data set. Suppose the haemoglobin levels in the blood of 10 hospital patients are as under :

9.1, 10.1, 10.7, 10.7, 10.9, 11.3, 11.3, 11.4, 11.4, 11.6.

In the above set of data, $x_1 = 9.1$, $x_2 = 10.1$, $x_3 = 10.7$ and so on. The sum of the ten values is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$.

The abbreviation for this sum is $\sum_{i=1}^{10} x_i$. The symbol Σ (which is read as 'sigma') is a Greek capital S, standing for sum. The '$i = 1$' at the bottom and the '10' at the top of Σ tell you that the sum starts at x_1 and finishes at x_{10}. This notation for a sum is called **Σ – notation.**

Thus, we can denote $1 + 2 + 3 + 4 ..... + n$ by $\sum_{r=1}^{n} r$,

$1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$ by $\sum_{r=1}^{n} r^2$ in sigma notation

Note : $\sum_{r=1}^{n} r$ is sometimes written as Σn for the sake of convenience. Though it may not be technically correct to write it as such but it should not cause any ambiguity if we know what it means.

Similarly $\sum_{r=1}^{n} r^2$ can simply be written as Σn^2 and $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ...+ n^3$ can be written as Σn^3.

Caution. We cannot write $\sum_{r=1}^{10}$ as $\Sigma\, 10$.

Test your understanding :

If $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4,$ evaluate:

1. $\sum_{i=1}^{3} x_i$ **2.** $\sum_{i=1}^{4} x_i^2$ **3.** $\sum_{i=1}^{3} x_i^3$

If $x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 10$, evaluate:

4. $\sum_{i=1}^{3} x_i$ **5.** $\sum_{i=1}^{4} x_i^2$ **6.** $\sum_{i=1}^{2} x_i^3$

ANSWERS

1. 6 **2.** 30 **3.** 36 **4.** 12 **5.** 156 **6.** 72.

SERIES INVOLVING NATURAL NUMBERS

1.19. To find the sum of first n natural numbers

Let $S = 1 + 2 + 3 + ... + n$.

This is an A.P. whose first term is 1 and common difference is 1, and nth term is n.

$$S = \frac{n}{2}(n+1), \text{ using } S = \frac{n}{2}(a+l).$$

Thus, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$

Using the sigma notation this may be written as $\boxed{\sum n = \dfrac{n(n+1)}{2}.}$

1.20. To find the sum of the squares of the first n natural numbers

Let $S = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$.

We know that $x^3 - (x-1)^3 = 3x^2 - 3x + 1$.

Putting $x = n, n-1, n-2, \ldots.., 3, 2, 1$ successively in this identity, we get

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$
$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$
$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$
$$1^3 - 0^3 = 3.1^2 - 3.1 + 1.$$

Adding, we get $n^3 = 3[1^2 + 2^2 + 3^2 + \ldots\ldots + n^2] - 3[1 + 2 + 3 + \ldots\ldots + n] + n$

$$\Rightarrow \quad n^3 = 3S - 3\frac{n(n+1)}{2} + n \quad \Rightarrow \quad 3S = (n^3 - n) + 3\frac{n(n+1)}{2} = n(n+1)(n-1) + \frac{3n(n+1)}{2}$$

$$\Rightarrow \quad 6S = 2n(n+1)(n-1) + 3n(n+1) = n(n+1)(2n - 2 + 3) = n(n+1)(2n+1)$$

$$\therefore \qquad S = \frac{n(n+1)(2n+1)}{6}$$

Using the sigma notation this may be written as $\boxed{\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}.}$

1.21. To find the sum of the cubes of the first n natural numbers

Let $\quad S = 1^3 + 2^3 + 3^3 + \ldots + n^3$.

We know that $\quad x^4 - (x-1)^4 \equiv 4x^3 - 6x^2 + 4x - 1$.

Putting $x = n, n-1, n-2, \ldots\ldots, 3, 2, 1$ successively in this identity, we get

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$
$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$
$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1.$$

Adding, we get

$$n^4 = 4(1^3 + 2^3 + 3^3 + \ldots. + n^3) - 6(1^2 + 2^2 + 3^2 + \ldots.. + n^2) + 4(1 + 2 + 3 + \ldots.. + n) - n$$

$$\Rightarrow \quad n^4 = 4S - 6\frac{n(n+1)(2n+1)}{6} + 4.\frac{n(n+1)}{2} - n$$

$$\therefore \quad 4S = (n^4 + n) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2) = n(n+1)(n^2 + n) = n^2(n+1)^2$$

$$S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Using the Sigma notation, this may be written as $\boxed{\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum n\right)^2.}$

It is clear that the sum of the cubes of the first n natural numbers is a perfect square, namely, the square of the sum of the first n natural numbers. ***(EAMCET 2007)***

1.22. A very important theorem

If $T_n = an^3 + bn^2 + cn + d$, *then* $S_n = a.\,\Sigma n^3 + b.\Sigma n^2 + c.\Sigma n + dn.$

Let S_n denote the sum.

$$T_n = an^3 + bn^2 + cn + d.$$

Put $n = 1, 2, 3, \ldots\ldots, n-2, n-1, n.$

$$T_1 = a.1^3 + b.1^2 + c.1 + d$$
$$T_2 = a.2^3 + b.2^2 + c.2 + d$$
$$T_3 = a.3^3 + b.3^2 + c.3 + d$$

....

$$\therefore \quad T_{n-2} = a.(n-2)^3 + b.(n-2)^2 + c(n-2) + d$$
$$T_{n-1} = a.(n-1)^3 + b.(n-1)^2 + c(n-1) + d$$
$$T_n = a.n^3 + b.\,n^2 + cn + d.$$

Adding, we get

$$S_n = a(1^3 + 2^3 + 3^3 + \ldots\ldots + n^3) + b(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2) + c\,(1 + 2 + 3 + \ldots\ldots + n) + dn$$
$$= a.\,\Sigma n^3 + b.\Sigma n^2 + c.\,\Sigma n + dn.$$

Ex. 42. ***Find the nth term and then the sum to n terms of the series 3.5 + 4.7 + 5.9 + ...***

Sol. Here each term is the product of two factors. The factors 3, 4, 5,...... are in A.P. having 3 as the first term and 1 as the common difference and therefore the nth factor is $3 + (n-1)$, *i.e.*, $2 + n$. Also the other factors 5, 7, 9,..... are in A.P., having 5 as the first term and 2 as the common difference and therefore the nth factor is $5 + (n-1)\,2$, *i.e.*, $2n + 3$.

$\therefore$ The nth term of the series is $(2 + n)(2n + 3)$

$$\Rightarrow \quad T_n = 2n^2 + 7n + 6$$
$$\therefore \quad \Sigma T_n = 2.\Sigma n^2 + 7.\Sigma n + 6n.$$
$$\Rightarrow \quad S_n = \frac{2n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + 6n$$
$$= \frac{2n(n+1)(2n+1) + 21n(n+1) + 36n}{6} = \frac{4n^3 + 27n^2 + 59n}{6}$$

Hence, the nth term is $2n^2 + 7n + 6$, and the sum of the n terms is $\frac{4n^3 + 27n^2 + 59n}{6}$.

Ex. 43. ***Find the nth term and the sum to n terms of the following series:***

(i) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots$ ***(ii)*** $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots$

(iii) $1.4.7 + 2.5.8 + 3.6.9 + \ldots\ldots$

Sol. (*i*) $$T_n = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

$$\therefore \qquad S_n = \frac{1}{3}.\sum n^3 + \frac{1}{2}.\sum n^2 + \frac{1}{6}.\sum n$$

$$= \frac{1}{3}.\frac{n^2(n+1)^2}{4} + \frac{1}{2}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}.\frac{n(n+1)}{2}$$

or
$$S_n = \frac{n(n+1)}{12}\{n(n+1)+(2n+1)+1\}$$

$$= \frac{n(n+1)(n^2+3n+2)}{12} = \frac{n(n+1)^2(n+2)}{12}$$

(ii)
$$T_n = \{1+(n-1)2\}^2 = (2n-1)^2 = 4n^2 - 4n + 1.$$

$$\therefore \qquad S_n = 4.\Sigma n^2 - 4.\Sigma n + n$$

$$= 4.\frac{n(n+1)(2n+1)}{6} - 4.\frac{n(n+1)}{2} + n = n\left[\frac{2}{3}(2n^2+3n+1) - 2n - 2 + 1\right]$$

$$= n\left[\frac{4}{3}n^2 + 2n + \frac{2}{3} - 2n - 1\right] = n\left[\frac{4}{3}n^2 - \frac{1}{3}\right] = \frac{n}{3}(4n^2-1).$$

(iii)
$$T_n = n(n+3)(n+6) = n^3 + 9n^2 + 18n.$$

$$\therefore \qquad S_n = \Sigma n^3 + 9.\Sigma n^2 + 18.\Sigma n$$

$$= \frac{n^2(n+1)^2}{4} + \frac{9n(n+1)(2n+1)}{6} + \frac{18n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 3(2n+1) + 18\right]$$

$$= \frac{n(n+1)}{4}(n^2+n+12n+6+36) = \frac{n(n+1)(n^2+13n+42)}{4}$$

Ex. 44. ***300 trees are planted in a regular pattern in rows in the shape of an isosceles, triangle, the number in the successive rows diminishing by one from the base to the apex. How many trees are there in the row which forms the base of the triangle ?***

Sol. Let there be n trees at the base of the triangle. There is one tree at the first row, *i.e.*, at the apex. Then, the trees in the successive rows from apex to the base are 1,2,3, ... n.

Total number of trees = 300

$\therefore \quad 1+2+3+ \ldots\ldots + n = 300$

$\Rightarrow \quad \dfrac{n(n+1)}{2} = 300 \qquad \Rightarrow \qquad n^2 + n - 600 = 0$

$\Rightarrow (n-24)(n+25) = 0 \qquad \Rightarrow \qquad n = 24, -25$

But the number of trees cannot be negative. $\therefore \; n = 24$

Hence, the number of trees in the row which forms the base of the triangle = **24**.

Ex. 45. ***Sum up the series***

$$\boldsymbol{1.n + 2.(n-1) + 3.(n-2) + \ldots\ldots\ldots n.1.}$$

Sol. Here find the rth term, instead of finding the nth term.

$$T_r = r.[n-(r-1)] = r(n-r+1) = (n+1).r - r^2.$$

$$\therefore \qquad S = (n+1).\sum_1^n r - \sum_1^n r^2$$

$$= (n+1).\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{2}\left[(n+1) - \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left(\frac{3n+3-2n-1}{3}\right) = \frac{n(n+1)(n+2)}{6}.$$

1.23. Method of differences

In certain series, which are neither A.P. nor G.P., it may happen that the successive difference like $(T_2 - T_1), (T_3 - T_2), (T_4 - T_3)$, etc. may form an A.P., or G.P. If it so happens, the nth term can be found and then the sum can be deduced.

Ex. 46. ***The natural numbers are grouped as follows :***

(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), ...

Find an expression for the first term of the nth group. **[I.S.C.]**

Sol. We have to find the nth term of the progression 1, 2, 4, 7.....

Let T_n be the nth term and S denote the sum of the first n terms, then

$$S = 1 + 2 + 4 + 7 + 11 + \ldots + T_{n-2} + T_{n-1} + T_n$$

Also $$S = 1 + 2 + 4 + 7 + \ldots\ldots + T_{n-2} + T_{n-1} + T_n$$

Subtracting, we get

$$0 = 1 + 1 + 2 + 3 + 4 + \ldots\ldots \text{ up to } (n \text{ terms}) - T_n$$

$$\therefore \quad T_n = 1 + 1 + 2 + 3 + 4 + \ldots\ldots \text{ up to } (n \text{ terms})$$

$$= 1 + [1 + 2 + 3 + 4 + \ldots\ldots \text{ up to } (n-1) \text{ terms}]$$

$$= 1 + \frac{n-1}{2}(1 + n - 1) = \frac{2 + n^2 - n}{2} \quad \Rightarrow \quad T_n = \frac{n^2 - n + 2}{2}$$

Ex. 47. ***Find the nth term and deduce the sum to n terms of the series***

4 + 11 + 22 + 37 + 56 + ...

Sol. Let $$S = 4 + 11 + 22 + 37 + 56 + \ldots\ldots + T_{n-1} + T_n$$

Also $$S = 4 + 11 + 22 + 37 + \ldots + T_{n-2} + T_{n-1} + T_n$$

Subtracting, we get $$0 = 4 + 7 + 11 + 15 + 19 + \ldots\ldots + (T_n - T_{n-1}) - T_n$$

$$T_n = 4 + \{7 + 11 + 15 + 19 + \ldots (n-1) \text{ terms}\}$$ (***Note this step.***)

$$= 4 + \frac{n-1}{2}[14 + (n-2)\,4] \quad \text{(A.P.)}$$

$$= 4 + \frac{n-1}{2}(4n + 6) = 4 + 2n^2 + n - 3 = 2n^2 + n + 1$$

$$\therefore \quad S_n = 2\Sigma n^2 + \Sigma n + n$$

$$= 2\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = n\left[\frac{(n+1)(2n+1)}{3} + \frac{n+1}{2} + 1\right]$$

$$= n\left[\frac{2n^2 + 3n + 1}{3} + \frac{n+1}{2} + 1\right] = \frac{n}{6}[4n^2 + 6n + 2 + 3n + 3 + 6]$$

$$= \frac{n}{6}(4n^2 + 9n + 11).$$

EXERCISE 1 (*g*)

1. Find the sum to n terms of the series whose nth term is

   (*i*) $n(n+2)$, (*ii*) $3n^2+2n$, (*iii*) $4n^3+6n^2+2n$.

2. Find the sum of the series

   (*i*) $3 \times 5 + 5 \times 7 + 7 \times 9 +$ ...... to n terms (*ii*) $1^2+3^2+5^2+$ ...... to n terms

   (*iii*) $2^2+4^2+6^2+$ ...... to n terms.

3. Find the nth term and the sum to n terms of the series $1.2 + 2.3 + 3.4 +$ ......
4. Sum up to n terms the series $1.2^2 + 2.3^2 + 3.4^2 +$......
5. Sum up $1 + (1+2) + (1+2+3) +$ ...... $+ (1+2+3+$ .... $+ n)$.
6. Sum up $1+4+8+13+$ .... to n terms.
7. Find the sum to n terms of the series $1.2.3 + 2.3.4 + 3.4.5 +$ .....

ANSWERS

1. (*i*) $\frac{n}{6}(n+1)(2n+7)$ (*ii*) $\frac{n}{2}(n+1)(2n+3)$ (*iii*) $n(n+1)^2(n+2)$

2. (*i*) $\frac{n(4n^2+18n+23)}{3}$ (*ii*) $\frac{n(2n+1)(2n-1)}{3}$ (*iii*) $\frac{2n(n+1)(2n+1)}{3}$

3. $n(n+1);\ \frac{n}{3}(n+2)(n+1)$ **4.** $\frac{1}{12}n(n+1)(n+2)(3n+5)$ **5.** $\frac{1}{6}n(n+1)(n+2)$

6. $\frac{n}{6}(n^2+6n-1)$ **7.** $\frac{n(n+1)(n+2)(n+3)}{4}$

REVISION EXERCISE

1. If in a geometric progression consisting of positive terms, each term equals the sum of the next two terms, then the common ratio of this progression equals.

   (*a*) $\sqrt{5}$ (*b*) $\frac{1}{2}(\sqrt{5}-1)$ (*c*) $\frac{1}{2}(1-\sqrt{5})$ (*d*) $\frac{1}{2}\sqrt{5}$

   [Hint. $a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0$]

2. If the first term of an infinite G.P. is 1 and each term is twice the sum of the succeeding terms, then the sum of the series is

   (*a*) 2 (*b*) 5/2 (*c*) 7/2 (*d*) 3/2 (*e*) 9/2.

3. If the sets $S_1, S_2, S_3,$ .... are given by $S_1 = \left\{\frac{2}{1}\right\}, S_2 = \left\{\frac{3}{2}, \frac{5}{2}\right\}, S_3 = \left\{\frac{4}{3}, \frac{7}{3}, \frac{10}{3}\right\}$,

   $S_4 = \left\{\frac{5}{4}, \frac{9}{4}, \frac{13}{4}, \frac{17}{4}\right\}$, ... then the sum of the numbers in the set S_{25} is

   (*a*) 320 (*b*) 322 (*c*) 324 (*d*) 325 (*e*) 326.

   [Hint. $S_{25} = \left\{\frac{26}{25}, \frac{26+25}{25}, \frac{26+50}{25}, \frac{26+75}{25}, \text{...... up to 25 terms}\right\}$.

   Add the A.P. by using $S_n = \frac{n}{2}[2a+(n-1)d]$.

4. If a, b, c are in G. P. with $1 < a < b < c$, and $n > 1$ is an integer. $\log_a n$, $\log_b n$, $\log_c n$ form a sequence. This sequence is which one of the following ?

(*a*) H. P. (*b*) A.P. (*c*) G. P. (*d*) none of these.

5. Fifth term of a G.P. is 2, then the product of its first 9 terms is

(*a*) 256 (*b*) 512 (*c*) 1024 (*d*) none of these.

6. If sum of n terms of an A.P. is $3n^2 + 5n$ and $T_m = 164$, then $m =$

(*a*) 26 (*b*) 27 (*c*) 28 (*d*) None of these.

7. Find the quadratic equation whose roots are the two numbers whose A.M. is 9 and G.M.4.

8. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

(*a*) $\frac{\alpha}{\beta}$ (*b*) $\frac{\beta}{\alpha}$ (*c*) $\sqrt{\frac{\alpha}{\beta}}$ (*d*) $\sqrt{\frac{\beta}{\alpha}}$.

[Hint. $\alpha = a_2 + a_4 + ...$ up to 100 terms $= ar + ar^3 + ...$ up to 100 terms]

$= ar\,(1 + r^2 + r^4 + ... + r^{198})$

Similarly, $\beta = a + ar^2 + ...$ up to 100 terms $= a\,(1 + r^2 + r^4 + ... + r^{198})$]

9. The sum of three decreasing numbers in A.P. is 27. If –1, –1, 3 are added to them respectively, the resultig series is in G.P. The numbers are

(*a*) 5, 9, 13 (*b*) 15, 9, 3 (*c*) 13, 9, 5 (*d*) 17, 9, 1

[Hint. Let the numbers be $a + d$, a, $a – d$. Then $(a + d) + a + (a – d) = 27 \Rightarrow a = 9$ and $(a + d) – 1$, $(a – 1)$, $(a – d) + 3$ are in G..P. $\Rightarrow$ $(a – 1)^2 = (a + d – 1)\,(a – d + 3)$]

10. The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is

(*a*) – 4 (*b*) – 12 (*c*) 12 (*d*) 4

[Hint. $\left.\begin{array}{ll} a + ar = 12 & .....(1) \\ ar^2 + ar^3 = 48 & ......(2) \end{array}\right\} \Rightarrow \frac{ar^2\,(1+r)}{a\,(1+r)} = \frac{48}{12} \Rightarrow r^2 = 4$

$\Rightarrow$ $r = –2$ (since the terms are +ve and –ve alternately)

$a = –12.$]

ANSWERS

1. (*b*)	**2.** (*d*)	**3.** (*e*)	**4.** (*b*)	**5.** (*b*)	**6.** (*b*)
7. $x^2 – 18x + 16 = 0$		**8.** (*a*)	**9.** (*b*)	**10.** (*d*)	

2

Binomial Theorem

Historical Note on Binomial Theorem

Binomial Theorem has got a long and chequered history. The earliest record, perhaps, is to be found in the Jaina work *Suryapajnapati* (500 B.C.) where the square root of *Kharab-yojanas* is given, correct to a high degree of approximation. It has the greatest constructive genius of Sir Isaac Newton, which discerned in the special cases given by his predecessors a general theorem of great value (1676). Newton has given no proof and it was only in the year 1826 that a complete rigorous proof was given by Abel, a Norwegian mathematician.

2.01. Factorial rotation

The product of n natural numbers from 1 to n is denoted by $n!$ or $|\underline{n}$ and is read as factorial n.

Thus, $n!$ or $|\underline{n} = 1.2.3......(n-1).n$

$4! = 1 \times 2 \times 3 \times 4 = 24$; $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$

$(n-1)! = 1 \times 2 \times 3......(n-1)$.

It is easily seen that 8! = 8 × (7!).

taken out

$n! = n\,(n-1)!$

One less

Note : Value of 0!

The value of 0! is assumed to be 1, *i.e.*, **0! = 1**.

2.02. Meaning of the symbol ${}^nC_r$

The symbol ${}^nC_r$ denotes the number of ways in which r things can be selected at a time out of n things. Thus, if there are 10 students in a group and out of these we have to select students taken 4 a time then ${}^{10}C_4$ would denote the number of ways in which it can be done. The value of ${}^{10}C_4$ is $\dfrac{10\times 9\times 8\times 7}{1\times 2\times 3\times 4} = 210$. So we can select 4 students out of 10 students in 210 ways.

In general, we have

$$\boxed{{}^nC_r = \frac{n(n-1)(n-2)\ldots r \text{ factors}}{r.(r-1)(r-2)\ldots 3.2.1}}$$

For example, ${}^5C_2 = \dfrac{5\times 4}{2\times 1} = 10$, ${}^7C_3 = \dfrac{7\times 6\times 5}{3\times 2\times 1} = 35$

Also, $^nC_r = \dfrac{n(n-1)(n-2).....r \text{ factors}}{1.2.3................r}$

$$= \frac{n(n-1)(n-2)........(n-\overline{r-1})}{1.2.3...........(r-1)r} \cdot \frac{(n-r)!}{(n-r)!}$$

$$= \frac{n(n-1)(n-2).....(n-\overline{r-1})\cdot(n-r)(n-r-1).....3.2.1.}{r!(n-r)!} = \frac{n!}{r!(n-r)!}$$

$\therefore$ $\boxed{\mathbf{{}^nC_r = \dfrac{n!}{r!(n-r)!}}}$

Cor. 1. Putting $r = n$, in the above result, we get

$$^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\,0!} = \frac{n!}{n!.1} = 1.$$

Cor. 2. $^nC_{n-r} = \dfrac{n!}{(n-r)!(n-\overline{n-r})!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r$.

Important results relating to the symbol ^nC_r.

1. $^nC_0 = 1$	**2.** $^nC_n = 1$	**3.** $^nC_r = {}^nC_{n-r}$	**4.** $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ $(1 \le r \le n)$
5. If $^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$			

You will learn the meaning in detail and the proof also in class 12 in the chapter on permutations and combinations.

Ex. 1. ***Find the value of (i)*** $^{15}C_3$ ***(ii)*** $^{10}C_8$ ***(iii)*** $^{51}C_{49}$

Sol. (*i*) $^{15}C_3 = \dfrac{15\times14\times13}{3\times2\times1} = \mathbf{455}$

(*ii*) $^{10}C_8 = {}^{10}C_2 = \dfrac{10\times9}{2\times1} = \mathbf{45}$. (Using $^nC_r = {}^nC_{n-r}$)

(*iii*) $^{51}C_{49} = {}^{51}C_{51-49} = {}^{51}C_2 = \dfrac{51\times50}{2\times1} = \mathbf{1275}$.

Ex. 2. ***If $^nC_{10} = {}^nC_{12}$, determine the value of n and hence $^nC_5$.***

Sol. $^nC_{10} = {}^nC_{12}$ $\Rightarrow$ $n = 10 + 12 = 22$

$\therefore$ $^nC_5 = {}^{22}C_5 = \dfrac{22\times21\times20\times19\times18}{5\times4\times3\times2\times1} = \mathbf{26334}$.

$^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

Here, $10 \neq 12$, so $10 + 12 = n$

Ex. 3. ***Evaluate:*** $^{31}C_{26} - {}^{30}C_{26}$.

Sol. Using $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, we have

$$^{30}C_{26} + {}^{30}C_{25} = {}^{31}C_{26}$$

$$\Rightarrow \quad {}^{31}C_{26} - {}^{30}C_{26} = {}^{30}C_{26} + {}^{30}C_{25} - {}^{30}C_{26} = {}^{30}C_{25}$$

$$= {}^{30}C_5 = \frac{30\times29\times28\times27\times26}{5\times4\times3\times2\times1} = \mathbf{142506.}$$

Test your understanding :

1. Evaluate:

(*i*) ${}^{9}C_4$ (*ii*) ${}^{20}C_{18}$ (*iii*) ${}^{15}C_6$ (*iv*) ${}^{25}C_{22} - {}^{24}C_{21}$

2. Verify that $2 \times {}^{7}C_4 = {}^{8}C_4$

3. Prove that (*i*) ${}^{2}C_1 + {}^{3}C_1 + {}^{4}C_1 = {}^{3}C_2 + {}^{4}C_2$

(*ii*) $1 + {}^{3}C_1 + {}^{4}C_2 = {}^{5}C_3$

4. If ${}^{n}C_8 = {}^{n}C_6$, find ${}^{n}C_2$.

5. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^{r}C_5$.

6. If $12.\,{}^{n}C_2 = {}^{2n}C_3$, find n.

[Hint. $12.\ \dfrac{n(n-1)}{2\times 1} = \dfrac{2n(2n-1)(2n-2)}{3\times 2\times 1} \Rightarrow 18.(n-1) = (2n-1).2(n-1)$]

ANSWERS

1. (*i*) 126 (*ii*) 190 (*iii*) 5005 (*iv*) 276

4. 91 **5.** 56 **6.** $n = 5$

2.03. Binomial expression

Def. An expression consisting of two terms is called a *binomial expression, e.g.*, $x + a$, $2x + 3y$, $5x^2 - 6y^2$, $2x - \dfrac{1}{3x}$ are all binomial expressions.

2.04. Binomial theorem

Binomial theorem helps us to expand any power of a given binomial expression. It was first established by Isaac Newton. First we take up the case when the index is a positive integer.

2.05. Development of binomial expansion

By actual multiplication, we may obtain the following expansions :

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = {}^{2}C_0\,x^2 + {}^{2}C_1\,xy + {}^{2}C_2\,y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = {}^{3}C_0\,x^3 + {}^{3}C_1\,x^2y + {}^{3}C_2\,x\,y^2 + {}^{3}C_3\,y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$= {}^{4}C_0\,x^4 + {}^{4}C_1\,x^3y + {}^{4}C_2\,x^2y^2 + {}^{4}C_3\,xy^3 + {}^{4}C_4\,y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10\,x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$= {}^{5}C_0\,x^5 + {}^{5}C_1\,x^4y + {}^{5}C_2\,x^3y^2 + {}^{5}C_3\,x^2y^3 + {}^{5}C_4\,x\,y^4 + {}^{5}C_5\,y^5.$$

A careful observation of these expansions shows that $(x + y)^n$ (where $n = 1, 2, 3, 4, 5$), when expanded, has the following properties :

1. The coefficients form a certain pattern as shown below :

Pascal's Triangle

Number n	*Coefficients in the expansion of* $(x + y)^n$
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Inspection will show that each term in the table is derived by adding together the two terms in the line above, which lie on either side of it. Thus in the line for $n = 5$, the term 10 is found by adding together the terms 4 and 6 in the line $n = 4$.

2. The first term in the expansion is (first term of the binomial)^{index}, *i.e.*, x^n, *i.e.*, ${}^nC_0\, x^n$.
3. The second term is $nx^{n-1}y$.
4. As the expansion progresses, the exponent of x decreases by one and the exponent of y increases by one.
5. The total number of terms in the expansion is one more than the index *i.e.*, $n + 1$.
6. The last term is the $(n + 1)$th term and is (second term of the binomial)^{index}, *i.e.*, y^n or ${}^nC_n y^n$.
7. The coefficients of the various terms in the expansion from the first term onwards are ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_n$.

Assuming that the above properties hold for all integral values of n, we may write the five terms in the expansion of $(x + y)^n$ as follows :

First term = ${}^nC_0\, x^n = x^n$

Second term = ${}^nC_1\, x^{n-1}\, y = n\, x^{n-1}\, y$

$$\text{Third term } = {}^nC_2\, x^{n-2}\, y^2 = \frac{n(n-1)}{1\times 2} x^{n-2} y^2$$

$$\text{Fourth term} = {}^nC_3\, x^{n-3}\, y^3 = \frac{n(n-1)\,(n-2)}{1\times 2\times 3} x^{n-3} y^3$$

$$\text{Fifth term } = {}^nC_4\, x^{n-4}\, y^4 = \frac{n(n-1)\,(n-2)\,(n-3)}{1\times 2\times 3\times 4} x^{n-4} y^4$$

Continuing the above process, it is easily seen that last but one term [*i.e.*, nth term] of the expansion $= {}^nC_{n-1}\, xy^{n-1} = nxy^{n-1}$

Last term [*i.e.*, $(n + 1)$th term] = ${}^nC_n\ y^n = y^n$.

Thus the expansion of $(x + y)^n$, where x and y are any two *real* numbers and n is any positive integer, may be stated as follows :

$$\boldsymbol{(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots.. + {}^nC_r x^{n-r} y^r \ldots.. + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n y^n} \quad ...(1)$$

$$\text{or} \quad \boldsymbol{x^n + nx^{n-1} y + \frac{n(n-1)}{1\times 2} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3} x^{n-3} y^3 + \ldots\ldots + y^n}.$$

In short form : $\boldsymbol{(x + y)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r} y^r}$.

Note. Observe that the **number of terms in the expansion of** $(x+y)^n$ **is** $(n+1)$, *i.e.*, **one more than the index.**

The numbers ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ etc. are called **binomial coefficients.**

Cor. Putting 1 for x and x for y

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^3 + \ldots\ldots + x^n.$$

Deductions

1. Replacing y by $-y$, we get

$$(x-y)^n = {}^nC_0 \,.\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 - \ldots. + (-1)^r\, {}^nC_r\, x^{n-r}\, y^r + \ldots. + (-1)^n \,.\, {}^nC_n\, y^n \quad \ldots(2)$$

$$i.e.,\ (x-y)^n = \sum_{r=0}^{n} (-1)^r \,.\, {}^nC_r\ x^{n-r} \,.\, y^r$$

2. Replacing x by 1 and y by x in (1) we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2 \,.\, x^2 + \ldots\ldots + {}^nC_r \,.\, x^r + \ldots. + {}^nC_n \,.\, x^n$$

$$i.e.,\ (1+x)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^r$$

3. Replacing x by 1 and y by $-x$ in (1) we get

$$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\ x^2 - \ldots\ldots + (-1)^r \,.\, {}^nC_r\ x^r + \ldots. + (-1)^n .\ {}^nC_n\ x^n$$

$$i.e.,\ (1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^n C_r \cdot x^r$$

4. Adding (1) and (2), we get

$(x+y)^n + (x-y)^n = 2[x^n + {}^nC_2\, x^{n-2}\, y^2 + {}^nC_4\, x^{n-4}\, y^4 + \ldots.]$ = **2 [sum of terms at odd places]**

The last term is ${}^nC_n \,.\, y^n$ or ${}^nC_{n-1}\,,\ xy^{n-1}$ according as n is even or odd respectively.

5. Subtracting (2) from (1), we get

$(x+y)^n - (x-y)^n = 2[{}^nC_1\, x^{n-1}\, y + {}^nC_3\, x^{n-3}\, y^3 + \ldots.]$ = **2 [sum of terms at even places]**

The last term is ${}^nC_{n-1}\ xy^{n-1}$ or ${}^nC_n\ y^n$ according as n is even or odd respectively.

Note. Pascal. Many Mathematicians have done their best work at an early age. One of the most celebrated of these was Blaise Pascal (1623 – 1662). He did not invent the triangular array of numbers that has been named after him. It was Pascal, however, who saw the relationship of the triangular array to the expansion of a binomial. He also found connections between the triangle and such problems as finding the number of different combinations possible from a given set of objects. Pascal made significant advances in algebra and was one of the founders of probability theory.

2.05. Proof of binomial theorem for positive integral index

To prove that when n is a positive integer

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + \ldots\ldots.. + {}^nC_r\, x^{n-r}a^r + \ldots\ldots.. + {}^nC_n\, a^n.$$

We shall prove it by the method of induction.

By actual multiplication we have

$$(x + a)^1 = {}^1C_0\, x^1 + {}^1C_1\, x^{1-1}\, a^1$$

$$(x + a)^2 = x^2 + 2ax + a^2 = {}^2C_0\, x^2 + {}^2C_1\, xa + {}^2C_2\, a^2$$

$$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 = {}^3C_0\, x^3 + {}^3C_1\, x^2a + {}^3C_2\, xa^2 + {}^3C_3\, a^3.$$

Thus we notice that the theorem is true for $n = 1$, $n = 2$ and $n = 3$.

Let us *assume* that it is true for a particular integral power $n = m$, say. Then

$$(x + a)^m = {}^mC_0\, x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a^2 + ... + {}^mC_r x^{m-r}\, a^r + ... + {}^mC_m\, a^m \quad ...(1)$$

Multiplying both sides by $(x + a)$, we get

$$(x + a)^{m+1} = {}^mC_0\, x^{m+1} + {}^mC_1 x^m a + {}^mC_2\, x^{m-1}\, a^2 + ... + {}^mC_r x^{m-r+1}\, a^r + ... + {}^mC_m x a^m$$

$$+ C_0\, x^m a + {}^mC_1\, x^{m-1}\, a^2 + ...+ {}^mC_{r-1}\, x^{m-r+1} a^r + ... + {}^mC_{m-1}\, xa^m + {}^mC_m\, a^{m+1}$$

$$(x + a)^{m+1} = {}^{m+1}C_0 x^{m+1} + {}^{m+1}C_1\, x^m\, a + {}^{m+1}C_2\, x^{m-1}\, a^2 + ... + {}^{m+1}C_r\, x^{m+1-r}\, a^r$$

$$+ .... + {}^{m+1}C_m x\, a^m + {}^{m+1}C_{m+1}\, a^{m+1} \quad ...(2)$$

$[\because\ {}^mC_0 = {}^{m+1}C_0,\ {}^mC_m = {}^{m+1}C_{m+1},\ {}^{m+1}C_r = {}^mC_r + {}^mC_{r-1}]$.

We notice that the successive terms in (2) are of the same form as in (1) except that m is changed into $m + 1$.

Therefore, if the theorem is true for $n = m$, it is also true for $n = m + 1$. But the theorem is true for $n = 3$.

Therefore, it is true for $n = 3 + 1$, *i.e.* 4. If it is true for $n = 4$, it is also true for $n = 4 + 1$, *i.e. 5.* Continuing this reasoning it follows that the theorem is true for n = 6, 7, 8, etc. *i.e.* true for all positive integral values of n.

Method II. The expansion of $(x + a)^n$ is obtained by finding the continued product of n binomial factors, each equal to $(x + a)$. Hence, every term in the expansion is formed by multiplying together n letters, *one taken from each of the n factors*, and, as such, is of *n dimensions.* Therefore, if the index of the power of 'x' in any term be $n - r$, the index of the power of 'a' in *that* term must be 'r', in order that the sum of the indices of the powers of 'x' and 'a' in the term may be equal to 'n'.

Thus each term in the product must be of the form $x^{n-r}\, a^r$, being obtained by multiplying together a's taken from any r of the factors and x's from the *remaining n–r* factors. Therefore, there will be as many terms involving $x^{n-r}\, a^r$ as there are ways of selecting *any r* of the factors out of n. Hence, the coefficient of $x^{n-r}\, a^r$ is ${}^nC_r$.

Since r denotes the number of factors, from which a's are taken out of n factors on the whole, r must be a *positive integer less than,* or at most *equal* to, *n;* also r may have the value *zero,* for by multiplying all the *first letters, x's,* together, we shall get a term in the expansion, in which the power of a is evidently *zero.*

Thus, ${}^nC_r x^{n-r}\, a^r$ denotes a term in the required expansion for all positive integral values of r from 0 to n. Hence, by giving to r the values 0, 1, 2, 3 .... , r, ... n, we have

$$(x + a)^n = {}^nC_0 x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2 x^{n-2}\, a^2 + ...+ {}^nC_r x^{n-r}\, a^r + ... + {}^nC_{n-1}\ x\, a^{n-1} + {}^nC_n\, a^n.$$

Ex. 4. ***Expand*** $(1 + 4x)^5$**.**

Sol. $(1 + 4x)^5 = 1 + {}^5C_1 4x + {}^5C_2(4x)^2 + {}^5C_3(4x)^3 + {}^5C_4(4x)^4 + (4x)^5$

$$= 1 + 5 \times 4x + \frac{5\times 4}{1\times 2} \times 16x^2 + \frac{5\times 4}{1\times 2} \times 64x^3 + 5 \times 256x^4 + 1024x^5$$

$$[\because\ {}^5C_3 = {}^5C_2\,;\ {}^5C_4 = {}^5C_1]$$

$$= 1 + 20x + 160x^2 + 640x^3 + 1280x^4 + 1024x^5.$$

Ex. 5. ***Expand*** $\boldsymbol{(3x - 2y)^4}$**.**

Sol. $(3x - 2y)^4 = (3x)^4 + {}^4C_1 \times (3x)^3 \times (-2y)$

$$+ {}^4C_2 \times (3x)^2 \times (-2y)^2 + {}^4C_3(3x)(-2y)^3 + (-2y)^4$$

$$= 81x^4 - 4 \times 27 \times 2 \times x^3y + \frac{4\times 3}{2\times 1}\ 9x^2 \times 4y^2 - 4 \times 3x \times 8y^3 + 16y^4$$

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.$$

Ex. 6. ***Expand*** $\left(x + \frac{1}{x}\right)^6$. $\quad (x \quad 0)$

Sol. $\left(x+\frac{1}{x}\right)^6 = x^6 + {}^6C_1 \times x^5 \times \frac{1}{x} + {}^6C_2 \times x^4 \times \frac{1}{x^2} + {}^6C_3 \times x^3 \times \frac{1}{x^3} + {}^6C_4 \times x^2 \times \frac{1}{x^4}$

$$+ {}^6C_5 \times x \times \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + \frac{6\times 5}{2\times 1}x^2 + \frac{6\times 5\times 4}{1\times 2\times 3} + \frac{6\times 5}{1\times 2}\times\frac{1}{x^2} + 6\times\frac{1}{x^4} + \frac{1}{x^6}$$

$$[\because\ {}^6C_4 = {}^6C_2,\ {}^6C_5 = {}^6C_1]$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}.$$

Ex. 7. ***Find the value of*** $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$ ***and show that the value of*** $(\sqrt{2}+1)^6$ ***lies between 197 and 198.*** **[I.S.C., I.I.T]**

Sol. We know that $(x+a)^n + (x-a)^n = 2\left[{}^nC_0\, x^n a^0 + {}^nC_2 x^{n-2}a^2 + ...\right]$. Here two binomials are given to us, which are raised to the same power and whose second terms are numerically equal but of opposite sign. On expanding the two, it is obvious that the second, fourth and sixth terms containing odd powers of 1 and –1 will cancel out and the remaining will be added up. Thus,

$$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[(\sqrt{2})^6 + {}^6C_2\,(\sqrt{2})^4 + {}^6C_4(\sqrt{2})^2 + 1]$$

$$2 \quad 8 \quad \frac{6 \quad 5}{1 \quad 2} \quad 4 \quad \frac{6 \quad 5}{1 \quad 2} \quad 2 \quad 1 \qquad 2\,(8 \quad 60 \quad 30 \quad 1) \quad 2 \quad 99 \quad 198.$$

Also, $\because (\sqrt{2}-1)^6 = (1.42-1)^6 = (0.42)^6 < 1$

$\therefore (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 198 \Rightarrow (\sqrt{2}+1)^6 = 198 - (\sqrt{2}-1)^6$

$\Rightarrow (\sqrt{2}+1)^6 = 198\ -$ A number between 0 and 1

$\Rightarrow (\sqrt{2}+1)^6 =$ A number between 197 and 198.

$\Rightarrow$ Integral part of $(\sqrt{2}+1)^6$ is 197.

Ex. 8. ***Use the binomial theorem to find the exact value of (10.1)^5.*** **[S.C.]**

Sol. In such questions, first we express the given quantity as a binomial and then expand using the rules of binomial expansion. Thus, $(10.1)^5 = (10 + 0.1)^5$

$$= (10)^5 + {}^5C_1 \times (10)^4 \times (0.1) + {}^5C_2 \times (10^3) \times (0.1)^2 + {}^5C_3 \times (10)^2 \times (0.1)^3 + {}^5C_4 \times 10 \times (0.1)^4 + (0.1)^5$$

$$= 100\,000 + 5 \times 10\,000 \times 0.1 + \frac{5 \times 4}{1 \times 2} \times 1000 \times 0.01 + \frac{5 \times 4}{1 \times 2} \times 100 \times 0.001 + 5 \times 10 \times 0.0001 + 0.00001$$

$$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501.$$

Ex. 9. ***Expand (2 + x + x^2)^3.*** **[S.C.]**

Sol. Regarding $(2 + x)$ as one term, let $2 + x = y$. Then

$$[2 + x + x^2]^3 = (y + x^2)^3 = {}^3C_0\, y^3 + {}^3C_1\, y^2 x^2 + {}^3C_2\, y^1 (x^2)^2 + {}^3C_3\, (x^2)^3$$

$$= y^3 + 3y^2 x^2 + 3yx^4 + x^6$$

$$= (2 + x)^3 + 3(2 + x)^2 x^2 + 3\,(2 + x)\, x^4 + x^6$$

$$= 2^3 + 3 \times 2^2 \times x + 3 \times 2x^2 + x^3 + 3x^2\,(4 + 4x + x^2) + 3x^4\,(2 + x) + x^6$$

$$= 8 + 12x + 18x^2 + 13x^3 + 9x^4 + 3x^5 + x^6.$$

EXERCISE 2 (*a*)

1. What is the number of terms in the expansion of $[(x - 2y)^3]^3$? **[S.C.]**

2. Write out the expansions of the following :

(*a*) $(3x - y)^4$ (*b*) $(3 + 2x^2)^4$ (*c*) $\left(x - \frac{y}{2}\right)^4$ (*d*) $\left(2x + \frac{y}{2}\right)^5$ (*e*) $(1 + 2x)^7$

3. Expand $(2 + x)^5 - (2 - x)^5$ in ascending powers of x and simplify your result. **[S.C.]**

4. Evaluate the following :

(*a*) $(2 + \sqrt{5})^5 + (2 - \sqrt{5})^5$ (*b*) $(\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5$.

Hence, show in (*b*), without using tables, that the value of $(\sqrt{3} + 1)^5$ lies between 152 and 153.

[Hint. Type Solved Ex. 4] **[*I.S.C. 1989 Type*]**

5. If the first three terms in the expansion of $(1 + ax)^n$ in ascending powers of x are $1 + 12x + 64x^2$, find n and a. **[S.C.]**

6. Find the first three terms in the expansion of $[2 + x\,(3 + 4x)]^5$ in ascending powers of x.

7. Expand $(1 + 2x + 3x^2)^n$ in a series of ascending powers of x up to and including the term in x^2. **[S.C.]**

8. Expand $\left(y + \frac{1}{10y}\right)^8$ and use your expansion to evaluate $(1.1)^8$ correct to four places of decimals. **[S.C.]**

9. Write down and simplify the first five terms of the expansion of $\left(1 - \frac{x}{2}\right)^{10}$.

Use your result to find the value of £ 1000 $(0.95)^{10}$, correct to the nearest pound. **[S.C.]**

10. Write down the expansion by the binomial theorem of $\left(3x - \frac{y}{2}\right)^4$. By giving x and y suitable values, deduce the value of $(29.5)^4$ correct to four significant figures. **[S.C.]**

11. Using binomial theorem, evaluate : $(999)^3$. **[I.S.C]**

12. Expand $(1 + x)(1 - x)^6$ in ascending powers of x ; show that the result is also the expansion of $(1 - x^2)(1 - x)^5$. **[S.C.]**

13. Expand $(a + b)^4$ by the binomial theorem.
Hence, or otherwise, prove that

$$(1 - 2x)^4 + 4x^2(1 - 2x)^3 + 6x^4(1 - 2x)^2 + 4x^6(1 - 2x) + x^8 = (1 - x)^8.$$ **[S.C.]**

14. Write down in terms of x and n, the term containing x^3 in the expansion of $\left(1 - \frac{x}{n}\right)^n$ by the binomial theorem. If this term equals $\frac{7}{8}$ when $x = -2$, and n is a positive integer, calculate the value of n. **[I.S.C.]**

15. (*i*) Obtain the binomial expansion of $(2-\sqrt{3})^6$ in the form $a + b\sqrt{3}$, where a and b are integers. State the corresponding result for the expansion $(2+\sqrt{3})^6$ and (*ii*) show that $(2-\sqrt{3})^6$ is the reciprocal of $(2+\sqrt{3})^6$.

[Hint. (*ii*) Show that $(2+\sqrt{3})^6\,(2-\sqrt{3})^6 = 1$] **[I.S.C.]**

16. Find the coefficient of x^5 in the expansion of $(1 + 2x)^6(1 - x)^7$.

[Sol. $(1 + 2x)^6(1 - x)^7 = [1 + {}^6C_1.(2x) + {}^6C_2.(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + (2x)^6]$

$$\times [1 - {}^7C_1 x + {}^7C_2 x^2 - {}^7C_3 x^3 + {}^7C_4 x^4 - {}^7C_5 x^5 + {}^7C_6 x^6 - x^7$$

$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)$$

$$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$$

Terms containing x^5 in the product are obtained on multiplying constant term by the term containing x^5, term containing x by the term containing x^4, and so on.

These products are $-21x^5 + 420x^5 - 2100x^5 + 3360x^5 - 1680x^5 + 192x^5$

coefft. of $x^5 = (-21 + 420 - 2100 + 3360 - 1680 + 192) = 171$]

17. (*a*) If the coefficients of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A. P., show that $2n^2 - 9n + 7 = 0$.

(*b*) Let n be a positive integer. If the coefficients of 2nd, 3rd, 4th terms in the expansion of $(1 + x)^n$ are in A.P., then find the value of n.

[Hint: (*a*) $T_2 = {}^{2n}C_1 x$, $T_3 = {}^{2n}C_2 . x^2$, $T_4 = {}^{2n}C_3 x^3$. These being in A.P.

$2.{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3 \quad 2n^2 - 9n + 7 = 0]$

(*b*) $n^2 - 9n + 14 = 0 \quad x = 7$, rejecting the other value $n = 2$ because when $n = 2$, then there are only three terms in the expansion of $(1 + x)^2$]

18. In the binomial expansion of $(\sqrt[3]{3} + \sqrt{2})^5$ find the term which does not contain Irrational expression. **(I.S.C. 2002)**

ANSWERS

1. 10. **2.** (*a*) $81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$

(*b*) $81 + 216x^2 + 216x^4 + 96x^6 + 16x^8$ (*c*) $x^4 - 2x^3y + \frac{3}{2}x^2y^2 - \frac{1}{2}xy^3 + \frac{1}{16}y^4$

(*d*) $32x^5 + 40x^4y + 20x^3y^2 + 5x^2y^3 + \frac{5}{8}xy^4 + \frac{1}{32}y^5$

(*e*) $1 + 14x + 84x^2 + 280x^3 + 560x^4 + 672x^5 + 448x^6 + 128x^7$

3. $160x + 80x^3 + 2x^5$ **4.** (*a*) 1364 (*b*) 152 **5.** $9, \frac{4}{3}$ **6.** $32 + 240x + 1040x^2$

7. $1 + 2nx + x^2(2n^2 + n)$ **8.** $y^8 + 0.8y^6 + 0.28y^4 + 0.056y^2 + \ldots\ldots$; 2.1436

9. $1 - 5x + \frac{45}{4}x^2 - 15x^3 + \frac{105}{8}x^4$; £ 599 **10.** $81x^4 - 54x^3y + \frac{27}{2}x^2y^2 - \frac{3}{2}xy^3 + \frac{y^4}{16}$; 757300

11. 997002999 **12.** $1 - 5x + 9x^2 - 5x^3 - 5x^4 + 9x^5 - 5x^6 + x^7$

13. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ **14.** $\frac{-(n-1)(n-2)}{6n^2}.x^3$; $n = 8$

15. $1351 - 780\sqrt{3}$, $1351 + 780\sqrt{3}$ **18.** $T_3 = 60$.

2.06. General term T_{r+1}

Suppose we have to find the $(r+1)^{th}$ term in the expansion of $(x + y)^n$. In its expansion

$$T_3 = {}^nC_2 \times x^{n-2}y^2,\ T_4 = {}^nC_3\, x^{n-3}y^3,\ T_5 = {}^nC_4\, x^{n-4}y^4.$$

We can see that all these terms have the following common properties :

(*i*) The suffix of C is one less than the number of the term.

(*ii*) The power of the first factor x is equal to the difference of the upper and lower suffixes of C.

(*iii*) The power of the second factor y and the suffix of C are the same.

(*iv*) The sum of the powers of x and y is equal to the index of the given binomial.

Hence the $(r + 1)$th term, T_{r+1} is given by

$$\boxed{T_{r+1} = {}^nC_r\, x^{n-r}y^r}$$

Note. In the expansion of $(x - y)^n$, $T_{r+1} = {}^nC_r \times x^{n-r} \times (-y)^r = (-1)^r \times {}^nC_r\, x^{n-r}y^r$.

Ex. 10. ***Find the tenth term in the expansion of $(2x - y)^{11}$.***

Sol. $T_{10} = T_{9+1} = {}^{11}C_9\,(2x)^{11-9}(-y)^9 = {}^{11}C_2\,(2x)^2(-y)^9$

$$= -\frac{11\times 10}{1\times 2}\times 4x^2y^9 = -220x^2y^9.$$

Middle term.

Ex. 11. ***Write the middle term or terms in the expansion of***

(i) $\left(x^2 - \frac{1}{x}\right)^6$ ***(ii)*** $\left(3a - \frac{a^3}{6}\right)^9$

Sol. (*i*) There are 7 terms in the expansion and so the 4th term will be the middle term.

$$T_4 = {}^6C_3(x^2)^{6-3}\times\left(-\frac{1}{x}\right)^3 = -20x^3.$$

(*ii*) There are 10 terms in the expansion and so the 5th and the 6th terms will be the middle terms.

$$T_5 = {}^9C_4\times(3a)^5\times\left(-\frac{a^3}{6}\right)^4 = \frac{189a^{17}}{8};\ T_6 = {}^9C_5\times(3a)^4\times\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}a^{19}.$$

Ex. 12. ***The number of integral terms in the expansion of*** $(\sqrt{3}+\sqrt[8]{5})^{256}$ ***is***

(a) 32 ***(b) 33*** ***(c) 34*** ***(d) 35***

Sol. $T_{r+1} = {}^{256}C_r \,.\, (\sqrt{3})^{256 \;\; r} \,.\, (\sqrt[8]{5})^r \quad {}^{256}C_r \,.\, (3)^{\frac{256 \;\; r}{2}} \,.\, (5)^{r/8}$

Terms would be integral if $\frac{256 \;\; r}{2}$ and $\frac{r}{8}$ both are +ve integers.

As $0 \le r \le 256$ so $r = 0, 8, 16, 24, \ldots\ldots, 256$.

For the above values of r, $\frac{256 \;\; r}{2}$ are also integers.

Total number of values of r = **33.** **Ans.** (*b*)

Coefficient of a particular power of *x*.

Procedure. 1. *Let the particular power occur in the* $(r + 1)^{th}$ *term.*

2. *Write the* $(r + 1)$ *th term of the given binomial.*

3. *Equate the power of x in the* $(r + 1)$ *th term and the given power.*

4. *Evaluate r and substitute it in* (2).

Note. The value of r should be a positive integer less than the index of the binomial.

Ex. 13. ***Find the coefficient of*** x^{15} ***in the expansion of*** $(x - x^2)^{10}$.

Sol. Let x^{15} occur in the $(r + 1)$th term. Then

$$T_{r+1} = {}^{10}C_r\, x^{10-r} \times (-x^2)^r = {}^{10}C_r\, x^{10-r} \times x^{2r} \times (-1)^r = (-1)^r \times {}^{10}C_r x^{10+r}$$

$$x^{10+r} = x^{15} \qquad 10 + r = 15 \qquad r = 5$$

The required coefficient $= {}^{10}C_5\,(-1)^5$

$$\frac{10 \quad 9 \quad 8 \quad 7 \quad 6}{1 \quad 2 \quad 3 \quad 4 \quad 5} \quad 252.$$

Ex. 14. ***If for positive integers r > 1, n > 2, the coefficients of the (3r)th and (r + 2)th powers of x in the expansion of*** $(1 + x)^{2n}$ ***are equal, then prove that n = 2r + 1.***

Sol. We have, $T_{3r} = {}^{2n}C_{3r-1}\, x^{3r-1}, \; T_{r+2} = {}^{2n}C_{r+1}\, x^{r+1}$

Given, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

Either $3r - 1 = r + 1$ or $3r - 1 + (r + 1) = 2n$

$\Rightarrow r = 1$ or $r = \frac{n}{2}$ $\therefore r = \frac{n}{2}$. But r is a positive integer greater than 1, so rejecting $r = 1$, we have, $r = \frac{n}{2}$, provided n is an even integer (> 2), otherwise r has no value.

Term independent of *x*.

Procedure. 1. *Let* $(r + 1)$*th term be the term independent of x.*

2. *Put the power of x in this term equal to zero and evaluate r.*

Ex. 15. ***Find the term independent of x in the expansion of***

$$\frac{3}{2}x^2 \quad \frac{1}{3x} \quad ^9 .$$

Sol. Let $(r+1)$th term be the term independent of x.

Then, $T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(-\frac{1}{3x}\right)^r = (-1)^r \, {}^9C_r \, \frac{3^{9-2r}}{2^{9-r}} \, x^{18-3r}$

Putting $18 - 3r = 0$, $\Rightarrow$ $r = 6$.

$$\text{The required term} = (-1)^6 \times {}^9C_6 \times \frac{3^{-3}}{2^3} = {}^9C_3 \times \frac{1}{3^3 \times 2^3} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} \times \frac{1}{27 \times 8} = \frac{7}{18}.$$

Ex. 16. ***The 2nd, 3rd and 4th terms in the expansion of $(x+y)^n$ are 240, 720 and 1080 respectively; find the values of, x, y and n.***

Sol. $nx^{n-1}y = 240$, $\frac{n(n-1)}{1.2}x^{n-2}y^2 = 720$ and $\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 = 1080$

Dividing. $\frac{n-2}{3}.\frac{y}{x} = \frac{3}{2}$ and $\frac{n-1}{2}.\frac{y}{x} = 3$ $\therefore$ $\frac{2(n-2)}{3} = \frac{n-1}{2} \Rightarrow n = 5$

Substituting the value of n, we get $x = 2, y = 3$.

EXERCISE 2 (*b*)

1. Find the specified term of the expression in each of the following binomials :

(*a*) Fifth term of $(2a+3b)^{12}$. Evaluate it when $a = \frac{1}{3}, b = \frac{1}{4}$.

(*b*) Sixth term of $\left(2x - \frac{1}{x^2}\right)^7$. **(*I.S.C. 2000*)**

(*c*) Middle term of $\left(2x - \frac{1}{y}\right)^8$.

(*d*) Middle term of $\left(x^4 - \frac{1}{x^3}\right)^{11}$.

(*e*) Middle term of $\left(\frac{x^2}{4} - \frac{4}{x^2}\right)^{10}$ **(*I.S.C 2003*)**

2. Find the term independent of x in the expansion of the following binomials :

(*a*) $x \quad \frac{1}{x} \quad ^{14}$

(*b*) $\left(\sqrt{\frac{x}{3}} - \frac{\sqrt{3}}{2x}\right)^{12}$ **[*I.S.C. 1990*]**

(*c*) $\left(2x^2 - \frac{1}{x}\right)^{12}$. What is its value ? **[*S.C.*]**

3. Find the coefficient of

(*a*) a^6b^3 in the expansion of $\left(2a - \frac{b}{3}\right)^9$ **[*S.C.*]**

(*b*) x^7 in the expansion of $x^2 \quad \frac{1}{x} \quad ^{11}$

(*c*) $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

(*d*) x^4 in the expansion of $\frac{x}{2} \quad \frac{3}{x^2} \quad ^{10}$.

(DCE 2000, Raj. PET 2001, UPSEAT 2002, I.I.T, EAMCET)

4. If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are the same, find the value of *a*. **[S.C.]**

5. Write down the fourth term in the binomial expansion of $px \quad \frac{1}{x} \quad ^{n}$. If this term is independent of *x*, find the value of *n*. With this value of *n*, calculate the value of *p* given that the fourth term is equal to $\frac{5}{2}$. **[I.S.C.]**

6. The expansion by the binomial theorem of $\left(2x+\frac{1}{8}\right)^{10}$ is $1024x^{10} + 640x^9 + ax^8 + bx^7 + ...$ Calculate

(*i*) the numerical value of *a* and *b;* (*ii*) coefficient of x^8 in $(3x – 2)\left(2x+\frac{1}{8}\right)^{10}$;

(*iii*) the value of *x*, for which the third and the fourth terms in the expansion of $\left(2x+\frac{1}{8}\right)^{10}$ are equal. **[I.S.C.]**

7. Obtain the first three terms in the binomial expansion of $\left(\frac{q}{2x}-px\right)^{n}$. If the fifth term is independent of *x*, find the value of *n*. With this value of *n* calculate the values of *p* and *q* given that the fifth term is equal to 1120, *p* is greater than *q* and $p + q = 5$. **[I.S.C.]**

8. Find the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in $\left(ax+\frac{1}{bx^2}\right)^{11}$. If these coefficients are equal, find the relation between *a* and *b*. **[I.S.C.]**

9. In a binomial expansion, $(x + a)^8$, the first three terms are 1, 56 and 1372 respectively. Find values of *x* and *a*.

[Hint. Type solved Ex. 13]

10. Write the 4th term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^{9}$.

[Hint. In the expansion of $(x + y)^n$, *r* th term from the end

$= [(n + 1) – r + 1]$th term from the beginning.

Here 4th term from the end = (10 – 4 + l)th, *i.e.*, 7th term from the beginning.]

11. The coefficients of $(2r + 1)$ th and $(r + 2)$th terms in the expansions of $(1 + x)^{43}$ are equal. Find the value of *r.*

12. The coeffiecint of the middle term in the binomial expansion in powers of *x* of $(1 + \quad x)^4$ and of $(1 – \quad x)^6$ is the same if equals (*a*) –3/10 (*b*) 10/3 (*c*) –5/3 (*d*) 3/5

[Hint. The expansion of $(1 + ax)^4$ contains 5 terms, so 3rd term will be the middle term. Similarly, in the expansion of $(1 – \quad x)^6$, 4th term will be the middle term.

Since the coeffts. in the two terms are given to be equal, we have

$^4C_2 . \quad ^2 = {}^6C_3 . (– \quad)^3 \qquad 6 = – 20 \qquad = \frac{3}{10}$.]

13. Find the sixth term of the expansion of $(y^{1/2} + x^{1/3})^n$, if the binomial coefficient of the third term from the end is 45. ***[I.S.C. 1999 Type, 1992]***

[Hint. The third term from the end is equal to $\{(n + 1) - 3 + 1\}$th, *i.e.* $(n - 1)$th term from the beginning.]

14. Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

[Sol. The expansion of $(1 + x)^{2n}$ contains $(2n + 1)$ terms and therefore, the middle term is the $(n + 1)$th term.

Middle term in the expansion of $(1 + x)^{2n}$ is given by

$$T_{n+1} = {}^{2n}C_n . x^n = \frac{(2n)!}{(n!).(n!)}.x^n = \frac{(2n)!}{(n!)^2}.x^n$$

Coefft.of middle term of $(1 + x)^{2n}$ is $\dfrac{(2n)!}{(n!)^2}$. ...(1)

Now, the expansion of $(1 + x)^{2n-1}$ contains $2n$ terms. So, it has two middle terms, *viz.* the nth and the $(n + 1)$th terms.

$$T_n = {}^{2n-1}C_{n-1} . x^{n-1},\ T_{n+1} = {}^{2n-1}C_n . x^n$$

Sum of the coeffts of middle terms of $(1 + x)^{2n-1} = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$$= {}^{2n-1+1}C_n = {}^{2n}C_n = \frac{(2n)!}{n!\times(2n-n)!} = \frac{(2n)!}{(n!)^2}\Bigg] \text{(Applying } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$$

= coefficient of middle term of $(1 + x)^{2n}$ [From (1)]]

15. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\dfrac{1.3.5 \ldots. (2n-1)}{n!}.2^n x^n$. where n N.

[Hint. See Q. 14 above. Middle term = $\dfrac{(2n)!}{(n!)\times(n!)}x^n = \dfrac{1.2.3.4 \ldots.. (2n-2).(2n-1)(2n)}{(n!)\times(n!)}.x^n$

$$= \frac{[1.3.5. \ldots (2n-3)(2n-1)]\times[2.4.6. \ldots (2n-2)(2n)]}{(n!)(n!)}.x^n$$

$$= \frac{[1.3.5. \ldots (2n-1)].2^n.[1.2.3. \ldots (n-1)n]}{(n!)(n!)}.x^n]$$

16. Find the coefficient of x^5 in the expansion of

$$1 + (1 + x) + (1 + x)^2 + \ldots.. + (1 + x)^{10}.$$

[Hint. Given exp. = $\dfrac{(1+x)^{11}-1}{(1+x)-1} = \dfrac{(1+x)^{11}-1}{x}$ [Sum of a G.P.]

Coefft. of x^5 in the given exp. = Coefft. of x^5 in $\left[\dfrac{(1+x)^{11}-1}{x}\right]$.

= coefft. of x^6 in $[(1 + x)^{11} - 1]$ $= {}^{11}C_6 = {}^{11}C_5 = 462$]

17. If x^p occurs in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{2n}$, prove that its coefficient is

$$\frac{(2n)!}{\left[\frac{1}{3}(4n-p)!\right]\left[\frac{1}{3}(2n+p)!\right]}$$

[Hint. $T_{r+1} = {}^{2n}C_r \,.\, x^{(4n-3r)}$. Putting $4n - 3r = p$, we get $r = \frac{1}{3}(4n-p)$

coefft.of $x^p = {}^{2n}C_{\frac{1}{3}(4n-p)}$.]

18. If P be the sum of odd terms and Q be the sum of even terms in the expansion of $(x + a)^n$, prove that (*i*) $P^2 - Q^2 = (x^2 - a^2)^n$, (*ii*) $4PQ = (x + a)^{2n} - (x - a)^{2n}$ and (*iii*) $2(P^2 + Q^2) = (x + a)^{2n} + (x-a)^{2n}$.

[Sol. We know that

$$(x + a)^n = T_1 + T_2 + T_3 + T_4 + ... + T_{n+1},\ (x - a)^n = T_1 - T_2 + T_3 - T_4 + ... + (-1)^n T_{n+1}$$

P = Sum of odd term = $T_1 + T_3 + T_5 + ...$; Q = Sum of even terms = $T_2 + T_4 + T_6 + ...$

$(x + a)^n = P + Q$ and $(x - a)^n = P - Q$

$P^2 - Q^2 = (P + Q)(P - Q) = (x + a)^n (x - a)^n = (x^2 - a^2)^n$.

$4PQ = (P + Q)^2 - (P - Q)^2 = (x + a)^{2n} - (x - a)^{2n}$].

19. If the coeffts. of the rth, $(r + 1)th$ and $(r + 2)th$ terms in the expansion of $(1 + x)^n$ are in *A.P.*, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

[Sol. Coefft. of $T_r = {}^nC_{r-1}$, coefft. of $T_{r+1} = {}^nC_r$, coefft. of $T_{r+2} = {}^nC_{r+1}$

Since they are in A.P., therefore, ${}^nC_{r-1} + {}^nC_{r+1} = 2.\,{}^nC_r$.

$$\frac{n!}{(r\quad 1)!(n\quad r\quad 1)!} \qquad \frac{n!}{(r\quad 1)!(n\quad r\quad 1)!} \quad 2.\frac{n!}{r!(n\quad r)!}$$

Multiplying both sides by $(r + 1)!\,(n - r + 1)!$

$r(r + 1) + (n - r)(n - r + 1) = 2(r + 1)(n - r + 1)$

$n^2 - n(4r + 1) + 4r^2 - 2 = 0$.]

20. In the expansion of $\left(x^2 + \frac{1}{x}\right)^n$, the coefficient of the fourth term is equal to the coefficient of the ninth term. Find n and the sixth term of the expansion. **[*I.S.C. 2000*]**

21. The cofft. of x^n in the expansion of $(1 + x)(1 - x)^n$ is

(*a*) $(-1)^{n-1} . (n - 1)^2$ (*b*) $(-1)^n (1 - n)$ (*c*) $n - 1$ (*d*) $(-1)^{n-1}.n$

[Hint. $(1 + x)(1 - x)^n = (1 + x)[{}^nC_0 + {}^nC_1 . (-x) + {}^nC_2 . (-x)^2 + ... + {}^nC_{n-2}(-x)^{n-2} + {}^nC_{n-1}(-x)^{n-1} + {}^nC_n(-x)^n$

coefft. of x^n in the expansion $= (-1)^n [1 + n(-1)] = (-1)^n . (1 - n)$]

ANSWERS

1. (*a*) $\frac{55}{9}$ (*b*) $\frac{-84}{x^8}$ (*c*) $\frac{1120x^4}{y^4}$ (*d*) $-462x^9, 462x^2$ (*e*) -252

2. (*a*) $T_8 \quad -3432$, (*b*) $T_5 = \dfrac{55}{16}$, (*c*) $T_9 = 7920$

3. (*a*) $\dfrac{-1792}{9}$ (*b*) 462 (*c*) –1365 (*d*) $\dfrac{405}{256}$. **4.** $\dfrac{9}{7}$

5. ${}^{n}C_3 P^{n-3}\ x^{n-6}, n = 6,\ p = \dfrac{1}{2}$ **6.** (*i*) $a = 180, b = 30$, (*ii*) – 270, (*iii*) $x = \dfrac{1}{6}$

7. $\dfrac{q^n}{2^n \,.\, x^n}, \dfrac{-np}{2^{n-1}} \,.\, \dfrac{q^{n-1}}{x^{n-2}}, \dfrac{n(n-1)}{2^{n-1} \,.\, x^{n-4}}\ p^2 q^{n-2}; n = 8,\ p = 4,\ q = 1$ or $p = \dfrac{5+\sqrt{41}}{2},\ q = \dfrac{5-\sqrt{41}}{2}$

8. Hint. ${}^{11}C_5 \dfrac{a^6}{b^5} \quad {}^{11}C_6 \dfrac{a^5}{b^6} \quad ab \quad 1$ **9.** $x = 1,\ \ a = 7$ **10.** $\dfrac{672}{x^3}$ **11.** 14

12. (*a*) **13.** $252 y^{5/2}\, x^{5/3}$ **20.** $n = 11;\ T_6 = 462\, x^7$. **21.** (*b*)

2.07. Binomial theorem for negative integer or fractional index

It can be shown that binomial theorem holds good even when *n* is a negative integer or a fraction. Thus, we have for all values of *n*, positive, or fractional or both

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{1\times 2}x^2 + \frac{n(n-1)\,(n-2)}{1\times 2\times 3}x^3 + \ldots\ldots \textit{ ad infinitum}, |\,x\,| < 1.$$

The following points should be noted in regard to the above expansion when *n* is a negative integer or a fraction :

1. The symbols ${}^{n}C_0, {}^{n}C_1, {}^{n}C_2$..... are not used as they become meaningless, because they have meaning only when *n* is a positive integer.

2. The expansion contains an *infinite* number of terms.

3. The expansion is valid only when *x* is numerically less than 1, *i.e.* the second term in the binomial expression 1 + *x* on L.H.S. is less than 1 numerically *i.e.*, | *x* | < 1.

The condition that the numerical value of x must be less than one is absolutely essential.

For, if it is not so, the series gives absurd results. For example, expanding $(1 - x)^{-2}$ by Binomial Theorem, we get

$$(1 \quad x)^{\ \ 2} \quad 1 \quad (\ \ 2)(\ \ x) \quad \frac{(\ \ 2)(\ \ 2 \quad 1)}{1.2}(\ \ x)^2 \quad \ldots\ldots \quad 1 \quad 2x \quad 3x^2 \quad \ldots\ldots$$

Putting *x* equal to 2, we have $(-1)^{-2} = 1 = 1 + 2.2 + 3.2^2 + \ldots\ldots\ldots$
which is evidently absurd.

Similarly, whenever *x* is numerically equal to or greater than 1, the expansion may be invalid.

4. In expansions like that of $(a + x)^n$ *the first term should be made unity*, if it is different from 1.

Thus, $$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n \text{ if } \frac{x}{a} < 1 \text{ numerically or } -1 < \frac{x}{a} < 1.$$

and $$(a+x)^n = x^n\left(1+\frac{a}{x}\right)^n \text{ if } \frac{a}{x} < 1 \text{ numerically or } -1 < \frac{a}{x} < 1.$$

5. The general term T_{r+1} is given by

$$T_{r+1} = \frac{n(n \quad 1)\,(n \quad 2) \ldots (n \quad r+1)}{r\,!} \times x^r$$

Ex. 17. ***Expand $(1-x)^{3/2}$ to four terms.***

Sol. $$(1-x)^{3/2} = 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1 \times 2}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1 \times 2 \times 3}(-x)^3 + \ldots$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

Ex. 18. ***Expand $(3-2x)^{-7}$ as far as the term containing x^4. Also mention the condition for validity of the expansion.***

Sol. $(3-2x)^{-7} = 3^{-7}\left(1 - \frac{2}{3}x\right)^{-7}$

$$= \frac{1}{3^7}\Bigg[1 + (-7)\left(-\frac{2}{3}x\right) + \frac{(-7)(-7-1)}{1 \times 2}\left(-\frac{2}{3}x\right)^2 + \frac{(-7)(-7-1)(-7-2)}{1 \times 2 \times 3}\cdot\left(-\frac{2}{3}x\right)^3$$

$$+ \frac{-7(-7-1)(-7-2)\,.\,(-7-3)}{1 \times 2 \times 3 \times 4}\left(-\frac{2}{3}x\right)^4 + \ldots\Bigg]$$

$$= \frac{1}{3^7} \quad 1 \quad \frac{14}{3}x \quad \frac{112}{9}x^2 \quad \frac{224}{9}x^3 \quad \frac{1120}{27}x^4 \quad \ldots$$

This expansion is valid only when $\left|\frac{2}{3}x\right| < 1$, *i.e.*, when $\left|\frac{2}{3}x\right| < 1$,

i.e., when $|x| < \frac{3}{2}$, *i.e.*, when $-\frac{3}{2} < x < \frac{3}{2}$.

Ex. 19. ***Expand $\frac{1}{(3-2x^2)^{2/3}}$ to four terms. For what values of x is the expansion valid ? Also write the general term.***

Sol. $$\frac{1}{(3-2x^2)^{2/3}} = (3-2x^2)^{-2/3} = (3)^{-2/3}\left(1 - \frac{2}{3}x^2\right)^{-2/3}$$ **(Note this step.)**

$$= \frac{1}{3^{2/3}}\Bigg[1 + \left(-\frac{2}{3}\right)\left(-\frac{2}{3}x^2\right) + \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}\left(-\frac{2}{3}x^2\right)^2$$

$$+ \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}\left(-\frac{2}{3}x^2\right)^3 + \ldots\Bigg]$$

$$= \frac{1}{3^{2/3}}\left[1 + \frac{4}{9}x^2 + \frac{20}{81}x^4 + \frac{320}{2187}x^6 + \ldots\ldots\right]$$

This expansion is valid only when $\left|\frac{2}{3}x^2\right| \quad 1$ when $\left|\frac{2}{3}x^2\right| \quad 1$ when $|x^2| < \frac{3}{2}$

when $|x| < \sqrt{\frac{3}{2}}$ when $\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$.

Ex. 20. ***Expand $(3 + 2x)^{-5}$ up to four terms in (i) ascending powers of x, (ii) descending powers of x. For what values of x is the expansion valid in each case?***

Sol. (*i*) $(3 + 2x)^{-5} = 3^{-5}\,(1 + \frac{2}{3}x)^{-5}$

$$= 3^{-5}\,[1 + (-5)\,(\frac{2}{3}x) + \frac{(\ 5)(\ 5\ \ 1)}{1\ \ 2}.\ \frac{2}{3}x\ \ ^2\ \ \frac{(\ 5)(\ 5\ \ 1)(\ 5\ \ 2)}{1\ \ 2\ \ 3}.\ \frac{2}{3}x\ \ ^3 + \ldots\ldots\ ]$$

$$= \frac{1}{3^5}\ \ 1\ \ \frac{10}{3}x\ \ \frac{20}{3}x^2\ \ \frac{280}{27}x^3\ \ \ldots\ldots$$

This expansion is valid only when $\left|\frac{2}{3}x\right| < 1 \quad |x| < \frac{3}{2} \quad \frac{3}{2} \quad x \quad \frac{3}{2}$.

(*ii*) $(3 + 2x)^{-5} = (2x)^{-5}\ (1 + \frac{3}{2x})^{-5}$

$$= \frac{1}{32x^5}\ \ 1\ \ (\ 5)\ \frac{3}{2x}\ \ \frac{(\ 5)(\ 5\ \ 1)}{1\ \ 2}\ \frac{3}{2x}\ \ ^2\ \ \frac{(\ 5)(\ 5\ \ 1)(\ 5\ \ 2)}{1\ \ 2\ \ 3}.\ \frac{3}{2x}\ \ ^3\ \ \ldots.$$

$$= \frac{1}{32x^5}\ \ 1\ \ \frac{15}{2x}\ \ \frac{135}{4x^2}\ \ \frac{945}{8x^3}\ \ \ldots\ldots$$

$$= \frac{1}{32}\left[\frac{1}{x^5} - \frac{15}{2x^6} + \frac{135}{4x^7} - \frac{945}{8x^8} + \ldots\ldots\right]$$

This expansion is valid only when $\left|\frac{3}{2x}\right| \quad 1 \quad \left|\frac{1}{x}\right| \quad \frac{2}{3} \quad |x| \quad \frac{3}{2}$. when $x \quad (-\ \ ,\ \frac{3}{2})\quad (\frac{3}{2},\ \ )$.

2.08. Approximations

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)\,(n-2)}{1\times 2\times 3}x^3 + \ldots\ldots$$

As $x < 1$, so the terms of the above expansion go on decreasing and if x be very small, a stage may reach when we may neglect the terms containing higher powers of x of the expansion. Thus if x be so small that its squares and higher powers may be neglected, then

$$(1+x)^n = 1 + nx.$$

This is an approximate value of $(1+x)^n$.

Ex. 21. ***If x be so small that its squares and higher powers may be neglected, find the approximate value of***

$$\frac{\sqrt[3]{1-\frac{3}{7}x} + \left(1-\frac{3}{5}x\right)^{-5}}{\sqrt[4]{1+\frac{1}{2}x} + \sqrt[7]{1-\frac{7}{3}x}}.$$

Sol. Neglecting squares and higher powers of x in each expansion, the given expression

$$= \frac{\left(1-\frac{3}{7}x\right)^{1/3} + \left(1-\frac{3}{5}x\right)^{-5}}{\left(1+\frac{1}{2}x\right)^{1/4} + \left(1-\frac{7}{3}x\right)^{\frac{1}{7}}} = \frac{\left(1-\frac{1}{3}\times\frac{3}{7}x+....\right) + \left[1+(-5)\times\left(-\frac{3}{5}x\right)+....\right]}{\left(1+\frac{1}{4}\times\frac{1}{2}x+....\right)+\left(1-\frac{1}{7}\times\frac{7}{3}x+....\right)}$$

$$= \frac{1-\frac{1}{7}x+1+3x}{1+\frac{1}{8}x+1-\frac{1}{3}x} = \frac{2+\frac{20}{7}x}{2-\frac{5}{24}x} = \frac{1+\frac{10}{7}x}{1-\frac{5}{48}x} = \left(1+\frac{10}{7}x\right)\left(1-\frac{5}{48}x\right)^{-1}$$

$$= \left(1+\frac{10}{7}x\right)\left(1+\frac{5}{48}x\right) = 1+\frac{10}{7}x+\frac{5}{48}x = 1+\frac{515}{336}x.$$

2.09. Evaluation of a root

Suppose we have to find the *nth* root of any number N. We express N in the form $a^n + b$, where a^n is nearest to N. Then, b which is either positive or negative is very small in comparison to a^n.

$$N^{1/n} = \left(a^n + b\right)^{1/n} \left(a^n\right)^{1/n}\left(1+\frac{b}{a^n}\right)^{1/n}.$$

Now on expanding $\left(1+\frac{b}{n^n}\right)^{1/n}$ an approximate value of the root can be obtained.

Ex. 22. ***Using the binomial theorem, evaluate $\frac{1}{\sqrt{0.9}}$ to four places of the decimal.***

[*I.S.C.1993*]

Sol. $\frac{1}{\sqrt{0.9}} = (0.9)^{-1/2} = (1-0.1)^{-1/2}$

$$= 1+\left(-\frac{1}{2}\right)(-0.1) + \frac{(-1/2)(-3/2)}{2\times 1}(-0.1)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3\times 2\times 1}(-0.1)^3 + ...$$

$= 1 + 0.05 + 0.00375 + 0.0003125 + ...$

$= 1.0540625 = 1.0541,$ correct to four places of the decimal.

Ex. 23. ***Find the cube root of 1001 correct to five decimal places.***

Sol. As $10^3 = 1000$ so we write $1001 = 10^3 + 1$

$$\sqrt[3]{1001} = (1000 + 1)^{1/3} = (10^3 + 1)^{1/3} = 10 \quad 1 \quad \frac{1}{10^3}^{1/3}$$ [Make first term unity and second term < 1.]

$$10 \quad 1 \quad \frac{1}{3} \quad \frac{1}{10^3} \quad \frac{\frac{1}{3} \quad \frac{1}{3} \quad 1}{1 \quad 2} \quad \frac{1}{10^3}^{2} \quad ....$$

$$= 10\left[1 + \frac{1}{3 \times 10^3} - \left(\frac{1}{3^2}\right) \times \frac{1}{10^6} + ...\right]$$

$$= 10 + \frac{1}{3} \times \frac{1}{10^2} - \frac{1}{3^2} \times \frac{1}{10^5} + ... = 10 + 0.00333 - 0.000001 + ...$$

$$= 10.00333 - 0.000001 = 10.003329 = 10.00333.$$ [*To 5 decimal places*]

Ex. 24. ***Write down and simplify the first four terms in the binomial expansion of $(1 - 2x)^{2/3}$. Calculate the value of x for which $(1 - 2x)^{2/3}$ becomes equal to $(0.99)^{2/3}$ and hence find the value of $(0.99)^{2/3}$, correct to 5 significant figures.*** ***[I.S.C. 1991]***

Sol. $$(1 - 2x)^{2/3} = 1 + \frac{2}{3}(-2x) + \frac{\frac{2}{3}\left(\frac{2}{3} - 1\right)}{2.1}(-2x)^2 + \frac{\frac{2}{3}\left(\frac{2}{3} - 1\right)\left(\frac{2}{3} - 2\right)}{3.2.1}(-2x)^3 + ...$$

$$= 1 - \frac{4}{3}x - \frac{4}{9}x^2 - \frac{32x^3}{81} + ... \qquad .... (1)$$

$(1 - 2x)^{2/3} = (0.99)^{2/3}$ $\qquad 1 - 2x = 0.99$ $\qquad x = 0.005$

Substituting $x = 0.005$ in (1), we get

$$(0.99)^{2/3} = 1 - \frac{4}{3}(0.005) - \frac{4}{9}(0\text{-}005)^2 - \frac{32}{81}(0.005)^3$$

$$= 1 - 0.0066667 - 0.0000111 - 0.0000000494$$

$$= 1 - 0.006677 = 0.99332, \text{ correct to 5 significant figures.}$$

EXERCISE 2 (*c*)

1. Write down the condition on y so that the expansion of the following binomials is valid :

(*i*) $(11 + \frac{1}{c}y)^{-3/4}$ (*ii*) $a^2 \quad \frac{1}{a}y^{\,1/4}$ (*iii*) $\frac{1}{(4 \quad 3y^2)^{1/3}}$

2. Expand the following up to three terms in ascending powers of x.

(*i*) $\sqrt{2 \quad 3x}$ (*ii*) $(27 - 6x)^{-2/3}$ (*iii*) $\frac{1}{x^2 \quad \frac{1}{x}^{4/3}}$.

3. Find the first four terms in the expansion of the following :

(*a*) $(1-x)^{-1}$. (*b*) $(1+x)^{-3}$. (*c*) $(1-x)^{4/3}$.

(*d*) $(2x+a)^{-2}$ in ascending powers of a. (*e*) $\left(x-\dfrac{1}{x}\right)^{-1}$.

(*f*) $\left(x^3-\dfrac{1}{x}\right)^{1/3}$. (*g*) $\dfrac{1}{2x-1}$ in ascending powers of x.

(*h*) $\dfrac{1}{(2x-1)^2}$ in ascending powers of x.

In (*g*) and (*h*) state the value of x for which the expansions are valid.

4. Expand $(2+3x)^{-5}$ up to four terms in (*i*) ascending powers of x, (*ii*) decending powers of x. For what values of x is the expansion valid in each case ?

5. Use the binomial theorem to expand $\dfrac{1}{1+x}+\sqrt{1+x}$ in a series of ascending powers of x as far as the term in x^3. **[S.C.]**

6. Expand $\dfrac{4+x}{\sqrt{(1+4x)}}$ in a series of ascending powers of x up to and including x^2. **[S.C.]**

7. Evaluate the following correct to three decimal places :

(*i*) $\sqrt{10}$ **[*I.S.C. 1985*]** (*ii*) $(63)^{1/6}$ (*iii*) $(627)^{1/4}$

(*iv*) 1.04^{-3} (*v*) 1.01^{-7} (*vi*) $\dfrac{1}{(0.97)^4}$.

8. Find the approximate value of the following correct to 4 places of decimals.

(*i*) $\sqrt[3]{1010}$ **[*I.S.C.*]** (*ii*) $\sqrt[4]{16.08}$. **[*I.S.C.*]** (*iii*) $(98)^{1/2}$ **[*I.S.C. 2003*]**

9. If x be so small that its squares and higher powers may be neglected, prove that

(*i*) $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}=1-\dfrac{41}{24}x.$ (*ii*) $\dfrac{(1+x)^{1/2}+(1-x)^{2/3}}{(1+x)+(1+x)^{1/2}}=1-\dfrac{5}{6}x.$

(*iii*) $\dfrac{\left(1-\dfrac{2x}{3}\right)^{-3}+(1+5x)^{-1/2}}{(1-x)^{3/2}}=2+\dfrac{5}{2}x.$ (*iv*) $(8+4x)^{1/3}(16-x)^{1/4}=4+\dfrac{29}{48}x.$

10. Assuming x to be so small that x^2 and higher powers of x can be neglected, find the value of $\dfrac{(1-2x)^{2/3}(4+5x)^{3/2}}{\sqrt{1-x}}$. **[*I.S.C.*]**

11. (*i*) Expand $(1-3x)^{2/3}$ by the binomial theorem as far as the term in x^2. For what value of x would this expansion give an approximation to $\sqrt[3]{0.49}$. ?

(*ii*) Express $\dfrac{1}{(3+x)^2}$ in a form suitable for expansion by the binomial theorem in ascending powers of x, and write down the first two terms of this expansion. **[*I.S.C.*]**

12. Express $(4 + 3x)^{1/2} - (1 - \frac{1}{2}x)^{-2}$ as a series of ascending power of x up to and including the term in x^2. **[I.S.C.]**

13. Write down and simplify the first four terms of the binomial expansion of $(1 + x)^{1/2}$, when $x = 0.08$. Show that $(1.08)^{1/2} = \frac{3}{5}\sqrt{3}$ and hence using your expansion, find an approximation for $\sqrt{3}$.

14. If x be nearly equal to 1, prove that $\frac{mx^m \quad nx^n}{m \quad n} \quad x^{m \quad n}$ nearly.

[Hint. Let $x = 1 + h$ where h is so small that h^2 and higher powers of h may be neglected.]

15. If x is nearly equal to 1, prove that $\frac{ax^b - bx^a}{x^b - x^a} = \frac{1}{1-x}$ approximately.

16. Write down and simplify the first four terms in the binomial expansion of $(1 + x)^{1/3}$. Use $x = \frac{1}{64}$ in the expansion of $(1 + x)^{1/3}$ to find the value of $\sqrt[3]{65}$ correct to 5 significant figures. **[I.S.C.]**

17. Express $\frac{x}{(2x+y)^{-6}}$ in the form suitable for expansion by the binomial theorem and obtain the first three terms in the expansion. If the fourth and fifth terms are equal and $x + 2y = 19$, then calculate the value of x and y. **[I.S.C.]**

18. Using the binomial theorem, evaluate $(0.99)^{15}$ correct to four decimal places. **[I.S.C.]**

19. $\frac{\sqrt{1+2x}+\sqrt[4]{16+3x}}{\sqrt[4]{1-2x}}$ is nearly equal to $a + bx$, find a and b. **[I.S.C.]**

ANSWERS

1. (*i*) $-11c < y < 11c$ (*ii*) $-a^3 < y < a^3$ (*iii*) $\frac{2}{\sqrt{3}} \quad y \quad \frac{2}{\sqrt{3}}$

2. (*i*) $\sqrt{x} \quad \frac{3\sqrt{2}}{4}x \quad \frac{9\sqrt{2}}{32}x^2 \ldots$ (*ii*) $\frac{1}{9} \quad \frac{4x}{243} \quad \frac{20x^2}{6561} \quad \ldots\ldots$

(*iii*) $x^{4/3} - \frac{4}{3}x^{13/3} \quad \frac{14}{9}x^{22/3}$ ..... (the expansion is valid if $|x^2| <$ $|x| < 1$ $-1 < x < 1$.

3. (*a*) $1 + x + x^2 + x^3$ (*b*) $1 - 3x + 6x^2 - 10x^3$ (*c*) $1 - \frac{4}{3}x + \frac{2}{9}x^2 - \frac{4}{81}x^3$

(*d*) $\frac{x^{-2}}{4} - \frac{x^{-3}a}{4} + \frac{3x^{-4}a^2}{16} - \frac{x^{-5}a^3}{8}$ (*e*) $\frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5} + \frac{1}{x^7}$ (*f*) $x - \frac{1}{3x^3} - \frac{1}{9x^7} - \frac{5}{81x^{11}}$

(*g*) $1 + 2x + 4x^2 + 8x^3 + \ldots \left(-\frac{1}{2} < x < \frac{1}{2}\right)$ (*h*) $1 \quad 4x \quad 12x^2 \quad 32x^3 \quad \ldots \quad \frac{1}{2} \quad x \quad \frac{1}{2}$

4. (*i*) $\frac{1}{32}[1 \quad \frac{15}{2}x \quad \frac{135}{4}x^2 \quad \frac{945}{8}x^3$ ......]; valid when $\frac{2}{3} \quad x \quad \frac{2}{3}$.

(*ii*) $\frac{1}{243}\quad \frac{1}{x^5}\quad \frac{10}{3}\cdot\frac{1}{x^6}\quad \frac{20}{3}\cdot\frac{1}{x^7}\quad \frac{280}{27}\cdot\frac{1}{x^8}$ ..... ; valid when $x\quad (\quad , \frac{2}{3})\ (\frac{2}{3}, \quad)$

5. $2-\frac{1}{2}x+\frac{7}{8}x^2-\frac{15}{16}x^3$ **6.** $4-7x+22x^2$

7. (*i*) 3.162 (*ii*) 1.995 (*iii*) 5.004 (*iv*) 0.889 (*v*) 0.933 (*vi*) 1.130

8. (*i*) 10.0332 (*ii*) 2.0025 (*iii*) 9.8995

10. $8+\frac{25}{3}x$ **11.** (*i*) $1-2x-x^2, x=0.1$ (*ii*) $\frac{1}{9}-\frac{2x}{27}$ **12.** $1-\frac{x}{4}-\frac{57}{64}x^2$

13. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3=1+0.04-0.0008+0.000032=1.039232;\ \sqrt{3}=1.73205$

16. $1+\frac{x}{3}-\frac{x^2}{9}+\frac{5}{81}x^3$........; 4.0207 **17.** $x\,(2x+y)^6$; $64x^7,\ 192x^6y,\ 240x^5y^2$; $x=3,\ y=8$

18. 0.8601 **19.** $a=3, b=\frac{83}{32}$.

2.11. General term of the expansion of $(1 \pm x)^n$

We have already found in Art. 6.05 the general term in the expansion of $(x+y)^n$ and $(x—y)^n$ where n is a positive integer. Now we shall find the general term in the expansions of $(1+x)^n$ and $(1-x)^n$. where n is a negative integer or a fraction, and $-1<x<1$.

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+.......+\frac{n\,(n-1)\,(n-2)\,...\,(n-r+1)}{r!}x^r+....\text{ to infinity}$$

The general term $T_{r+1}=\dfrac{n\,(n-1)\,(n-2)\,...\,(n-r+1)}{r!}x^r$

Similarly, the general term T_{r+1} in the expansion of $(1-x)^n$

$$=\frac{n\,(n-1)\,(n-2)\,...\,(n-r+1)}{r!}(-x)^r=\frac{(-1)^r\,n\,(n-1)\,(n-2)\,...\,(n-r+1)}{r!}x^r.$$

The general term T_{r+1} in the expansion of $(1+x)^{-n}$

$$=\frac{(-n)\,(-n-1)\,(-n-2)\,...\,(-n-r+1)}{r!}x^r=\frac{(-1)^r\,n\,(n+1)\,(n+2)\,....\,(n+r-1)}{r!}x^r.$$

The general term T_{r+1} in the expansion of $(1-x)^{-n}$

$$\frac{(\quad n)\,(\quad n\quad 1)\,(\quad n\quad 2)\,...\,(\quad n\quad r\quad 1)}{r!}(\quad x)^r$$

$$=(-1)^{2r}\frac{n(n+1)\,(n+2)\,...\,(n+r-1)}{r!}x^r=\frac{n(n+1)\,(n+2)\,....\,(n+r-1)}{r!}x^r.$$

The following particular cases of the above expansions, which can be easily deduced, may be remembered.

$$\mathbf{(1-x)^{-1}=1+x+x^2+....+x^r+......}$$

$$(1 + x)^{-1} = 1 - x + x^2 - .... + (-1)^r x^r + ...$$

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + ... + (r + 1)x^r + ...$$

$$(1 + x)^{-2} = 1 - 2x + 3x^2 - .... + (-1)^r (r + 1) x^r + ...$$

$$(1 - x)^{-3} = 1 + 3x + 6x^2 + ... + \frac{(r + 1)(r + 2)}{2} x^r + ...$$

$$(1 + x)^{-3} = 1 - 3x + 6x^2 - .... + \frac{(-1)^r (r + 1)(r + 2)}{2} x^r + ...$$

$$(1 - x)^{-4} = 1 + 4x + 10x^2 + ... + \frac{(r + 1)(r + 2)(r + 3)}{6} x^r + ...$$

$$(1 + x)^{-4} = 1 - 4x + 10x^2 - .... + \frac{(-1)^r (r + 1)(r + 2)(r + 3)}{6} x^r + ...$$

Note. It should be observed that when x is positive and n is negative, the terms are alternately positive and negative, but when x and n are both negative, then the terms are all positive.

Ex. 25. ***Find the general term in the expansion of $(1 - x)^{-4}$.***

Sol. Let T_{r+1} denote the general term. Then

$$T_{r+1} = \frac{(-4)(-4-1)(-4-2).......(-4-r+1)}{r!}(-x)^r$$

$$= \frac{(-4)(-5)(-6).......(-r-3)}{1.2.3.4.5.....r}(-1)^r x^r$$

$$= (-1)^r \frac{4.5.6.......r(r+1)(r+2)(r+3)}{1.2.3.4.5.....r}(-1)^r x^r$$

$$= (-1)^{2r} \frac{(r+1)(r+2)(r+3)}{1.2.3.} x^r = \frac{(r+1)(r+2)(r+3)}{1.2.3.} x^r \qquad [\because (-1)^{2r} = 1]$$

Ex. 26. ***Find the general term in the expansion of $(2 - 3x^2)^{-2/3}$.***

Sol. $(2 - 3x^2)^{-2/3} = 2^{-2/3}\left(1 - \frac{3x^2}{2}\right)^{-2/3}$

Let T_{r+1} denote the general term in the expansion of $\left(1 - \frac{3x^2}{2}\right)^{-2/3}$

$$T_{r+1} = \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)......\left(-\frac{2}{3}-(r-1)\right)}{r!}\left(-\frac{3}{2}x^2\right)^r$$

$$= (-1)^r \frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)......\left(\frac{3r-1}{3}\right)}{r!}.(-)^r.\frac{3^r}{2^r}.x^{2r}$$

$$= (-1)^{2r} \frac{2.5.8.....(3r-1)}{r!.3^r}.\frac{3^r}{2^r}.x^{2r} = \frac{2.5.8.....(3r-1)}{r!.2^r}.x^{2r}$$

Hence, the required general term = $2^{-2/3} \cdot \dfrac{2.5.8.....(3r \quad 1)}{r!\,.\,2^r} \cdot x^{2r}$.

Ex. 27. ***Find the first negative term in the expansion of $(1 + x)^{7/2}$.***

Sol. Let $(r + 1)$ th term be the first negative term in the expasion of $(1 + x)^{7/2}$.

$$T_{r+1} = \frac{\frac{7}{2}\ \frac{7}{2} \quad 1 \quad \frac{7}{2} \quad 2 \ ..... \ \frac{7}{2} \quad r \quad 1}{r!}\ x^r$$

This will be the first negative term if $\frac{7}{2} - r + 1 < 0$, *i.e.*, $\frac{9}{2} - r <$, *i.e.*, $r > \frac{9}{2}$.

Hence, we get the first negative term when $r = 5$.

$$\text{First negative term is } T_6 = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-3\right)\left(\frac{7}{2}-4\right)}{5!}\,x^5 = \mathbf{\frac{-7}{256}\,x^5}.$$

Coefficient of a particular power.

Type I

Ex. 28. ***Find the coeff of x^{2r} in the expansion of $(1 - 4x^2)^{-1/2}$.***

Sol. $T_{p+1} = \dfrac{\frac{1}{2} \quad \frac{1}{2} \quad 1 \quad \frac{1}{2} \quad 2 \ .... \ \frac{1}{2} \quad p \quad 1}{p!}\,(\quad 4x^2)^p$

Since the power of x should be $2r$, therefore, $2p = 2r \quad p = r$

$$\text{Reqd. coefft.} = \frac{\frac{1}{2} \quad \frac{1}{2} \quad 1 \quad \frac{1}{2} \quad 2 \ .... \ \frac{1}{2} \quad r \quad 1}{r!}\,(\quad 4)^r$$

$$= (-1)^{2r} \cdot \frac{1.3.5......(2r-1)}{2.2.2....r \times r!}\,(2)^{2r} = (-1)^{2r} \cdot \frac{1.3.5.....(2r-1)}{2^r \ r!} \cdot 2^{2r}$$

$$= \frac{1.3.5.....(2r \quad 1)}{r!} \cdot 2^r$$

Ex. 29. ***Find the coefficient of x^{10} in the expansion of $\dfrac{1 + 2x}{(1 - 2x)^2}$.***

Sol. $\dfrac{1+2x}{(1-2x)^2} = (1 + 2x)(1 - 2x)^{-2}$

$= (1 + 2x)\,[1 + 2\,(2x) + 3\,(2x)^2 + ... + 10\,(2x)^9 + 11\,(2x)^{10} + ...]$

$= (1 + 2x)\,[1 + 4x + 12x^2 + ... + 10.2^9 \,.\, x^9 + 11.2^{10} \,.\, x^{10} + ...]$

Required coefficent $= 11 \times 2^{10} + 2 \times 10 \times 2^9 = \mathbf{21{,}504.}$

Ex. 30. ***Find the coefficient of x^n in the expansion of $\dfrac{(1+x)^2}{(1-x)^2}$, (x is positive and less than 1).***

Sol. $\dfrac{(1+x)^2}{(1-x)^2} = (1 + x)^2\,(1 - x)^{-2} = (1 + 2x + x^2)\,[1 + 2x + 3x^2 + ... + (r + 1)\,x^r + ...]$

$= (1 + 2x + x^2)\,[1 + 2x + 3x^2 + ... + (n - 1)x^{n-2} + nx^{n-1} + (n + 1)\,x^n + ...]$

On multiplying we get the terms containing x^n

$= (n + 1)\,x^n + 2nx^n + (n - 1)\,x^n = (n + 1 + 2n + n - 1)x^n = 4n \,.\, x^n$

Therefore, the coefficient of $x^n = 4n$.

Ex. 31. ***Prove that the coefficient of x^n in the expansion of $(1-2x)^{-1/2}$ is $\dfrac{(2n)!}{2^n\,(n!)^2}$.***

Sol. $$T_{r+1} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)....\left(-\frac{1}{2}-r+1\right)}{r\,!}(-2x)^r$$

$$= \frac{(\ \ 1)^r\ \frac{1}{2}\ \ \frac{3}{2}\ \ \frac{5}{2}\ ....\ \frac{2r\quad 1}{2}}{r\,!}(\ \ 1)^r\,.\,2^r\,.\,x^r$$

$$= (-1)^{2r}\,.\,\frac{1.3.5.7......(2r-1)}{r\,!\,.\,2^r}2^r\,.\,x^r = \frac{1.3.5.7....(2r-1)}{r\,!}x^r$$

For this term to contain x^n , r should be equal to n. Hence, putting $r = n$, we get

$$T_{n+1} = \frac{1.3.5.7....(2n\quad 1)}{n!}x^n \quad \frac{1.2.3.4.5.6....(2n\quad 2)\,(2n\quad 1)\,(2n)}{2.4.6.....(2n\quad 2)\,(2n)\,n!}x^n$$

$$= \frac{(2n)!}{2^n\,[1.2.3......(n-1)\,.\,n]\,n!}x^n = \frac{(2n)!}{(n!)^2\,2^n}x^n$$

Hence, coefft. of $x^n = \dfrac{(2n)!}{(n!)^2\,2^n}$.

Ex. 32. ***Prove that $(1+x+x^2+x^3+.......)\,(1-x+x^2-x^3+......) = 1+x^2+x^4+x^5+......$***

Sol. $(1-x)^{-1} = 1+(-1)\,(-x) + \dfrac{(\ \ 1)(\ \ 2)}{2!}(\ \ x)^2 \quad \dfrac{(\ \ 1)(\ \ 2)(\ \ 3)}{3!}(\ \ x)^3 \quad ....$

$= 1+x+x^2+x^3+.....$

Similarly $(1+x)^{-1} = 1-x+x^2-x^3+.....$

L.H.S. $= (1-x)^{-1}\,(1+x)^{-1} = (1-x^2)^{-1}$

$= 1+(-1)\,(-x^2) + \dfrac{(\ \ 1)(\ \ 1\ \ \ 1)}{2!}(\ \ x^2)^2 \quad \dfrac{(\ \ 1)(\ \ 1\ \ \ 1)(\ \ 1\ \ \ 2)}{3!}(\ \ x^2)^3 \quad ....$

$= 1+x^2+x^4+x^6+....$

Ex. 33. ***Find the coefficient of x^n in the expansion of $(x+1)^n\,(1+x)^{-2}$. Use this to prove $C_0 - 2C_1 + 3C_2 - 4C_3 + ...+ (-1)^n\,(n+1)\,C_n = 0$.***

Sol. $(x+1)^n\,(1+x)^{-2} = (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_{n-1}x + C_n) \times (1-2x+3x^2-4x^3+.......$
$+ (-1)^n\,(n+1)x^n + ...)$

Coefft. of $x^n = C_0 \times 1 - 2 \times C_1 + 3\,C_2 - 4 \times C_3 + ... + (-1)^n\,.\,(n+1)\,C_n$
$= C_0 - 2C_1 + 3C_2 - 4C_3 + ... + (-1)^n\,(n+1)C_n$

Now, $(x+1)^n\,(1+x)^{-2} = (x+1)^{\,n-2}$, Since the 'highest' power of x in this expansion will be x^{n-2}, therefore, coefft. of $x^n = 0$.

From (i) and (ii), we infer that

$$C_0 - 2.C_1 + 3.C_2 - 4.C_3 + ... + (-1)^n (n+1)\, C_n = 0.$$

Ex. 34. ***Identify the series*** $1 + \frac{15}{8} + \frac{15}{8} \cdot \frac{21}{16} + \frac{15}{8} \cdot \frac{21}{16} \cdot \frac{27}{24} + ...$ ***as a binomial expansion and hence find its sum.*** **[*I.S.C. 1998 Type*]**

Sol. Let the given series be identical with

$$1 + nx + \frac{n(n-1)}{2!} x^2 + ... \quad i.e., \text{ with } (1+x)^n.$$

Comparing this with the given series, we get $nx = \frac{15}{8}$ ..... (i) $\quad \frac{n(n-1)}{2!} x^2 = \frac{15}{8} \cdot \frac{21}{16}$ .....(ii)

Dividing (ii) by the square of (i), we get $\frac{n(n-1)\, x^2}{2n^2 x^2} = \frac{15}{8} \cdot \frac{21}{16} \times \left(\frac{8}{15}\right)^2$

$$\Rightarrow \frac{n-1}{2n} = \frac{7}{10} \qquad \Rightarrow \quad n = \frac{-5}{2}; \quad \text{From } (i), \quad \frac{-5}{2} x = \frac{15}{8}, \quad \Rightarrow \quad x = -\frac{3}{4}$$

Hence, the given series is the expansion of $\left(1 - \frac{3}{4}\right)^{-5/2} = \left(\frac{1}{4}\right)^{-5/2} = 4^{5/2} = 32.$

Note. It may be noted that all series cannot be identified as binomial expansions.

EXERCISE 2 (*d*)

1. Find the 5th term in the expansion of $(1 - 2x^3)^{11/2}$.

2. Write the general term in each of the following expansions

(i) $(1 - 2x)^{3/4}$ (ii) $(1 - x^2)^{-4}$ (iii) $1 \quad \frac{2x}{3}$ $^{1/2}$ (iv) $(1 + 2x)^{5/3}$

3. Assuming the expansion possible in the following, find the coefft. of x^{10} in each of these :

(i) $\frac{1 \quad 3x \quad 2x^2}{(1 \quad x)^4}$ (ii) $\frac{1 \quad 3x^2}{(1 \quad x)^3}$ (iii) $\frac{1 \quad 3x^2}{(1 \quad x^2)^3}$ (iv) $\frac{1 \quad x \quad 2x^2}{(1 \quad x)^4}$

4. Find the coefft. of

(i) x^3 in the expansion of $\frac{(1 \quad 3x)^2}{1 \quad 2x}$ (ii) x^5 in the expansion of $\frac{x^2}{1 \quad \frac{x}{2}}$ ^{5} $\frac{x}{1 \quad 3x}$

5. Find the term involving x^2 in the expansion of $(1 - 2x^{1/3})^{-1}$

6. Prove that $(1 + x + x^2 + ...)(1 + 2x + 3x^2 + ...) = 1 + 3x + 6x^2 + ...$

7. If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is

(a) 7th term (b) 5th term (c) 8th term (d) 6th term

8. Prove that the coefft. of x^n in the expansion of $(1-4x)^{-1/2}$ is $\dfrac{(2n)!}{(n!)^2}$.

9. Find the coefficients of x^n in the expansion of

(*i*) $\dfrac{(1+x)^3}{(1-x)^2}$ (*ii*) $\dfrac{(1+x)^2}{(1-x)^3}$ (*iii*) $\dfrac{1+x-2x^2}{(1-x)^3}$ (*iv*) $\dfrac{1+4x^2+x^4}{(1+x)^4}$.

10. Identify the following series as a binomial expansion and hence find their sums :

(*i*) $1+\dfrac{3}{8}+\dfrac{3}{8}\cdot\dfrac{9}{16}+\dfrac{3}{8}\cdot\dfrac{9}{16}\cdot\dfrac{15}{24}\ldots\ldots+$ (*ii*) $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5.}{4.8.12}+\ldots\ldots\ldots$

(*iii*) $1+\dfrac{7}{18}+\dfrac{7.9}{18.36}+\dfrac{7.9.11}{18.36.54}+\ldots\ldots\ldots\infty.$

[Hint. $nx=\dfrac{7}{18}\Rightarrow n^2x^2=\dfrac{49}{18\times 18}$ .... (1), $\dfrac{n(n-1)}{2}x^2=\dfrac{7.9}{18.36}$ ... (2)]

11. Show that $x^n=1+n\left(1-\dfrac{1}{x}\right)+\dfrac{n(n+1)}{2\,!}\left(1-\dfrac{1}{x}\right)^2+\ldots\ldots$ $\left[\textbf{Hint. } x^n=\left[1-\left(1-\dfrac{1}{x}\right)\right]^{-n}.\right]$

12. Find the general term in the expansion of $(1+x+x^2+x^3+\ldots..$ to infinity$)^n$. When is the expansion valid ?

[Hint. The given expansion $=[(1-x)^{-1}]^n=(1-x)^{-n}$.]

13. Show that $\left(\dfrac{1+x}{1-x}\right)^{1/3}=1+\dfrac{2}{3}\dfrac{x}{1+x}+\dfrac{8}{9}\left(\dfrac{x}{1+x}\right)^2+\ldots..$

$\left[\textbf{Hint. } \left(\dfrac{1+x}{1-x}\right)^{\frac{1}{3}}=\left(\dfrac{1-x}{1+x}\right)^{-\frac{1}{3}}=\left(1-\dfrac{2x}{1+x}\right)^{-\frac{1}{3}}.\right]$

14. Find the coefficient of x^n in the expansion of

(*i*) $(1+x+x^2+\ldots.$ to infinity$)^{-1}$ (*ii*) $(1+2x+3x^2+\ldots.$ to infinity$)^{1/2}$

[Hint. (*ii*) The given expansion $=[(1-x)^{-2}]^{1/2}=(1-x)^{-1}$.]

15. Prove that the coefficient of x^n in the expansion of $(1-2x)^{-1/2}$ is $\dfrac{(2n)\,!}{(n\,!)^2\,.\,2^n}$.

16. The sum of the series

$\dfrac{3}{4.8}-\dfrac{3.5}{4.8.12}+\dfrac{3.5.7}{4.8.12.16}-\ldots\ldots$ up to ∞, is

(*a*) $\sqrt{\dfrac{3}{2}}\ \dfrac{3}{4}$ (*b*) $\sqrt{\dfrac{2}{3}}\ \dfrac{3}{4}$ (*c*) $\sqrt{\dfrac{3}{2}}\ \dfrac{1}{4}$ (*d*) $\sqrt{\dfrac{2}{3}}\ \dfrac{1}{4}$

17. If $y = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \ldots\ldots\ldots$ prove that $y^2 + 2y - 2 = 0$.

[Sol. $y + 1 = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \ldots\ldots$

$$= 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2} \cdot \frac{2^2}{3^2} + \ldots\ldots\ldots = \left(1 - \frac{2}{3}\right)^{-1/2} = \sqrt{3}$$

Squaring both sides, we get $y^2 + 2y + 1 = 3 \Rightarrow y^2 + 2y - 2 = 0$.]

18. If $y = x - x^2 + x^3 - x^4 + \ldots$ to ∞ where $|x| < 1$, prove that

$x = y + y^2 + y^3 + y^4 + \ldots\ldots$ to

What is the condition for the expansion to be true? **[*I.S.C. 1996*]**

19. Identify the following series as a binomial expansion and hence find the sum of the series

$$1 + \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots \text{ up to } \infty$$ **[*I.S.C. 1998*]**

ANSWERS

1. $\frac{1155}{8} x^{12}$ **2.** (*i*) $\frac{3}{2^r} \cdot \frac{1.5.9\ldots\ldots(7 \quad 4r)}{(r!)} x^r$ (*ii*) $\frac{(r \quad 1)(r \quad 2)(r \quad 3)}{1.2.3} x^{2r}$

(*iii*) $\frac{(2r)!}{6^r (r!)^2} x^r$ (*iv*) $(-1)^{r-2}\, 5.2^r \frac{1.4.7\ldots(3r-8)}{r!} x^r$

3. (*i*) 1276 (*ii*) 201 (*iii*) 66 (*iv*) 396 **4.** (*i*) 50 (*ii*) $85\frac{3}{8}$ **5.** $64x^2$ **7.** (*c*)

9. (*i*) $8n - 4$ (*ii*) $2n^2 + 2n + 1$ (*iii*) $3n + 1$ (*iv*) $(-1)^n (n^3 + 3n)$

10. (*i*) Sum = 2 (*ii*) Sum = $\sqrt{2}$ (*iii*) Sum = $\left(\frac{9}{8}\right)^{7/2}$ **12.** $\frac{n(n \quad 1)\ldots\; n \quad r \quad 1}{r!} x^r, |x| \quad 1.$

14. (*i*) $(-1)^n$. (*ii*) 1 **16.** (*b*) **18.** $|y| < 1$ **19.** $\sqrt{5}$

REVISION EXERCISE

1. The 9th term of the expansion $3x \quad \frac{1}{2x} \quad ^8$ is

(*i*) $\frac{1}{512x^9}$ (*ii*) $\frac{1}{512x^9}$ (*iii*) $\frac{1}{256x^8}$ (*iv*) $\frac{1}{256x^8}$.

2. Find the general term in the expansion of $(2 - 3x^2)^{-2/3}$.

3. In the binomal expansion of $(a - b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ equals

(*i*) $\frac{n \quad 5}{6}$ (*ii*) $\frac{n-4}{5}$ (*iii*) $\frac{5}{n \quad 4}$ (*iv*) $\frac{6}{n \quad 5}$.

[Hint. ${}^nC_4\, a^{n-4}\, (-b)^4 = -({}^nC_5\, a^{n-5}\, (-b)^5) \quad \frac{a}{b} \quad \frac{n \quad 4}{5}$]

4. What is the middle term in the expansion of $\frac{x\sqrt{y}}{3} \quad \frac{3}{y\sqrt{x}}^{12}$?

(*a*) $C(12, 7)\, x^3\, y^{-3}$ (*b*) $C(12, 6)\, x^{-3}\, y^3$ (*c*) $C(12, 7)\, x^{-3}\, y^3$
(*d*) $C(12, 6)\, x^3\, y^{-3}$

5. For natural numbers m, n if
$(1 - y)^m\, (1 + y)^n = 1 + a_1 y + a_2 y^2 + \ldots\ldots$ and $a_1 = a_2 = 10$, then (m, n) is
(*a*) (35, 45) (*b*) (20, 45) (*c*) (35, 20) (*d*) (45 30)
[Hint. $(1 - {}^mC_1 y + {}^mC_2 y^2 - \ldots\ldots.)\,(1 + {}^nC_1\, y + {}^nC_2\, y^2 + \ldots\ldots.) = 1 + a_1 y + a_2\, y^2 + \ldots\ldots.$
Coefft. of $y = a_1 = {}^nC_1 - {}^mC_1 = n - m \quad 10 = n - m$, which is satisfied by $n = 45$, $m = 35$
coefft. of $y^2 = a_2 = {}^nC_2 - {}^mC_1 \,.\, {}^nC_1 + {}^mC_2 \quad 20 = (m - n)^2 - (m + n)$. This is also satisfied by $n = 45$, $m = 35$.]

6. If the expression in powers of x of the function $\frac{1}{(1 \quad ax)\,(1 \quad bx)}$ is $a_0 + a_1\, x + a_2\, x^2 + \ldots\ldots.$, then a_n is

(*i*) $\frac{b^{n \ 1} \quad a^{n \ 1}}{b \quad a}$ (*ii*) $\frac{b^n \quad a^n}{b \quad a}$ (*iii*) $\frac{a^n \quad b^n}{b \quad a}$ (*iv*) $\frac{a^{n \ 1} \quad b^{n \ 1}}{b \quad a}$.

[Hint. $\frac{1}{(1 - ax)\,(1 - bx)} = \frac{\frac{1}{1 - b/a}}{1 - ax} + \frac{\frac{1}{1 - a/b}}{1 - bx} = \frac{a}{a - b}(1 - ax)^{-1} + \frac{b}{b - a}(1 - bx)^{-1}$

a_n which is the coefft. of $x^n = \frac{a}{a \quad b} \,.\, a^n \quad \frac{b}{b \quad a} \,.\, b^n$]

7. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1 \quad x)^{3/2} \quad 1 \quad \frac{1}{2}x^{\,3}}{(1 \quad x)^{1/2}}$ may be approximated as

(*i*) $1 \quad \frac{3}{8}x^2$ (*ii*) $3x \quad \frac{3}{8}x^2$ (*iii*) $\frac{3}{8}x^2$ (*iv*) $\frac{x}{2} - \frac{3}{8}x^2$

8. $1 \quad \frac{2}{4} \quad \frac{2.5}{4.8} \quad \frac{2.5.8}{4.8.12}$ .... to is
(*a*) $4^{-2/3}$ (*b*) $\sqrt[3]{16}$ (*c*) $\sqrt[3]{4}$ (*d*) $4^{3/2}$.

9. If the coefft. of rth and $(r + 4)$th terms are equal in the expansion of $(1 + x)^{20}$, then the value of r will be (*a*) 7 (*b*) 8 (*c*) 9 (*d*) 10.

10. (*i*) If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_{2n} x^{2n}$, then find $a_0 + a_2 + a_4 + \ldots\ldots. + a_{2n}$.

[Hint. Put $x = 1$ and $x = -1$. then add]

(*ii*) If $(1 + x - 2x^2)^6 = 1 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_{12} x^{12}$, then find $a_2 + a_4 + \ldots + a_{12}$.

ANSWERS

1. (*iv*) **2.** $\frac{2.5.8\ldots\ldots(3r-1)}{r!} . 2^{-\frac{2}{3}-r} . x^{2r}$ **3.** (*ii*) **4.** (*d*)

5. (*a*) **6.** (*i*) **7.** (*iii*) **8.** (*b*) **9.** (*c*) **10.** (*i*) $\frac{3^n \quad 1}{2}$ (*ii*) 31.

3

Logarithms

3.01. The laws of exponents (Revision)

Recall the following laws of exponents which you have learnt in earlier classes.

If x and y are any two real numbers and m and n are any two rational numbers, then

1. $x^m . x^n = x^{m+n}$ **2.** $\left(x^m\right)^n = x^{mn}$ **3.** $\left(xy\right)^m = x^m . y^m$

4. $\dfrac{x^m}{x^n} = x^{m-n}, \; x \neq 0$ **5.** $x \quad ^{n} \quad \dfrac{1}{x^n}, \; x \neq 0$ **6.** $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}, \; y \neq 0$

7. $a^m = a^n \Rightarrow m = n$ **provided** $a > 0$ **and** $a \neq 1$.

8. $a^0 = 1$ **where** a **is any real number other than 0.**

9. If p **denotes an integer,** q **a positive integer, and** x **a positive real number, then**

$$x^{p/q} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p.$$

i.e., when any number has a fractional exponent, the numerator of the exponent indicates the index to which the base is to be raised, and the denominator indicates the root to be taken. It may be noted that the two operations indicated by a fractional index can be performed in either order without changing the result. It is usually easier to take the root first.

In particular, if $p = 1, \; x^{1/q} = \sqrt[q]{x}$

Moreover, if p is a positive integer as well as q. we define

$$0^{p/q} = 0.$$

Note: In defining exponent with fractional indices, we have restricted the base x to be a positive real number. Without this restriction, some of the familiar laws of indices do not hold.

Illustrative solved examples

Ex. 1. ***Find the values of*** *(i)* $36^{1/2}$, *(ii)* $16^{3/4}$, *(iii)* $\left(\dfrac{27}{64}\right)^{-2/3}$ *and* *(iv)* $\dfrac{3x^0 - 1}{3x^0 + 1}$

Sol. (*i*) $36^{1/2} = \sqrt{36} = \mathbf{6}$

(*ii*) $16^{-3/4} = (2^4)^{-3/4}$

$= 2^{-3} = \mathbf{\dfrac{1}{8}}$

First express the base as the exponent of a prime number, and then multiply the exponents.

$(iii)\ \left(\frac{27}{64}\right)^{-2/3} = \left(\frac{3^3}{4^3}\right)^{-2/3} = \frac{3^{3\times(-2/3)}}{4^{3\times(-2/3)}} = \frac{3^{-2}}{4^{-2}} = \frac{4^2}{3^2} = \mathbf{\frac{16}{9}}.$

$(iv)\ \frac{3x^0-1}{3x^0+1} = \frac{3\times1-1}{3\times1+1} = \frac{3-1}{3+1} = \mathbf{\frac{1}{2}}.$

Ex. 2. *Simplify: (i)* $x^{2a+b-c} \cdot x^{2c+a-b} \cdot x^{2b+c-a}$ ***(ii)*** $\left(\frac{125}{8}\right)^{-2/3}$

Sol. $(i)\ x^{2a+b-c} \cdot x^{2c+a-b} \cdot x^{2b+c-a}\quad x^{2a\ \ b-c\ \ 2c\ \ a-b\ \ 2b\ \ c-a}\quad \mathbf{x^{2a+2b+2c}}.$

$(ii)\ \left(\frac{125}{8}\right)^{-2/3} = \left[\left(\frac{5}{2}\right)^3\right]^{-2/3} = \left[\left(\frac{5}{2}\right)^{3\times\frac{1}{3}}\right]^{-2} = \left(\frac{5}{2}\right)^{-2} = \left(\frac{2}{5}\right)^2 = \mathbf{\frac{4}{25}}.$

Ex. 3. *Simplify: (i)* $(32)^{4/5} + \left(\frac{1}{81}\right)^{-3/4} - \left(\frac{1}{125}\right)^{-2/3} - 6^0 \times 16^{3/2}$ ***(ii)*** $\left[\left\{4\sqrt{x^{-3/4}}\right\}^{-4/3}\right]^4$

Sol. $(i)\ (32)^{4/5} + \left(\frac{1}{81}\right)^{-3/4} - \left(\frac{1}{125}\right)^{-2/3} - 6^0 \times 16^{3/2}$

$(32)^{4/5}\quad (81)^{3/4}\quad (125)^{2/3}\quad 6^0\quad 16^{3/2}$

$(2^5)^{4/5}\quad (3^4)^{3/4}\quad (5^3)^{2/3}\quad 6^0\quad (4^2)^{3/2}$

$2^{(5\ \ 4/5)}\quad 3^{(4\ \ 3/4)}\quad 5^{(3\ \ 2/3)}\quad 6^0\quad 4^{(2\ \ 3/2)}$

$2^4\quad 3^3\quad 5^2\quad 1\quad 4^3\quad 16\quad 27\quad 25\quad 64\qquad \mathbf{46.}$

$(ii)\ \left[\left\{\sqrt[4]{x^{-3/4}}\right\}^{-4/3}\right]^4 = \left[\left\{(x^{-3/4})^{1/4}\right\}^{-4/3}\right]^4 = \left[\left\{x^{-3/16}\right\}^{-4/3}\right]^4$

$= x^{\ 3/16\quad 4/3\quad 4} = (x^{1/4})^4\quad x^{1/4\times4} = \mathbf{x}.$

Ex. 4. *Prove that :*

$$\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \cdot \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \cdot \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2} = 1$$

Sol. L.H.S. $= \left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \cdot \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \cdot \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$

$= (x^{a-b})^{a^2+ab+b^2}\ (x^{b-c})^{b^2+bc+c^2}\ (x^{c-a})^{c^2+ca+a^2}$

$= x^{(a-b)(a^2+ab+b^2)} \cdot x^{(b-c)(b^2+bc+c^2)} \cdot x^{(c-a)(c^2+ca+c^2)}$

$= x^{a^3-b^3} \cdot x^{b^3-c^3} \cdot x^{c^3-a^3}$ $\quad \boxed{\because\ x^3 - y^3 = (x-y)(x^2+xy+y^2)}$

$= x^{a^3-b^3+b^3-c^3+c^3-a^3} = x^0 = \mathbf{1}.$

Questions for practice

1. **Evaluate :**

   (*i*) $5^0 \times 4^{-1} + 8^{1/3}$ (*ii*) $\sqrt[3]{(64)^{-4}\,(125)^{-2}}$ (*iii*) $(9^{-3} \times 16^{3/2})^{1/6}$

   (*iv*) $\sqrt[3]{(16)^{-3/4} \times (125)^{-2}}$ (*v*) $(32)^{-2/5}\quad(216)^{-2/3}$

2. **Evaluate :**

   (*i*) $\left(\frac{64}{125}\right)^{-2/3} \div \frac{1}{\left(\frac{256}{625}\right)^{1/4}} + \left(\frac{\sqrt{25}}{\sqrt[3]{64}}\right)^0.$

   (*ii*) $(81)^{3/4} - \left(\frac{1}{32}\right)^{-2/5} + (8)^{1/3}\left(\frac{1}{2}\right)^{-2}.(2)^0$

ANSWERS

1. (*i*) $2\frac{1}{4}$ (*ii*) $\frac{1}{6400}$ (*iii*) $\frac{2}{3}$ (*iv*) $\frac{1}{50}$ (*v*) 9
2. (*i*) $2\frac{1}{4}$ (*ii*) 31

3.02. Logarithm of a number – Definition

If three numbers a, x, n are so related that $a^x = n$, ...(1)

then x is said to be the **logarithm** of the number n to the **base** a and is written as $\log_a n = x$ ...(2)

and is read as x = log of n to the base a.

Note. Both (1) and (2) express a relation between the three numbers a, x and n. The relation (1) is in the index form and the relation (2) expresses the same thing in the log form. Stated in words, the logarithm may be defined as follows :

Def. ***The logarithm of any number of a given base is equal to the power to which the base should be raised to get the given number.***

Thus,

If	*then*	*i.e.*
$4^3 = 64$	log of 64 to the base 4 = 3	$\log_4 64 = 3$
$3^4 = 81$	log of 81 to the base 3 = 4	$\log_3 81 = 4$
$5^{-3} = \frac{1}{125}$	log of $\frac{1}{125}$ to the base 5 = –3	$\log_5 \frac{1}{125} = -3$
$10^{-1} = \frac{1}{10} = 0.1$	log of 0.1 to the base 10 = –1	$\log_{10} 0.1 = -1$
$a^0 = 1$	log of 1 to the base a = 0	$\mathbf{\log_a 1 = 0}$
$a^1 = a$	log of a to the base a = 1	$\mathbf{\log_a a = 1}$

The last two illustrations give the following two important results :

1. The logarithm of 1 to any base is zero.

2. The logarithm of any number to the same base is unity.

Ex. 5. ***Find the logarithm of (i) 16 to the base 2 (ii) 1000 to the base 10.***

Sol. (*i*) Let $\log_2 16 = x$. Then by definition $2^x = 16 = 2^4$ $\quad$ $\mathbf{x = 4.}$

Therefore, $\log_2 16 = 4$.

(*ii*) Let $\log_{10} 1000 = x$. Then,

$10^x = 1000$ (by definition) $\quad 10^x = 10^3 \quad$ $\mathbf{x = 3.}$

Ex. 6. ***If*** $\boldsymbol{\log_5 a = 3}$***, find the value of a.***

Sol. $\log_5 a = 3$.

By definition $a = 5^3 = \mathbf{125.}$

Ex. 7. ***Determine the value of x if*** $\boldsymbol{\log_2 (x^2 - 1) = \log_2 8}$***.***

Sol. $\log_2 (x^2 - 1) = \log_2 8 \quad x^2 - 1 = 8 \quad x^2 = 9 \quad \mathbf{x = 3, -3}$

Ex. 8. ***Find the logarithm of 5832 to the base*** $\boldsymbol{3\sqrt{2}}$ ***.***

Sol. Let $\log_{3\sqrt{2}} 5832 = x$.

Then by def., $(3\sqrt{2})^x = 5832 = 8 \times 729 = 2^3 \times 3^6 = [(\sqrt{2})^2]^3 \times 3^6$

$= (\sqrt{2})^6 \times 3^6 = (3\sqrt{2})^6 \qquad x = 6.$

Hence, $\log_{3\sqrt{2}} 5832 = 6$.

Ex. 9. ***If*** $\boldsymbol{\log_a \sqrt{2} = \frac{1}{6}}$***, find a.***

Sol. $\because \log_a \sqrt{2} = \frac{1}{6} \qquad a^{\frac{1}{6}} = \sqrt{2}$ ***$\log_5 a$***

Taking 6th power of both sides, we have $a = (\sqrt{2})^6 = 2^3 = 8$.

Ex. 10. ***Solve :*** $\boldsymbol{\log_2 (\log_2 16) = \log_7 x}$***.***

Sol. Let $(\log_2 16) = n$; then , $2^n = 16 = 2^4 \quad n = 4$

$\log_2 (\log_2 16) = \log_2 4$

Let $\log_2 4 = m \quad 2^m = 4 = 2^2 \quad m = 2 \quad \log_2 4 = 2$

$2 = \log_7 x \quad x = 7^2 = 49.$

3.03. A few results

Theorem 1. Let *a* and *x* be positive real numbers where *a* 1. Then $\boldsymbol{a^{\log_a x} = x}$**.**

Proof. Let $\log_a x = z$. *Then* $a^z = x$ (*by definition*)

${}_a\log_a x = x$ $\qquad$ [Putting the value of z in a^z]

The above is a very useful result and you should remember it.

Illustration : $3^{\log_3 7} = 7$, $2^{\log_2 9} = \mathbf{9.}$

Ex. 11. ***Find the value of (i)*** $\boldsymbol{2^{2-\log_2 5}}$ ***(ii)*** $\boldsymbol{27^{(\log_3 \sqrt[3]{125} + \frac{1}{3})}}$

Sol . (*i*) $2^{2-\log_2 5} = \dfrac{2^2}{2^{\log_2 5}} = \dfrac{4}{5}$. $\qquad [\because a^{\log_a x} = x]$

(ii) $27^{(\log_3 \sqrt[3]{125}+\frac{1}{3})} = 27^{\log_3 5+\frac{1}{3}} = (3^3)^{\log_3 5} . (3^3)^{1/3}$

$= 3^{\log_3 (5)^3} . 3 = (5^3)(3) = 125 \times 3 =$ **375**.

Ex. 12. ***If*** $\boldsymbol{\log_a [1+\log_b \{1+\log_c x\}] = 0}$***, find x.***

Sol . Given $\log_a [1 + \log_b \{1 + \log_c x\}] =$ $\log_b\{1 + \log_c x\}= a^0$

$1 + \log_b \{1 + \log_c x\} = 1$ $\log_b\{1 + \log_c x\}= 0$ $1 + \log_c x = b^0$

$1+ \log_c x = 1$ $\log_c x = 0$ $x = c^0$ $\boldsymbol{x = 1}$.

Theorem 2. For $a > 0$, a 1, $\log_a x_1 = \log_a x_2$ if $x_1 = x_2$ $(x_1, x_2 > 0)$.

It follows from the fact that for $a > 0$, a 1, $a^{y_1} = a^{y_2}$ if $y_1 = y_2$.

Theorem 3. If $a > 1$ and $x > y$, then $\log_a x > \log_a y$.

Proof. $x > y$ $a^{\log_a x} > a^{\log_a y} \Rightarrow \log_a x > \log_a y$, *i.e.*, $\log_a x$ is an increasing function.

Ex. 13. ***15 > 7*** $\boldsymbol{\log_{10} 15 > \log_{10} 7}$.

Theorem 4. If $0 < a < 1$ and $x > y$, then $\log_a x < \log_a y$.

Proof. $x > y$ $a^{\log_a x} > a^{\log_a y}$

$a < 1$ $\log_a x < \log_a y$, *i.e.*, $\log_a x$ is a decreasing function.

Thus $15 > 7$ $\log_{0.1} 15 < \log_{0.1} 7$.

[Notation. lg x represents $\log_{10} x$]

EXERCISE 3 (*a*)

1. Change the following from exponential to logarithmic form :

(*a*) $2^3 = 8$ (*b*) $7^{-1} = \frac{1}{7}$ (*c*) $x^0 = 1$ (*d*) $\sqrt[3]{27} = 3$ (*e*) $32^{\frac{3}{5}} = 8$ (*f*) $(\sqrt{3})^6 = 27$

2. Change the following from logarithmic to exponential form :

(*a*) $\log_7 49 = 2$ (*b*) $\log_4 64 = 3$ (*c*) $\log_5 \frac{1}{625} = -4$.

(*d*) $\log_{\frac{1}{3}} \frac{1}{27} = 3$ (*e*) $\log_6 1 = 0$ (*f*) $\left[\log_5 \sqrt[4]{5} = \frac{1}{4}\right]$.

3. Find the log of the following numbers to base 10 :

(*a*) 100 (*b*) 1000 (*c*) 0.1 (*d*) 0.000001

4. Find the value of

(*a*) $\log_5 25$ (*b*) $\log_4 256$ (*c*) $\log_{\sqrt{2}} 16$ (*d*) $\log_{\sqrt{2}} \left[\frac{1}{32}\right]$ (*e*) $\log_{2\sqrt{3}} 1728$

(*f*) $\log_{\sqrt{6}} 36\sqrt{6}$ (*g*) $\log_{\sqrt{a}} \sqrt{a^{-8/5}}$ (*h*) $\left[8^{\log_2 \sqrt[3]{11}}\right]^{\frac{1}{3}}$ (*i*) $\log_{0.11} (0.0121)$

5. Find m, if

(*i*) $\log_5 m = 3$ (*ii*) $\log_{10} m = -2$ (*iii*) $m = \log_2 \frac{1}{32}$ (*iv*) $\log_a m = 0$

(*v*) $\log_m 125 = 6$ (*vi*) $\log_m \frac{1}{36} = \frac{-2}{3}$ (*vii*) $\log_{32} 8 = m$ (*viii*) $\log_5 (2m + 5) = 3$

(*ix*) $\log_3 (m^2 + 17) = 4$ (*x*) $\log_{m^3} 64 = \frac{2}{3}$

6. Evaluate

(*a*) $\log_7 7^{10}$ (*b*) $\log_3 9^7$ (*c*) $\log_{1000} 10^{12}$ (*d*) $\log_5 25^9$

7. Solve for *x*.

(*a*) $\log_2 (\log_9 3) = \log_x 6$ (*b*) $\log_{10} [\log_2 (\log_3 9)] = x$ (*c*) $\log_4 (\log_8 64) = \log_5 x$.

[Hint. Type Solved Ex. 3]

8. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$ and $z = 1 + \log_c ab$, prove that $xyz = xy + yz + zx$.

[Hint. $x = 1 + \log_a bc$ x $\log_a a + \log_a bc = \log_a abc$ ***(Type EAMCET 98)***

Similarly, $y = \log_b abc$, $z = \log_c abc$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc} abc = 1 \Rightarrow \frac{yz + zx + xy}{xyz} = 1]$$

ANSWERS

1. (*a*) $\log_2 8 = 3$ (*b*) $\log_7 \frac{1}{7} = -1$ (*c*) $\log_x 1 = 0$ (*d*) $\log_{27} 3 = \frac{1}{3}$

(*e*) $\log_{32} 8 = \frac{3}{5}$ (*f*) $\log_{\sqrt{3}} 27 = 6$

2. (*a*) $7^2 = 49$ (*b*) $4^3 = 64$ (*c*) $5^{-4} = \frac{1}{625}$ (*d*) $\left[\frac{1}{3}\right]^3 = \frac{1}{27}$ (*e*) $6^0 = 1$

(*f*) $5^{1/4} = \sqrt[4]{5}$

3. (*a*) 2 (*b*) 3 (*c*) –1 (*d*) –6

4. (*a*) 2 (*b*) 4 (*c*) 8 (*d*) –10 (*e*) 6 (*f*) 5

(*g*) $\frac{-8}{5}$ (*h*) $\sqrt[3]{11}$ (*i*) 2

5. (*i*) 125 (*ii*) 0.01 (*iii*) –5 (*iv*) 1 (*v*) $\sqrt{5}$ (*vi*) 216

(*vii*) $\frac{3}{5}$ (*viii*) 60 (*ix*) ± 8 (*x*) 8

6. (*b*) 14 (*c*) 4 (*d*) 18 **7.** (*a*) $\frac{1}{6}$ (*b*) 0 (*c*) $\sqrt{5}$.

3.04. Laws of logarithms

Let *a* be a positive number not equal to 1 (*i.e.*, $a > 0$ and a 1).

Let x_1 and x_2 be any positive numbers (*i.e.* $x_1 > 0$, $x_2 > 0$) and *n* be any real number (*i.e.*, *n* *R*). Then

1. $\log_a x_1 x_2 = \log_a x_1 + \log_a x_2$ **2.** $\log_a \frac{x_1}{x_2} = \log_a x_1 - \log_a x_2$

3. $\log_a (x_1)^n = n \log_a x_1$.

Proof.

Law 1. Let $\log_a x_1 = p$ ... (1) and $\log_a x_2 = q$ ... (2). Then, by def.

$x_1 = a^p$ and $x_2 = a^q$ $\quad x_1 x_2 = a^p . a^q = a^{p+q}$

$\log_a x_1 x_2 = p + q$ (*By def.*) $\quad \log_a x_1 x_2 = \log_a x_1 + \log_a x_2$ [From (1) and (2)]

Extension. $\log_a (x_1 x_2 x_3 ....) = \log_a x_1 + \log_a x_2 + \log_a x_3 + ...$

Aid to memory : Log of a product to any base = sums of logs of the factors to the same base

For example,

Note. $\log_a 210 = \log_a (2 \times 3 \times 5 \times 7) = \log_a 2 + \log_a 3 + \log_a 5 + \log_a 7$

Law 2. $\dfrac{x_1}{x_2} = \dfrac{a^p}{a^q} = a^{p-q} \quad \log_a \dfrac{x_1}{x_2} = p - q$ (By def.)

$\log_a \dfrac{x_1}{x_2} = \log_a x_1 - \log_a x_2$ [From (1) and (2)]

Cor. $\log_a \dfrac{x_1 x_2 x_3 .....}{y_1 y_2 y_3 .....} = (\log_a x_1 + \log_a x_2 + \log_a x_3 + ...) - (\log_a y_1 + \log_a y_2 + \log_a y_3 + ...)$

For example, $\log_{10} \dfrac{30}{1001} = \log_{10} \dfrac{2 \times 3 \times 5}{7 \times 11 \times 13}$

$= (\log_{10} 2 + \log_{10} 3 + \log_{10} 5) - (\log_{10} 7 + \log_{10} 11 + \log_{10} 13)$

Aid to memory : $\log_a$ (fraction) = $\log_a$ (numerator) – $\log_a$ (denominator)

Law 3. Let $\log_a x_1 = p$ .... (1) Then , by def. $a^p = x_1$

Raising both sides to the power n, we have $(a^p)^n = x_1^n$

$a^{np} = x_1^n \quad$ by def. $\log_a x_1^n = np = n \log_a x_1$ [From (1)]

For example, $\log_a x^3 = 3 \log_a x$, $\log_{10} 32 = \log_{10} 2^5 = 5 \log_{10} 2$,

$\log_m \sqrt{(a+b)} = \log_m (a+b)^{1/2} = \dfrac{1}{2} \log_m (a+b)$.

Cor. $\log_{a^n} x = \dfrac{1}{n} \log_a x$.

Proof. Let $\log_{a^n} x = p$ .... (1). Then $(a^n)^p = x$ (By def.)

$a^{np} = x \quad \log_a x = np \quad p = \dfrac{1}{n} \log_a x$

$\log_{a^n} x = \dfrac{1}{n} \log_a x$ [using (1)]

For example, $\log_9 16 = \log_{3^2} 16 = \dfrac{1}{2} \log_3 16$.

$\log_8 (x+1) = \log_{2^3} (x+1) = \dfrac{1}{3} \log_2 (x+1)$.

3.05. Generalisation

If x_1 and x_2 are real numbers such that $x_1 x_2 > 0$ and a is a positive real number $a \neq 1$, then,

(*i*) $\log_a (x_1 x_2) = \log_a |x_1| + \log_a |x_2|$

(*ii*) $\log_a \left|\dfrac{x_1}{x_2}\right| = \log_a |x_1| - \log_a |x_2|$

Ex. 14. ***Express*** $\log_b \dfrac{x\sqrt{y}}{z^3}$ ***in terms of*** $\log_b x$, $\log_b y$, ***and*** $\log_b z$.

Sol. $\log_b \dfrac{x\sqrt{y}}{z^3} = \log_b \left[x.y^{1/2}\right] - \log_b z^3$ ***by Law 2***

$= (\log_b x + \log_b y^{1/2}) - \log_b z^3$ ***by Law 1***

$= \log_b x + \dfrac{1}{2}\log_b y - 3\log_b z.$ ***by Law 3***

Ex. 15. ***Evaluate : (a)*** $\log_2 40 - \log_2 5$ ***(b)*** $\left(5^{\log_{25}7^2}\right)$ ***(c)*** $8^{\log_2 5}$.

Sol. (*a*) $\log_2 40 - \log_2 5 = \log_2 \dfrac{40}{5}$ **[*Law 2*]** $= \log_2 8 = \log_2 2^3$ **[*Law 3*]**

$= 3\log_2 2 = 3 \times 1 = 3\ (\because \log_a a = 1)$

(*b*) $\left(5^{\log_{25}7^2}\right) = 5^{2\log_{25}7}$ **[*Law 3*]** $= (5^2)^{\log_{25}7} = 25^{\log_{25}7} = 7\ \left[\because a^{\log_a x} = x\right]$

(*c*) $8^{\log_2 5} = (2^3)^{\log_2 5} = 2^{3\log_2 5} = 2^{\log_2 5^3}$ **[*Law 3*]** $= 2^{\log_2 125} = 125.$

Ex. 16. ***Express*** $\dfrac{3}{4}\log a - 7\log b + \dfrac{2}{3}\log c$ ***as a single logarithm.***

Sol. $\dfrac{3}{4}\log a - 7\log b + \dfrac{2}{3}\log c = \log a^{3/4} - \log b^7 + \log c^{2/3}$ **[*by Law 3*]**

$= \log\left(\dfrac{a^{3/4}}{b^7}\right) + \log c^{2/3}$ **[*by Law 2*]**

$= \log\left(\dfrac{a^{3/4}.c^{2/3}}{b^7}\right)$ **[*by Law 1*]**

Ex. 17. ***Solve the equation :*** $\log_4 x + \log_4 (x - 6) = 2$.

Sol. $\log_4 x + \log_4 (x-6) = 2$

$\log_4 x\,(x-6) = 2$

$\log_4 (x^2 - 6x) = 2 \qquad x^2 - 6x = 4^2$

$x^2 - 6x - 16 = 0 \qquad (x-8)(x+2) = 0 \qquad x = 8 \text{ or } x = -2$ **[*by Law 1*]**

Check : $x = 8$: L.H.S. $= \log_4 8 + \log_4 (8-6)$ [*by def.*]

$= \log_4 8 + \log_4 2 = \log_4 (8 \times 2) = \log_4 16 = \log_4 4^2$

$= 2\log_4 4 = 2 =$ R.H.S.

$x = -2$: Since $\log_4 (-2)$ is not defined, therefore, -2 is not a solution.

Hence, $x = 8$.

Ex. 18. ***Find the value of*** xyz ***if*** $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$.

Sol. Let $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}. = k$. Then,

$\log x = k(y-z);\ \log y = k(z-x);\ \log z = k(x-y)$

$\therefore \quad \log x + \log y + \log z = k[y - z + z - x + x - y] = k \times 0 = 0$

$\log xyz = 0 \qquad xyz = 1.$ $[\because \log_a 1 = 0]$

Ex. 19. ***Simplify :*** $\frac{1}{2}\log_{10} 25 - 2\log_{10} 3 + \log_{10} 18.$ **[S.C.]**

Sol. The given exp. $= \log_{10}(25)^{\frac{1}{2}} - \log_{10} 3^2 + \log_{10} 18$

$$= \log_{10} 5 - \log_{10} 9 + \log_{10} 18 = \log_{10}\frac{5\times 18}{9} = \log_{10} 10 = 1.$$

Ex. 20. ***If*** $a^2 + b^2 = 7ab$, ***prove that*** $\log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b).$ **[I.S.C.]**

Sol. $a^2 + b^2 + 2ab = 9ab$ or $\frac{(a+b)^2}{9} = ab$ $\quad \left[\frac{1}{3}(a+b)\right]^2 = ab$

$\log\left[\frac{1}{3}(a+b)\right]^2 = \log ab \qquad 2\log\left[\frac{1}{3}(a+b)\right] = \log a + \log b$

$\log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b).$

Ex. 21. ***Find the value of***

(i) $\log_x x + \log_x x^3 + \log_x x^5 + \ldots\ldots\ldots + \log_x x^{2n-1}$

(ii) $\log_3\left(1+\frac{1}{3}\right) + \log_3\left(1+\frac{1}{4}\right) + \ldots. + \log_3\left(1+\frac{1}{80}\right).$

Sol. (*i*) Given expression $= \log_x x + 3\log_x x + 5\log_x x + \ldots + (2n-1)\log_x x$

$$= 1 + 3 + 5 + \ldots. + (2n-1)$$

$$= \frac{n}{2}[1 + (2n-1)] = n^2.$$

(*ii*) Given expression $= \log_3\left(\frac{4}{3}\right) + \log_3\left(\frac{5}{4}\right) + \log_3\left(\frac{6}{5}\right) + \ldots..\log_3\left(\frac{81}{80}\right)$

$$= \log_3\left(\frac{4}{3}.\frac{5}{4}.\frac{6}{5}\ldots\ldots.\frac{81}{80}\right) = \log_3\frac{81}{3}$$

$$= \log_3 27 = \log_3(3^3) = 3\log_3 3 = 3\times 1 = \mathbf{3}.$$

Ex. 22. ***Find the value of x, if*** $\log_2(5.2^x + 1)$, $\log_4(2^{1-x} + 1)$ ***and 1 are in A.P.***

(*Type AIEEE 2002*)

Sol. $\log_2(5.2^x + 1), \log_4(2^{1-x} + 1)$, 1 are in A.P.

$\log_2(5.2^x + 1) + 1 = 2\log_4(2^{1-x} + 1)$ $\quad$ [a, b, c are in A.P. $\quad a + c = 2b$]

$\log_2(5.2^x + 1) + \log_2 2 = 2\log_{2^2}(2^{1-x} + 1)$

$\log_2(5.2^x + 1).2 = 2\times\frac{1}{2}\log_2(2^{1-x} + 1) = \log_2(2^{1-x} + 1)$ $\quad$ [$\log_{a^n} x = \frac{1}{n}\log_a x$, Cor. Law 3]

$10.2^x + 2 = 2^{1-x} + 1 = \frac{2^1}{2^x} + 1$. Let $2^x = a$. Then,

$10 \,.\, a + 2 = \frac{2}{a} + 1 \quad 10a + 1 \quad \frac{2}{a} \quad 0 \quad 10a^2 + a - 2 = 0$

$(5a-2)(2a+1) = 0 \quad a = \frac{2}{5}, \frac{-1}{2}, \quad 2^x = \frac{2}{5}$ ($\because 2^x > 0$ reject $a = \frac{1}{2}$)

$\log 2^x = \log \frac{2}{5} \quad x \log_2 2 = \log_2 2 - \log_2 5$

$x = 1 - \log_2 5. \quad [\because \log_2 2 = 1]$.

Ex. 23. ***Find the value of x if***

$$5^{\log x} - 3^{\log x-1} = 3^{\log x+1} - 5^{\log x-1} \qquad \textit{(I.S.C.)}$$

Sol. $5^{\log x} - 3^{\log x} \times 3^{-1} = 3^{\log x} \times 3 - 5^{\log x} \times 5^{-1}$

$5^{\log x} - 3^{\log x} \times \frac{1}{3} = 3^{\log x} \times 3 - 5^{\log x} \times \frac{1}{5}$

$\left(3+\frac{1}{3}\right)3^{\log x} = \left(1+\frac{1}{5}\right)5^{\log x} \Rightarrow \frac{10}{3}.3^{\log x} = \frac{6}{5}.5^{\log x} \Rightarrow \frac{3^{\log x}}{5^{\log x}} = \frac{6}{5} \times \frac{3}{10} = \frac{9}{25}$

$\left(\frac{3}{5}\right)^{\log x} = \left(\frac{3}{5}\right)^2 \quad \log x = 2 \quad x =$ 100, if the base is 10. $(\log_{10} x = 2 \quad 10^2 = x)$

3.05. Change of base

To prove that $\log_a n = \log_b n \,.\, \log_a b$

Proof. Let $\log_a n = x;\ \log_b n = y$ and $\log_a b = z$. Then

$n = a^x,\ n = b^y,\ b = a^z$ [*By def.*]

$a^x = n = b^y = (a^z)^y = a^{yz}$ [Putting $b = a^z$]

$x = yz$, *i.e.* $\log_a n = \log_b n \,.\, \log_a b$

The above formula changes the base from a to b.

Note 1. It can be proved in the same manner that $\log_b n = \log_a n \,.\, \log_b a$.

Note 2. From the above formula, we have

Aid to Memory

$\frac{n}{b} \quad \frac{n}{a} \quad \frac{a}{b}$

$\log_b n = \frac{\log_a n}{\log_a b}$. Also, $\log_b a = \frac{\log_b n}{\log_a n}$, $\log_a b = \frac{\log_a n}{\log_b n}$

Note 3. Put $n = a$ in the formula $\log_a n = \log_b n \,.\, \log_a b$, we have

$\log_a a = \log_b a \,.\, \log_a b \quad 1 = \log_b a \,.\, \log_a b$

$$\boxed{\log_b a \,.\, \log_a b = 1 \text{ or } \log_b a = \frac{1}{\log_a b},\quad \log_a n = \log_b n.\ \log_a b.} \qquad [\textit{Remember}]$$

Cor. 1. $\log_x y = \frac{\log_x a}{\log_y a}$ [See Note 2] $= \frac{\log_a y}{\log_a x}$ [See Note 3] $= \frac{\log y}{\log x}$, it being understood that base of both $\log y$ and $\log x$ *is the same* ($y > 0, x > 0, a > 0, a \quad 1, x \quad 1, y \quad 1$).

For example, $\log_3 11 = \frac{\log_{10} 11}{\log_{10} 3} \quad \frac{\log 11}{\log 3}$ (It is understood that base is the same)

Cor 2. $\log_b a \,.\, \log_c b \,.\, \log_a c = \dfrac{\log_m a}{\log_m b} \cdot \dfrac{\log_m b}{\log_m c} \cdot \dfrac{\log_m c}{\log_m a} = 1.$

$$\boxed{\log_b a \,.\, \log_c b \,.\, \log_a c = 1}$$ **(I.S.C.)**

Cor. 3. $\log_{x^n}(y^m) = \dfrac{\log_a y^m}{\log_a x^n} = \dfrac{m \log_a y}{n \log_a x} = \dfrac{m}{n} \log_x y.$

In particular, $\log_{a^n} x = \dfrac{1}{n} \log_a x$ [Same as Cor. in Law 3]

For example, $\log_{16} 125 = \log_{2^4} 5^3 = \dfrac{3}{4} \log_2 5,\ \log_{a^2}(m^3) = \dfrac{3}{2} \log_a m.$

$\log_9 7 \quad \log_{3^2} 7 \quad \dfrac{1}{2} \log_3 7.$

Cor. 4. $x^{\log_a y} \quad x^{\log_x y . \log_a x} \quad x^{\log_x y}\,{}^{\log_a x} \quad y^{\log_a x} \quad \because x^{\log_x y} \quad y$ by Th.1on page 4

Ex. 24. *Prove that* $\dfrac{1}{\log_a abc} + \dfrac{1}{\log_b abc} + \dfrac{1}{\log_c abc} = 1.$

Sol. L.H.S. $= \dfrac{1}{\log_a abc} + \dfrac{1}{\log_b abc} + \dfrac{1}{\log_c abc}$

$= \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc} abc = 1 = \text{R.H.S.}$ [Using $\log_b a = \dfrac{1}{\log_a b}$]

Ex. 25. *Using change of base formula, prove each of the following:*

(*a*) $(\log_{25} 2) = \dfrac{\log_5 2}{2}$, (*b*) $\log_{ab} x = \dfrac{\log_a x}{1 + \log_a b}$.

(*c*) $(\log_a x)(\log_b y) = (\log_b x)(\log_a y)$.

Sol. (*a*) $\log_{25} 2 = \dfrac{\log_5 2}{\log_5 25} \quad \dfrac{\log_5 2}{\log_5 5^2} \quad \dfrac{\log_5 2}{2 \log_5 5} \quad \dfrac{\log_5 2}{2}$

(*b*) $\log_{ab} x = \dfrac{\log_a x}{\log_a ab} = \dfrac{\log_a x}{\log_a a + \log_a b} = \dfrac{\log_a x}{1 + \log_a b}$

(*c*) Consider $\log_x y$

$\log_x y = \dfrac{\log_a y}{\log_a x} = \dfrac{\log_b y}{\log_b x}$

Cross-multiplying, we get, $(\log_a y)(\log_b x) = (\log_b y)(\log_a x)$.

Ex. 26. *If $a^x = b^y = c^z$ and $y^2 = xz$, prove that $\log_b a = \log_c b$.* **[A.P.]**

Sol. Let $a^x = b^y = c^z = k \Rightarrow x = \log_a k,\ y = \log_b k,\ z = \log_c k$

Given : $y^2 = zx \Rightarrow (\log_b k)^2 = \log_a k . \log_c k$

$\Rightarrow \dfrac{1}{(\log_k b)^2} = \dfrac{1}{(\log_k a)(\log_k c)} \Rightarrow \dfrac{\log_k a}{\log_k b} = \dfrac{\log_k b}{\log_k c} \Rightarrow \log_b a = \log_c b.$ [Cor. 1]

Ex. 27. ***Prove that*** $\dfrac{\log_3 8}{\log_9 16 \log_4 10} = 3\log_{10} 2.$ **[L.U.]**

Sol. Change all logarithms on left side to base 10.

$$\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} = \frac{\log_{10} 2^3}{\log_{10} 3} = \frac{3\log_{10} 2}{\log_{10} 3}$$

$$\log_9 16 = \frac{\log_{10} 16}{\log_{10} 9} = \frac{\log_{10} 2^4}{\log_{10} 3^2} = \frac{4\log_{10} 2}{2\log_{10} 3}$$

$$\log_4 10 \quad \frac{\log_{10} 10}{\log_{10} 4} \quad \frac{1}{\log_{10} 2^2} \quad \frac{1}{2\log_{10} 2}$$

Formula used.

$$\log_b n \quad \frac{\log_a n}{\log_a b} \quad \text{(Cor. 1)}$$

Left side $= \dfrac{3\log_{10} 2}{\log_{10} 3} \times \dfrac{2\log_{10} 3}{4\log_{10} 2} \times \dfrac{2\log_{10} 2}{1} = 3\log_{10} 2.$ = R.H.S. Hence, proved.

EXERCISE 3 (*b*)

1. Without using tables, find the value of

(*a*) $\dfrac{\log_{40} 1000}{\log_{40} 100}$ (*b*) $\dfrac{\log_a 32}{\log_a 4}$ (*c*) $\log_2 8$ (*d*) $5^{\log_5 2 + \log_5 3}$

(*e*) $25^{\log_5 7}$ (*f*) $2^{\frac{1}{3}\log_2 27}$ (*g*) $\log_2 (\log_2 4)$ **[A.P.]**

(*h*) $(b^2)^{5\log_b x}$ (*i*) $(7^3)^{-2\log_7 8}$ **[A.P.]**

2. Simplify :

(*a*) $\log (m^2) - \log m$ (*b*) $\log (y^2) + \log y$

(*c*) $\dfrac{1}{3}\log 27 - 2\log \dfrac{1}{3}$ (*d*) $\log (a - b) + \log (a^2 + ab + b^2)$

(*e*) $\log 256 - \log 1024$ (*f*) $\log 256 \div \log 1024$

3. Prove that:

(*i*) $\log 7 + \log \dfrac{1}{7} = 0$ (*ii*) $\log (\log x^2) - \log (\log x) = \log 2$

(*iii*) $\log \dfrac{a^2}{bc} + \log \dfrac{b^2}{ac} + \log \dfrac{c^2}{ab} = 0$ (*iv*) $\log \dfrac{26}{33} - \log \dfrac{65}{69} + \log \dfrac{55}{46} = 0.$ **[S.C.]**

(*v*) $\log_{10} 2 + 16\log_{10} \dfrac{16}{15} + 12\log_{10} \dfrac{25}{24} + 7\log_{10} \dfrac{81}{80} \quad 1$ (*vi*) $\dfrac{2\log 6 + 6\log 2}{4\log 2 + \log 27 - \log 9} = 2$

(*vii*) $2\log (a + b) = 2\log a + \log \left(1 + \dfrac{2b}{a} + \dfrac{b^2}{a^2}\right)$ (*viii*) $(\log a)^2 - (\log b)^2 = \log (ab) \log \left(\dfrac{a}{b}\right)$.

4. If $\log \left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log a + \log b)$, show that $a = b$.

5. If $\log_{10} x^2 - \log_{10} \sqrt{y} = 1$, find, without using the tables, the value of y when $x = 2$. **[S.C.]**

6. If $(x + y)^2 = 125\, xy$, show that $2 \log (x + y) = 3 \log 5 + \log x + \log y$.

7. Given that $h = \log 2$ and $k = \log 7$, express $\log \sqrt[3]{392}$ in terms of h and k. Express as a surd the number x such that $\log x = \left(\frac{4h-k}{3}\right)$. **[S.C.]**

8. Given that $3 \log_{10} x^2 y = 5 + \log_{10} x - 2 \log_{10} y$, where x and y are both positive, express y in terms of x. **[I.S.C.]**

9. If $3 + \log_{10} x = 2 \log_{10} y$, express x in terms of y. **[S.C.]**
[**Hint.** $3 = 3 \times 1 = 3 \times \log_{10} 10 = \log_{10} 10^3 = \log_{10} 1000$.]

10. If $\log \sqrt{x} = \frac{1}{2} \log 8$, $\log y^2 = 2 \log 2$, find the value of $x+y$.

11. Without using table, find x, y, if

(*i*) $\frac{\log x}{\log 5} = \frac{\log 36}{\log 6} = \frac{\log 64}{\log y}$.

(*ii*) $\log x = \log 3 + 2 \log 2 - \frac{3}{4} \log 16$. **[S.C.]**

(*iii*) $\log_{10}(10x+5) - \log_{10}(x+4) = \log 2$. **[S.C.]**

(*iv*) $\log(4y-3) = \log(y+1) + \log 3$. **[S.C.]**

(*v*) $\log_8(x^2-1) - \log_8(3x+9) = 0$.

(*vi*) $\log_7(x^3+27) - \log_7(x+3) = 2$.

12. Given that $\log_b (xy^3) = m$ and $\log_b(x^3 y^2) = p$, find $\log_b \sqrt{xy}$ in terms of m and p. **[S.C.]**

13. If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $xyz + 1 = 2yz$.

[**Hint.** $x = \frac{\log a}{\log 2a}$. Similarly, $y = \frac{\log 2a}{\log 3a}$, $z = \frac{\log 3a}{\log 4a}$]

14. Given that $u = \log_9 x$, find, in terms of u

(*a*) x, (*b*) $\log_9(3x)$, (*c*) $\log_x 81$.

(Table may not be used and answers may not contain the word log.) **[G.C.E.]**

15. (*a*) If $\log_8 x = p$, express $\log_2 x$ in terms of p.
(*b*) Given that $\log_q (xy) = 3$ and $\log_q (x^2y^3) = 4$, calculate the values of $\log_q x$ and $\log_q y$.
Hence, calculate the values of x and y when $q = 2$. **[I.S.C.]**

16. Given that $2 \lg (x^2y) = 3 + \lg x - \lg y$
where x and y are both positive, express y in terms of x. If $x - y = 3$, find the values of x and y.
[lg x represents $\log_{10} x$]. **[I.S.C.]**

17. If $f(x) = \log \left(\frac{1+x}{1-x}\right)$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

18. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, show that $a^a\, b^b\, c^c = 1$. **[I.I.T.]**

[**Hint:** $\log a = k(b-c)$, $\log b = k(c-a)$, $\log c = k(a-b)$
Let $a^a\, b^b\, c^c = p$, then $\log p = a \log a + b \log b + c \log c$
$\log p = 0 = \log 1$ $p = 1$.]

19. (*i*) Calculate the value of $\log_8 128$.
(*ii*) Evaluate x if $\log_3(3+x) + \log_3(8-x) - \log_3(9x-8) = 2 - \log_3 9$. **[I.S.C.]**

20. Given that $\log_e x^4 y = a$ and $\log_e x^2 y^2 = b$, find $\log_e \sqrt{y}$ in terms of a and b. **[I.S.C.]**

21. If $a = \log_{12} m$, and $b = \log_{18} m$, prove that $\frac{a-2b}{b-2a} = \log_3 2$. **[I.S.C.]**

22. Given that $3\log_{10}(x^2y) = 4 + 2\log_{10}x - \log_{10}y$,

where x and y are both positive, express y in terms of x. If $x - y = 2\sqrt{6}$, find the values of x and y.

[I.S.C. 1990]

23. Given that $\log_a x = \dfrac{1}{\alpha}$, $\log_b x = \dfrac{1}{\beta}$, $\log_c x = \dfrac{1}{\gamma}$, find $\log_{abc} x$. **[I.S.C. 1991]**

[Hint. $\dfrac{1}{\log_a x}$ $\log_x a$, $\dfrac{1}{\log_b x}$ $\log_x b$, $\dfrac{1}{\log_c x}$ $\log_x c$

$\Rightarrow \alpha + \beta + \gamma = \log_x a + \log_x b + \log_x c = \log_x abc$

$\dfrac{1}{}$ $\dfrac{1}{\log_x abc}$ $\log_{abc} x$]

> **To solve the following questions involving base change, recall the following formulas :**
>
> $\log_y x = \dfrac{1}{\log_x y}$, $\log_y x = \dfrac{\log x}{\log y}$, $\log_{a^n} x = \dfrac{1}{n}\log_a x$.]

24. Prove that

(*i*) $\dfrac{1}{\log_a ab}$ $\dfrac{1}{\log_b ab}$ 1 **[Hint.** L.H.S $\dfrac{1}{\dfrac{\log ab}{\log a}}$ $\dfrac{1}{\dfrac{\log ab}{\log b}}$ $\dfrac{\log a}{\log ab}$ $\dfrac{\log b}{\log ab}$]

(*ii*) $\dfrac{\log_a n}{\log_{ab} n} = 1 + \log_a b$

[Hint. L.H.S $= \dfrac{\log n}{\log a} / \dfrac{\log n}{\log ab} = \dfrac{\log ab}{\log a} = \dfrac{\log a + \log b}{\log a} = 1 + \dfrac{\log b}{\log a} = 1 + \log_a b$].

(*iii*) $\log_{y^2} x^3 \log_{z^2} y^3 \log_{x^2} z^3 = \dfrac{27}{8}$ **[Hint.** Use the formula $\log_{a^n} x^m = \dfrac{m}{n}\log_a x$]

(*iv*) $\log_{\sqrt{b}} a . \log_{\sqrt[3]{c}} b \log_{\sqrt[4]{a}} c = 24$ **[Hint.** L.H.S $= 2\log_b a \,.\, 3\log_c b \,.\, 4\log_a c$] [Cor. Law 3] **[A.P.]**

(*v*) $\dfrac{1}{1+\log_a(bc)} + \dfrac{1}{1+\log_b(ca)} + \dfrac{1}{1+\log_c(ab)} = 1$ **[A.P.]**

[Hint. $\dfrac{1}{1+\log_a(bc)} = \dfrac{1}{\log_a a + \log_a(bc)} = \dfrac{1}{\log_a(abc)} = \log_{abc} a$. Similarly for others]

(*vi*) $\dfrac{\log_9 11}{\log_5 13}$ $\dfrac{\log_3 11}{\log_{\sqrt{5}} 13} = 0$.

25. If $\log_a M = x$, show that (*i*) $\log_{1/a} M = \log_a\left(\dfrac{1}{M}\right) = -x$ and (*ii*) $\log_{1/a}\left(\dfrac{1}{M}\right) = x$.

26. If $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, then prove that $1 + abc = 2bc$.

[Hint. $1 + abc = 1 + \log_{24} 12 \,.\, \log_{36} 24 \,.\, \log_{48} 36$

$= 1 + \log_{48} 12 = \log_{48} 48 + \log_{48} 12 = \log_{48} (48 \times 12) = \log_{48} (24 \times 24)$

$= \log_{48} (24)^2 = 2 \log_{48} 24$. Similary find $2bc$.]

27. If x, y, z are distinct positive numbers different from 1 such that

$(\log_y x \,.\, \log_z x - \log_x x) + (\log_x y \,.\, \log_z y - \log_y y) + (\log_x z \,.\, \log_y z - \log_z z) = 0$

prove that $xyz = 1$.

[Hint. $\log_y x \,.\, \log_z x - \log_x x = \dfrac{\log x}{\log y} . \dfrac{\log x}{\log z} - 1 = \dfrac{(\log x)^2}{\log y . \log z} - 1$

Similarly do for other terms.

Then L.H.S. $= \dfrac{(\log x)^2}{\log y . \log z} - 1 + \dfrac{(\log y)^2}{\log z . \log x} - 1 + \dfrac{(\log z)^2}{\log x . \log y} - 1$

$$= \frac{(\log x)^3 + (\log y)^3 + (\log z)^3 - 3 \log x . \log y . \log z}{\log x . \log y . \log z} = 0 \text{ (given)}$$

$(\log x)^3 + (\log y)^3 + (\log z)^3 - 3 \log x \,.\, \log y \,.\, \log z = 0$

$\log x + \log y + \log z = 0$ [if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.]

$\log xyz = 0$ $\quad xyz = 1$.]

28. Solve for x if $a > 0$ and $2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0$

[Hint. $2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0 \qquad \dfrac{2}{\log_a x} + \dfrac{1}{1 + \log_a x} + \dfrac{3}{2 + \log_a x} = 0$.

[$\because \log_{a^2x} a \quad \dfrac{\log_a a}{\log_a a^2 x} \quad \dfrac{1}{\log_a a^2 \quad \log_a x} \quad \dfrac{1}{2 \log_a a \quad \log_a x} \quad \dfrac{1}{2 \quad \log_a x}$]

Put $\log_a x = t$ then $\dfrac{2}{t} + \dfrac{1}{1+t} + \dfrac{3}{2+t} = 0 \Rightarrow t = -1/2, -4/3$.

$t = -1/2 \qquad \log_a x = -\dfrac{1}{2} \Rightarrow x = a^{\frac{-1}{2}}, t = \dfrac{-4}{3} \Rightarrow \log_a x = \dfrac{-4}{3} \Rightarrow x = a^{-4/3}$.]

29. Solve: $\log_{2x+3} (6x^2 + 23x + 21) = 4 - \log_{3x+7} (4x^2 + 12x + 9)$.

[Hint. $1 + \log_{2x+3} (3x + 7) = 4 - 2 \log_{3x+7} (2x + 3) \qquad \log_{2x+3} (3x + 7) + 2 \log_{3x+7} (2x + 3) = 3$

Put $t = \log_{2x+3} (3x + 7)$. Then $t + \dfrac{2}{t} = 3 \qquad t = 1, 2$

$t = 1 \qquad \log_{2x+3} (3x + 7) = 1 \qquad 3x + 7 = 2x + 3 \qquad x = -4$

$t = 2 \qquad \log_{2x+3} (3x + 7) = 2 \qquad 3x + 7 = (2x + 3)^2 \qquad x = -2, -1/4$

But $x = -4, -2$ are extraneous solutions $\qquad x = -1/4$.]

30. If $2^{\log_{10} 3\sqrt{3}} = 3^{k \log_{10} 2}$, then find k.

[Hint. $2^{\log_{10} 3\sqrt{3}} \quad 3^{k \log_{10} 2} \qquad 2^{\log_{10} (3^{3/2})} \quad 3^{k \log_{10} 2} \qquad 2^{\log_2 3^{3/2} . \log_{10} 2} \quad 3^{k \log_{10} 2}$

$\left(2^{\log_2 3^{3/2}}\right)^{\log_{10} 2} \quad \left(3^k\right)^{\log_{10} 2} \qquad 2^{\log_2 3^{3/2}} \quad 3^k \qquad 3^{3/2} \quad 3^k$ (Using $a^{\log_a x} \quad x$)

$k \quad \dfrac{3}{2}$.]

31. $5^{3x^2 \log_{10} 2} \quad 2^{(x \;\; \frac{1}{2})\log_{10} 25}$, then find the value of x. ***(EAMCET)***

[Hint. $5^{3x^2 \log_{10} 2} = 2^{(x+\frac{1}{2})\log_{10} 25} \Rightarrow 5^{3x^2 \log_{10} 2} = 2^{(2x+1)\log_{10} 5}$

Þ $5^{3x^2 \log_{10} 2} = 2^{(2x+1)\log_2 5 . \log_{10} 2} = [2^{\log_2 5^{2x+1}}]^{\log_{10} 2} = (5^{2x+1})^{\log_{10} 2} = 5^{(2x+1)\log_{10} 2}$

$5^{3x^2 \log_{10} 2} \quad 5^{(2x \;\; 1)\log_{10} 2} \quad 3x^2 \quad 2x \quad 1.]$

32. Find the sum of the series

$$\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \ldots\ldots + \frac{1}{\log_{43} n}.$$ ***(I.I.T)***

[Hint. $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \ldots\ldots + \frac{1}{\log_{43} n}. = \log_n 2 + \log_n 3 + \log_n 4 + \ldots\ldots + \log_n 43$

$= \log_n (2.3.4 \ldots.43) = \log_n (1.2.3 \ldots..43) = \log_n 43!.]$*

ANSWERS

1. (*a*) $\frac{3}{2}$ (*b*) $\frac{5}{2}$ (*c*) 3 (*d*) 6 (*e*) 49 (*f*) 3

(*g*) 1 (*h*) x^{10} (*i*) $\frac{1}{8^6}$

2. (*a*) $\log m$ (*b*) 2 (*c*) $\log 27$ (*d*) $\log (a^3 - b^3)$ (*e*) $-\log 4$ (*f*) $\frac{4}{5}$

5. $\frac{4}{25}$ **7.** $\frac{1}{3}(3h + 2k), x = \sqrt[3]{\frac{16}{7}}$ **8.** $y = \frac{10}{x}$ **9.** $x = \frac{y^2}{1000}$

10. 10, 6 **11.** (*i*) $x = 25, y = 8$ (*ii*) $x = \frac{3}{2}$ (*iii*) $x = \frac{3}{8}$ (*iv*) $y = 6$

(*v*) 5, –2 (*vi*) $x = 8$ **12.** $\frac{2p+m}{14}$ **14.** (*a*) 9^u (*b*) $\frac{1}{2} + u$ (*c*) $\frac{2}{u}$

15. (*a*) $\log_2 x = 3p$ (*b*) $\log_q x = 5$, $\log_q y = -2$; $x = 32, y = \frac{1}{4}$

16. $y = 10x^{-1}$; $x = 5, y = 2$ **19.** (*i*) $\frac{7}{3}$ (*ii*) $x = 4$ **20.** $\frac{2b-a}{6}$

22. $y = \frac{10}{x}, x \;\; 4 \;\; \sqrt{6}, y \;\; 4 \;\; \sqrt{6}$ **23.** $\frac{1}{\alpha+\beta+\gamma}$ **30.** $k = \frac{3}{2}$

31. $x = -\frac{1}{3}, 1$ **32.** $\log_n (43)!$.

REVISION EXERCISE

1. $7^{2\log_7 5}$ is equal to

(*a*) $\log_7 35$ (*b*) 5 (*c*) 25 (*d*) $\log_7 25$

---

* $n!$ = Product of n natural numbers

= 1. 2.3 ...... n $(n - 1)$

2. If $\log_{\sqrt{3}} 5 = a$ and $\log_{\sqrt{3}} 2 = b$, then find $\log_{\sqrt{3}} 300$.

3. If 1, $\log_9 (3^{1-x} + 2)$, $\log_3 (4.3^x - 1)$ are in A. P., then x =

(a) $\log_3 4$ (b) $1 - \log_3 4$ (c) $1 - \log_4 3$ (d) $\log_4 3$.

[Hint. Type solved Ex. 12]

4. If a, b, c 0 belong to the set {0, 1, 2, 3, ..., 9}, then $\log_{10} \dfrac{a \quad 10b \quad 100c}{10^{\ 4} a \quad 10^{\ 3} b \quad 10^{\ 2} c}$ =

(i) 1 (ii) 2 (iii) 3 (iv) 4

[Hint. Given exp. = $\log_{10} \dfrac{1}{10^{\ 4}} \cdot \dfrac{a \quad 10b \quad 100c}{a \quad 10b \quad 100c}$.

5. $\log_{27} (\log_3 x) = \dfrac{1}{3}$ x =

(a) 3 (b) 6 (c) 9 (d) 27.

[Hint. $\log_{27} (\log_3 x) = \dfrac{1}{3}$ $\log_{3^3} (\log_3 x) = \dfrac{1}{3}$ $\dfrac{1}{3} \log_3 (\log_3 x) = \dfrac{1}{3}$

$\because \log_{a^n} x \quad \dfrac{1}{n} \log_a x$

$\log_3 (\log_3 x) = 1$ $\log_3 x = 3^1 = 3$ $x = 3^3 = 27$. ($\log_a n = x$ $a^x = n$ by def.)]

6. If $x = \log_{0.1}$ (0.001), $y = \log_9 81$, then $\sqrt{x \quad 2\sqrt{y}}$ =

(a) $3 - 2\sqrt{2}$ (b) $\sqrt{3} - 2$ (c) $\sqrt{2} - 1$ (d) $\sqrt{2} - 2$

ANSWERS

1. (c) **2.** 2 ($a + b + 1$) **3.** (b) **4.** (iv) **5.** (d) **6.** (c)

4

Remainder and Factor Theorem

4.01. Rational integral function

A polynomial in x, in which no term contains a fractional power of x, or a negative power of x, is called a **rational integral function of x**, *and is denoted by* $f(x)$ or $F(x)$, or $Q(x)$ etc.

For example, $3x^4 - 5x^3 + 2x + 1$, $2x^3 + 7x^2 - x + 4$ are rational integral functions of x. They can be written as $f(x) = 3x^4 - 5x^3 + 2x + 1$, $F(x) = 2x^3 + 7x^2 - x + 4$, etc.

4.02. Value of a function

The value of a function $f(x)$ or $F(x)$ for $x = a$ is denoted by $f(a)$ or $F(a)$, as the case may be, and is obtained by putting $x = a$ in the polynomial.

If $\quad f(x) = 3x^4 - 5x^3 + 2x + 1,$

then $\quad f(2) = (3 \times 2^4) - (5 \times 2^3) + (2 \times 2) + 1 = 48 - 40 + 4 + 1 = 13$

If $\quad F(x) = 2x^3 + 7x^2 - x + 4,$

then $\quad F(-1) = 2 \times (-1)^3 + 7(-1)^2 - (-1) + 4 = -2 + 7 + 1 + 4 = 10.$

4.03. Remainder theorem

Let $f(x) = 2x^2 - 5x + 7$. Find the remainder when $f(x)$ is divided by $(x - 1)$.

$$
\begin{array}{r|l}
x-1 \,\big)\; 2x^2 - 5x + 7 \;\big(\; 2x - 3 \\
2x^2 - 2x \qquad \\
- \quad + \qquad \\
\hline
-3x + 7 \\
-3x + 3 \\
+ \quad - \\
\hline
4
\end{array}
$$

Also

$f(1) = 2.1^2 - 5.1 + 7$

$= 2 - 5 + 7$

$= \mathbf{4}$

Thus, we see that the remainder obtained on dividing $f(x)$ by $(x - 1)$, is equal to the value of $f(x)$ at $x = 1$, *i.e.*, $f(1)$.

Similarly, find the value of the remainder when $f(x) = x^3 + 4x^2 - 3x + 10$ is divided by $x + 4$.

Also find the value of $f(x)$ at $x = -4$ and check whether remainder obtained $= f(-4)$ or not !

You will find that both the values are equal. Therefore, it follows that the remainder obtained when $f(x)$ is divided by $x - a$ is equal to the value of $f(x)$ at $x = a$, *i.e.*, $f(a)$.

This fact can be stated as a theorem called the Remainder Theorem.

Theorem. Let $f(x)$ be any rational integral function of x or polynomial of degree greater than or equal to 1 (≥ 1) and 'a' be any real number. If $f(x)$ is divided by $(x - a)$, then the remainder is always equal to $f(a)$.

Let $q(x)$ be the quotient and $r(x)$ the remainder when the given function is divided by $x - a$. Then

$$f(x) = (x-a)\,q(x) + r(x) \qquad ...(i)$$

Putting $x = a$, we get

$$f(a) = r(x), \quad \text{or} \quad r(x) = f(a) \qquad ...(ii)$$

In other words, the remainder is equal to the value of $f(x)$ when x is put equal to a.

Here $r(x) = 0$ or degree of $r(x) <$ degree of $(x - a)$. Degree of $(x - a)$ is 1   degree of $r(x) < 1$ and a polynomial of degree less than 1 is a constant. Therefore, either $r(x) = 0$ or $r(x) =$ constant.

Cor. 1. If $f(x)$ be divided by $x + a$, the remainder is $f(-a)$, $\left[\because x + a = x - (-a) \text{ or } x + a = 0 \Rightarrow x = -a\right]$

Cor. 2. If $f(x)$ be divided by $ax - b$, the remainder is $f\left(\dfrac{b}{a}\right)$ $[\because ax - b = 0 \quad x = b/a]$

Cor. 3. If $f(x)$ be divided by $ax + b$, the remainder is $f(-b/a)$ $[\because ax + b = 0 \quad x = -b/a]$

Ex. 1. ***Find the remainder when $x^3 - 7x + 4$ is divided by $x - 1$, $x + 2$ and $2x + 1$.***

Sol. (*i*) Putting $x = 1$, in the given expression, we get remainder $= 1^3 - 7 \quad 1 + 4 = \mathbf{-2.}$

(*ii*) Putting $x = -2$, we get

$$\text{Remainder} = (-2)^3 - 7(-2) + 4 = -8 + 14 + 4 = \mathbf{10.}$$

(*iii*) Putting $x = \dfrac{-1}{2}$, we get

$$\text{Remainder} = \left(\frac{-1}{2}\right)^3 - 7\left(\frac{-1}{2}\right) + 4 = \frac{-1}{8} + \frac{7}{2} + 4 = \frac{-1 + 28 + 32}{8}$$

$$= \frac{59}{8} = \mathbf{7\frac{3}{8}}.$$

Ex. 2. ***Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $1 - 3x$.***

Sol. By remainder theorem, when $f(x)$ is divided by $1 - 3x$, the remainder $= f(1/3)$

$[\because 1 - 3x = 0 \quad x = 1/3]$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 6\times\left(\frac{1}{3}\right)^2 + 2\times\frac{1}{3} - 4$$

$$= \frac{1}{27} - \frac{6}{9} + \frac{2}{3} - 4 = \frac{1}{27} \quad \frac{2}{3} \quad \frac{2}{3} \quad 4$$

$$\frac{1 \quad 4 \quad 27}{27} \quad \frac{1 \quad 108}{27} \quad \frac{107}{27}$$

The required remainder is $-\mathbf{\dfrac{107}{27}}$.

Ex. 3. ***If the polynomial $f(x) = 2x^3 - ax^2 + 4x - 1$, leaves a remainder –37 when divided by $x + 2$, find the value of a.***

Sol. By remainder theorem, when $f(x)$ is divided by $x + 2$, the remainder $= f(-2)$.

$$f(-2) = 2(-2)^3 - a(-2)^2 + 4(-2) - 1$$
$$= -16 - 4a - 8 - 1$$
$$= -25 - 4a$$

Given Remainder $= -37$ $\qquad -25 - 4a = -37$

$4a = 37 - 25 = 12$ $\qquad$ $\boldsymbol{a = 3.}$

Ex. 4. ***The polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $x + 2$. If the remainder is same in each case, find the value of a.***

Sol. Let $f(x) = ax^3 + 3x^2 - 13$ and $g(x) = 2x^3 - 5x + a$.

The remainders when $f(x)$ and $g(x)$ are divided by $x + 2$ are $f(-2)$ and $g(-2)$.

$$f(-2) = a.(-2)^3 + 3.(-2)^2 - 13$$
$$= -8a + 12 - 13 = -8a - 1$$
$$g(-2) = 2.(-2)^3 - 5.(-2) + a = -16 + 10 + a = -6 + a.$$

Given, $\quad f(-2) = g(-2)$

$-8a - 1 = -6 + a \qquad 6 - 1 = a + 8a \qquad 5 = 9a \qquad \boldsymbol{a = \frac{5}{9}}.$

Hence the value of $\boldsymbol{a = \frac{5}{9}}$.

4.04. Factor theorem

If $f(x)$ be a polynomial of degree greater than or equal to one (≥ 1) and 'a' be a real number such that $f(a) = 0$, then $x - a$ is a factor of $f(x)$. Conversely if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

Alternative form of the Factor Theorem

If $f(a) = 0$, *then* $(x - a)$ *is a factor of* $f(x)$.

Cor. 1. If $f(-a) = 0$, then $(x + a)$ is a factor of $f(x)$.

Cor. 2. If $f\left(\frac{-a}{b}\right) = 0$, then $(bx + a)$ is a factor of $f(x)$. $\left[x - \left(\frac{-a}{b}\right) = \frac{bx + a}{b}\right]$

Cor. 3. If polynomial $f(x)$ vanishes when $x = a$ and also $x = b$, then $f(x)$ is exactly divisible by $(x - a)(x - b)$.

> **Note.** Only those values of x will make $f(x)$ zero which are factors of the constant term, *e.g.*, if $f(x) = x^3 - 19x - 30$, x can only have the values $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$ and no other values for making $f(x)$ zero. The reason is that the constant term is always the product of the roots of the equation $f(x) = 0$.

Ex. 5. ***Use factor theorem to determine whether***

(*a*) ***$x - 1$ is a factor of $f(x) = 2x^3 + 5x^2 - 3x - 4$***

(*b*) ***$2x - 3$ is a factor of $f(x) = 2x^3 - 9x^2 + x + 12$.***

Sol. (*a*) $f(x) = 2x^3 + 5x^2 - 3x - 4$

$x - 1$ is a factor of $f(x)$ if $f(1) = 0$

Now $f(1) = 2.1^3 + 5.1^2 - 3.1 - 4 = 2 + 5 - 3 - 4 = 0$

Hence $(x - 1)$ is a factor of $f(x) = 2x^3 + 5x^2 - 3x - 4$

(b) $f(x) = 2x^3 - 9x^2 + x + 12$

$(2x - 3)$ is a factor of $f(x)$ if $f\left(\frac{3}{2}\right) = 0$

$$\because 2x - 3 = 0 \quad \Rightarrow x = \frac{3}{2}$$

Now $f\left(\frac{3}{2}\right) = 2.\left(\frac{3}{2}\right)^3 - 9.\left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$

$$\frac{2\quad 27}{8} \quad \frac{9\quad 9}{4} \quad \frac{3}{2} \quad 12 = \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12$$

$$\frac{27\quad 81\quad 6\quad 48}{4} \quad \frac{81\quad 81}{4} \quad 0$$

Hence $(2x - 3)$ is a factor of $f(x) = 2x^3 - 9x^2 + x + 12$.

Ex. 6. ***Find the value of a if x + a is a factor of the polynomial $x^3 + ax^2 - 2x + a + 4$.***

Sol. Let $f(x) = x^3 + ax^2 - 2x + a + 4$

$(x + a)$ is a factor of $f(x)$ if $f(-a) = 0$

$f(-a) = (-a)^3 + a.(-a)^2 - 2.(-a) + a + 4$

$= -a^3 + a^3 + 2a + a + 4 = 3a + 4$

$f(-a) = 0 \qquad 3a + 4 = 0 \qquad a = -\frac{\mathbf{4}}{\mathbf{3}}$

Hence $\boldsymbol{a = -\frac{4}{3}}$.

Ex. 7. ***If $x^3 + ax^2 - x + b$ has $(x - 2)$ as a factor and leaves a remainder 3 when divided by $(x - 3)$, find a and b.***

Sol. Let $f(x) = x^3 + ax^2 - x + b$

Since $(x - 2)$ is a factor of $f(x)$, therefore,

$f(2) = (2)^3 + a(2)^2 - 2 + b = 0 \qquad 4a + b = -6$ ...(1)

Since $f(x)$ divided by $(x - 3)$, leaves a remainder 3, therefore

$f(3) = 3$, *i.e.*, $(3)^3 + a.(3)^2 - 3 + b = 3 \qquad 9a + b = -21$ ...(2)

Subtracting eqn. (2) from eqn. (1), we get

$-5a = 15 \qquad a = -3$

Putting $a = -3$ in (1), we get $4(-3) + b = -6 \qquad b = 6$

Hence $\boldsymbol{a = -3, b = 6}$.

Ex. 8. ***Without actual division prove that $x^4 + 2x^3 - 2x^2 + 2x - 3$ is exactly divisible by $x^2 + 2x - 3$.***

Sol. Let $f(x) = x^4 + 2x^3 - 2x^2 + 2x - 3$

$g(x) = x^2 + 2x - 3 = x^2 + 3x - x - 3$

$= x(x + 3) - 1(x + 3) = (x - 1)(x + 3)$.

Now $f(x)$ will be exactly divisible by $g(x)$ if it is exactly divisible by $(x - 1)$ as well as $(x + 3)$.

i.e., if $f(1) = 0$ and $f(-3) = 0$.

Now, $f(1) = (1)^4 + 2.1^3 - 2.1^2 + 2.1 - 3$

$= 1 + 2 - 2 + 2 - 3 = 0 \qquad (x - 1)$ is a factor of $f(x)$

$f(-3) = (-3)^4 + 2.(-3)^3 - 2.(-3)^2 + 2.(-3) - 3$

$= 81 - 54 - 18 - 6 - 3 = 0$.

$(x + 3)$ is a factor of $f(x)$

$(x-1)(x+3)$ divides $f(x)$ exactly

Hence, $x^2 + 2x - 3$ is a factor of $f(x)$.

Ex. 9. ***Factorize $x^3 + 13x^3 + 31x - 45$, given that $x + 9$ is a factor of it.***

Sol. To get the other factors of $f(x) = x^3 + 13x^2 + 31x - 45$, we divide it by $(x + 9)$, by performing long division.

$$
\begin{array}{r|l}
 & x^2 + 4x - 5 \\
\hline
x + 9 & x^3 + 13x^2 + 31x - 45 \\
 & \underline{x^3 + 9x^2} \\
 & \quad\; 4x^2 + 31x \\
 & \quad\; \underline{4x^2 + 36x} \\
 & \qquad\quad - 5x - 45 \\
 & \qquad\quad \underline{- 5x - 45} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$$
\begin{aligned}
\therefore \quad x^3 + 13x^2 + 31x - 45 &= (x + 9)(x^2 + 4x - 5) \\
&= (x + 9)(x + 5x - x - 5) \\
&= (x + 9)\{x(x + 5) - 1(x + 5)\} \\
&= (x + 9)(x + 5)(x - 1).
\end{aligned}
$$

Ex. 10. ***Factorize $x^3 - 7x + 6$, using factor theorem.***

Note: For factorising the polynomial $f(x) = x^3 - 7x + 6$, we have to first find one value of x, for which $f(x) = 0$ by hit and trial method.

For this we see the constant term + 6 and consider its factors $\pm 1, \pm 2, \pm 3, \pm 6$. Putting these factors as the values of x, one by one in $f(x)$, we find the value of x for which $f(x) = 0$. That gives us the first factor of $f(x)$. The rest we find either by continuing this process or by long division by the factor that we have got.

Sol. The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

By putting $x = 1$ in $f(x) = x^3 - 7x + 6$, we get

$f(1) = 1 - 7 + 6 = 0$ $\qquad$ $(x - 1)$ is a factor of $x^3 - 7x + 6$.

By putting $x = 2$ in $f(x)$, we get

$f(2) = 8 - 14 + 6 = 0$ $\qquad$ $(x - 2)$ is a factor of $x^3 - 7x + 6$.

By putting $x = -3$ in $f(x)$, we get

$f(-3) = -27 + 21 + 6 = 0$ $\qquad$ $(x + 3)$ is a factor of $x^3 - 7x + 6$.

Since the highest power of x in the expression is 3, there cannot be more than three factors. Therefore

$$x^3 - 7x + 6 = \mathbf{(x - 1)(x - 2)(x + 3).}$$

Method II. After getting one factor, say $x - 1$, we divide $x^3 - 7x + 6$ by $x - 1$ and we get the quotient as $x^2 + x - 6$, which can be easily factorised as $(x + 3)(x - 2)$.

Hence, $\quad x^3 - 7x + 6 = \mathbf{(x - 1)(x + 3)(x - 2)}$

$$
\begin{array}{r|l}
 & x^2 + x \quad 6 \\
\hline
x \quad 1 & x^3 \qquad\quad 7x + 6 \\
 & \underline{x^3 \quad x^2} \\
 & \quad\;\; + \\
 & \qquad x^2 \quad 7x + 6 \\
 & \qquad \underline{x^2 \quad x} \\
 & \qquad\quad + \\
 & \qquad\qquad 6x + 6 \\
 & \qquad\qquad \underline{6x + 6} \\
 & \qquad\qquad\; + \\
 & \qquad\qquad\quad 0
\end{array}
$$

Ex. 11. ***If the expression $x^3 + 3x^2 + 4x + p$ has $(x + 6)$ as a factor, find p.***

Sol. As $(x + 6)$ is its factor, so $\quad f(-6) = 0$,

i.e., $\quad (-6)^3 + 3(-6)^2 + 4(-6) + p = 0$

$-216 + 108 - 24 + p = 0$ $\qquad$ $\mathbf{p = 132.}$

Ex. 12. ***If*** $(x + a)$ ***be the HCF of*** $x^2 + mx + n$ ***and*** $x^2 + rx + s$, ***then show that*** $a = \dfrac{n - s}{m - r}$.

Sol. Let $P(x) = x^2 + mx + n$ and $Q(x) = x^2 + rx + s$,

$(x + a)$ is the HCF of both $P(x)$ and $Q(x)$ means, $(x + a)$ is a factor of both $P(x)$ and $Q(x)$. Therefore,

$$P(-a) = 0 \qquad (-a)^2 + m(-a) + n = 0 \qquad a^2 - am + n = 0 \qquad ...(1)$$

and

$$Q(-a) = 0 \qquad (-a)^2 + r(-a) + s = 0 \qquad a^2 - ar + s = 0 \qquad ...(2)$$

eqn. (1) – eqn. (2) gives $ar - am + n - s = 0$

$$am - ar = n - s \qquad a(m - r) = n - s \qquad a = \frac{n - s}{m - r}.$$

Ex. 13. ***Find the values of a and b so that the polynomials*** $P(x)$ ***and*** $Q(x)$ ***have*** $(x + 1)(x + 3)$ ***as their HCF, where*** $P(x) = (x^2 + 3x + 2)(x^2 + 2x + a)$, ***and*** $Q(x) = (x^2 + 7x + 12)(x^2 + 7x + b)$.

Sol. $P(x) = (x^2 + 3x + 2)(x^2 + 2x + a) = (x + 1)(x + 2)(x^2 + 2x + a)$

(Factorising the trinomial $x^2 + 3x + 2$)

Similarly, $Q(x) = (x^2 + 7x + 12)(x^2 + 7x + b) = (x + 3)(x + 4)(x^2 + 7x + b)$

HCF $= (x + 1)(x + 3)$ $\quad$ $(x + 1)$ and $(x + 3)$ are factors of $P(x)$ and $Q(x)$.

$(x + 3)$ is a factor of $P(x)$, $\quad P(-3) = 0$

$(-3 + 1)(-3 + 2)(9 - 6 + a) = 0 \qquad (-2)(-1)(3 + a) = 0 \qquad 2(3 + a) = 0 \qquad a = -3.$

Now since $(x + 1)$ is a factor of $Q(x)$ therefore $Q(-1) = 0$

$(-1 + 3)(-1 + 4)(1 - 7 + b) = 0 \qquad (2)(3)(b - 6) = 0 \qquad b = 6.$

Hence, $\boldsymbol{a = -3,\ b = 6}$.

Ex.14. ***What number must be added to*** $3x^3 + x^2 - 22x + 9$, ***so that the result becomes exactly divisible by*** $x + 3$.

Sol. Let the number to be added be a. Then the expression becomes:

$f(x) = 3x^3 + x^2 - 22x + 9 + a.$

For $(x + 3)$ to be a factor of $f(x)$, $f(-3) = 0$.

$f(-3) = 3.(-3)^3 + (-3)^2 - 22(-3) + 9 + a$

$= -81 + 9 + 66 + 9 + a = 3 + a$

Given that $3 + a = 0 \qquad \boldsymbol{a = -3.}$

$\therefore$ **–3 should be added** to the given expression to make the result exactly divisible by $x + 3$.

EXERCISE 4 (*a*)

1. **Find the remainder when the expression**
   (*i*) $3x^3 + 8x^2 - 6x + 1$ is divided by $x + 3$.
   (*ii*) $5x^3 - 8x^2 + 3x - 4$ is divided by $x - 1$.
   (*iii*) $x^3 + 3x^2 - 1$ is divided by $3x + 2$.
   (*iv*) $4x^3 - 12x^2 + 14x - 3$ is divided by $2x - 1$.
2. When $x^3 + 3x^2 - kx + 4$ is divided by $x - 2$, the remainder is k. Find the value of the constant k.
3. Find the value of a if the division of $ax^3 + 9x^2 + 4x - 10$ by $x + 3$ leaves a remainder 5.
4. If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x - a$ leave the same remainder when divided by $x - 2$, find the value of a.
5. Let A and B be the remainders when the polynomial $y^3 + 2y^2 - 5ay - 7$ and $y^3 + ay^2 - 12y + 16$ are divided by $y - 1$ and $y + 2$ respectively. If $2A + B = 0$, find the value of a.

6. Use factor theorem in each of the following to find whether $g(x)$ is a factor of $f(x)$ or not:

(*i*) $f(x) = x^3 - 6x^2 + 11x - 6$; $g(x) = x - 3$

(*ii*) $f(x) = 2x^3 - 9x^2 + x + 12$; $g(x) = x + 1$

(*iii*) $f(x) = 7x^2 - 2\sqrt{8}\,x - 6$; $g(x) = x - \sqrt{2}$.

(*iv*) $f(x) = 3x^3 + x^2 - 20x + 12$; $g(x) = 3x - 2$.

7. Find the value of a if $x^3 + ax + 2a - 2$ is exactly divisible by $x + 1$.

8. Use the remainder theorem to determine the value of k for which $x + 2$ is a factor of $(x + 1)^7 + (2x + k)^3$.

9. (*a*) Find the values of a and b so that the expression $x^3 + 10x^2 + ax + b$ is exactly divisible by $x - 1$ as well as $x - 2$.

(*b*) If $(x - 2)$ is a factor of $x^2 + ax - 6 = 0$ and $x^2 - 9x + b = 0$, find the values of a and b.

10. If both $x - 2$ and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that $p = r$.

11. Without actual division show that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

12. Find the value of a, if $x + a$ is a factor of $x^4 - a^2x^2 + 3x - a$.

13. If $x^3 + ax^2 + bx + 6$ has $(x - 2)$ as a factor and leaves a remainder 3 when divided by $(x - 3)$, find the values of a and b.

14. Use remainder theorem to find which of the expressions $(x - 1)$, $(x + 1)$, $(x - 2)$, $(x + 2)$ are the factors of $3x^3 - 7x^2 - 2x + 8$.

Hence factorise completely.

15. Factorise:

(*a*) $x^3 + 13x^2 + 32x + 20$, if it is given that $x + 2$ is its factor.

(*b*) $4x^3 + 20x^2 + 33x + 18$, if it is given that $2x + 3$ is its factor.

16. Show that (*a*) $(x - 10)$ is a factor of $x^3 - 23x^2 + 142x - 120$ and hence factorise it completely.

(*b*) $(3z + 10)$ is a factor of $9z^3 - 27z^2 - 100z + 300$ and factorise it completely.

17. Given that $(x - 2)$ and $(x + 1)$ are factors of $x^3 + 3x^2 + ax + b$, calculate the values of a and b, and hence find the remaining factor.

18. The expression $4x^3 - bx^2 + x - c$ leaves remainders 0 and 30 when divided by $(x + 1)$ and $(2x - 3)$ respectively. Calculate the values of b and c and hence factorise the expression completely.

19. Given that $(x + 2)$ and $(x - 3)$ are factors of $x^3 + ax + b$, calculate the values of a and b, and find the remaining factor.

20. Factorise, using remainder theorem :

(*a*) $x^3 - 19x - 30$ (*b*) $x^3 + 7x^2 - 21x - 27$ (*c*) $x^3 - 3x^2 - 9x - 5$

(*d*) $2x^3 + 9x^2 + 7x - 6$ (*e*) $y^3 - 2y^2 - 29y - 42$ (*f*) $3x^3 - 4x^2 - 12x + 16$

21. Find the values of a and b such that the polynomials $P(x)$ and $Q(x)$ have $(x + 3)(x - 2)$ as their H C F : $P(x) = (x^2 - 4x - 21)(x^2 - 4x + a)$, $Q(x) = (x^2 - 5x + 6)(x^2 - 4x + b)$

22. $(x - 3)$ is the H.C.F of $x^3 - 2x^2 + px + 6$ and $x^2 - 5x + q$. Find $6p + 5q$.

23. Find the values of a and b so that the polynomials $P(x)$ and $Q(x)$ have $(x - 1)(x + 4)$ as their HCF : $P(x) = (x^2 - 3x + 2)(x^2 + 7x + a)$, $Q(x) = (x^2 + 5x + 4)(x^2 - 5x + b)$

24. If the H.C.F. of $(x - 5)(x^2 - x - a)$ and $(x - 4)(x^2 - 2x + b)$ is $(x - 4)(x - 5)$, find the values of a and b.

25. Using factor theorem, show that $x - y$, $y - z$, $z - x$ are the factors of

$$x^2\,(y - z) + y^2\,(z - x) + z^2\,(x - y).$$

26. If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.

[Hint. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$x^2 - 1$ is a factor of $f(x)$ $\quad (x - 1)$ and $(x + 1)$ and $(x + 1)$ are factors of $f(x)$.

$f(1) = 0$ and $f(-1) = 0$.

$a + b + c + d + e = 0$.......... (1) and $a - b + c - d + e = 0$ ...(2)

Adding and subtracting (1) and (2) we get the desired result.]

27. (*a*) What number must be subtracted from $x^3 - 6x^2 - 15x + 80$, so that the result is exactly divisible by $x + 4$.

(*b*) What number must be added to $x^3 - 3x^2 - 12x + 19$, so that the result is exactly divisible by $x - 2$.

ANSWERS

1. (*i*) 10 (*ii*) – 4 (*iii*) $\dfrac{1}{27}$ (*iv*) $\dfrac{3}{2}$ **2.** $k = 8$ **3.** $a = 2$

4. $a = -2$ **5.** $a = 4$ **6.** (*i*) yes (*ii*) yes (*iii*) yes (*iv*) yes

7. $a = 3$ **8.** $k = 5$ **9.** (*a*) $a = -37, b = 26$ (*b*) $a = 1, b = 14$

12. $a = 0$ **13.** $a = -3, b = -1$ **14.** $(x + 1)(x - 2)(3x + 4)$

15. (*a*) $(x + 2)(x + 10)(x + 1)$ (*b*) $(x + 2)(2x + 3)^2$

16. (*a*) $(x - 1)(x - 10)(x - 12)$ (*b*) $(3z + 10)(3z - 10)(z - 3)$

17. $a = -6, b = -8$; $(x + 4)$ **18.** $b = -8, c = 3$; $(x + 1)(2x + 3)(2x - 1)$

19. $a = -7, b = -6$; $(x + 1)$

20. (*a*) $(x + 2)(x + 3)(x - 5)$ (*b*) $(x + 1)(x - 3)(x + 9)$ (*c*) $(x + 1)^2 (x - 5)$

(*d*) $(x + 2)(x + 3)(2x - 1)$ (*e*) $(y + 2)(y + 3)(y - 7)$ (*f*) $(3x - 4)(x - 2)(x + 2)$

21. $a = 4, b = -21$ **22.** 0 **23.** $a = 12, b = 4$ **24.** $a = 12, b = -15$

27. (*a*) –20 (*b*) 9

REVISION EXERCISE

1. Find the remainder when $2x^3 - 3x^2 + 7x - 8$ is divided by $x - 1$.

2. Find the value of the constants a and b if $(x - 2)$ and $(x + 3)$ are both factors of the expression $x^3 + ax^2 + bx - 12$.

3. Using factor theorem, show that $(x - 3)$ is a factor of $x^3 - 7x^2 + 15x - 9$. Hence factorise the given expression completely.

4. Find the value of a, if $(x - a)$ is a factor of $x^3 - a^2x + x + 2$.

5. Use the factor theorem to factorise completely :
$x^3 + x^2 - 4x - 4$.

6. $(x - 2)$ is factor of the expression $x^3 + ax^2 + bx + 6$. When this expression is divided by $(x - 3)$, it leaves a remainder 3. Find the values of a and b.

7. Show that $2x + 7$ is a factor of $2x^3 + 5x^2 - 11x - 14$. Hence factorise the given expression completely, using factor theorem.

8. Show that $(x - 1)$ is a factor of $x^3 - 7x^2 + 14x - 8$. Hence completely factorise the above expression.

9. If $(x-2)$ is a factor of $2x^3 - x^2 - px - 2$.

(*i*) Find the value of p.

(*ii*) With the value of p, factorise the above expression completely.

ANSWERS

1. -2 **2.** $a = 3, b = -4$ **3.** $(x-3)^2 (x-1)$ **4.** $a = -2$

5. $(x+1)(x+2)(x-2)$ **6.** $a = -3, b = -1$ **7.** $(2x+7)(x-2)(x+1)$

8. $(x-1)(x-4)(x-2)$ **9.** (*i*) $p = 5$ (*ii*) $(x-2)(x+1)(2x+1)$

5

Quadratic Equations

5.01. What is a quadratic equation ?

A quadratic equation in one variable is an equation in which the highest power of the variable is two.

Thus $3x^2 + 2x - 1 = 0$ is a quadratic equation in x. The **standard form of a quadratic equation** in one variable is

$$ax^2 + bx + c = 0 \; ; \; a, b, c \in R, a \neq 0.$$

Thus, $5x^2 - 7x + 8 = 0$ is in standard form. Here, $a = 5$, $b = -7$, $c = 8$.

5.02. Solving quadratic equations by factorisation

We follow the following two rules :

Rule 1. ***Every quadratic equation has two roots.***

Thus, $x^2 = 16$ has two roots, 4 and -4, that is $x = \pm 4$.

In the case of the quadratic equation $x^2 = 0$, the equation is considered to have two equal roots, each equal to 0, while the quadratic equation $(x - 4)^2 = 0$ is considered to have two equal roots, each equal to 4.

Rule 2. ***If the product of two factors is zero, then one or the other of the factors equals zero.***

Thus, if $x(3x - 5) = 0$, then $x = 0$ or $3x - 5 = 0$.

Also, if $(x - 2)(4x + 9) = 0$, then $x - 2 = 0$ or $4x + 9 = 0$.

To solve a quadratic equation, we work as under :

Step 1. *Express the given equation in the form* $ax^2 + bx + c = 0$.

Step 2. *Factorise* $ax^2 + bx + c$.

Step 3. *Put each factor* $= 0$.

Step 4. *Solve each resulting equation.*

Ex. 1. ***Solve : (i)*** $x^2 - 3x = 0$, ***(ii)*** $x^2 + 2x - 15 = 0$, ***(iii)*** $x + \frac{16}{x} = 8$.

Sol. (*i*)

$x^2 - 3x = 0 \Rightarrow x(x - 3) = 0$

$x = 0$ or $x - 3 = 0 \Rightarrow$ $\mathbf{x = 0}$ **or** $\mathbf{x = 3}$

(*ii*)

$x^2 + 2x - 15 = 0 \Rightarrow x^2 + 5x - 3x - 15 = 0$

$x(x + 5) - 3(x + 5) = 0 \Rightarrow (x + 5)(x - 3) = 0$

$x + 5 = 0$ or $(x - 3) = 0 \Rightarrow$ $\mathbf{x = -5}$ **or** $\mathbf{x = 3}$**.**

(*iii*)

$x + \frac{16}{x} = 8$

$x^2 + 16 = 8x \Rightarrow x^2 - 8x + 16 = 0$

$(x - 4)^2 = 0 \Rightarrow (x - 4)(x - 4) = 0$

$x - 4 = 0$ or $x - 4 = 0 \Rightarrow$ $\mathbf{x = 4, 4}$ **(equal roots).**

Ex. 2. ***Solve the following quadratic equation by factorisation method :***

$$\text{(i)}\ \frac{1}{x-2}+\frac{2}{x-1}=\frac{6}{x} \qquad \text{(ii)}\ 4\sqrt{3}\,x^2+5x\quad 2\sqrt{3}=0$$

Sol. (*i*) $\dfrac{1}{x\quad 2}\qquad \dfrac{2}{x\quad 1}\qquad \dfrac{6}{x}\qquad \dfrac{x\quad 1\quad 2\,(x\quad 2)}{(x\quad 2)\,(x\quad 1)}\qquad \dfrac{6}{x}$

$\dfrac{3x\quad 5}{x^2\quad 3x\quad 2}\qquad \dfrac{6}{x}\qquad\qquad 3x^2\quad 5x\quad 6x^2\quad 18x\quad 12$

$3x^2 - 13x + 12 = 0 \qquad 3x^2 - 9x - 4x + 12 = 0$

$3x\,(x-3) - 4(x-3) = 0 \qquad (3x-4)\,(x-3) = 0$

$3x - 4 = 0$ or $x - 3 = 0 \qquad x = \mathbf{\dfrac{4}{3}}$ **or** $\mathbf{x = 3}$.

(*ii*) $4\sqrt{3}\,x^2\quad 5x\quad 2\sqrt{3}\quad 0 \qquad 4\sqrt{3}\,x^2\quad 8x\quad 3x\quad 2\sqrt{3}\quad 0$

$4x\ \ \sqrt{3}x\quad 2\qquad \sqrt{3}\ \ \sqrt{3}x\quad 2\ \ = 0$

$(4x-\sqrt{3})\,(\sqrt{3}x+2) = 0$

$4x-\sqrt{3}=0$ or $\sqrt{3}x+2=0 \qquad x\quad \mathbf{\dfrac{\sqrt{3}}{4}}$ or $x = \mathbf{\dfrac{-2}{\sqrt{3}}}$.

> $\because 4\sqrt{3}\times -2\sqrt{3} = -24$
> Think of two factors whose product is – 24 and difference is 5. Factors are 8 and –3

EXERCISE 5(*a*)

Solve :

1. (*i*) $(x\quad 3)\,(x+7)=0$ (*ii*) $(3x+4)\,(2x\quad 11)=0$

2. $x^2\quad 4x$

3. $\left(\frac{1}{3}x-1\right)\left(\frac{1}{2}x+7\right)=0$

4. $\dfrac{x^2-5x}{2}=0$

5. $x^2\quad 3x\quad 10=0$

6. $x^2+x\quad 12=0$

7. $2\,(x^2+1)=5x$

8. $x\,(2x+5)=3$

9. $4x^2\quad 3x\quad 1=0$

10. $6x^2\quad 13x+5=0$

11. $3x^2\quad 5x\quad 12=0$

12. $2x^2\quad 11x+5=0$

13. $\dfrac{x}{2}+\dfrac{6}{x}=4.$

14. $10x-\dfrac{1}{x}=3$

15. $9x+\dfrac{1}{x}=6$

16. $\dfrac{x}{5}+\dfrac{28}{x+2}=5$

17. $\dfrac{x}{x-1}+\dfrac{x-1}{x}=2\tfrac{1}{2}$

18. $\dfrac{x+3}{x-2}-\dfrac{1-x}{x}=\dfrac{17}{4},\ x\neq 0,\ x\neq 2$

19. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}=\dfrac{5}{6},\ x\neq 1,-1$

20. $\sqrt{2}\,x^2-3x-2\sqrt{2}=0$

ANSWERS

1. (*i*) 3, –7 (*ii*) $-\frac{4}{3}, \frac{11}{2}$ **2.** 0, 4 **3.** 3, – 14 **4.** 0, 5

5. 5, – 2 **6.** – 4, 3 **7.** 2, $\frac{1}{2}$ **8.** $\frac{1}{2}$, – 3 **9.** 1, $-\frac{1}{4}$

10. $\frac{1}{2}, \frac{5}{3}$ **11.** 3, $-\frac{4}{3}$ **12.** 5, $\frac{1}{2}$ **13.** 2, 6 **14.** $\frac{1}{2}, -\frac{1}{5}$

15. $\frac{1}{3}, \frac{1}{3}$ **16.** 5, 18 **17.** 2, – 1

18. 4, $-\frac{2}{9}$ **19.** 5, $-\frac{1}{5}$ **20.** $-\frac{1}{\sqrt{2}}, 2\sqrt{2}$

5.03. Introduction to the concept of imaginary and complex numbers

Necessity is the mother of invention. It was found out that the real numbers were not sufficient to meet the demands of civilization in general and of mathematics in particular. When mathematicians came across such equations as $x^2 + 1 = 0$, $x^2 + 4 = 0$, etc., they found themselves unable to solve them, as there is no real number whose square is a negative real number. Thus the real numbers were found inadequate. To obtain solutions of such equations it became necessary to extend the number system. About 400 years ago mathematicians proposed the introduction of a number i with the property that

$$\boldsymbol{i^2 + 1 = 0} \quad \text{or} \quad \boldsymbol{i^2 = \quad 1.}$$

The solution of the equation $x^2 + 1 = 0$ is then $x = i$ or $\quad i$. Also since $i^2 = \quad 1$, we may write $\sqrt{-1} = i$ and call i "a square root of 1". Using i we can now introduce numbers that are square roots of any negative number. The necessary definition is as follows:

$$\sqrt{-a} = i\sqrt{a}\text{, where '}a\text{' is any positive real number.}$$

For example,

$$\sqrt{-9} = i\sqrt{9} = 3i \qquad (3i)^2 = \quad 9$$

$$\sqrt{-36} = i\sqrt{36} = 6i \qquad (6i)^2 = \quad 36$$

$$\sqrt{-12} = i\sqrt{12} = \sqrt{4}\cdot\sqrt{3}\cdot i = 2\sqrt{3}\,i \quad \therefore \left(2\sqrt{3}\,i\right)^2 = -12.$$

The above examples suggest that *for every non-zero real number b, ib is a number whose square is b^2, that is $(ib)^2 = \quad b^2$. For b 0, ib is called a* **pure imaginary number.** *We define i.o to be 0.*

Ex. 3. ***Simplify the following:***

(i) ***(5i) 7*** ***(ii) (3i) (4i)*** ***(iii)*** $\sqrt{-9} + \sqrt{-16}$

(iv) $\frac{21}{4}\sqrt{-48} - 5\sqrt{-27}$ ***(v)*** $\sqrt{-18}\cdot\sqrt{-2}$

Sol. (*i*) $(5i) \quad 7 = (5 \quad 7)i = 35i$

(*ii*) $(3i)\,(4i) = (3 \quad 4)\,(i\cdot i) = 12\,i^2 = 12\,(\quad 1) = \quad 12$

(*iii*) $\sqrt{-9} + \sqrt{-16} = i\sqrt{9} + i\sqrt{16} = 3i + 4i = 7i$

(*iv*) $\frac{21}{4}\sqrt{-48} - 5\sqrt{-27} = \frac{21}{4}\cdot i\cdot\sqrt{48} - 5\cdot i\cdot\sqrt{27}$

$$= \frac{21}{4} \quad i \quad 4\sqrt{3} \quad 5 \quad i \quad 3\sqrt{3} \quad (21 \quad 15) \quad i \quad \sqrt{3} \quad 6\sqrt{3}\,i$$

(*v*) $\sqrt{-18}\cdot\sqrt{-2} = i\cdot\sqrt{18}\cdot i\,\sqrt{2} = \sqrt{18\times 2}\cdot i^2 = 6\,(-1) = -6$

Caution. In (*vii*) above the student should not commit the mistake of writing $\sqrt{-18}\cdot\sqrt{-2}$ $= \sqrt{(-18)\,(-2)} = \sqrt{36} = 6$. This is an incorrect result obtained by applying the property $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ which holds good only if $\sqrt{a}, \sqrt{b}$ are real numbers. It does not hold good if a, b are negative numbers *i.e.*, $\sqrt{a}, \sqrt{b}$ are imaginary numbers. In this case $\sqrt{a}\,.\sqrt{b} = -\sqrt{ab}$.

To avoid difficulty with this point you should write all expressions of the form $\sqrt{-b}$, $(b > 0)$, in the form $i\sqrt{b}$ before writing expressions involving imaginary numbers.

5.04. Complex numbers

A number of the form $7+\sqrt{-5}, -6+\sqrt{-11}, 23-\sqrt{-7}$ etc., which is the sum of a real number and a pure imaginary number is called a **complex number**.

The standard form of a complex number is $\boldsymbol{a + bi}$, where a, b are real numbers. Notice that when $b = 0$, then complex number is $a + 0i$, *i*.e., the real number a. We call a the *real part*, and b the *imaginary part* of $a + bi$.

Definition. ***A complex number is a number of the form a + bi where a and b are real numbers and i =*** $\boldsymbol{\sqrt{-1}}$**.**

Any complex number for which b 0 such as 2 $7i$, $3i$, $5 + i$, etc. is called an imaginary number.

An imaginary number is a complex number whose imaginary part is not zero. A pure imaginary number is an imaginary number whose real part is zero.

EXERCISE 5(*b*)

Simplify:

1. $\sqrt{-25}$ **2.** $\sqrt{-8}$ **3.** $\sqrt{\dfrac{-1}{3}}$ **4.** $\dfrac{1}{2}\sqrt{\dfrac{-3}{4}}$

5. $\sqrt{-144}$ **6.** $\sqrt{-4}+\sqrt{-16}-\sqrt{-25}$ **7.** $\sqrt{-20}+\sqrt{-12}$ **8.** $-\sqrt{\dfrac{-7}{4}}-\sqrt{\dfrac{-1}{7}}$

9. $\sqrt{\dfrac{-x}{4}}+\sqrt{\dfrac{-x}{16}}-\sqrt{\dfrac{-x}{64}}$, where x is a positive real number.

10. $\sqrt{-5x^8}-\sqrt{-20x^8}+\sqrt{-45x^8}$, where x is a positive real number.

11. *If* $i=\sqrt{-1}$, prove the following :

$(x+1+i)\,(x+1-i)\,(x-1+i)\,(x-1-i) = x^2+4.$ ***(I.S.C. 2004)***

ANSWERS

1. $5i$ **2.** $2\sqrt{2}\,i$ **3.** $\dfrac{1}{3}\sqrt{3}\,i$ **4.** $\dfrac{1}{4}\sqrt{3}\,i$ **5.** $12i$

6. i **7.** $2(\sqrt{5}+\sqrt{3})i$ **8.** $\dfrac{-9\sqrt{7}i}{14}$ **9.** $\dfrac{5}{8}\sqrt{x}\,i$ **10.** $2\sqrt{5}\,x^4\,i$

5.05. Solutions of quadratic equations by the formula method

Let the quadratic equation be $ax^2 + bx + c = 0$ $(a \neq 0)$.

$$ax^2 + bx + c = 0 \quad (a \quad 0)$$

or $\quad ax^2 + bx = -c$ (Transposing the constant term)

or $\quad x^2 \quad \frac{b}{a}x = \frac{-c}{a}$ (Dividing by the coefficient of x^2)

or $\quad x^2 \quad \frac{b}{a}x \quad \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

(Adding $\frac{b^2}{4a^2}$ to both sides to make L.H.S. a perfect square.)

or $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ or $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

or $\quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{x \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Hence, the roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textbf{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The roots of the equation are also called the *zeros of the function* defined by $f(x) = ax^2 + bx + c$.

Aid to Memory : The formula is

$$\frac{-(\text{coeff. of } x) \pm \sqrt{(\text{coeff. of } x)^2 - 4\,(\text{coeff. of } x^2)\,(\text{constant term})}}{2\,(\text{coeff. of } x^2)}$$

Ex. 4. ***Solve the following equations :***

(i) $3x^2 \quad 10x + 3 = 0$ ***(ii)*** $6y^2 \quad 35y + 50 = 0$

(iii) $3y + \frac{5}{16y} \quad = 2$ ***(iv)*** $x^2 \quad 4x \quad 1 = 0.$

Sol.

(*i*) $3x^2 \quad 10x + 3 = 0.$ Here $a = 3$, $b = \quad 10$, $c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)\,(3)}}{2 \times 3}$$

$$= \frac{10 \pm \sqrt{100 - 36}}{6} = \frac{10 \pm \sqrt{64}}{6} = \frac{10 \pm 8}{6} = \frac{10 + 8}{6} \text{ or } \frac{10 - 8}{6}$$

$$= \frac{18}{6} \text{ or } \frac{2}{6} = \mathbf{3 \text{ or } \frac{1}{3}}.$$

(*ii*) $6y^2 \quad 35y + 50 = 0$, Here $a = 6$, $b = \quad 35$, $c = 50$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-35) \pm \sqrt{(-35)^2 - 4\,(6)\,(50)}}{2 \times 6}$$

$$= \frac{35 \pm \sqrt{1225 - 1200}}{12} = \frac{35 \pm \sqrt{25}}{12} = \frac{35 \pm 5}{12}$$

$$= \frac{35+5}{12} \text{ or } \frac{35-5}{12} = \frac{40}{12} \text{ or } \frac{30}{12} = \mathbf{\frac{10}{3}} \textbf{ or } \mathbf{\frac{5}{2}}.$$

(*iii*) $3y + \frac{5}{16y} = 2 \qquad \frac{48y^2 + 5}{16y} = 2 \qquad 48y^2 + 5 = 32\,y$

$48y^2 \quad 32y + 5 = 0.$ Here $a = 48$, $b = \quad 32$, $c = 5$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-32) \pm \sqrt{(-32)^2 - 4\,(48)\,(5)}}{2 \times 48}$$

$$= \frac{32 \pm \sqrt{1024 - 960}}{96} = \frac{32 \pm \sqrt{64}}{96} = \frac{32 \pm 8}{96}$$

$$= \frac{40}{96} \text{ or } \frac{24}{96} = \mathbf{\frac{5}{12}} \textbf{ or } \mathbf{\frac{1}{4}}.$$

(*iv*) $x^2 \quad 4x \quad 1 = 0.$ Here $a = 1$, $b = \quad 4$, $c = \quad 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\,(1)\,(-1)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm \sqrt{4 \times 5}}{2} = \frac{4 \pm 2\sqrt{5}}{2}$$

$$= \frac{2\left(2 \pm \sqrt{5}\right)}{2} = 2 \pm \sqrt{5} = \mathbf{2 + \sqrt{5}} \textbf{ or } \mathbf{2 - \sqrt{5}}.$$

Ex. 5. ***Solve : $2x^2 + 2x - 3 = 0$, giving your answer correct to one decimal place.***

Sol. Here $a = 2, b = 2, c = -3$

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we get

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times (-3)}}{2 \times 2} = \frac{-2 \pm \sqrt{4 + 24}}{4}$$

$$= \frac{2 \quad \sqrt{28}}{4} \quad \frac{2 \quad 2\sqrt{7}}{4} \quad \frac{1 \quad \sqrt{7}}{2} \quad \frac{1 \quad 2.645}{2} \quad \frac{3.645}{2}, \frac{1.645}{2}$$

$= -1.8225, 0.8225 = -1.8, 0.8$, correct to one decimal place.

Ex. 6. ***Solve***

(i) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

(ii) $x^2 + \frac{5}{2}ix + 1 = 0$

(iii) $x^2 + (1+3i)x + \left(\frac{3}{2}i + 2\right) = 0$

(iv) $ix^2 + 4x - 5i = 0$

Sol. (*i*) Comparing with $ax^2 + bx + c = 0$, here, $a = \sqrt{3}$, $b = -\sqrt{2}$, $c = 3\sqrt{3}$

$$\therefore \qquad x = \frac{b \quad \sqrt{b^2 \quad 4ac}}{2a} \quad \frac{\sqrt{2} \quad \sqrt{\quad \sqrt{2}^{\;2} \quad 4.\ \sqrt{3} \quad 3\sqrt{3}}}{2 \quad \sqrt{3}}$$

$$= \frac{\sqrt{2} \quad \sqrt{2 \quad 36}}{2\sqrt{3}} \quad \frac{\sqrt{2} \quad \sqrt{\quad 34}}{2\sqrt{3}} \quad \frac{\sqrt{2} \quad \sqrt{34}i}{2\sqrt{3}}$$

(*ii*) Here, $a = 1$, $b = \frac{5}{2}i$, $c = -1$

$$\therefore \qquad x = \frac{b \quad \sqrt{b^2 \quad 4ac}}{2a} \quad \frac{\frac{5}{2}i \quad \sqrt{\quad \frac{5}{2}i^{\;2} \quad 4 \quad 1(\quad 1)}}{2 \quad 1}$$

$$= \frac{-\frac{5}{2}i \pm \sqrt{-\frac{25}{4}+4}}{2} = \frac{\frac{-5}{2}i \pm \sqrt{-\frac{9}{4}}}{2} \quad (\because i^2 = -1)$$

$$= \frac{\frac{5}{2}i \quad \frac{3}{2}i}{2} \quad \frac{\frac{5}{2}i \quad \frac{3}{2}i}{2}, \quad \frac{\frac{-5}{2}i - \frac{3}{2}i}{2}$$

$$= -\frac{i}{2} \text{ and } -2i.$$

(*iii*) Here $a = 1$, $b = (1 + 3i)$, $c = \frac{3}{2}i + 2$

$$\therefore \qquad x = \frac{(1 \quad 3i) \quad \sqrt{(1 \quad 3i)^2 \quad 4(1) \quad \frac{3}{2}i \quad 2}}{2}$$

$$= \frac{-(1+3i) \pm \sqrt{1+9i^2+6i-6i-8}}{2}$$

$$= \frac{(1 \quad 3i) \quad \sqrt{1 \quad 9 \quad 8}}{2} \quad \frac{(1 \quad 3i) \quad \sqrt{\quad 16}}{2}$$

$$= \frac{-1-3i \pm 4i}{2} = \frac{-i+i}{2} \text{ and } \frac{1 \quad 7i}{2}.$$

(*iv*) Here $a = i$, $b = 4$, $c = -5i$

$$x \quad \frac{4 \quad \sqrt{(4)^2 \quad 4(i)(\quad 5i)}}{2(i)} \quad \frac{4 \quad \sqrt{16 \quad 20i^2}}{2i}.$$

$$= \frac{-4 \pm \sqrt{16-20}}{2i} = \frac{-4 \pm \sqrt{-4}}{2i} = \frac{-4 \pm 2i}{2i}$$

$$= \frac{-4+2i}{2i} \text{ and } \frac{-4-2i}{2i}$$

$$= \frac{(-4+2i)i}{2i^2} \text{ and } \frac{(-4-2i)i}{2i^2}$$

(Multiplying the num. and den. by i to rid the den. of i)

$$= \frac{4i \quad 2i^2}{2(\quad 1)} \text{ and } \frac{-4i - 2i^2}{2(-1)}$$

$$= \frac{-4i - 2}{-2} \text{ and } \frac{-4i + 2}{-2}$$

$= 2i + 1$ and $2i - 1$.

EXERCISE 5 (*c*)

Find the roots of the following equations :

1. $2x^2 + x - 3 = 0$ **2.** $6x^2 + 7x - 20 = 0$ **3.** $9x^2 + 6x = 35$ **4.** $36x^2 + 23 = 60x$

5. $x^2 - 2x + 5 = 0$ **6.** $3x^2 - 17x + 25 = 0$ **7.** $15x^2 - 28 = x.$

8. $\sqrt{2}x^2 + x + \sqrt{2} = 0$ **9.** $27x^2 - 10x + 1 = 0$ **10.** $x^2 + 4ix - 4 = 0$

11. $x^2 + 3x - 3 = 0$, giving your answer correct to two decimal places.

12. $-3x^2 - 1lx + 4 = 0$ **13.** $\frac{x^2 + 8}{11} = 5x - x^2 - 5$ **14.** $\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\frac{1}{3}$

15. $\frac{x \quad 6}{x \quad 7} \quad \frac{x \quad 1}{x \quad 2} \quad \frac{1}{3x \quad 1}$

18. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$ is (*a*) 3 (*b*) 4 (*c*) 1 (*d*) 3.

[Hint. $x^2 - 3|x| + 2 = 0 \quad (|x|)^2 - 3|x| + 2 = 0 \quad (|x| - 2)(|x| - 1)] = 0$

$|x| = 1, 2 \quad x = \pm 1, \pm 2$].

ANSWERS

1. $1, \frac{-3}{2}.$ **2.** $1\frac{1}{3}, -2\frac{1}{2}$ **3.** $1\frac{2}{3}, \quad 2\frac{1}{3}$ **4.** $\frac{5 \pm \sqrt{2}}{6}$

5. $1 \pm 2\sqrt{-1}$ **6.** $\frac{17 \pm \sqrt{-11}}{6}$ **7.** $1\frac{2}{5}, -1\frac{1}{3}$ **8.** $\frac{-1 \quad \sqrt{7}i}{2\sqrt{2}}$

9. $\frac{5 \quad \sqrt{2}i}{27}$ **10.** $-2i, -2i$ **11.** $-3.79, 0.79$ **12.** $\frac{1}{3}, -4$

13. $2\frac{1}{3}, 2\frac{1}{4}$ **14.** $6, 3\frac{1}{13}$ **15.** 3, 3 **16.** (*b*)

5.06. Graphical solution of a quadratic equation

You have studied in class X that a graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. The graph is concave upward or downward depending on the sign of 'a'.

The **'vertex'** or 'turning point of the graph is the point at which the direction of the graph changes . We will begin with plotting the graph of a quadratic function.

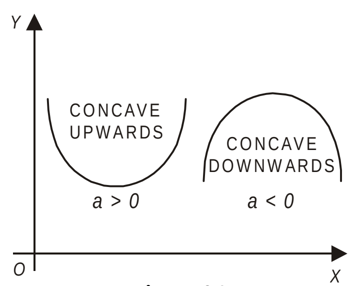


Fig. 5.01

Procedure :

1. *Make a table of corresponding values for x and the function.*
2. *Plot these points, on a pair of x, f(x) axes.*
3. *Draw a smooth curve joining the plotted points.*

Ex. 7. ***Graph the expression*** $x^2 + x - 6$.

Let $y = x^2 + x - 6$.

Table of values:

x	y
3	6
2	0
1	– 4
0	– 6
– 1	– 6
– 2	– 4
– 3	0
– 4	6

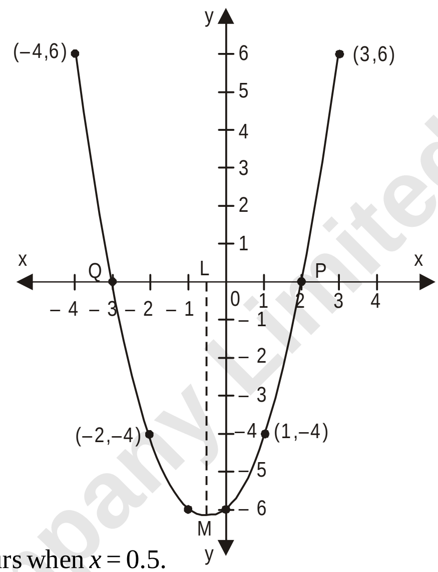


Fig. 5.02

Note. Real roots of the equation $x^2 + x - 6 = 0$ are $x = OP$ and $x = OQ$, *i.e.*, $x = 2$ and $x = -3$. (On reading from the graph).

Mininum value of the function = $LM = -6.25$ and occurs when $x = 0.5$.

Ex. 8. ***Graph the expression*** $4 - 5x - x^2$.

Let $y = 4 - 5x - x^2$

Tables of values:

x	y
– 6	– 2
– 5	4
– 4	8
– 3	10
– 2	10
– 1	8
0	4
1	– 2

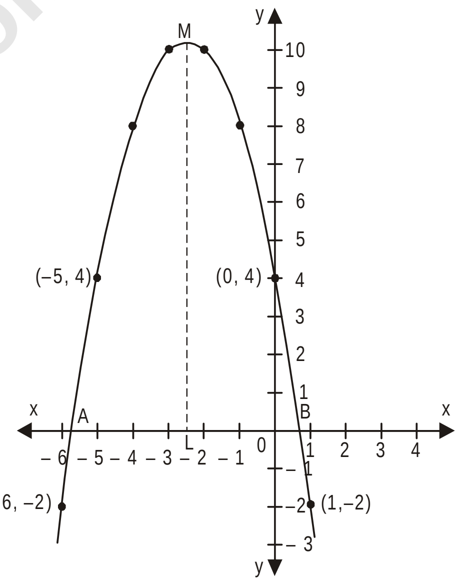


Fig. 5.03

Note: Real roots of equation $4 - 5x - x^2 = 0$ are $x = OA$ and $x = OB$, *i.e.*, $x = -5.71$, $x = 0.71$.

Maximum value of the function = $LM = 10.25$, when $x = OL = -2.5$.

Now we determine the roots of the quadratic equations from their respective graphs.

> **The roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are the x-intercepts of the graph of $y = f(x) = ax^2 + bx + c$.**

Ex. 9. ***The roots x = 0 and x = 3 of the equation 2x (x – 3) = 0 are the x-intercepts of the graph of y = 2x (x – 3) as shown in fig. 5.04.***

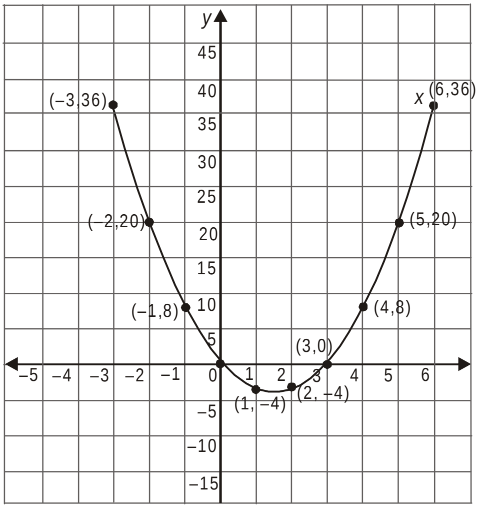


Fig. 5.04

In this case the graph cuts the x-axis at two points because there are two real and distinct roots. In the formula the part $b^2 - 4ac > 0$

Ex. 10. ***The two roots x = –5, –5 of the equation $(x + 5)^2 = 0$ is the x-intercept of the graph of $y = (x + 5)^2$ as shown below.***

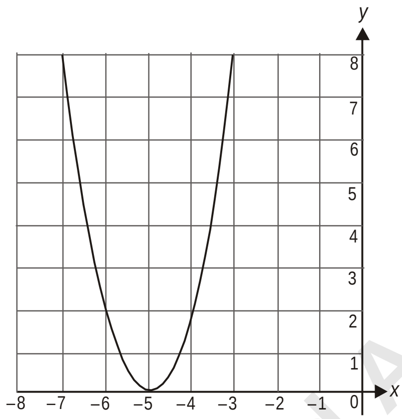


Fig. 5.05

In this case the vertex of the graph cuts the x-axis at only one point because the roots are equal. In the formula the part $b^2 - 4ac = 0$

Ex. 11. ***The two roots of the quadratic equation $x^2 - 2x + 2 = 0$ are $1 + i$ and $1 - i$. Therefore the graph of the function $f(x) = x^2 - 2x + 2$ does not touch the x-axis as shown below because the roots are imaginary.***

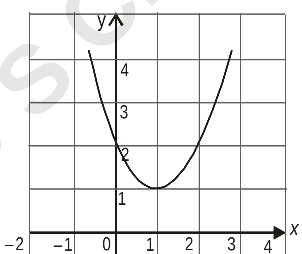


Fig. 5.06

In this case in the formula the part $b^2 - 4ac < 0$

Ex. 12. ***(i) Draw a graph of $y = x^2 - 4x + 3$ for $-2 \le x \le 5$.***

(ii) Use the graph to solve the equation $x^2 - 4x + 3 = 0$.

Sol: The table of values is

x	– 2	– 1	0	1	2	3	4	5
y	15	8	3	0	– 1	0	3	8

To solve the equation it is necessary to find the values of x when $y = 0$, i.e., where the graph crosses the x-axis. These points occur when $x = 1$ and $x = 3$, therefore these are the solutions.

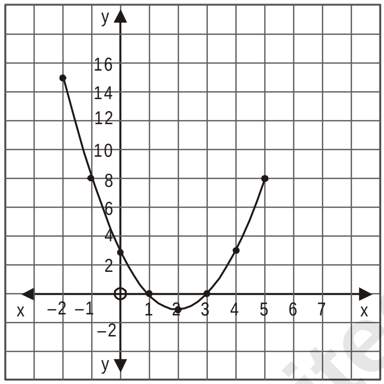


Fig. 5.07

EXERCISE 5(*d*)

Solve the following equations graphically and compare your answer with algebraic solution either by factorization or formula method

(*i*) $y = x^2 - 5x + 6$

(*ii*) $y = -x^2 + 2x + 3$

(*iii*) $y = x^2 - 4x + 4$

(*iv*) $y = x^2 - x - 6$

(*v*) $y = x^2 - 6x + 9$

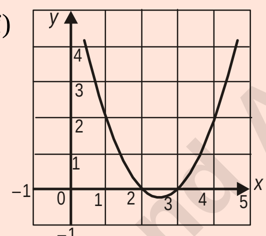
(*vi*) $y = -x^2 - x + 12$

(*vii*) $y = x^2 - 4x + 5 = 0$

(*viii*) $y = x^2 + 2x + 2 = 0$

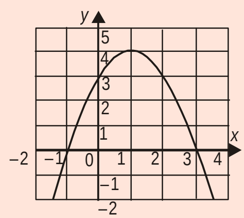
ANSWERS

(*i*)



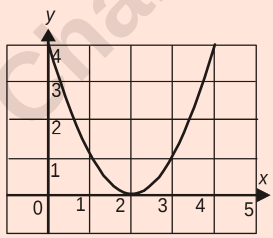
$x = 2, 3$

(*ii*)



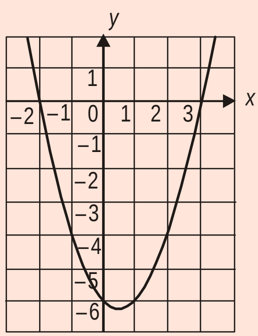
$x = -1, 3$

(*iii*)



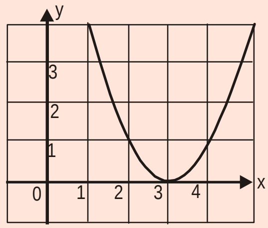
$x = 2, 2$

(*iv*)



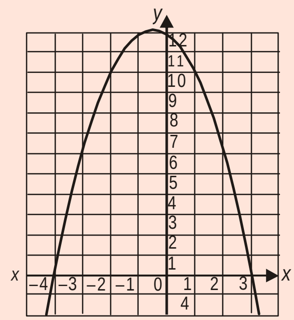
$x = -2, 3$

(*v*)



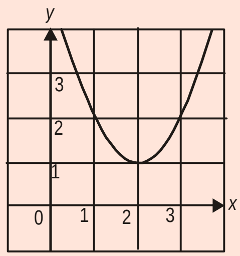
$x = 3, 3$

(*vi*)



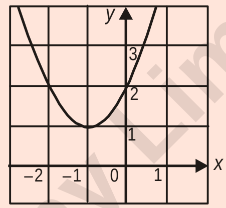
$x = -4, 3$

(*vii*)



$x = 2 + i, 2 - i$

(*viii*)



$x = -1 + i, -1 - i$

5.07. Nature or character of the roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Examining the nature of the roots means to see what type of roots the equation has, that is, whether they are real or imaginary, rational or irrational, equal or unequal. The nature of the roots depends entirely on the value of the expression $b^2 - 4ac$ which is called the **discriminant** (Abbrev : **disct.**) of the equation. Thus : if a, b, c are real and $a \neq 0$, then

1. The roots are real and unequal if $b^2 - 4ac$ is positive.

Further,

(*a*) The roots are rational if $b^2 - 4ac$ is a perfect square.

(*b*) The roots are irrational if $b^2 - 4ac$ *is* not a perfect square.

(*c*) The roots are equal if $b^2 - 4ac$ is zero, each root being equal to $-\dfrac{b}{2a}$. It follows that $ax^2 + bx + c$ is a perfect square if $b^2 - 4ac = 0$.

2. The roots are imaginary if $b^2 - 4ac$ *is* negative, *i.e.*, the roots are complex.

Ex. 13. ***Examine the nature of the roots of the equations***

(i) $2x^2 + 2x + 3 = 0$, ***(ii)*** $2x^2 - 7x + 3 = 0$.

(iii) $x^2 - 5x - 2 = 0$, ***(iv)*** $4x^2 - 4x + 1 = 0$.

Sol. (*i*) $2x^2 + 2x + 3 = 0$

$$\text{Disct.} = (2)^2 - 4 \times 2 \times 3 \qquad [a = 2, b = 2, c = 3]$$

$$= -20, \text{ is negative.}$$

$\Rightarrow$ The roots are imaginary,

(*ii*) $2x^2 - 7x + 3 = 0$

Disct. $= (-7)^2 - 4 \times 2 \times 3 = 25$, is + ve and a perfect square.

The roots are real and rational,

(*iii*) $x^2 - 5x - 2 = 0$

Disct. $= (-5)^2 - 4 \times 1 \times (-2)$

$= 25 + 8 = 33$, is +ve and not a perfect square.

The roots are real and irrational,

(*iv*) $4x^2 - 4x + 1 = 0$

Disct. $= (-4)^2 - 4 \times 4 \times 1 = 16 - 16 = 0$.

The roots are real and equal.

Ex. 14. ***For what value of m, are the roots of the equation (3m + 1) x^2 + (11 + m) x + 9 = 0 equal ? real and unequal ? imaginary ?***

Sol. $b^2 - 4ac = (11 + m)^2 - 4(3m + 1)\,9 = m^2 - 86m + 85$

(*a*) For equal roots, $b^2 - 4ac = 0$ *i.e.*, $m^2 - 86m + 85 = 0$

or $(m - 85)(m - 1) = 0$ $m = 85$ or 1.

(*b*) For real and unequal roots, $b^2 - 4ac > 0$, *i.e.* $(m - 85)(m - 1) > 0$

Putting each factor equal to 0, we get $m = 1$ or 85, which are called the critical points.

(*c*) For imaginary roots, $b^2 - 4ac < 0$, *i.e.*, $(m - 85)(m - 1) < 0$

Make a table showing the signs of the factors in the product $(m - 85)(m - 1)$

	$m < 1$	$m = 1$	$1 < m < 85$	$m = 85$	$m > 85$
$m - 1$	–	0	+	+	+
$m - 85$	–	–	–	0	+
$(m - 1)(m - 85)$	**+**	**0**	**–**	**0**	**+**

From the table we can see that

(*b*) $(m - 85)(m - 1) > 0$ when $m < 1$ or $m > 85$.

(*c*) $(m - 85)(m - 1) < 0$ when $1 < m < 85$.

Alternatively, you can find the values by the method of intervals discussed in Art. 5.13.

5.08. Sum and product of the roots

If the two roots of the quadratic equation $ax^2 + bx + c = 0$, obtained in Art. 5.05, be denoted by and , we have

$$+ = \frac{b \quad \sqrt{b^2 \quad 4ac}}{2a} \quad \frac{b \quad \sqrt{b^2 \quad 4ac}}{2a}$$

i.e. $$+ = \frac{-b}{a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(\ b)^2 \quad \left(\sqrt{b^2 \quad 4ac}\right)^2}{4a^2} \quad \frac{b^2 \quad (b^2 \quad 4ac)}{4a^2} \quad \frac{4ac}{4a^2}; \text{ i.e., } \quad \frac{c}{a}.$$

$$\boxed{\alpha + \beta = \frac{-b}{a} \qquad \alpha\beta = \frac{c}{a}}$$

Aid to Memory : In a given quadratic equation,

(1) Sum of the roots $= -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } x^2}$

(2) Product of the roots $= \dfrac{\text{Absolute term}}{\text{Coeff. of } x^2}$

Thus, if , be the roots of the equation $6x^2 - 5x + 7 = 0$,

$+ \quad = -(-5/6) = 5/6, \qquad = 7/6.$

5.09. Values of the symmetric expressions of the roots

If be the roots of a given quadratic equation and we wish to find the value of the function of and , we can do so by proceeding as follows:

Procedure : 1. *Write the values of . + and from the given equation.*

2. *Express the given function in terms of + and .*

3. *Substitute the values of + and from I.*

Caution. We do not find the values of and separately.

We use the following relations :

$^2 + \ ^2 = (\ +\ )^2 - 2$ ...(*i*) ; $(\ -\ )^2 = (\ +\ )^2 - 4$ ...(*ii*)

$^2 - \ ^2 = (\ +\ )(\ -\ ) = (\ +\ )\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$ ...(*iii*)

$^3 + \ ^3 = (\ +\ )(\ ^2 - \quad +\ ^2) = (\ +\ )[(\ +\ )^2 - 3\quad]$

$= (\ +\ )^3 - 3\quad(\ +\ )$ ...(*iv*)

$^3 - \ ^3 = (\ -\ )(\ ^2 + \quad +\ ^2) = (\ -\ )[(\ -\ )^2 + 3\quad]$

$= (\ -\ )^3 + 3\quad(\ -\ )$ ....(*v*)

$^4 + \ ^4 = (\ ^2 + \ ^2)^2 - 2\ ^2\ ^2$

$= [(\ +\ )^2 - 2\quad]^2 - 2(\quad)^2$ ....(*vi*)

$^4 - \ ^4 = (\ ^2 - \ ^2)(\ ^2 + \ ^2) = (\ -\ )(\ +\ )(\ ^2 + \ ^2)$

$= \sqrt{(\quad)^2 \quad 4}\quad(\ +\ )[(\ +\ )^2 - 2\quad]$ ...(*vii*)

Ex. 15. ***If , are roots of $x^2 - px + q = 0$, find the values of $^2 + \ ^2$, $^3 + \ ^3$, – and $^4 + \ ^4$.***

Sol. $\because$, are the roots of $x^2 - px + q = 0 \Rightarrow \ +\ = p, \quad = q$

(*a*) $^2 + \ ^2 = (\ +\ )^2 - 2\quad = p^2 - 2q.$

(*b*) $^3 + \ ^3 = (\ +\ )^3 - 3\quad(\ +\ ) = p^3 - 3pq.$

(*c*) $- \quad = \sqrt{(\quad)^2 \quad 4} \quad \sqrt{p^2 \quad 4q}$, if ; $\sqrt{p^2 \quad 4q}$, if

(*d*) $^4 + \ ^4 = (\ ^2 + \ ^2)^2 - 2\ ^2\ ^2 = \{(\ +\ )^2 - 2\quad\}^2 - 2\ ^2\ ^2$

$= (p^2 - 2q)^2 - 2q^2 = p^4 - 4p^2q + 4q^2 - 2q^2 = p^4 - 4p^2q + 2q^2.$

5.10. Formation of equations

Suppose we have to form the equation whose roots are   and .

As $x =$, $x =$, are the roots of the equation, so $x -$ $= 0$ and $x -$ $= 0$

$(x -\ \ )\ (x -\ \ ) = 0$

or $x^2 - (\ \ +\ \ )\ x +\ \ \ \ = 0$

i.e., $\boxed{\textbf{x}^2\textbf{– (sum of the roots) } x \textbf{ + product of the roots = 0.}}$

The **procedure** for forming an equation whose roots are given is as follows :

1. *Find the sum of the given roots.* 2. *Find their product.*

3. *Then, the required equation is* x^2 *– (sum of the roots)x + product of the roots = 0.*

Thus, the equation whose roots are 5 and 7 is $x^2 - (7 + 5)\ x + 7 \times 5 = 0$

i.e., $x^2 - 12x + 35 = 0.$

To form an equation whose roots are some functions of the roots of the given equation.

Ex. 16. ***If   and   are the roots of*** $ax^2 + bx + c = 0$***, form the equation whose roots are 1/ and 1/ .***

Sol. ,   are the roots of $ax^2 + bx + c = 0$

Note that 'a' and 'c' have interchanged their position

$$+\ \ = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$$

Sum of the roots of the required equation $= \frac{1}{\alpha}\ \ \frac{1}{\beta}\ \ \frac{\alpha\ \ \ \beta}{\alpha\beta}\ \ \frac{b/a}{c/a}\ \ \frac{b}{c}$

Product of the roots of the required equation $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{a}{c}$

The equation required is $x^2 - \left(-\frac{b}{c}\right) x + \frac{a}{c} = 0$

or $c\,x^2 + bx + a = 0.$

Note. The equation found above has roots which are the reciprocals of the roots of the equation $ax^2 + bx + c = 0$.

Ex. 17. ***If ,   are the roots of*** $ax^2 + bx + c = 0$***, form that equation whose roots are***

$$(\alpha^2 + \beta^2),\ \ \frac{1}{\alpha^2} + \frac{1}{\beta^2}\ .$$

Sol. $+\ \ = -\frac{b}{a},\ \ \alpha\beta = \frac{c}{a}$

Sum of the roots of the reqired equation $= \alpha^2 + \beta^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} = (\alpha^2 + \beta^2) + \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$$= (\alpha + \beta)^2 - 2\alpha\beta + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = \frac{b^2}{a^2} - \frac{2c}{a} + \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}}$$

$$= \frac{b^2 - 2ac}{a^2} + \frac{b^2 - 2ac}{c^2} = \frac{(a^2 + c^2)(b^2 - 2ac)}{a^2c^2}$$

Product of the roots of the required equation $= (\alpha^2+\beta^2)\left(\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}\right) = (\alpha^2+\beta^2)\left(\dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right)$

$$\frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2} = \frac{[(\alpha+\beta)^2-2\alpha\beta]^2}{\alpha^2\beta^2} = \frac{\left(\dfrac{b^2}{a^2}-\dfrac{2c}{a}\right)^2}{\dfrac{c^2}{a^2}} = \frac{(b^2-2ac)^2}{a^2c^2}$$

Hence, the required equation is

$$x^2 - \frac{(a^2+c^2)(b^2-2ac)}{a^2c^2}x + \frac{(b^2-2ac)^2}{a^2c^2} = 0$$

or $$a^2c^2x^2 - (a^2+c^2)(b^2-2ac)x + (b^2-2ac)^2 = 0.$$

5.11. To find the condition when a relation between the two roots is given

Procedure. 1. *Let one root be . Write the other root using the given relation.*

2. *Write the sum and the product of the roots.*

3. *Eliminate from the two relations so obtained.*

The eliminate is the condition required.

Ex. 18. ***Find the condition that one root of $ax^2+bx+c=0$ may be four times the other.***

Sol. Let the root be and 4 . Then

$$+ 4 \quad = 5 \quad = -\frac{b}{a} \quad ...(1) \quad \text{and} \quad \times 4 \quad = 4 \quad ^2 = \frac{c}{a} \quad ...(2)$$

From (1), $$\alpha = -\frac{b}{5a}$$

Substituting the value of in (2),

$$4 \times \frac{b^2}{25a^2} = \frac{c}{a} \quad \textit{i.e.,}\ 4b^2 = 25\,ac,$$

which is the condition required.

EXERCISE 5(*e*)

1. Without solving, find the nature of the roots of the following equations:

(*i*) $3x^2-7x+5=0$. (*ii*) $4x^2+4x+1=0$. (*iii*) $3x^2+7x+2=0$. (*iv*) $x^2+px-q^2=0$.

2. If the equation $(1+m^2)x^2+2mcx+c^2-a^2=0$ has equal roots, show that $c^2=a^2(1+m^2)$.

3. Find the value of m so that the roots of the equation $(4-m)x^2+(2m+4)x+(8m+1)=0$ may be equal.

4. If the roots of $ax^2+x+b=0$ be real and unequal, show that the roots of $\dfrac{x^2+1}{x}=4\sqrt{ab}$ are imaginary.

5. Find a so that the sum of the roots of the equation $ax^2+2x+3a=0$ may be equal to their product.

6. If , are the roots of the equation $x^2 + x + 1 = 0$, find the value of $^3 + {}^3$.

7. If , are the roots of the equation $x^2 + px + q = 0$, find the value of

(*a*) $^3 \quad + \quad ^3$ (*b*) $^4 + {}^2\,{}^2 + {}^4$.

8. If the roots of the equation $x^2 + px + 7 = 0$ are denoted by and , and $^2 + {}^2 = 22$, find the possible values of p. **[S.C.]**

9. If , are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of

$$\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta.$$

10. If , are the roots of $ax^2 + bx + c = 0$, find the value of

(*i*) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ (*ii*) $\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha}$

11. If the sum of the roots of the equation $x^2 - px + q = 0$ be m times their difference, prove $p^2 (m^2 - 1) = 4m^2 q$.

12. If one root of the equation $x^2 + ax + 8 = 0$ is 4 while the equation $x^2 + ax + b = 0$ has equal roots, find b.

13. Find the value of a for which one root of the quadratic equation $(a^2 - 5a + 3)\, x^2 + (3a - 1)\, x + 2 = 0$ is twice as large as the other is.

[Hint. $+ 2 \quad = 3 \quad = \frac{1 \quad 3a}{a^2 \quad 5a \quad 3}$, $2 \quad ^2 = \frac{2}{a^2 \quad 5a \quad 3}$. Eliminate].

14. If are the roots of the equation $ax^2 - bx + b = 0$, prove that

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{b}{a}} = 0.$$

15. If and are the roots of the equation $x^2 + x - 7 = 0$, form the equation whose roots are ^{2} and ^{2} **[S.C.]**

16. If and are the roots of the equation $2x^2 + 3x + 2 = 0$, find the equation whose roots are $+ 1$ and $+ 1$. **[S.C.]**

17. Find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$, where and are the roots of the equation $x^2 + 2x + 3 = 0$. **[S.C.]**

18. If and are the roots of the equation $2x^2 - 3x + 1 = 0$, form the equation whose roots are $\frac{\alpha}{2\beta + 3}$ and $\frac{\beta}{2\alpha + 3}$.

19. If $a \quad b$ and $a^2 = 5a - 3$, $b^2 = 5b - 3$, then form that equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$.

[Hint. a, b are the roots of the equation $x^2 = 5x - 3$.]

20. Given that and are the roots of the equation $x^2 = x + 7$.

(*i*) Prove that (*a*) $\frac{1}{\alpha} = \frac{\alpha - 1}{7}$ and (*b*) $^3 = 8 \quad + 7$.

(*ii*) Find the numerical value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$. **[S.C.]**

21. Given that and are the roots of the equation $x^2 - x + 7 = 0$, find

(a) the numerical value of $\frac{\alpha}{\beta+3} + \frac{\beta}{\alpha+3}$;

(b) an equation whose roots are $\frac{\alpha}{\beta+3}$ and $\frac{\beta}{\alpha+3}$

***22.** It is given that tan A and tan B are the roots of the equation $t^2 - pt + q = 0$. find, in terms of p and q expression for

(a) tan (A + B) (b) $\sin^2$ (A + B) (c) cos 2 (A + B). **[G.C.E.]**

23. If are the roots of the equation $x^2 = x + 1$, prove that

(i) $\alpha + 1 = \frac{\alpha}{\alpha - 1}$ (ii) $^5 = 5 \quad + 3$ (iii) $\frac{\alpha^2}{\beta} - \frac{\beta^2}{\alpha} = -2\sqrt{5}$.

24. Given that and are the roots of the equation $2x^2 - 3x + 4 = 0$, find an equation whose roots are $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$. **[I.S.C.]**

25. The roots of the quadratic equation $x^2 + px + 8 = 0$ are and . Obtain the values of p, if

(i) $= \ ^2$ (ii) $- \quad = 2$. **[I.S.C.]**

26. If the roots of $x^2 - bx + c = 0$ be two consecutive integers, then find the value of $b^2 - 4c$.

[Hint. Let the roots be , + 1. Then + (+ 1) = b, . (+ 1) = c]

27. The roots of the equation $px^2 - 2(p + 2)x + 3p = 0$ are and . If – = 2, calculate the value of and p. **[I.S.C.]**

28. The roots of the equation $ax^2 + bx + c = 0$ are and . Form the quadratic equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.

29. Two candidates attempt to solve a quadratic equation of the form $x^2 + px + q = 0$. One starts with a wrong value of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots to be 2 and – 9. Find the correct roots and the equation. **[I.S.C.]**

30. Given that and are the roots of the equation $x^2 = 7x + 4$,

(i) Show that $a^3 = 53 \quad + 28$ (ii) Find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$. **[I.S.C.]**

31. The ratio of the roots of the equation $x^2 + \quad x + \quad + 2 = 0$ is 2. Find the values of the parameter .

[I.S.C.]

32. If $(1 - p)$ is a root of the quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are

(a) 0, – 1 (b) – 1, 1 (c) 0, 1 (d) – 1, 2

33. Find the condition that one root of $ax^2 + bx + c = 0$ may be

(i) three times the other, (ii) n times the other, (iii) more than the other by h.

34. Find the condition that the ratio between the roots of the equation $ax^2 + bx + c = 0$ may be $m : n$.

35. If the ratio of the roots of the equation $x^2 + px + q = 0$ is equal to the ratio of the roots of $x^2 + lx + m = 0$, prove that $mp^2 = ql^2$.

* Question No. 22 may be done after doing Trigonometry.

ANSWERS

1. (*i*) Imaginary. (*ii*) Real and equal and rational. (*iii*) Real and unequal and rational. (*iv*) Real and unequal if $p \quad 0, q \quad 0$; Real and equal if $p = q = 0$. **3.** $m = 0, 3$.

5. $-\frac{2}{3}$. **6.** 2. **7.** $q(p^2 - 2q), p^4 + 3q^2 - 4p^2q$. **8.** ± 6 **9.** 8.

10. (*i*) $\frac{b^2(b^2 - 4ac)}{a^2c^2}$ (*ii*) $\frac{b^4 + 2a^2c^2 - 4ab^2c}{a^3c}$.

12. 9. **13.** $\frac{2}{3}$ **15.** $x^2 - 15x + 49 = 0$. **16.** $2x^2 - x + 1 = 0$.

17. $3x^2 + 2x + 3 = 0$. **18.** $40x^2 - 14x + 1 = 0$. **19.** $3x^2 - 19x + 3 = 0$.

20. (*ii*) $\frac{-15}{7}$. **21.** (*a*) $\frac{-10}{19}$. (*b*) $19x^2 + 10x + 7 = 0$.

22. (*a*) $\frac{p}{1-q}$; (*b*) $\frac{p^2}{p^2 + (1-q)^2}$; (*c*) $\frac{(1-q)^2 - p^2}{(1-q)^2 + p^2}$ **24.** $8x^2 - 18x + 13 = 0$.

25. (*i*) $p = -6$, (*ii*) $p = \pm 6$. **26.** 1

27. $\ = 3, \ = 1, p = 2$, or $\ = -1, \ = -3, p = \frac{-2}{3}$.

28. $acx^2 + b(c + a)x + (c + a)^2 = 0$. **29.** $-3, -4$; $x^2 + 7x + 12 = 0$.

30. (*ii*) $\frac{-57}{4}$ **31.** $\alpha = 6, \frac{-3}{2}$. **32.** a

33. (*i*) $3b^2 = 16ac$, (*ii*) $b^2n = ac(1 + n)^2$, (*iii*) $a^2h^2 = b^2 - 4ac$. **34.** $(m + n)^2 ac = mnb^2$.

5.12. Sign of the quadratic function $ax^2 + bx + c$, where $n \qquad R$

Since the discriminant $D = b^2 - bac$ of $ax^2 + 4x + c = 0$ can be positive, zero or negative. We shall examine what the sign of the expression $ax^2 + bx + c$ would be in these three cases.

Let $\ ,\ $ be the roots of the equation $ax^2 + bx + c = 0$. Then $ax^2 + bx + c \quad a(x - \ )(x - \ )$.

Three cases arise :

Case I. $b^2 - 4ac$ is negative ($D < 0$), *i.e.*, when the roots are complex.

We have,

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \qquad \textit{Step I. Taking out the coeff. of } x^2.$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \qquad \textit{Step II. Completing the square.}$$

As $b^2 - 4ac$ is negative, so $-\dfrac{b^2 - 4ac}{4a^2}$ is positive and $\left(x + \dfrac{b}{2a}\right)^2$ is also positive for real values of x.

Therefore the expression has the same sign as a.

Case II. $b^2 - 4ac = 0$ $(D = 0)$, *i.e.*, when the roots are real and equal (=).

$$ax^2 + bx + c = a\,(x - \quad)^2 = a \times \text{a positive expression}.$$

Hence the expression has the same sign as a.

Case III. $b^2 - 4ac$ is positive $(D > 0)$, *i.e.*, when the roots are real and unequal.

In this case the roots are real and unequal. Let > .

(*i*) We assume that x does not lie between and . Then,

If $x >$, $(x -\quad)$ and $(x -\quad)$ both will be positive. [Fig. 5.08 (*i*)]

If $x <$, $(x -\quad)$ and $(x -\quad)$ both will be negative. [Fig. 5.08 (*ii*)]

The product $(x -\quad)(x -\quad)$ will always be positive.

$ax^2 + bx + c \quad a\ (x -\quad)(x -\quad) \quad a \times$ a positive expression.

(*i*) (*ii*)

Fig. 5.08

Fig. 5.09

Hence, the expression has the same sign as a.

(*ii*) If x lies between and , then $(x -\quad)$ and $(x -\quad)$ will be of opposite signs.

The product $(x -\quad)(x -\quad)$ will be negative

$ax^2 + bx + c = a \times$ a negative expression.

$ax^2 + bx + c$ and a will be of opposite signs.

From the above discussion it follows that the sign of the expression $ax^2 + bx + c$ is always the same as that of a, except when the roots of the equation $ax^2 + bx + c = 0$ are real and distinct and x lies between them.

Value of x :	**Less than**	**between and**	**greater than**
Sign of $f(x)$:	same as that of a	opposite to that of a	same as that of a

Tests

(*i*) A quadratic $ax^2 + bx + c$ is positive for all real x if $a > 0$ and $D < 0$.

(*ii*) A quadratic $ax^2 + bx + c$ is negative if $a < 0$ and $D < 0$.

Illustrations :

1. The sign of $x^2 + 6x + 12 = 0$ is positive for all real x since $D = b^2 - 4ac = 36 - 48 < 0$ and $a = 1 > 0$.

2. The sign of $-3x^2 + 5x - 12$ is negative for all real x since $D = b^2 - 4ac = 25 - 4(-3)(-12) = -119 < 0$ and $a = -3 < 0$.

Ex. 19. ***Find the sign of $6x^2 - 5x + 1$ for all real values of x.***

Sol. $D = b^2 - 4ac = 5^2 - 4 \times 6 \times 1 = 1$, is positive.

The roots are real and different.

By solving the equation, the roots are $\dfrac{5 \pm \sqrt{25 - 24}}{12}$, *i.e.*, $\dfrac{1}{2}$ and $\dfrac{1}{3}$.

The given expression has the same sign as the coefficient of x^2, *i.e.*, positive for all real values of x except for those which lie between $\frac{1}{3}$ and $\frac{1}{2}$.

Ex. 20. ***Determine the sign of the function $3x^2 - 2x + 1$ for real values of x.***

Sol. $$3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right) = 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{1}{3} - \frac{1}{9}\right\} \quad \text{(Completing the square)}$$

$$= 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right\}.$$

As x is real, the term $\left(x - \frac{1}{3}\right)^2$ is always non-negative, and hence the function is positive for all real values of x.

Method II. $D = b^2 - 4ac = (-2)^2 - 4(3)(1) = -8$. Therefore, the roots of $3x^2 - 2x + 1 = 0$ are imaginary. Hence $3x^2 - 2x + 1$ has the same sign as 'a', *i.e.* 3, *i.e.* positive. **(Case I)**

Ex. 21. ***Find the ranges of the values of x for which $x^2 - 4x + 2$ lies between -1 and $+1$.*** **[*G.C.E.*]**

Sol. $x^2 - 4x + 2 > -1 \Rightarrow x^2 - 4x + 3 > 0$.

Since the coefficient of x^2 is positive and the roots of $x^2 - 4x + 3 = 0$ are 1 and 3, therefore,

$$x^2 - 4x + 3 > 0 \text{ for } x < 1 \text{ or } x > 3^* \qquad ...(1)$$

Again, $x^2 - 4x + 2 < 1 \Rightarrow x^2 - 4x + 1 < 0$

*Since the coefficient of x^2 is positive and the roots of $x^2 - 4x + 1 = 0$ are $2 - \sqrt{3}$ and $2 + \sqrt{3}$, therefore $x^2 - 4x + 1 < 0$, for $2 - \sqrt{3} < x < 2 + \sqrt{3}$. ...(2)

Combining (1) and (2), we get $2 - \sqrt{3} < x < 1$ or $3 < x < 2 + \sqrt{3}$, which are the required ranges of the values of x.

Ex. 22. ***Determine the values of 'a' so that the expression $x^2 - 2(a+1)\,x + 4$ is always positive.***

Sol. Given : $x^2 - 2\,(a + 1)\,x + 4 > 0$ for all x

Since coeffi. of $x^2 = 1$ is $+\,ve$, therefore, sign of $x^2 - 2\,(a + 1)\,x + 4$ is positive if $D < 0$,

i.e., if $4(a + 1)^2 - 16 < 0$

$\Rightarrow (a + 1)^2 - 2^2 < 0 \Rightarrow [(a + 1) + 2]\,[(a + 1) - 2] < 0$

$\Rightarrow (a + 3)\,(a - 1) < 0$

$\Rightarrow \{a - (-3)\}\,(a - 1) < 0 \Rightarrow -3 < a < 1 \Rightarrow a \quad (-3, 1).$

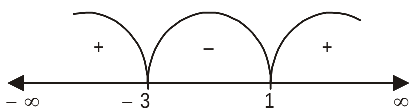


Fig. 5.10

5.13. Method of intervals

Suppose we have to solve the inequality

$$(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_{n-2})(x - x_{n-1})(x - x_n) < 0 \qquad ...(1)$$

where $x_1, x_2, x_3, \ldots\ldots x_{n-2}, x_{n-1}, x_n$ are *distinct real* numbers.

We will assume that $(x_1 < x_2 < x_3 < \ldots\ldots < x_{n-2} < x_{n-1} < x_n)$.

Plot these points on the real number line (Fig. 5.09) and consider the polynomial

* Alternatively use Art. 5.13.

$$F(x) = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_{n-2})(x - x_{n-1})(x - x_n) \qquad \ldots(2)$$

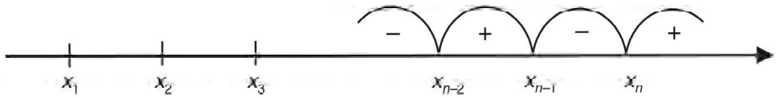


Fig. 5.11

Always start with + sign from the right towards the left.

It is clear that for all $x > x_n$ all the parenthetic expressions in (2) are positive and hence, for $x > x_n$ we have $F(x) > 0$, *i.e.* positive. Since $x_{n-1} < x < x_n$, the last parenthesis in the expression $F(x)$ is negative, and all other parentheses are positive, it follows that for $x_{n-1} < x < x_n$ we have $F(x) < 0$. Similarly we obtain $F(x) > 0$ for $x_{n-2} < x < x_{n-1}$, and so on, alternately positive and negative, as is shown in Fig. 5.09. Before putting the signs +, –, ensure that the highest degree term is +ve.

This is the underlying idea of the method of intervals.

> On the number line, the numbers $x_1, x_2, x_3, \ldots\ldots x_{n-2}, x_{n-1}, x_n$ must be arranged in order of increasing magnitude. Then place the plus sign in the interval to the right of the largest number x_n. In the next interval (from right to left) place the minus sign, then plus sign, then minus sign, etc. The solution of the inequality (1), *i.e.*, $F(x) < 0$ will then consist of intervals having the minus sign, and the solution of the inequality $F(x) > 0$ will consist of intervals having the plus sign.

5.14. Maximum and minimum values

To find the limits within which the value of $ax^2 + bx + c$ will lie for real values of x.

Let $\qquad ax^2 + bx + c = y. \qquad$ Then, $ax^2 + bx + c - y = 0$

$$x = \frac{-b \pm \sqrt{\{b^2 - 4a(c - y)\}}}{2a}$$

As x is real, so the disct. $b^2 - 4a(c - y)$ will be positive or zero, *i.e.*, $b^2 - 4a(c - y) \quad 0$

$$b^2 - 4ac + 4ay \geq 0 \Rightarrow 4a\left[y - \frac{4ac - b^2}{4a}\right] \geq 0.$$

This condition will be satisfied if a and $y - \left(\frac{4ac - b^2}{4a}\right)$ are of the same sign. Thus

(*i*) If a is positive, then $y - \left(\frac{4ac - b^2}{4a}\right) \geq 0$ or $y \geq \frac{(4ac - b^2)}{4a}$,

i.e., y is either equal to or greater than $\frac{(4ac - b^2)}{4a}$.

(*ii*) If a is negative, then, $y - \frac{(4ac - b^2)}{4a} \leq 0$ or $y \leq \frac{(4ac - b^2)}{4a}$,

i.e., y is to remain less than $\frac{(4ac - b^2)}{4a}$.

This shows that y has the minimum value when $a > 0$ and the maximum value when $a < 0$ and the same is $\frac{(4ac - b^2)}{4a}$.

Further, as $\quad ax^2 \quad bx \quad c \quad a \quad x \quad \frac{b}{2a}^{\;2} \quad \frac{b^2 \quad 4ac}{4a^2} \quad a \quad x \quad \frac{b}{2a}^{\;2} \quad \frac{4ac \quad b^2}{4a^2}$,

the maximum and minimum values are obtained at $x = \frac{-b}{2a}$.

Ex. 23. ***If x be real, find the maximum value of*** $7 + 10x - 5x^2$.

Sol. Let $7 + 10x - 5x^2 = y$. Then, $5x^2 - 10x + (y - 7) = 0$.

If x be real, then $b^2 - 4ac \quad 0$, *i.e.*, $100 - 20(y - 7) \quad 0$ or $20(12 - y) \quad 0$.

This shows that y cannot be greater than 12 as otherwise $20(12 - y)$ will become negative.

Hence, the maximum value of y, *i.e.*, the expression is 12.

5.22. The quadratic fraction

Just like $ax^2 + bx + c$, a fraction of the form $\dfrac{ax^2 \quad bx \quad c}{px^2 \quad qx \quad r}$ also varies in magnitude and sign. We can determine this variation in numerical case as illustrated in the following examples.

Ex. 24. ***If x is real, prove that the value of the expression*** $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ ***cannot be between*** $\dfrac{4}{9}$ ***and 1.***

Sol. Let $\dfrac{(x-1)(x+3)}{(x-2)(x+4)} = y$. Then $x^2 + 2x - 3 = y(x^2 + 2x - 8)$

or $\quad x^2(1 - y) + 2x(1 - y) + (8y - 3) = 0.$

For x to be real, its discriminant $\quad 0$

or $\quad [2(1-y)]^2 - 4(1-y)(8y-3) \geq 0 \Rightarrow 9y^2 - 13y + 4 \geq 0$

$$(9y \quad 4)(y \quad 1) \quad 0 \qquad 9\left(y \quad \frac{4}{9}\right)(y \quad 1) \quad 0 \qquad \left(y \quad \frac{4}{9}\right)(y \quad 1) \quad 0$$

The critical points are $\frac{4}{9}$ and 1. By the Method of Intervals, (Art. 5.13), since $\left(y \quad \frac{4}{9}\right)(y \quad 1)$ is positive, therefore, y cannot lie between $\frac{4}{9}$ and 1. In other words the given expression cannot lie between $\frac{4}{9}$ and 1. Its range is $\left(\quad , \frac{4}{9}\right] \quad [1, \quad)$.

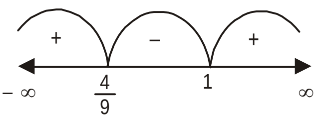


Fig. 5.12

Procedure. (Method of intervals explained in Art. 5.13)

Step 1. *Make the coefficient of x^2 positive, if it is not so, by multiplying both sides of the inequation by – 1.*

On doing so, the inequality will be reversed, i.e., > will change to < and vice-versa

Step 2. *Factorise the quadratic expression and express the L.H.S. of the inequality in the form* $(x - \alpha)(x - \beta)$, *where* $\alpha < \beta$.

Step 3. *Plot the points and on the number line thus, dividing the number line into three parts. Starting from the right most region put +, –, + signs as shown as the expression* $(x - \alpha)(x - \beta)$ *is non-negative in the region on the right of .*

Step 4. Now reason out as under :

(i) ***If*** $(x - \alpha)(x - \beta) > 0$, then x, lies outside the interval $[\ ,\ ]$, *i.e.*, $x \in (-\infty, \alpha) \cup (\beta, \infty)$

(ii) ***If*** $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside the interval $(\ ,\ )$, *i.e.*, $x \in (-\infty, \alpha] \cup [\beta, \infty)$.

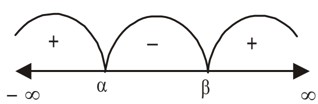


Fig. 5.13

(iii) ***If*** $(x - \alpha)(x - \beta) < 0$, then x lies in the interval $\ ,\ $

(iv) ***If*** $(x - \alpha)(x - \beta) \leq 0$, then x lies inside the interval $\ ,\ $ *i.e.*, $x \in [\alpha, \beta]$.

Note on use of brackets for open and closed intervals.

Sometimes the domains of variation, *i.e.*, the intervals are denoted as follows:

(*i*) '$a < x < b$' is denoted by (a, b) and is called an open interval of the variable x,

(*ii*) '$a \quad x \quad b$' is denoted by $[a, b]$ and is called as closed interval of the variable x, as x can take up values 'a' and 'b' also;

(*iii*) Any 'x' is denoted by $(\quad , \quad)$. Here, it should be noted that the symbols $-\infty, \infty$ are not mumbers in any sense whatsoever.

(*iv*) '$x \geq a$' is denoted by $[a, \quad)$, '$x \leq b$' is denoted by $(\quad , b]$.

(*v*) '$a \quad x \quad b$' is denoted by $[a, b)$, '$a < x \leq b$' is denoted by $(a, b]$. These are called semiclosed intervals.

Ex. 25. ***If x be real, find the maximum value of*** $\dfrac{(x+2)}{2x^2+3x+6}$.

Sol. Let $\dfrac{x+2}{2x^2+3x+6} = y$. Then, $2x^2y + (3y-1)x + 6y - 2 = 0$.

For x to be real, $(3y-1)^2 - 8y(6y-2) \quad 0$ or $(1+13y)(1-3y) \quad 0$.

$$y \quad \frac{1}{13} \quad \frac{1}{3} \quad y \quad 0 \qquad \left(y+\frac{1}{13}\right)\left(y-\frac{1}{3}\right) \leq 0 \qquad y \quad \frac{1}{3} \quad y \quad \frac{1}{3} \quad 0 \quad \ldots (i)$$

The critical points are $\dfrac{1}{13}$ and $\dfrac{1}{3}$.

By the Method of Intervals, y lies in the interval $\left[\dfrac{-1}{13}, \dfrac{1}{3}\right]$ as $\left(y+\dfrac{1}{13}\right)\left(y-\dfrac{1}{3}\right)$ is negative. The range of y, *i.e.*, the given expression is $\left[\dfrac{-1}{13}, \dfrac{1}{3}\right]$. Hence the maximum value of the given expression is $\dfrac{1}{3}$.

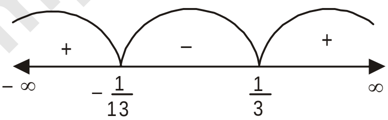


Fig. 5.14

Ex. 26. ***If x is real, then find the minimum value of*** $\dfrac{x^2 \quad x+1}{x^2+x+1}$.

Sol. Let $y = \dfrac{x^2 \quad x \quad 1}{x^2 \quad x \quad 1}$ $\quad (y-1)x^2 + (y+1)x + y - 1 = 0$

For x to be real, $(y+1)^2 - 4(y-1)^2 \geq 0$ $\qquad$ Using $b^2 - 4ac \geq 0$

$-3y^2 + 10y - 3 \geq 0 \quad 3y^2 - 10y + 3 \leq 0$ $\qquad$ (If $a > b$, then $-a < -b$)

$(3y-1)\ (y-3) \leq 0 \quad (y - \dfrac{1}{3})\ (y-3) \leq 0$

Plot the critical points $\dfrac{1}{3}$ and 3 on the number line and place the signs in the intervals.

Since $y \quad \dfrac{1}{3} \quad (y \quad 3)$ is –ve or 0, therefore, $y \quad [\dfrac{1}{3}, 3]$.

Minimum value of $\dfrac{x^2 \quad x \quad 1}{x^2 \quad x \quad 1}$ is $\dfrac{1}{3}$.

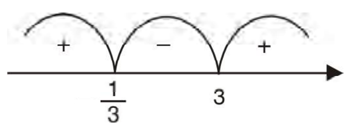


Fig. 5.15

Ex. 27. ***If x is real, the maximum value of*** $\dfrac{3x^2+9+17}{3x^2+9x+7}$ ***is***

(a) $\dfrac{17}{7}$ *(b)* $\dfrac{1}{4}$ *(c) 41* *(d) 1*

Sol. $y = \dfrac{3x^2 \quad 9x \quad 17}{3x^2 \quad 9x \quad 7} \quad 1 \quad \dfrac{10}{3x^2 \quad 9x \quad 7} \quad 1 \quad \dfrac{10}{3 \ (x \ \frac{3}{2})^2 \ \frac{1}{12}}$

Obviously, y will be maximum when the denominator is minimum.

Now, minimum value of $3 \ (x \ \frac{3}{2})^2 \ \frac{1}{12}$ is $3 \ \frac{1}{12} \ \frac{1}{4}$ when $x = \frac{3}{2}$.

the max. value of $\dfrac{10}{3x^2 \quad 9x \quad 7}$ is $\dfrac{10}{1/4} = 40$ and max. value of $\dfrac{3x^2 \quad 9x \quad 17}{3x^2 \quad 9x \quad 7}$ is $1 + 40 =$ **41.**

EXERCISE 5 (*f*)

1. Show that

(*a*) $x^2 - 3x + 6 > 0$ for all x

(*b*) $4x - x^2 - 6 < 0$ for all x

(*c*) $2x^2 - 4x + 7$ is always positive

(*d*) $-2x^2 + 3x - 4$ is always negative

(*e*) $-x^2 + 3x - 3$ is always negative

2. Explain why $3x^2 + kx - 1$ is never always positive for any value of k.

3. Under what conditions is $2x^2 + kx + 2$ always positive.

4. Find the values of a so that the expression $x^2 - (a + 2)\, x + 4$ is always positive.

5. Find the range of values of x for which the expression $12x^2 + 7x - 10$ is negative.

6. Find the range of values of x for which the expression $x^2 + 4x + c$ is always positive.

[Sol. Discriminant $\geq 0 \ \therefore \ 16 - 4c \geq 0 \quad \Rightarrow \quad c \leq 4$.

$x^2 + 4x + c = 0$ gives the critical points $\dfrac{-4 \pm \sqrt{16-4c}}{2}$, i.e., $-2-\sqrt{4-c}$, $-2+\sqrt{4-c}$,

It is given that $x^2 + 4x + c \geq 0$. Therefore, by the Method of Intervals (Art 5.13), we have $x \leq -2-\sqrt{-4-c}$ and $x \geq -2+\sqrt{4-c}\,(c \leq 4)$. Hence, the required ranges of values of x are $-\infty < x < -2\sqrt{4-c}$ and $-2+\sqrt{4-c} < x < \infty$, or $\left[-\infty, x, -2\sqrt{4-c}\right]$ and $2 \ \sqrt{4 \ \ c}, x,$.

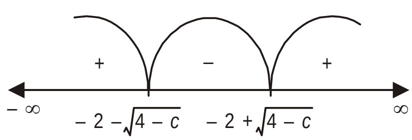


Fig. 5.16

7. Find the greatest value of $3 + 5x - 2x^2$ for all real values of x.

8. Find the least value of $\dfrac{6x^2 \quad 22x \quad 21}{5x^2 \quad 18x \quad 17}$ for real values of x.

9. If x be real, prove that the value of $\dfrac{11x^2 \quad 12x \quad 6}{x^2 \quad 4x \quad 2}$ cannot lie between – 5 and 3.

10. $\frac{h^2}{a^2}+\frac{k^2}{b^2}=1$, prove that $(h^2 - a^2)\,x^2 - 2hkx + (k^2 - b^2)$ is a perfect square.

11. Draw graphs of the functions in the following equations. For each graph, determine the maximum and minimum points of the curves (as the case may be) as well as the real roots of the equation. If your graph shows the roots to be imaginary, state it.
(*a*) $3x^2 + 7x - 6 = 0$ from $x = -4$ to $x = 3$; (*b*) $3x - 2x^2 + 7 = 0$ from $x = -3$ to $x = 3$;
(*c*) $x^2 - 6x + 9 = 0$ from $x = 0$ to $x = 6$.

12. Trace the changes in sign and value of the expression $-5x^2 + 5x - 6$ for real values of x. Illustrate by a graph.

13. Both the roots of the quadratic equation $x^2 - (a + 1)x + a + 4 = 0$ are negative. Calculate the values of a. **[*I.S.C. 1992*]**

[Sol. $x^2 - (a + 1)\,x + a + 4 = 0$

Let the roots be – , – , where are positive.

– (+) = a + 1, and (–) (–) = a + 4, *i.e.* = a + 4

$a + 1 < 0$, and $a + 4 > 0$ $a < -1$ and $a > -4$ $-4 < a < -1$ ...(1)

The roots are also supposed to be real, therefore the discriminant 0.

$[-(a+1)]^2 - 4\times 1\times(a+4)$ 0 $a^2 + 2a + 1 - 4a - 16 \geq 0$

$a^2 - 2a - 15$ 0, $(a-5)(a+3)$ 0

$(a-5)\,\{a-(-3)\}$

This is possible if a 5 or a – 3 ...(2)

Combining the two results of (1) and (2), we get $-4 < a \leq -3$.]

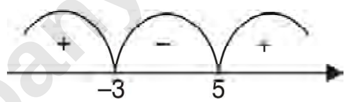


Fig. 5.17

ANSWERS

2. $a = 3$ which is > 0 and $D = k^2 + 12$ which is > 0 (as $k^2 > 0$) for all k)

3. $a = 2$ which is > 0 and $D = k^2 - 16$ The given quadratic will always be positive when $k^2 - 16 < 0$, *i.e.*, $k^2 < 16$, *i.e.*; $-4 < k < 4$.

4. $-6 < a < 2$. **5.** $\frac{-5}{4} < x < \frac{2}{3}$. **6.** $6\frac{1}{8}$. **7.** 1

11. (*a*) $\frac{+2}{3}, -3$; Minimum value $\frac{-121}{12}$ at $x = \frac{-7}{6}$. (*b*) 2.77, – 1.27, max. value $\frac{65}{8}$ at $x = \frac{3}{4}$.

(*c*) 3, 3 ; Min. value 0 at $x = 3$.

5.16. Quadratic inequalities

By a quadratic inequality we mean an inequality of one of the following kinds :

(*i*) $ax^2 + bx + c > 0$ (*ii*) $ax^2 + bx + c < 0$. $(a > 0)$

The range of such an inequality can be found by the method of intervals explained below.

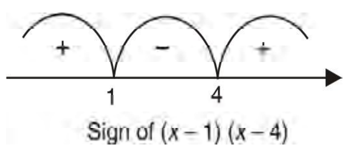


Fig. 5.18

Ex. 28. ***Solve :*** $x^2 - 5x + 4 > 0$.

Sol. $x^2 - 5x + 4 = (x - 1)(x - 4)$.

Plot 1 and 4 on the number line and place + sign in the interval $x > 4$, – sign in the interval $1 < x < 4$, and + sign in the interval $x < 1$.

The required range is, therefore, $x < 1$ or $x > 4$.

[$\because$ $x^2 - 5x + 4$ is positive.]

Note. In case L.H.S, is a perfect square like $x^2 - 6x + 9 > 0$, then the inequality $(x-3)^2 > 0$ is true for x R. The range is (– ,). If $(x-3)^2 = 0$, then $x = 3$.

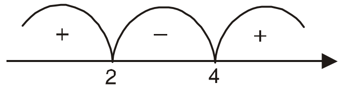


Fig. 5.19

Ex. 29. ***Solve :*** $-x^2 + 6x - 8 > 0$.

Sol. Multiplying the given inequality by a minus sign and reversing the sign of inequality, we get

$$x^2 - 6x + 8 < 0.$$

Note this step. The highest degree term is made positive.

Now $x^2 - 6x + 8 = (x-2)(x-4)$

Plot 2 and 4 on the number line and place the signs in the intervals.

The required range is $2 < x < 4$. [$\because$ $x^2 - 6x + 8$ is negative.]

Ex. 30. ***If*** $\sqrt{9x^2 + 6x + 1} < 2 \quad x$, ***then***

(a) $x \quad \frac{3}{2}, \frac{1}{4}$ ***(b)*** $x \quad \frac{3}{2}, \frac{1}{4}$ ***(c)*** $x \quad \frac{3}{2}, \frac{1}{4}$ ***(d)*** $x < \frac{1}{4}$.

Sol. L.H.S. < R.H.S. and L.H.S. is non-negative,

R.H.S. > 0

Squaring $9x^2 + 6x + 1 < 4 - 4x + x^2$

$$8x^2 + 10x - 3 < 0 \qquad (4x-1)(2x+3) < 0$$

$$8\left(x - \frac{1}{4}\right)\left(x - \quad \frac{3}{2}\right) < 0$$

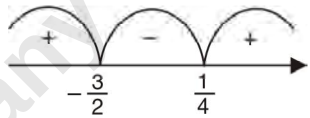


Fig. 5.20

Plot $\frac{3}{2}$ and $\frac{1}{4}$ on the number line and place the signs in the intervals.

$x \quad \frac{3}{2}, \frac{1}{4}$. **Ans.** (*b*).

5.17. Inequalities in fractions

Inequalities in fractions are different from equations in fractions. In the latter we did not make any mistake in clearing fractions by cross-multiplying, but in the inequalities in fractions we are not justified always if we cross-multiply. Consider the following example :

$$\frac{-1}{2} < 0 \text{ or } -1 < 0, \text{ (By cross-multiplying) which is true.}$$

But if we write the given inequality as $\frac{1}{-2} < 0$, then $1 < 0$ (By cross-multiplying) which is not true.

Now consider the inequality. $\frac{1}{x} < 1$ ...(1) $1 < x$ ...(2)

The inequality (2) gives all *positive* values of x greater than 1, whereas the inequality (1) holds true for all *negative* values of x as well.

From the above it is clear that clearing of fractions by the method of cross-multiplication in equations is quite different from that operation in inequalities.

Actually, clearing of fractions in an equation (or inequality) consists in multiplying both members of the equation (or inequality) by the expression in the denominator. In this operation, equations remain equivalent if they are multiplied by a non-zero expression, but for inequalities this property is more involved : *multiplication of both members of an inequality by a positive expression does not change the sense of the inequality, whereas multiplication by a negative expression reverses the sense of the inequality.*

Therefore, when multiplying both members of the inequality at hand by x, one should have taken into account that the x could have assumed negative values as well as positive values, and then one should have reversed the sense of the inequality in the latter case.

Thus, in every case when we wish to multiply both members of an inequality by an expression that is dependent on x and assumes both positive and negative values, the student should examine the two appropriate cases. This rule is often forgotten and is the cause of a lot of trouble.

Let us consider the inequality $\dfrac{x-2}{x+2} > \dfrac{2x-3}{4x-1}$ ...(1)

The range of the variable x in this inequality consists of all values of x except $x = -2$ and $x = \dfrac{1}{4}$. Hence, we cannot cross-multiply, *i.e.* cannot multiply both sides by the L.C.M, $(x + 2)(4x - 1)$ unless we are sure that $(x + 2)(4x - 1)$ is a positive quantity. Since we are not sure, we adopt another method. Transpose all terms of the original inequality to the left side and reduce it to the common denominator.

$$\frac{x-2}{x+2} - \frac{2x-3}{4x-1} > 0 \Rightarrow \frac{(x-2)(4x-1)-(x+2)(2x-3)}{(x+2)(4x-1)} > 0$$

$$\frac{2\,(x^2-5x+4)}{(x+2)\,(4x-1)} > 0 \Rightarrow \frac{2\,(x-1)(x-4)}{4\,(x+2)\left(x-\frac{1}{4}\right)} > 0 \Rightarrow \frac{(x-1)(x-4)}{(x+2)\left(x-\frac{1}{4}\right)} > 0 \qquad ....(2)$$

Multiply both sides of (2) by the expression $(x+2)^2\left(x-\frac{1}{4}\right)^2$, which is *positive* for the x under consideration.

$$\frac{(x-1)(x-4)(x+2)^2\left(x-\frac{1}{4}\right)^2}{(x+2)\left(x-\frac{1}{4}\right)} > 0 \Rightarrow (x-1)(x-4)(x+2)\left(x-\frac{1}{4}\right) > 0 \qquad ....(3)$$

Putting each factor equal to zero, we get $x = -2, \dfrac{1}{4}, 1, 4$. Now we use the method of intervals.

Thus the range is

$$x < -2, \text{ or } \frac{1}{4} < x < 1, \text{ or } x > 4.$$

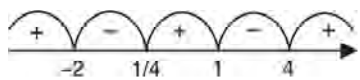


Fig. 5.21

Generally $\dfrac{F(x)}{f(x)} > 0$ and $F(x)\,.\,f(x) > 0$ are equivalent. Therefore to solve the inequality $\dfrac{F(x)}{f(x)} > 0$, where $F(x)$ and $f(x)$ are polynomials, one applies the method of intervals to the inequality $F(x) f(x) > 0$, which need not even be written out explicitly, it being sufficient to locate the roots of the polynomials $F(x)$ and $f(x)$ on the number line and affix the appropriate sign to each of the resulting intervals.

We now solve a few examples.

Ex. 31. ***Find the range of values of x for which***

$$\boldsymbol{\frac{x^2+x+1}{x^2+2} < \frac{1}{3}}, \textbf{\textit{ x being real.}}$$ **[I.S.C.]**

Sol. $\dfrac{x^2+x+1}{x^2+2} - \dfrac{1}{3} < 0$ or $\dfrac{2x^2+3x+1}{3\,(x^2+2)} < 0$

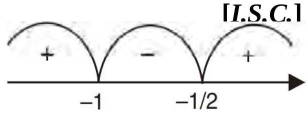


Fig. 5.22

$2x^2 + 3x + 1 < 0$, as $x^2 + 2$ is positive for all real values of x.

$(2x+1)(x+1) < 0 \qquad 2(x + \frac{1}{2})\ (x+1) < 0 \Rightarrow \left(x + \frac{1}{2}\right)(x+1) < 0.$

By using the method of intervals, we get the required range as $-1 < x < -\frac{1}{2}$.

$$\left[\because \left(x + \frac{1}{2}\right)(x+1) \text{ is negative.}\right]$$

Ex. 32. ***Determine the range of values of x for which*** $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}$. **[I.S.C.]**

Sol. $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}$ or $\dfrac{x^2-2x+5}{3x^2-2x-5} - \dfrac{1}{2} > 0$

Fig. 5.23

or $\dfrac{-x^2-2x+15}{3x^2-2x-5} > 0$ or $\dfrac{x^2+2x-15}{3x^2-2x-5} < 0$

or $\dfrac{(x+5)(x-3)}{(x+1)(3x-5)} < 0$ or $\dfrac{(x+5)(x-3)(x+1)^2(3x-5)^2}{(x+1)(3x-5)} < 0$

(Multiplying by the square of the denominator, which is positive)

or $(x + 5)(x - 3)(x + 1)(3x - 5) < 0$.

Putting each factor equal to zero, we get

$$x = -5, -1, \frac{5}{3}, 3.$$

By using the method of intervals, we get the range as $-5 < x < -1$ or $\frac{5}{3} < x < 3$.

Ex. 33. ***Find the range of real values of x for which*** $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$. ***[I.S.C. 1990]***

Sol. $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$ or $\dfrac{x-1}{4x+5} - \dfrac{x-3}{4x-3} < 0$

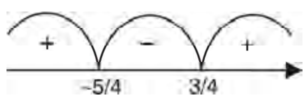


Fig. 5.24

or $\dfrac{18}{(4x\quad 5)(4x\quad 3)}\quad 0$ or $(4x\quad 5)(4x\quad 3)\quad 0$

or $16\left(x + \frac{5}{4}\right)\left(x - \frac{3}{4}\right) < 0.$

Putting each factor equal to zero, we get $x \quad \frac{5}{4}, \frac{3}{4}$.

By the method of intervals, we get the required range $-\frac{5}{4} < x < \frac{3}{4}$.

Ex. 34. ***For what value of a is the inequality*** $\dfrac{x^2+ax-2}{x^2-x+1} < 2$ ***satisfied for all real values of x ?*** ***[I.S.C. 1991]***

Sol. $\dfrac{x^2+ax-2}{x^2-x+1} < 2$ or $\dfrac{x^2+ax-2}{x^2-x+1} - 2 < 0$ *or* $\dfrac{-x^2+x(a+2)-4}{x^2-x+1} < 0$ ....(*i*)

Now $x^2 \quad x \quad 1 \quad x^2 \quad 2\;\frac{1}{2}\;x \quad \frac{1}{2}^{\,2} \quad \frac{3}{4} \quad x \quad \frac{1}{2}^{\,2} \quad \frac{3}{4}$,

which is positive for all real values of x.

From (i) it follows that $-x^2 + x(a+2) - 4 < 0$

or $x^2 - x(a+2) + 4 > 0$, *i.e.*, $x^2 - x(a + 2) + 4$ is positive.

We know from Art. 5.12 (Case I and Case II) that if $ax^2 + bx + c$ and 'a' have the same sign, then $b^2 - 4ac < 0$. Here $x^2 - x(a+2) + 4$ and the coefficient of x^2, namely 1 are positive, *i.e.*, have the same sign, therefore

$$(a+2)^2 - 4 \times 4 < 0 \qquad \text{(Discriminant} < 0)$$

or $a^2 + 4a - 12 < 0$

or $(a+6)(a-2) < 0$.

Now by the method of intervals (Art. 5.13), we get

$$-6 < a < 2,$$

which is the required range of values of a.

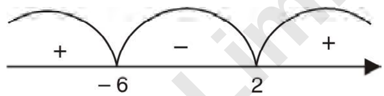


Fig. 5.25

Ex. 35. ***Determine the range of values of x for which*** $\dfrac{3x^2 - 2x - 5}{x^2 - 2x + 5} < 2.$ **[*I.S.C. 1993*]**

Sol. $\dfrac{3x^2 - 2x - 5}{(x-1)^2 \quad 4} \quad 2$ $\qquad [\because x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x-1)^2 + 4]$

Since $(x - 1)^2$ is positive, being a square, for all real values of x, therefore the denominator $(x-1)^2 + 4$ is positive and we can thus cross-multiply making no effect on inequality sign.

$3x^2 - 2x - 5 < 2(x^2 - 2x + 5) \qquad 3x^2 - 2x - 5 < 2x^2 - 4x + 10$

$x^2 + 2x - 15 < 0 \qquad (x+5)(x-3) < 0.$

Putting each factor equal to zero,

we get $x = -5, 3$.

By using the method of intervals, the required range is $-5 < x < 3$.

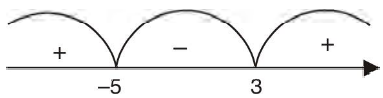


Fig. 5.26

Ex. 36. ***Show that the maximum value of*** $\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$ ***is 3, for real values of x.***

Sol. Let $y = \dfrac{x^2 - 2x \quad 4}{x^2 \quad 2x \quad 4}$

$y(x^2 + 2x + 4) = x^2 - 2x + 4 \qquad (y-1)x^2 + 2(y+1)x + (4y - 4) = 0.$

For real values of x the discriminant $\quad 0$.

$4(y+1)^2 - 4(y-1)(4y-4) \quad 0.$

$(y+1)^2 - 4(y-1)^2 \quad 0$

$-3y^2 + 10y - 3 \quad 0 \qquad 3y^2 - 10y + 3 \quad 0$

$(3y-1)(y-3) \quad 0.$

Putting each factor equal to zero,

we get $y = \dfrac{1}{3}, 3.$

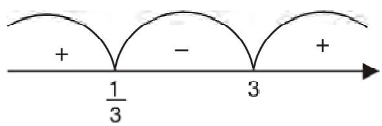


Fig. 5.27

By using the method of intervals,

we get $\frac{1}{3} \le y \le 3.$

The maximum value of y is 3.

In other words, the maximum value of the given fraction is 3.

Ex. 37. ***Obtain, by graphical method, the solution set of the inequation $x^2 + 3x - 18 \quad 0$.*** **[*I.S.C. 1999*]**

Sol. Method I. $x^2 + 3x - 18 \quad 0 \qquad (x + 6)(x - 3) \quad 0$

$x + 6 = 0$ and $x - 3 = 0$ gives $x = -6, x = 3$.

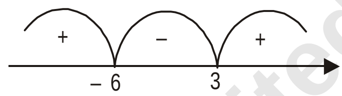


Fig. 5.28

By the method of intervals, we get

$x \quad 3$ and $x \quad -6$.

Hence, the solution set is

$\{x \quad -6 \text{ or } x \quad 3; x \quad R\}$.

Method II. $x^2 + 3x - 18 \quad 0$.

Discriminant $= b^2 - 4ac = (3)^2 - 4 \times 1 \times (-18) = 9 + 72 = 81 > 0$.

If we put $y = x^2 + 3x - 18$, then

$y = (x + 6)(x - 3)$.

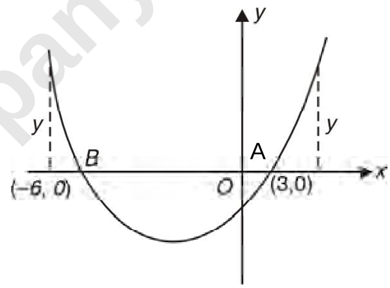


Fig. 5.29

It cuts the x-axis when $y = 0$, therefore

$x + 6 = 0$ or $x - 3 = 0$, *i.e.*, $x = -6, + 3$.

It cuts the x-axis at $(-6, 0)$, and $(3, 0)$.

Also when $b^2 - 4ac > 0$, the shape of the curve is as drawn here. It is clear that y is positive either to the right of A $(3, 0)$, or to the left of B.

In other words, $x^2 + 3x - 18 \quad 0$

for $x \quad 3$ and $x \quad -6$.

Hence, the solution set is

$\{x \quad -6 \text{ or } x \quad 3 ; x \quad R\}$.

Note. The student will study in Class XII that $y = x^2 + 3x - 18$ is an upward parabola, because

$$y = \left[x^2 + 2 \times \frac{3}{2} \times x + \left(\frac{3}{2}\right)^2\right] - \frac{81}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{81}{4}$$

or $\left(x + \frac{3}{2}\right)^2 = y + \frac{81}{4}$, which is an upward parabola with vertex $\left(\frac{-3}{2}, \frac{-81}{4}\right)$, as is drawn in Fig. 5.29.

Ex. 38. ***Find the range of values of x which satisfy $x^2 + 6x - 27 > 0, - x^2 + 3x + 4 > 0$ simultaneously.*** **[*I.S.C. 1997*]**

Sol. $x^2 + 6x - 27 > 0 \qquad (x + 9)(x - 3) > 0$

$(x + 9)(x - 3) = 0$ gives $x = -9, 3$.

Fig. 5.30

By the method of intervals, $x > 3, x < -9$ ....(*i*)

$-x^2 + 3x + 4 > 0 \qquad x^2 - 3x - 4 < 0 \qquad (x - 4)(x + 1) < 0$

$(x - 4)(x + 1) = 0$ gives $x = -1, 4$

By the method of intervals we get $-1 < x < 4$ .....(*ii*)

(*i*) and (*ii*) combined give

$$3 < x < 4,$$

which is the required range.

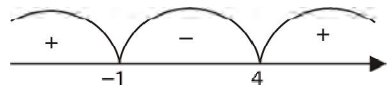


Fig. 5.31

Ex. 39. ***Find the range of real value of x for which*** $\dfrac{x-2}{3x+4} < \dfrac{x-4}{3x-2}$. **[*I.S.C. 1996*]**

Sol. $\dfrac{x-2}{3x+4} - \dfrac{x-4}{3x-2} < 0 \Rightarrow \dfrac{(x-2)(3x-2)-(x-4)(3x+4)}{(3x+4)\,(3x-2)} < 0$

$$\Rightarrow \frac{(3x^2-8x+4)-(3x^2-8x-16)}{(3x+4)\,(3x-2)} < 0 \Rightarrow \frac{20}{(3x+4)\,(3x-2)} < 0$$

$$\Rightarrow (3x+4)\,(3x-2) < 0 \Rightarrow 3\left(x+\frac{4}{3}\right) \times 3\left(x-\frac{2}{3}\right) < 0$$

$$\Rightarrow 9\left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) < 0 \Rightarrow \left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) < 0$$

$$\left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) = 0 \text{ gives } x = -\frac{4}{3}, \frac{2}{3}.$$

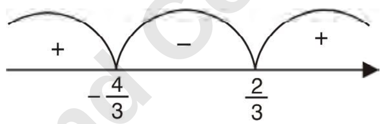


Fig. 5.32

By the method of intervals, we get $-\dfrac{4}{3} < x < \dfrac{2}{3}$, which is the required range.

Ex. 40. ***Determine the range of values of x for which*** $\mathbf{2x^2 + 3x - 9 \leq 0}$. **[*I.S.C. 1995*]**

Sol. $2x^2 + 3x - 9 \quad 0 = 2x^2 + 6x - 3x - 9 \quad 0 \quad 2x\,(x+3) - 3\,(x+3) \quad 0.$

$$\Rightarrow 2\,(x+3)\left(x-\frac{3}{2}\right) \leq 0$$

$$\Rightarrow 2\,(x+3)\left(x-\frac{3}{2}\right) = 0 \text{ gives } x = -3, \frac{3}{2} \text{ as critical points.}$$

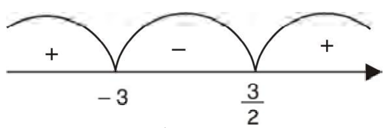


Fig. 5.33

By the method of intervals, we get $-3 \leq x \leq \dfrac{3}{2}$,

which is the required range.

EXERCISE 5 (*g*)

Find the range of x in each of the following inequalities:

1. $x^2 - 4x + 3 < 0$. **2.** $x^2 + 5x + 4 > 0$. **3.** $x^2 + x - 6$ _ 0. **4.** $x^2 - 16 < 0$.

5. $x^2 - 6x + 9 \geq 0$. **6.** $-x^2 + 2x + 3 < 0$. **7.** $5x < 2 - 3x^2$. **8.** $-x^2 - 4x - 5 < 0$.

9. $4x^2 + 1 > 4x$. **10.** $-x^2 + x > 0$. **11.** $6 + x < 2x^2$. **[I.S.C.]**

12. (*i*) $(x - 4)(x + 6) > 0$; (*ii*) $3 - 2x^2 > 5x$. **[I.S.C.]**

13. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$. **[I.S.C.]**

[Hint. The solution of the first inequality is $x < 1$, or $x > 2$. The solution of the second inequality is $-1 \leq x \leq 4$. The common solution is -1 $x < 1$, or $2 < x$ 4.]

14. The set of values of x for which the inequalities $x^2 - 3x - 10 < 0$, $10x - x^2 - 16 > 0$ hold simultaneously is

(*a*) (–2, 5) (*b*) (2, 8) (*c*) (–2, 8) (*d*) (2, 5).

[Hint. $x^2 - 3x - 10 < 0$ $(x + 2)(x - 5) < 0$ x $(-2, 5)$]

$10x - x^2 - 16 > 0$ $x^2 - 10x + 16 < 0$ $(x - 2)(x - 8) < 0$ x $(2, 8)$

x $(-2, 5)$ $(2, 8)$ x $(2, 5)$, *i.e.*, (*d*).

Solve the following inequalities :

15. $\dfrac{x+3}{x-1} > x$. **16.** $x + 4 < -\dfrac{2}{x+1}$. **17.** $\dfrac{x^2 - 2x + 3}{x^2 - 4x + 3} > -3$.

18. $\dfrac{x^2 + 6x - 11}{x+3} < -1$. **[I.S.C.]** **19.** $\dfrac{x^2 - 3x + 24}{x^2 - 4x + 3} > -4$. **[I.S.C.]**

ANSWERS

1. $1 < x < 3$ **2.** $x < -4$, or $x > -1$ **3.** x -3 or x _ 2 **4.** $-4 < x < 4$.

5. All real values of x **6.** $x < -1$, or $x > 3$ **7.** $-2 < x < \dfrac{1}{3}$

8. All real values of x. **9.** $x \in R - \left\{\dfrac{1}{2}\right\}$ **10.** $0 < x < 1$

11. $x < \dfrac{-3}{2}$ or $x > 2$ **12.** (*i*) $x < -6$ or $x > 4$ (*ii*) $-3 < x < \dfrac{1}{2}$ **15.** $x < -1$ or $1 < x < 3$

16. $x < -3$ or $-2 < x < -1$ **17.** $x < 1$ or $\dfrac{3}{2} < x < 2$, $x > 3$

18. $x < -8$ or $-3 < x < 1$ **19.** $x <$ 1 or $x > 3$.

REVISION EXERCISE

1. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values q of a is

(*i*) (3,) (*ii*) (– ,–3) (*iii*) (–3, 3) (*iv*) (–3,)

[Hint. $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\,\alpha\beta} = \sqrt{a^2 - 4}$

Since $|\alpha - \beta| < \sqrt{5} \Rightarrow \sqrt{a^2 - 4} < \sqrt{5} \Rightarrow a^2 < 9 \Rightarrow -3 < a < 3.$]

2. Let , be the roots of the equation $x^2 - px + r = 0$ and /2, 2 be the roots of the equation $x^2 - qx + r = 0$, then the value of r is

(*a*) $\frac{2}{9}(p-q)(2q-p)$ (*b*) $\frac{2}{9}(q-p)(2p-q)$

(*c*) $\frac{2}{9}(q-2p)(2q-p)$ (*d*) $\frac{2}{9}(2p-q)(2q-p)$

[Hint. + $= p$, $= r$, $\frac{\ }{2}$ + 2 $= q$, $= r$

+ $= p$, + 4 $= 2q$. Solve for and].

3. , are the roots of $ax^2 + 2bx + c = 0$ and + , + are the roots of $Ax^2 + 2Bx + C = 0$, then what is $(b^2 - ac)/(B^2 - AC)$ equal to ?

(*i*) $(b/B)^2$ (*ii*) $(a/A)^2$ (*iii*) $(a^2 b^2)/(A^2B^2)$ (*iv*) ab/AB

[Hint. $\frac{2b}{a}$, $\frac{c}{a}$, ( ) ( ) $\frac{2B}{A}$, ( ) ( ) $\frac{C}{A}$

Now, $\alpha - \beta = (\alpha + \delta) - (\beta + \delta) \Rightarrow (\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$

$(\alpha + \beta)^2 - 4\alpha\beta = (\overline{\alpha + \delta} + \overline{\beta + \delta})^2 - 4(\alpha + \delta)(\beta + \delta).$

4. If are the roots of the equation $x^2 - 2x - 1 = 0$, then what is the value of $^{2}\ ^{-2} + \ ^{-2}\ ^{2}$?

(*a*) –2 (*b*) 0 (*c*) 30 (*d*) 34

5. If , are the roots of the equation $ax^2 + bx + c = 0$, then what is the value of $(a + b)^{-1} + (a + b)^{-1}$?

(*i*) a/bc (*ii*) b/ac (*iii*) $-b/ac$ (*iv*) $-a/bc$

6. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then value of $2 + q - p$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0

[Hint. $\frac{\tan 30 \quad \tan 15}{1 \quad \tan 30 \quad \tan 15}$ tan (30 15) tan 45 $\frac{p}{1 \quad q}$ 1**]**

7. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval

(*a*) [5, 6] (*b*) (6,) (*c*) (– , 4) (*d*) [4, 5]

[Hint. Sum of the roots < 10 $2k < 10$ $k < 5$.

Solving, $x \quad k \quad \sqrt{5 \quad k}, \ k \quad \sqrt{5 \quad k}$

Since each root is less than 5, $k \quad \sqrt{5 \quad k} \quad 5 \quad \sqrt{5 \quad k} \quad 5 \quad k$

Since $k < 5,\ 5 - k < 25 - 10\,k + k^2 \quad k^2 - 9k + 20 > 0$

$$(k-4)\,(k-5) > 0 \quad k \in (- \quad , 4).]$$

8. If and are the roots of $ax^2 + bx + c = 0$ and if $px^2 + qx + r = 0$ has roots $\frac{1}{\quad}$ and $\frac{1}{\quad}$ then $r =$ (1) $a + 2b$ (2) $a + b + c$ (3) $ab + bc + ca$ (4) abc

[Hint. $\frac{b}{a}$, $\frac{c}{a}$. ....(i)

The equation whose roots are $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ is

$$x^2 \quad \frac{1}{\quad} \quad \frac{1}{\quad} \quad x \quad \frac{1}{\quad} . \frac{1}{\quad} \quad 0.$$

$cx^2 + (b + 2c)\,x + (a + b + c) = 0$ [using (i)]

Comparing with $px^2 + qx + r = 0$, we get $r = a + b + c$.]

9. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then the common root is

(a) 1 (b) 4 (c) 3 (d) 2

[Hint. Let the roots of the equation $x^2 - 6x + a = 0$ be and 4 and those of the equation $x^2 - cx + 6 = 0$ be and 3 . Then

$$\left.\begin{array}{l}\alpha + 4\beta = 6 \quad ...(1)\\ 4\alpha\beta = a \quad ...(2)\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l}\alpha + 3\beta = c \quad ...(3)\\ 3\alpha\beta = 6 \quad ...(4)\end{array}\right\}$$

(2) and (4) $a =$ The eqn. becomes $x^2 - 6x +\ \ = 0 \quad (x-2)\,(x-4) = 0$

The roots are 2 and 4 $= 2, \ = 1$

Hence, the common root is 2.]

ANSWERS

1. (iii)	**2.** (d)	**3.** (ii)	**4.** (d)	**5.** (ii)
6. (c)	**7.** (c)	**8.** (2)		