BHSEC MATHEMATICS

BOOK – II

FOR CLASS XII

For Class XII Students of Bhutan

O.P. MALHOTRA
(Gold Medallist)
Head of Mathematics Department (Retd.)
The Doon School, Dehradun

S.K. GUPTA
Principal (Retd.)
Birla Vidya Mandir
Nainital

ANUBHUTI GANGAL
M.A. (Gold Medallist), M.Ed.
Formerly, Senior Faculty Member
The Daly College, Indore

S. CHAND
School

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(An imprint of S Chand Publishing)
A Division of S Chand And Company Limited
(An ISO 9001 Certified Company)
Head Office : B-1/D-1, Ground Floor, Mohan Co-operative Industrial Estate, Mathura Road, New Delhi–110 044; **Phone :** 011–6667 2000, **e-mail :** info@schandpublishing.com
Registered Office : A-27, 2nd Floor, Mohan Co-operative Industrial Estate, New Delhi–110 044
www.schandpublishing.com; **e-mail :** helpdesk@schandpublishing.com

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Ranchi	:	ranchi@schandpublishing.com
Sahibabad	:	Ph: 0120-2771 238, info@schandpublishing.com

First Published 2011
Revised Edition 2013
Reprints 2012, 2014, 2016, 2017, 2018, 2019, 2020

ISBN : 978-81-21935-79-1 **Product Code :** SCS3BHM120MATAB13SBN

PRINTED IN INDIA
By Vikas Publishing House Private Limited, Plot 20/4, Site-IV, Industrial Area Sahibabad, Ghaziabad–201 010
and Published by S Chand And Company Limited, A-27, 2nd Floor, Mohan Co-operative Industrial Estate,
New Delhi–110 044.

PREFACE

We feel happy in presenting the revised version of our immensely popular book **ISC Mathematics** strictly in accordance with the *Mathematics syllabus for class 12 for 2011 and after* released by the Curriculum and Professional Support Division (CAPSD) of the Ministry of Education, Royal Government of Bhutan, as part of its Bhutan Higher Secondary Education Certificate (BHSEC) course.

The special features of this book are:

1. **It follows the prescribed syllabus strictly** and incorporates the latest trends in the teaching of Mathematics.
2. For the convenience of the teachers and students the detailed syllabus has been given right in the beginning.
3. The chapters are in the same sequence as given in the syllabus to facilitate teaching in the class and coverage of the syllabus.
4. The development is logical, and the preparation of each new idea is based on the preceding material.
5. Great pains have been taken to present the subject matter in a very easy to understand manner. To achieve this, the authors had sometimes to sacrifice brevity and give detailed explanation to bring home to the students the finer points of every topic. A sincere effort has been made to explain the '**How and Why**' of every concept to make the fundamentals clear. The authors are of the view that a textbook is not just a collection of formulae and questions but much more than this. A textbook should help in making an in-depth study of the subject and lay solid and sound foundation for further study.
6. The clearly development textual explanations are followed by appropriate solved **examples** which are **large in number** and include almost all types of questions possible on a particular topic or concept.
7. Effort has been made to include **quality questions** in exercises for practice, keeping in mind the latest trend and style of questions. The questions are ample in number and well-graded and would cater to the needs of all types of students—***average***, ***above average*** and ***brilliant***. **Hints have been provided to difficult and tricky questions** so that the student does not get stuck up and is able to maintain his pace.
8. Revision exercises containing **multi-choice questions** will, we hope expose the students to a variety of problems, requiring intelligent approach and help them in acquainting themselves with the latest trends and getting firm grasp of the fundamentals and thorough knowledge of different topics.
9. A sincere effort has been made to maintain **Mathematical accuracy and rigour.**
10. Historical notes have been interspersed throughout the text.

For proper feedback as per the requirement of the BHSEC syllabus, the authors are thankful to Mr. Karma Yeshey and Mr. Geewanath Sharma, Curriculum Officers, CAPSD, Ministry of Education, Bhutan.

Feedback and suggestions for further improvement would be most welcome.

AUTHORS

SYLLABUS FOR CLASS XII (2011 Onwards)

The syllabus for both the Pure Mathematics course and the Business Mathematics course are as given below. It will be noticed that while the students taking the Business Mathematics will study comparatively less content under certain units like Calculus and Coordinate Geometry, they will study an additional unit called Commercial Mathematics. The Pure Mathematics students will study two additional units called Trigonometry and Complex Numbers. The commonalties and the differences of the contents between the Pure Mathematics and Business Mathematics are clearly indicated below. A good estimate of the expected times that should be spent in the formal teaching of each topic is given in hours with the topics.

UNIT 1 – ALGEBRA	Pure Mathematics	Business Mathematics
1. Permutations and Combinations (12 hrs) ▪ Factorial Notation ▪ Concept of Permutation (^nP_r): Permutation of alike things; restricted permutation; circular permutations ▪ Concept of Combination (^nC_r): Restricted combinations; Distribution of different things into groups; Open selection of items from different things and from alike things ▪ Mixed problems on permutations and combinations **(Note: problems should be fairly simple ones)**	All	All
2. Determinants and Matrices **Determinants: (8 hrs)** ▪ Of order 2 and 3 ▪ Minors and Co-factors of a determinant ▪ Expansion of a determinant ▪ Properties of a determinant and their use in the evaluation of a determinant ▪ Product of determinants (without proof); ▪ Solution of simultaneous equations in 2 or 3 variables using Cramer's rule ▪ Conditions for consistency of 3 equations in two variables **Matrices: (6 hrs)** ▪ Of order $m \times n$, where $m, n \leq 3$, including case $m = n$; Types of matrices ▪ Operations: Addition/Subtraction (Compatibility); Multiplication by a scalar; Multiplication of two matrices (Compatibility) ▪ Application of matrix multiplications ▪ Adjoint and inverse of a matrix ▪ Use of matrices to solve simultaneous linear equations in 2 or 3 unknowns	All	All

UNIT 2 – TRIGONOMETRY	Pure Mathematics	Business Mathematics
1. Inverse Trigonometric Functions (10 hrs) ▪ Meaning of inverse trigonometric functions ($\sin^{-1}x$, $\cot^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}x$) ▪ Principal values (use of graphs in explanation) ▪ Properties of inverse trigonometric functions (without proof)	All	This unit is NOT for B/Maths
UNIT 3 – CALCULUS	**Pure Mathematics**	**Business Mathematics**
1. Differential Calculus (20 hrs) ▪ Revision of the topics done in class XI ▪ Derivatives of trigonometric, logarithmic, and exponential functions ▪ Derivatives of composite, absolute value, implicit and parametric functions ▪ Interchange of independent & dependent variable ▪ Differentiating function with respect to another function ▪ Logarithmic differentiation ▪ Successive differentiation up to 2nd order ▪ Maxima and minima ▪ Application of maxima and minima to practical problems ▪ *Derivatives of inverse trigonometric functions reducible to simple form by substitution*	All	All EXCEPT (i) Derivative of *Inverse trigonometric* function is NOT included for B/Maths (ii) The application of maxima & minima in Mensuration
2. Integral Calculus (20 hrs) ▪ Revision of formula of integration form class XI ▪ Standard method of integration of $1/x$, e^x, $\tan x$, $\cot x$, $\operatorname{cosec} x$, $(ax+b)^n$, where $n \in Q$ ▪ Integration using substitution ▪ Integration by using partial fractions ▪ Integration by parts ▪ Integrals of the $\sin^2 x\,dx$, $\sin^3 x\,dx$, $\cos^2 x\,dx$, $\cos^3 x\,dx$, $f'(x)[f(x)]^n\,dx$ – ▪ Definite integral – as a limit of sum (only algebraic functions) ▪ Properties of Definite Integrals (without proof) ▪ Application of definite integrals – area of a curve included between x or y axis, volume of revolution about the x-axis or y-axis or about a line	All	All EXCEPT, the portion on *Definite integrals*, *Properties of definite integrals* and *Applications of definite integrals* are NOT included for B/Maths
3. Differential Equations (10 hrs) ▪ Meaning of differential equations; order and degree of a differential equation ▪ Solution of differential equation of 1st order and 1st degree	All	This chapter is NOT for B/Maths

	▪ Variable separable ▪ Homogenous equations and equations reducible to homogenous form; $(dy/dx) + Py = Q$, where P and Q are function of x only ▪ Solution of differential equations of second order; $(d^2y/d^2x) = f(x)$		
	UNIT 4 – COORDINATE GEOMETRY	**Pure Mathematics**	**Business Mathematics**
1.	**Pairs of Straight Lines** **(10 hrs)** ▪ General equation of a family of lines passing through the intersection of two lines L_1 and L_2: $L_1 + kL_2 = 0$, $k \in R$; finding k using additional condition ▪ General equation of second degree in x and y representing a pair of lines ▪ Conditions for general second degree equation to represent a pair of staight line ▪ Reduction of general equation to individual equation ▪ Conditions for perpendicularity and parallel of the two lines ▪ Point of intersection and angle between two lines represented by a second degree equation in x and y ▪ Equation of the bisector of the angle between a pair of given straight lines	All	This chapter is NOT for B/Maths
2.	**Conics** **(15 hrs)** ▪ As a section of a cone ▪ Definition and understanding of Foci, Directrix, Latus Rectum ▪ Recognition and Equation of a Parabola, Ellipse and hyperbola in standard form ▪ Finding the equation for a conic when focus, directrix, and eccentricity or ralted data are given ▪ Finding basic information like foci, directrix, etc from a given equation.	All	All
3.	**Points and their coordinates in 3 dimensions** **(10 hrs)** Distance between two points; section and mid-point formulas; direction cosines and direction ratios of a line; angle between two lines; conditions of line to be parallel or perpendicular	All	All
4.	**Plane** **(10 hrs)** ▪ General equation of a plane, as $ax + by + cz + d = 0$, where a, b, c are direction ratios of the normal to the plane ▪ Equation of a plane: One point form; Normal form; Intercept form ▪ Distance of point from a plane	All	This chapter is NOT for B/Maths

		Pure Mathematics	Business Mathematics
	▪ Angle between two planes, and angle between a line and a plane ▪ Equation of plane though the intersection of two planes ▪ Finding the equation of a plane given a point and direction cosine/ratios of the normal and other sufficient data		
	UNIT 5 – DATA AND PROBABILITY	**Pure Mathematics**	**Business Mathematics**
1.	**Measures of Dispersion (4 hrs)** ▪ Meaning of dispersion; quartile deviation; standard deviation, coefficient of variation; Mean deviation from the mean or median ▪ Combined mean and standard deviation of two groups only	All	All
2.	**Correlation and Regression (15 hrs)** ▪ Definition and meaning of correlations coefficient ▪ Use of scatter diagram and Line of best fit ▪ Calculation of coefficient of correlation by Karl Pearson's method for ungroup data ▪ Calculation of rank correlation coefficient by Spearman's method (for both repeating and non-repeating ranks) ▪ Calculation of regression; coefficient and the two lines of regression by the method of least squares; use of lines of regression for prediction	All	All
3.	**Probability (15 hrs)** ▪ Random experiment and their outcomes ▪ Events: sure events, impossible events, mutually exclusive events, independent and dependent events ▪ Definition of probability of an event ▪ Laws of probability: addition and multiplication laws; conditional probability	All	All
	UNIT 6 – COMMERCIAL MATHEMATICS	**Pure Mathematics**	**Business Mathematics**
1.	**Annuities (10 hrs)** ▪ Meaning, Present Value, Annuity Certain, Contingent Annuity, Perpetual Annuity, Immediate Annuity, Annuity Due, PV of Immediate and Perpetual Annuity	This unit is NOT for P/Maths	All

2.	**Application of Derivatives in Commerce and Economics (10 hrs)** ▪ Cost Function ▪ Average cost ▪ Marginal cost ▪ Revenue function and break-even point	This unit is NOT for P/Maths	All
	UNIT 7 – COMPLEX NUMBERS	**Pure Mathematics**	**Business Mathematics**
	(15 hrs) ▪ Meaning and as an ordered pair of real number in the form $a + bi$ ▪ Geometrical representation in complex plane -Argand diagram for z (a complex number), $1/z$, z and $\bar{z}$; equality of two complex numbers; absolute value (modulus), properties (without proof) ▪ Argument (conjugate of complex numbers), polar form ▪ Operations: Sum/Difference, product and quotient of two complex numbers; additive and multiplicative inverse of a complex number ▪ Simple locus equation on complex numbers; proving using $\bar{z}.z = \|z\|^2$ and $z_1 + z_2 = z_1 \pm z_2$ ▪ Square root of a complex number ▪ De Moivre's theorem and its application ▪ Cube root of unity: $1, \omega, \omega^2$	All	This unit is NOT for B/Maths

ASSESSMENT

The final assessment for class 12, which will determine the students' result, will be 100% external examination conducted by the Bhutan Board of Examination (BBE) at the end of the academic session. The BBE examination format will be as per the specification provided herein for the trial examination.

However, for the purposes of assessing the students' learning process and progress, and for school's internal records, the schools must conduct their assessments on class 12 students based on the following structure, till the trial examinations, which is similar to that of class XI.

Mode of summative assessment
Class XII

Year beginning – Mid year 40%

- **Class Participation 5%**
- **Assignments 5%**
- **Units Tests 5%**
- **Mid-term Exam 25%**

Mid year – Year end 60%

- **Class Participation 5%**
- **Assignments 5%**
- **Units Tests 5%**
- **Trial Exam 45%**

A brief rationale on each of the components of the assessment above follows:

Year beginning to Mid year

Class Participation: Student's active involvement in the class is important for his/her learning. Class participation would consist of student's positive attitude and behaviours towards learning: his/her ability to follow instructions, cooperation displayed in doing group works, confidence in asking questions and answering the questions asked, etc to mention a few. Teacher should develop criteria to assess students for the class participation. A better alternative would be to work out the criteria with the students in the beginning of the year. It is important that the students know the criteria and are reminded of them from time to time. This would force the students to be active, cooperative, critical thinkers and confident communicators in the class. This would also force the teachers to drive students towards these qualities. These are desirable and healthy disposition we would want in our children. Whatever reasonable assessment tools and marking scheme the teacher has chosen to use for the class participation up to the mid term should be worked out to be worth 5% of the whole year assessment, for entering into the student progress Report Form.

Assignment: Reasonable amounts of assignment, which we normally called home works, should be assigned quite regularly. More importantly, they should be checked, and prompt feedback provided to the students on their works. The teacher will award marks at least two times to each student's homework during the first half term of the year; they can devise their own marking scheme. The average mark from the total should be worked out to be worth 5% for entering onto the students' Progress Report card.

Unit Tests: A unit test should be conducted at the end of teaching a unit. It should be carried out during one of the class periods. The teacher should keep proper record of the students' achievement in the series of unit tests. A minimum of two unit tests should be conducted before the mid-term exams. The total marks obtained in the unit tests should be worked out to be worth 5% for entering onto the student's Progress Report Card.

Mid-term Examination: The mid-term examination may be based on the specifications provided for the Trial examination/Board examination as below. The mark obtained in it should be brought down to 25% for entering into the Progress Report Card.

Mid year to Year end

Class Participation: To be done similarly as during the first term of the year.

Assignments: To be done similarly as during the first term of the year.

Unit Tests: To be done similarly as during the first half term of the year, but with the units covered after the mid-term examination.

Trial Examination/Board Examination: The annual examination paper will be set for 100 marks, with writing time of **Three hours**. The paper will consist of two sections:

- **Section A** will be composed of 15 multiple choice questions, covering the entire syllabus. Each MCQ will carry 2 marks, making the section worth 30 marks in total. Each MCQ should have one Key/Correct Answer and three distracters.
- **Section B** will be made up of about 13 open answer type questions set from the entire syllabus, out of which the student will have to attempt 10 questions. Each question will carry 7 marks, making the section worth 70 marks in total.

NOTE:

1. For Pure Mathematics, the weighting accorded for each of the units for the annual examination is as given below:

	UNITS	% MARKS
1.	Algebra	15%
2.	Trigonometry	7%
3.	Calculus	30%
4.	Coordinate Geometry	25%
5.	Data and Probability	15%
6.	Complex Number	8%
	Total	**100%**

2. For the Business Mathematics the weighting accorded for each of the units for the annual examination is as given below:

	UNITS	% MARKS
1.	Algebra	20%
2.	Calculus	25%
3.	Coordinate Geometry	15%
4.	Data and Probability	20%
5.	Commercial Mathematics	20%
	Total	**100%**

3. Care should also be taken in the preparation of questions ons having a balance of them requiring conceptual understanding, problem solving, communication, reasoning, and applications of procedural knowledge and skills. Some questions should cross strands or units. Along with these, test blue print based on Blooms Taxonomy would also be needed to be used in the preparation of the paper.
4. The marks obtained out of 100 in this examination should be worked out to be worth 45% for entering in to the student' progress report card.

CONTENTS

UNIT 1
ALGEBRA

- Permutations and Combinations
- Determinants
- Matrices

PERMUTATIONS AND COMBINATIONS

Syllabus

- Factorial notation
- Concept of permutation (^nP_r); permutation of alike things; restricted permutation; circular permutations.
- Concept of combination (^nC_r); restricted combinations; distribution of different things into groups; open selection of items from different things and from alike things.
- Mixed, problems on permutations and combinations.

(**Note.** Problems should be fairly simple ones)

HISTORICAL NOTE

In India, Jains were acquainted with concepts of permutations and combinations under the name Vikalpa. In the Vedic period we find the computations of the number of ways in which the poetic rhythms of verses can be altered. Mahavira is the world's first mathematician who provided the general formulae for permutations and combinations. Bhaskara treated the subject matter of permutations and combinations under the name Anka Pasha in his famous work Lilavati. In addition to general formulae for ${}^nC_r$ and ${}^nP_r$, already provided by Mahavira, Bhaskara gave many important theorems and results concerning the subject. The first book touching on the subject was Pacioli's Suma (1494) wherein is discussed the problem of how many ways a group of individuals can sit around a table. The first treatise on the subject, however, did not appear until 1713, it was Bernoulli's Ars Conjectandi.

1

Permutations and Combinations

1.01. Introduction

Suppose we have 40 books of different subjects and a shelf which can hold only 20 books. We have to choose 20 books out of these 40 books and then arrange them on the shelf. We can take any 20 books we like and then arrange them on the shelf in any manner we like. Thus, the process of arranging 20 books out of 40 books involves the following two different processes :

1. First, we select 20 books out of 40 books. Suppose these are Algebra, Trigonometry, Calculus, Geometry, etc. It is our sweet will whether we select a particular book or not.

2. After having selected 20 books out of 40 books, we proceed to arrange them on the shelf. We can arrange them on the shelf in any manner we like, for example, it is up to us whether we put a book on trigonometry or physics in the first place.

The above discussion shows that while considering the alternatives of things or acts, we come across two types of problems :

(*a*) Selection, (*b*) Arrangement.

The process of selecting things is called **combination** and that of arranging things is called **permutation.**

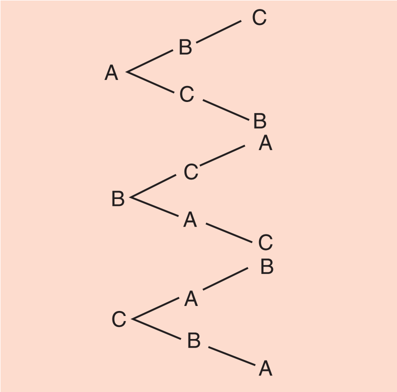
Difference between a Permutation and a Combination

Combination	***Permutation***
1. Concerns only with selection. 2. Ordering of the selected items is immaterial.	1. Concerns selection as well as arrangement. 2. Ordering is essential.

Thus, if we have 4 objects *A*, *B*, *C* and *D* the possible selections (or combinations) and arrangements (or permutations) of 3 objects out of 4 are given below. This will help you to understand clearly the difference between permutation and combination, clearly.

Selection ↓ ***Combination***	***Arrangement*** ↓ ***Permutation***
ABC	**ABC, ACB, BAC, BCA, CAB, CBA*
ABD	*ABD, ADB, BAD, BDA, DAB, DBA*
ACD	*ACD, ADC, CAD, CDA, DAC, DCA*
BCD	*BCD, BDC, CBD, CDB, DBC, DCB*
Total 4 combinations	24 permutations

***Note.** The possible arrangements corresponding to the selection of *A*, *B* and *C* can be easily written with the help of a tree diagram shown below:



A few problems relating to combination are :

1. Formation of a team from a number of players.
2. Formation of a particular committee from a number of members.

A few problems relating to permutation are :

1. Arrangements of books on a shelf.
2. Formation of numbers with the given digits.
3. Formation of words with the given letters.

1.02. Fundamental principle of counting

To discover the fundamental principle of counting, study the following examples:

1. Suppose you have 3 full-sleeve and 4 half-sleeve shirts. Since you have the choice of wearing any of these shirts, you can wear one shirt in $3 + 4 = 7$ ways. If in addition, you have 5 T-shirts, then you can wear one shirt in $3 + 4 + 5 = 12$ ways.

The above example illustrates one way of counting, which we may call the sum rule and applies when **one event** has to happen out of given disjoint events.

Here, if wearing a full-sleeve shirt is the event *A* and wearing a half-sleeve shirt is the event *B*, then *A* can happen in 3 ways and *B* can happen in 4 ways. Also, $A \cap B = \phi$, *i.e.*, *A* and *B* can't happen together. We also say that the two events are exclusive.

Now, suppose you have 3 shirts and 4 pairs of pants. In how many possible ways can you dress up by wearing a shirt and a pair of pants?

In the above case, you can wear any of the 3 shirts and after wearing one of these shirts any of these pairs of pants with it. If we label the shirts as S_1, S_2, S_3 and the pants as P_1, P_2, P_3 and P_4, then the different ways of dressing up can be as under :

$S_1 P_1$	$S_2 P_1$	$S_3 P_1$
$S_1 P_2$	$S_2 P_2$	$S_3 P_2$
$S_1 P_3$	$S_2 P_3$	$S_3 P_3$
$S_1 P_4$	$S_2 P_4$	$S_3 P_4$

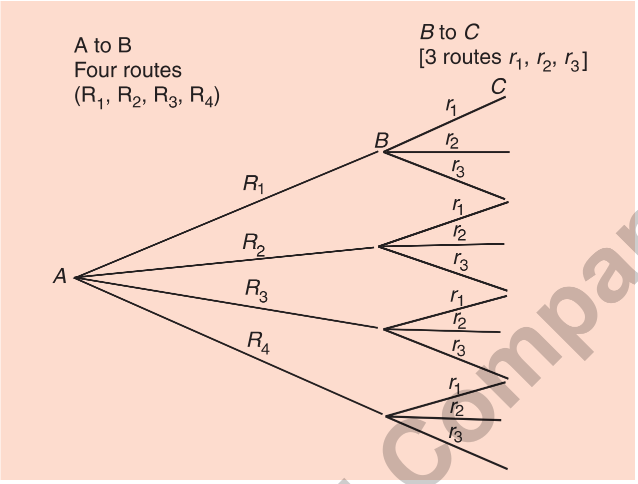
Total number of ways $= 12 = 3 \times 4$.

In this illustration, we **multiply** the number of ways in which you can wear a shirt and the number of ways in which you can wear a pair of pants.

2. There are 4 different routes between cities A and B, and 3 different routes between cities B and C. How many different routes are there from city A to city C through city B?

Discussion: Obviously, one can go from city A to city B by any of the 4 routes, *i.e.*, in 4 ways. After having gone to B by any of the different 4 routes, one can go to city C by any of the three routes. Thus, corresponding to one route taken, from A to B, he has 3 choices from B to C. Therefore, corresponding to 4 routes, there are 12 choices in all.

Therefore, he can go from A to C via B, in $4 \times 3 = 12$ ways as depicted in the following tree diagram.



The possible routes taken are :

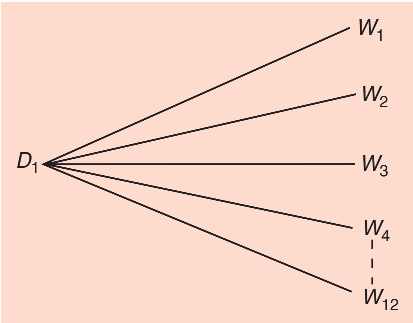
$R_1 r_1$	$R_2 r_1$	$R_3 r_1$	$R_4 r_1$
$R_1 r_2$	$R_2 r_2$	$R_3 r_2$	$R_4 r_2$
$R_1 r_3$	$R_2 r_3$	$R_3 r_3$	$R_4 r_3$

Total number of ways = 12.

3. (*a*) A house has 5 doors and 12 windows. In how many ways can a person run out of the house during emergency through a door or a window?

(*b*) In how many ways can a person enter the house through a door and exit through a window?

Discussion: (*a*) The person has the choice of entering by any one of the windows or doors. Thus the person can enter the house in 5 + 12 = 17 ways (sum rule).



(*b*) Since the person has to enter through a door and there are 5 doors, he can do so in 5 ways. After having entered through one of the doors, say, D_1, he can exit by any of the 12 windows, say, W_1, W_2, W_3, ... , and W_{12}. So, corresponding to entry through a particular door, say, D_1, there are 12 ways of exit as shown below:

D_1W_1, D_1W_2, D_1W_3, ..., and $D_1 W_{12}$.

Since, for each of the 5 doors D_1, D_2, D_3, D_4, D_5, ... , there are 12 ways of exit, so in all he can exit in $5 \times 12 = 60$ ways.

The examples discussed above illustrate the use of a general principle, called the product rule or the fundamental principle of counting , which is stated below.

1.03. A fundamental principle

If one operation can be performed in m ways, and if corresponding to each of the m ways of performing this operation, there are n ways of performing a second operation, then the number of ways of performing the two operations together is m × n. (This AND That).

Suppose that the first operation is performed in any one of the m ways, the second operation can then be performed in n ways and with the particular first operation, we can associate any one of the n ways of performing the second operation. This means that if the first operation could have been performed only in this one way, there would have been $1 \times n$, *i.e.*, n ways of performing both the operations. But it is given that the first operation can be performed in m ways and there are n ways of performing the second operation for every one way of performing the first operation. Therefore, there are $m \times n$ ways of performing both the operations.

Generalisation. The above principle can be extended to the case in which the different operations can be performed in $m, n, p,$......ways. In this case the number of ways of performing all the operations together would be $m \times n \times p$ ....

Ex. 1. There are 10 buses running between two towns *X* and *Y*. In how many ways can a man go from *X* to *Y* and return by a different bus?

Sol. The man can go from X *to* Y in 10 ways and as he is not to return by the same bus that he took while going, corresponding to each of the 10 ways of going, there are 9 ways of returning. Hence the total number of ways in which he can go to Y and be back is $10 \times 9 = 90$.

Ex. 2. How many different numbers of three digits can be formed with the digits 1, 2, 3, 4, 5, no digit is being repeated?

Sol. The unit's place can be filled with either of these 5 digits and so the unit's place can be filled in 5 ways. The ten's place can be filled in 4 ways corresponding to each way of filling up the unit's place, for we can have any digit here except the one used in the unit's place. Similarly, the hundredth's place can be filled in 3 ways as here we have any of the remaining three digits. Therefore, there are $5 \times 4 \times 3 = 60$ ways of forming a number of three digits with the five given digits.

Ex. 3. Each section in first year of plus two course has exactly 30 students. If there are 3 sections, in how many ways can a set of 3 student representatives be selected from each section?

Sol. 1st representative can be selected from first section in 30 ways.

2nd representative can be selected from second section in 30 ways.

3rd representative can be selected from third section in 30 ways.

$\therefore$ Required number of ways $= 30 \times 30 \times 30 = 27000$.

Ex. 4. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

Sol. Any number between 100 and 1000 is of 3 digits. The unit's place can be filled by 2 or 9 in 2 ways.

Similarly ten's place can be filled in 2 ways.

The hundred's place can also be filled in 2 ways.

$\therefore$ Required no. of numbers $= 2 \times 2 \times 2 = 8$.

Ex. 5. How many odd numbers less than 1000 can be formed using the digits 0, 2, 5, 7 when repetition of digits are allowed?

Sol. Since the required numbers are less than 1000 therefore, they are 1-digit, 2-digit or 3-digit numbers.

One-digit numbers. Only two odd one-digit numbers are possible, namely, 5 and 7.

Two-digit numbers. For two-digit odd numbers the unit place can be filled up by 5 or 7 *i.e.* in two ways and ten's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 2-digit odd numbers $= 2 \times 3 = 6$.

Three-digit numbers. For three-digit odd numbers, the unit place can be filled up by 5 or 7 in 2 ways. The ten's place can be filled up by any one of the digits 0, 2, 5, 7 in 4 ways. The hundred's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

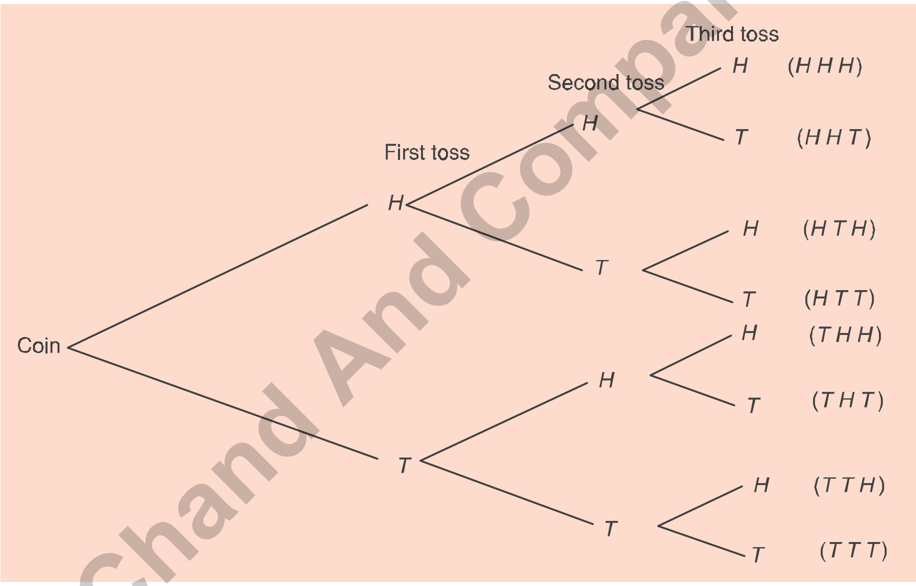
$\therefore$ No. of possible 3-digit numbers $= 2 \times 4 \times 3 = 24$

Hence the total no. of possible odd numbers $= 2 + 6 + 24 = 32$.

Ex. 6. A coin is tossed three times and outcomes are recorded. Use the product rule to determine the number of possible outcomes. Then list all the outcomes.

Sol. For each toss of coin we have 2 choices– a head (H) or a tail (T). Therefore, by the product rule, the number of possible outcomes of three tosses is $2 \times 2 \times 2 = 8$.

For a listing of these outcomes, it is convenient to draw a tree diagram



Thus, the possible outcomes are

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.$$

Note. Can you guess how many possible outcomes would be if the coin is tossed four times? Five times? n times?

Arranging in the same manner as above, and applying the product rule, we find that

For 4 tosses, the number of possible outcomes are $2 \times 2 \times 2 \times 2 = 2^4 = 16$

For 5 tosses, the number of possible outcomes are $2^5 = 32$

For n tosses, the number of possible outcomes are 2^n.

EXERCISE 1 (*a*)

1. Two persons go in a railway carriage where there are 6 vacant seats. In how many different ways can they seat themselves ?

2. In how many ways can 2 prizes be awarded to 9 contestants provided no contestant gets both the prizes ?

3. There are three mathematics teachers in a college in which there are 6 classes. In how many different ways can they choose the classes, provided one teaches one class only ?

4. How many words (with or without meaning) of three distinct letters of the English alphabets are there?

5. How many numbers are there between 100 and 1000 such that 7 is in the units place ?

6. How many integers of four digits each can be formed with the digits 0, 1, 3, 5, 6 (assuming no repetitions)?

7. How many automobile licence plates can be made if the inscription on each contains two different letters followed by three different digits ?

[Hint. There are 26 letters and 10 digits out of which the inscriptions are to be made. Also, the digit 0 cannot be used at the hundred's place.]

8. Find the number of ways of arranging 6 players to throw the cricket ball so that the oldest player may not throw first.

9. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6 ?

10. Find the number of even positive integers which have three digits.

11. How many 2-digit numbers can be formed from the digits 8, 1, 3, 5 and 4 assuming

(*a*) repetition of digits is allowed?

(*b*) repetition of digits is not allowed?

12. How many four-digit even integers can be formed using the digits 0, 1, 2, 3, 4, 5?

13. To pass an examination a student has to pass in each of the 3 papers. In how many ways can a student fail in the examination?

[Hint. For each of the 3 papers, there are two choices–pass (*P*) or fail (*F*). By product rule, there are $2 \times 2 \times 2 = 8$ choices. But he will pass only if he passes in all the papers (*PPP*).]

14. How many seven-digit phone numbers are possible if 0 and 1 cannot be used as the first digit and the first three digits cannot be 555, 411, or 936?.

[Hint. First three digits can be filled in $(8 \times 10 \times 10 - 3)$ ways. Last four digits can be filled in $(10 \times 10 \times 10 \times 10)$ ways.]

15. There are five routes for a journey from station *A* to station *B*. In how many different ways can a man go from *A* to *B* and return, if for returning

(*i*) any of the routes is taken, (*ii*) the same route is taken,

(*iii*) the same route is not taken ?

16. How many 9-digit numbers of different digits can be formed ?

ANSWERS

1. 30 **2.** 72 **3.** 120 **4.** 15600 **5.** 90

6. 96 **7.** Number of licence plates $= 26 \times 25 \times 9 \times 9 \times 8 =$ 4,21,200

8. 600 **[Hint.** For first place 5 players (excluding the oldest) and for the remaining places 5 (including the oldest) players are available $\therefore$ The no. of ways $= 5 \times 5 \times 4 \times 3 \times 2 \times 1$].

9. 64 **10.** 450 **11.** (*a*) 25, (*b*) 20 **12.** 540

13. $2^3 - 1 = 7$ ways **14.** $[8 \times 10 \times 10 - 3] \times 10 \times 10 \times 10 \times 10 = 7970000$

15. (*i*) $5 \times 5 = 25$, (*ii*) $5 \times 1 = 5$, (*iii*) $5 \times 4 = 20$

16. $9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3265920$

1.04. Permutations

Def. ***Each of the different arrangements which can be made by taking some or all of a number of things at a time is called a permutation.***

Notation. The number of permutations of n things taken r at a time is denoted by $^{n}P_{r}$ ***or*** $P(n, r)$. The letter P is an abbreviation of the word 'permutation'.

Thus $^{6}P_{4}$ denotes the number of permutations or arrangements of 6 things taken 4 at a time.

1.05. The value of $^{n}P_{r}$

To find the number of permutations of n different things, taken r at a time or to determine $^{n}P_{r}$.

The number of permutations of n things taken r at a time will be the same as the number of ways in which r blank places can be filled up with n given things.

As the first place can be filled in by any one of the n things so there are n ways of filling up the first place.

After having filled in the first place by any one of the n things, there are $(n – 1)$ things left. Hence the second place can be filled in $(n – 1)$ ways. Now, as for every one way of filling up the first place, there are $(n – 1)$ ways of filling up the second place, so the first two places can be filled in $n\,(n – 1)$ ways.

After having filled in the first two places in any one of the above ways, there are $(n – 2)$ things left and so the third place can be filled in $(n – 2)$ ways. Now for every one way of filling up the first two places, there are $(n – 2)$ ways of filling up the third place and so the first three places can be filled up in $n\,(n – 1)\,(n – 2)$ ways.

It may be observed that

(a) At every stage the number of factors is equal to the number of places filled up.

(b) Every factor is by one less than its preceding factor.

Thus, we can conclude that the first $(r – 1)$ places can be filled in $n(n – 1)(n – 2)....\{n–(r – 2)\}$ ways. After filling up first $(r – 1)$ places the rth place can be filled in $n(n – 1)(n – 2)....\{n – (r – 1)\}$ ways.

Position of the object	1st	2nd	...	$(r – 1)$th	rth
Number of ways	n	$n – 1$	...	$n – (r – 2)$	$n – (r – 1)$

Hence the number of ways of filling up all the r places, *i.e.*, the number of permutations of n different things taken r at a time is $n\,(n – 1)\,(n – 2)......r$ factors

$$= n(n-1)(n-2)......(n-\overline{r-1})$$

Hence $$^{n}P_{r} = n\,(n – 1)\,(n – 2)\,......\,(n – r + 1)$$

Thus, $^{7}P_{2} = 7 \times 6$; $^{10}P_{4} = 10 \times 9 \times 8 \times 7$, $^{20}P_{3} = 20 \times 19 \times 18$.

Cor. The number of permutations of n things taken all at a time is

$$^{n}P_{n} = n\,(n – 1)\,(n – 2)\,......\,3.\,2.\,1.$$ [Putting n for r]

Ex. 7. In how many ways can 5 persons occupy 3 vacant seats?

Sol. Total number of ways = $^{5}P_{3} = 5 \times 4 \times 3 = 60$.

Ex. 8. If $^{12}P_{r} = 1320$, find r.

Sol. $^{12}P_{r} = 12 \times 11 \times ......$ to r factors $= 1320 = 12 \times 11 \times 10$ $\therefore$ $r = 3$.

1.06. Factorial notation

The product of n natural numbers from 1 to n is denoted by n ! or $|\underline{n}$ and is read as factorial n.

Thus, $n!$ *or* $|\underline{n} = 1.\,2.\,3 \ldots\ldots (n-1)\,.\,n$

$4! = 1\times 2\times 3\times 4 = 24;\ 6! = 1\times 2\times 3\times 4\times 5\times 6 = 720$

$(n-1)! = 1\times 2\times 3 \ldots\ldots (n-1).$

It is easily seen that $8! = 8\times(7!)$ or $7! = 7\times(6!)$ or $6! = 6\times(5!)$, etc.,

In general, $\quad \boldsymbol{n! = n\,(n-1)!}$

taken out
$n! = n\,(n-1)!$
One less

1.07. Values of ^nP_r in terms of factorial notation

$$^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n(n-1)\ldots\ldots(n-r+1)}{|\underline{(n-r)}}\cdot |\underline{(n-r)}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)(n-r-1)\ldots 3.2.1}{|\underline{(n-r)}}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots 3.2.1}{|\underline{(n-r)}} = \frac{|\underline{n}}{|\underline{(n-r)}}$$

$$\therefore \qquad ^np_r = \frac{|\underline{n}}{|\underline{(n-r)}}$$

$$\text{Thus,} \qquad ^{18}P_5 = \frac{|\underline{18}}{|\underline{13}}$$

Cor. 1. **Putting $r = 0$, $^nP_0 = \dfrac{|\underline{n}}{|\underline{n}} = 1$**

Cor. 2. **Value of 0 !**

Putting $r = n$, $^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$; But $^nP_n = n!$ $\therefore$ $n! = \dfrac{n!}{0!}$ [Art.1.05, Cor.] $\therefore$ **0 ! = 1**

Note. In fact, 0 ! is meaningless but in order to avoid contradiction in the results, we suppose that 0 ! = 1.

Ex. 9. **Find the value of n if $^nP_{13} : {}^{n+1}P_{12} = \dfrac{3}{4}$.**

Sol. Here, $^nP_{13} = \dfrac{n!}{(n-13)!}$ and $^{n+1}P_{12} = \dfrac{(n+1)!}{(n-11)!}$

$$^nP_{13} : {}^{n+1}P_{12} = \frac{n!}{(n-13)!}\times\frac{(n-11)!}{(n+1)!} = \frac{3}{4} \Rightarrow \frac{n!}{(n-13)!}\times\frac{(n-11)\,(n-12)\cdot(n-13)!}{(n+1).n!} = \frac{3}{4}$$

i.e., $\dfrac{(n-11)(n-12)}{(n+1)} = \dfrac{3}{4} \Rightarrow 4n^2 - 95n + 525 = 0$ or $(n-15)\,(4n-35) = 0$

$\therefore$ $n = 15$ (Rejecting the fractional value of n).

Ex. 10. Show that $^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$ where the symbols have their usual meanings.

Sol. L.H.S. $= {}^nP_r = n(n-1)(n-2)\,.........(n-r+1)$ ...(*i*)

R.H.S. $= {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

$$= (n-1)(n-2)\,.......\,(n-1-r+1) + r.(n-1)(n-2)\,......\,(n-1-\overline{r-1}+1)$$

$$= (n-1)(n-2)\,........\,(n-r+1)(n-r) + r(n-1)(n-2)\,......\,(n-r+1)$$

$$= (n-1)(n-2)\,........\,(n-r+1)(n-r+r)$$

$$= n(n-1)(n-2)\,........\,(n-r+1) = \text{LHS} \quad ...(ii)$$

From (*i*) and (*ii*), we have $^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$.

EXERCISE 1 (*b*)

1. Evaluate $\dfrac{4!}{2!2!}$
2. Give the meaning and value of the symbol in the following : **(*SC*)**

   (*a*) $^5P_2$ (*b*) $^7P_3$ (*c*) $^{10}P_4$.

Find *n* if

3. $^nP_2 = 30$
4. $^nP_4 : {}^{n-1}P_3 = 9 : 1$
5. $^{2n}P_{n+1} : {}^{2n-2}P_n = 56 : 3$
6. $^{2n}P_3 = 100.\ {}^nP_2$
7. $P(n, 6) = 3P(n, 5)$
8. $2P(n, 3) = P(n+1, 3)$
9. Find r if $5P(4, r) = 6P(5, r-1), r \geq 1$.
10. If $^{n_1+n_2}P_2 = 90$ and $^{n_1-n_2}P_2 = 30$, find the values of n_1 and n_2.

Prove that

11. $P(n, n) = 2P(n, n-2)$
12. $P(10, 3) = P(9, 3) + 3P(9, 2)$
13. $P(n, r) = (n-r+1)P(n, r-1)$
14. $P(n, n) = P(n, n-1)$
15. If $\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{x}{11!}$, find x. **(*SC*)**
16. If $\dfrac{n!}{2!(n-2)!}$ and $\dfrac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, find the value of n.

Solve for *n*.

17. $\dfrac{(2n)!}{3!(2n-3)!} : \dfrac{n!}{2!(n-2)!} = 44 : 3$
18. $(n+1)! = 56.(n-1)!$
19. Convert into factorial : 7 . 8 . 9 . 10 . 11 . 12 . 13 . 14 . 15.

ANSWERS

1. 6.	**2.** (*a*) 20, (*b*) 210, (*c*) 5040.		**3.** 6.	**4.** 9	**5.** 4.
6. 13.	**7.** 8.	**8.** 5.	**9.** 3.	**10.** $n_1 = 8, n_2 = 2$.	
15. 121.	**16.** 5.	**17.** $n = 6$.	**18.** $n = 7$.	**19.** $\dfrac{15!}{6!}$	

1.08. Restricted permutations

Type I.

Ex. 11. In how many of the permutations of 10 things taken 4 at a time will (*a*) one thing always occur, (*b*) never occur?

Sol. (*a*) Keeping aside the particular thing which will always occur, the number of permutations of 9 things taken 3 at a time is $^9P_3$. Now this particular thing can take up any one of the four places and so can be arranged in 4 ways. Hence the total number of permutations = $^9P_3 \times 4 = 9 \times 8 \times 7 \times 4 = 2016$.

(*b*) Leaving aside the particular thing which has never to occur, the number of permutations of 9 things taken 4 at a time is $^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$.

Ex. 12. In how many of the permutations of *n* things taken *r* at a time will 5 things (*i*) always occur, (*ii*) never occur ?

Sol. (*i*) Keeping aside the 5 things, the number of permutations of $(r - 5)$ things taken out of $(n - 5)$ things is $^{n-5}P_{r-5}$. Now these 5 things can be arranged in *r* places in $^rP_5$ ways. Hence, the total number of permutations is $^rP_5 \times {}^{n-5}P_{r-5}$.

(*ii*) Total number of permutations = $^{n-5}P_r = \dfrac{(n-5)!}{(n-r-5)!}$

Type II. When certain things are not to occur together.

Case I. When the number of things not occurring together is two.

Procedure

1. *Find the total number of permutations when no restriction is imposed on the manner of arrangement.*

2. *Then find the number of permutations when the two things occur together.*

3. *The difference of the two results gives the number of permutations in which the two things do not occur together.*

Ex. 13. Prove that the number of ways in which n books can be placed on a shelf when two particular books are never together is $(n - 2) \times (n - 1)$!.

Sol. Regarding the two particular books as one book, there are $(n - 1)$ books now which can be arranged in $^{n-1}P_{n-1}$, *i.e.*, $(n - 1)$! ways. Now, these two books can be arranged amongst themselves in 2 ! ways. Hence the total number of permutations in which these two books are placed together is 2 ! . $(n - 1)$!. The number of permutations of *n* books without any restriction is *n* !.

Therefore, the number of arrangements in which these two books never occur together

$$= n! - 2!\,.\,(n-1)! = n\,.\,(n-1)! - 2\,.\,(n-1)! = (n-2)\,.\,(n-1)!$$

Case II. When the number of things not occurring together is more than two.

Ex. 14. In how many ways can 6 boys and 4 girls be arranged in a straight line so that no two girls are ever together ?

Sol. The seating arrangement may be done as desired in two operations.

(*i*) First we fix the positions of 6 boys. Their positions are indicated by $B_1, B_2, \ldots, B_6$.

$$\times\; B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times$$

This can be done in 6 ! ways.

(*ii*) Now if the positions of girls are fixed at places (including those at the two ends) shown by the crosses, the four girls will never come together. In any one of these arrangements there are 7 places for 4 girls and so the girls can sit in $^7P_4$ ways.

Hence the required number of ways of seating 6 boys and 4 girls under the given condition

$= {}^7P_4 \times 6! = 7 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **604800.**

Type III. Formation of numbers with digits.

Ex. 15. Suppose the six digits 1, 2, 4, 5, 6, 7 are given to us and we have to find the total number of numbers with no repetition of digits which can be formed under different conditions,

1. There is no restriction. The number of 6-digit numbers .

$= {}^6P_6 = 6\,! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

2. Numbers in which a particular digit occupies a particular place.

Suppose we have to form numbers in which 5 always occurs in the ten's place. In this case the ten's place is fixed and the remaining five places can be filled in by the remaining 5 digits in ${}^5P_5$, *i.e.*, $5\,! = 120$ ways.

The number of numbers in which 5 occurs in the ten's place = 120.

3. Numbers divisible by a particular number. Suppose we have to form numbers which may be divisible by 2. These numbers will have 2 or 4 or 6 in the unit's place. Thus the unit's place can be filled in 3 ways. After having filled up the unit's place in any one of the above ways, the remaining five places can be filled in ${}^5P_5 = 5\,! = 120$ ways.

∴ The total number of numbers divisible by 2 = 120 × 3 = 360.

4. Numbers having particular digits in the beginning and the end. Suppose we have to form numbers which begin with 1 and end with 5. Here, the first and the last places are fixed and the remaining four places can be filled in 4 !, *i.e.*, 24 ways by the remaining four digits.

Therefore, the total number of numbers beginning with 1 and ending with 5 = 24.

Note. If the numbers could have 1 or 5 in the beginning or the end, the number would have been 2 ! . 4 !, *i.e.*, 48.

5. Numbers which are smaller than or greater than a particular number. Suppose we have to form numbers which are greater than 4,00,000. In these numbers there will be 4 or a digit greater than 4, *i.e.*, 5, 6 or 7 in the lac's place. Thus this place can be filled in 4 ways. The remaining 5 places can then be filled in 5 ! = 120 ways.

∴ The total number of numbers = 4 × 120 = 480.

Ex. 16. How many numbers can be formed by using any number of the digits 3, 1, 0, 5, 7, 2, 9, no digit being repeated in any number ?

Sol. The number of single digit numbers is ${}^7P_1$.

The permutations of 7 digits taken 2 at a time are ${}^7P_2$. But ${}^6P_1$ of these have zero in the ten's place and so reduce to one digit numbers.

Hence the number of two-digit numbers is ${}^7P_2 - {}^6P_1$

Similarly the number of the three-digit numbers is ${}^7P_3 - {}^6P_2$ and so on.

∴ The total number required

$= {}^7P_1 + ({}^7P_2 - {}^6P_1) + ({}^7P_3 - {}^6P_2) + ({}^7P_4 - {}^6P_3) + ({}^7P_5 - {}^6P_4) + ({}^7P_6 - {}^6P_5) + ({}^7P_7 - {}^6P_6) = \mathbf{11743}.$

Ex. 17. How many different numbers can be formed with the digits 1, 3, 5, 7, 9, when taken all at a time, and what is their sum!

Sol. The total number of numbers = 5 ! = 120. Suppose we have 9 in the unit's place. We will have 4 ! = 24 such numbers. The number of numbers in which we have 1, 3, 5 or 7 in the unit's place is also 4 ! = 24 in each case.

Hence the sum of the digits in the unit's place in all the 120 numbers

$= 24\,(1 + 3 + 5 + 7 + 9) = 600.$

The number of numbers when we have any one of the given digits in ten's place is also 4 ! = 24 in each case. Hence the sum of the digits in the ten's place

$= 24\,(1 + 3 + 5 + 7 + 9)$ tens = 600 tens = 600×10.

Proceeding similarly, the required sum

= 600 units + 600 tens + 600 hundreds + 600 thousands + 600 ten thousands

$= 600\,(1 + 10 + 100 + 1000 + 10000) = 600 \times 11111 =$ **6666600**

Type IV. Word Building.

The following cases may arise :—

1. No letter may be repeated.
2. Some letters may be repeated.*
3. There may be a particular letter in the beginning or the end.
4. Some letters may occur together.

Words

Vowels occur together | Consonants occur together

Ex. 18. Suppose the word 'PENCIL' is given to us and we have to form words with the letters of this word.

1. There is no restriction on the arrangement of the letters.

The six different letters can be arranged in $^6P_6 = 6\,! = 720$ ways.

Hence the total number of words formed = 720.

2. All words begin with a particular letter.

Suppose all words begin with E. The remaining 5 places can be filled with remaining 5 letters in $5\,! = 120$ ways.

3. All words begin and end with particular letters.

Suppose all words begin with L, and end with P. The remaining 4 places can then be filled in 4 ! ways.

$\therefore$ The total number of words formed $= 4\,! = 24$.

Note. If the words were to begin or end with E or L, these two positions could have been filled in $^2P_2 = 2$ ways. Hence the number of words in this case would have been $= 2 \times 24 = 48$.

4. *N* is always after *E*.

P	EN	C	I	L

Since N is always after E, therefore 'EN' is considered to be one letter.

$\therefore$ Required no. of permutations $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

5. Vowels occur together.

The vowels are E and I. Regarding them as one letter, the 5 letters can be arranged in $5\,! = 120$ ways. These two vowels can be arranged amongst themselves in $2\,! = 2$ ways.

$\therefore$ The total number of words $= 2 \times 120 = 240$.

6. Consonants occur together.

Regarding these consonants as one letter the three letters E, I, $(PNCL)$ can be arranged in 3 !, *i.e.*, 6 ways. The letters $PNCL$ can be arranged among themselves in $4\,! = 24$ ways.

$\therefore$ The number of words in which consonants occur together $= 6 \times 24 = 144$.

7. Vowels occupy even places.

×	×	×	×	×	×
1	2	3	4	5	6

* This case is taken up in the next article.

There are 6 letters and 3 even places. E can be placed in any one of the three even places in ${}^3P_1$, *i.e.,* 3 ways. Having placed E in any one of these places, I can be placed in any one of the remaining 2 places in ${}^2P_1$, *i.e.,* 2 ways. Thus, the vowels can occupy even places in 2 × 3 = 6 ways. After the vowels have been placed the remaining 4 letters can take up their positions in ${}^4P_4$, *i.e.,* 24 ways.

∴ The total number of words = 6 × 24 = 144.

Ex. 19. How many ways are there to arrange the letter in the word GARDEN with the vowels in alphabetical order ?

Sol. The word *GARDEN* contains 6 letters - 4 consonents (G, R, D, N) and 2 vowels A, E. The 4 consonents can be arranged in 6 places in ${}^6P_4$ ways. In each of these arrangements two places will remain blank in which the first place will be filled by A and the place after this in only way by E as per the given condition (vowels in alphabetical order)

∴ Required number of ways = ${}^6P_4 \times 1 = 6 \times 5 \times 4 \times 3 =$ **360.**

1.09. Permutations of alike things

The number of permutations of n things taken all at a time where p of the things are alike and of one kind, q others are alike and of another kind, r others are alike and of another kind, and so on is

$$x = \frac{n!}{p!\,q!\,r!\,...}.$$

Suppose the things are letters of which p are a's, q are b's, r are c's and so on. In any one of the required x permutations, replace the p a's by p new letters (say $a_1, a_2, ...a_p$) different from each other and different from the remaining $(n - p)$ letters. Then, permuting these p new letters among themselves without disturbing the rest, we get p ! new permutations. If this change is made in each of the x permutations, we will obtain $x \times p$! new permutations in which p letters a's are now all different. Now, in any one of the $x \times p$! permutations, we replace the q b's by q different letters $b_1, b_2, ...b_q$. Permuting them without disturbing the rest, we get q ! permutations in all. By doing this in each of the $x \times p$! permutations we would obtain $x \times p!\, q!$ permutations in which all p letters 'a's' and all q letters 'b's' are different. Similarly on replacing the r c's by r different letters and permuting them, we will form $x \times p!\, q!\, r!$......permutations in which all the n letters are now different. But n different letters can be permuted in n ! ways. Hence $x \,.\, p!\, q!\, r! ...... = n!$

$$x = \frac{n!}{p!\,q!\,r!\,...}.$$

Ex. 20. In how many ways can the letters of the word (*i*) BHUTAN (*ii*) INDIA be arranged ?

Sol. (*i*) The word BHUTAN contains all different letters, *i.e.,* the letters are not repeated.

∴ The no. of possible arrangements is 6! = **720 ways.**

Sol. (*ii*) The word INDIA contains 5 letters of which 2 are '*I*'s.

The number of possible arrangements = $\frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$ = **60 ways.**

Ex. 21. How many signals can be made by hoisting 2 blue, 2 red and 5 yellow flags on a pole at the same time?

Sol. The number of signals = $\frac{9!}{2!2!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 5 \times 4 \times 3 \times 2 \times 1}$ = **756.**

Ex. 22. A coin is tossed 6 times. In how many different ways can we obtain 4 heads and 2 tails?

Sol. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

∴ The number of arrangements of 4 heads and 2 tails out of 6 is $\frac{6!}{4!2!} = 15$.

Ex. 23. How many numbers can be formed with digits 1, 2, 3, 4, 3, 2, 1, so that odd digits always occupy the odd places?

Sol. The odd digits having two 1's alike and two 3's alike can be arranged in four odd places in $\frac{4!}{2!2!} = 6$ ways.

The three even digits having two 2's alike can be arranged in the three even places in $\frac{3!}{2!} = 3$ ways.

∴ The number of numbers = 6 × 3 = 18.

Ex. 24. There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf.

Sol. Total number of books = 3 × 4 = 12

Each of the 4 different titles has 3 copies each

∴ Required number of ways of arranging them on a shelf = $\frac{12!}{3!3!3!3!} = \frac{12!}{(3!)^4} = 369600$.

Ex. 25. Find the number of arrangements of the letters of the word 'BANANA' in which the two N's do not appear adjacently.

Sol. Considering the two *N's* as one letter, the number of letters to be arranged = 5.

Therefore, the number of arrangements = $\frac{5!}{3!} = 20$ (∵ *A* is repeated 3 times)

Total number of arrangements if there were no restriction imposed = $\frac{6!}{3!2!} = 60$.

(*A* repeated 3 times and *N* repeated 2 times)

∴ Required number of arrangements = 60 – 20 = **40.**

Ex. 26. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Sol. A million is a 7-digit number. So any number greater than 1 million will contain all the seven digits. Since the digit 2 occurs twice and digit 3 occurs thrice and the rest are different, therefore, number of possible numbers which can be formed with the given seven digits = $\frac{7!}{(2!)(3!)} = 420$. These possible numbers include those which have 0 at the millions place. Keeping 0 fixed at the millions place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are different can be arranged in $\frac{6!}{(2!)(3!)} = 60$ ways.

∴ Number of numbers greater than 1 million made from the given digits = 420 – 60 = **360.**

Ex. 27. In how many ways can the letters of the word 'ARRANGE' be arranged such that the two r's do not occur together?

Sol. (*a*) There are two *a*'s, two *r*'s in the word 'arrange', therefore the number of arrangements

$$= \frac{7!}{2!2!} = 1260. \quad ...(1)$$

The number of arrangements in which the two *a*'s occur together $= \frac{6!}{2!} = 360.$ ...(2)

$\therefore$ The number of arrangements in which 2 *r*'s do not occur together = (1) – (2) = 1260 – 360 = **900**.

1.10. Permutations of repeated things

The number of permutations of n different things taken r at a time, when each thing may occur any number of times is n^r.

Suppose *r* places are to be filled with *n* things. The first place can be filled in *n* ways and when this has been filled up in any one of these ways, the second place can also be filled in *n* ways for the thing occupying the first place may occupy the second place also. Thus the first two places can be filled in $n \times n = n^2$ ways. Similarly the third place can also be filled in *n* ways.

Arguing in the same manner, we conclude that the *r* places can be filled in $n \times n \times n$.....*r* times, *i.e.*, n^r ways.

Ex. 28. In how many ways can 3 prizes be distributed among 4 boys, when

(*i*) no boy gets more than one prize;

(*ii*) a boy may get any number of prizes;

(*iii*) no boy gets all the prizes.

Sol. (*i*) The first prize can be given to any of the four boys. Then, the second prize can be given to any of the three boys. Lastly, the third prize can be given to any one of the remaining 2 boys.

$\therefore$ The number of ways in which all the 3 prizes can be given = $4 \times 3 \times 2$ = **24**.

(*ii*) In this case, each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given = $4 \times 4 \times 4 = 4^3$ = **64**.

(*iii*) Since anyone of the 4 boys can get all the prizes, therefore, the number of ways in which a boy gets all the 3 prizes = 4.

$\therefore$ Number of ways in which a boy does not get all the prizes = 64 – 4 = **60**.

Ex. 29. How many numbers of 3-digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

Sol. Since repetition is allowed, each of the 3 places in a 3-digit number can be filled in 5 ways.

H	T	O
5 ways	5 ways	5 ways

$\therefore$ Required number of 3-digit numbers

$= 5 \times 5 \times 5 = 5^3 = 125$

Ex. 30. How many numbers each containing four digits can be formed, when a digit may be repeated any number of times?

Sol. There are in all 10 digits, including zero. As the first digit of the number cannot be zero, so it can be chosen in 9 ways. Again, as a digit may occur any number of times in a number, the second, third and fourth digits of the numbers can be any one of the ten digits and so each of the remaining three places can be filled in 10 ways.

Hence the total number of 4-digit numbers = 9×10^3 = 9000.

Verification. All the 4-digit numbers will be between 1000 and 9999 and so their number is 9999 – 999 = 9000.

Ex. 31. Eight different letters of an alphabet are given. Words of 4 letters from these are formed. Find the number of such words with at least one letter repeated.

Sol. If any letter can be used any number of times, then the number of words of 4 Letters with 8 different letters is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$

Number of words of 4 letters with at least one letter repetition not allowed $= {}^8P_4 = 8 \times 7 \times 6 \times 5$
$= 1680$

$\therefore$ No of 4 letter words with at least one letter repeated is $8^4 - {}^8P_4 = 4096 - 1680 =$ **2416.**

1.11. Circular permutations

If we have to arrange the five letters *A, B, C, D, E,* two of the arrangements would be *ABCDE, EABCD* (Fig. 1.01) which are two distinct arrangements. Now, if these arrangements are written along the circumference of a circle, the two arrangements are one and the same. Thus, we conclude that circular permutations are different only when the relative order of the objects is changed otherwise they are the same. Thus the arrangements in Fig. 1.02 are different.

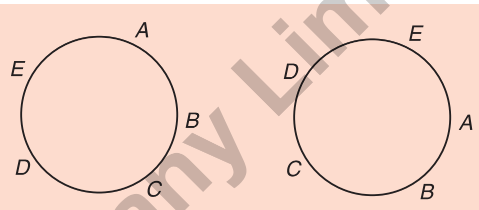


Fig. 1.01

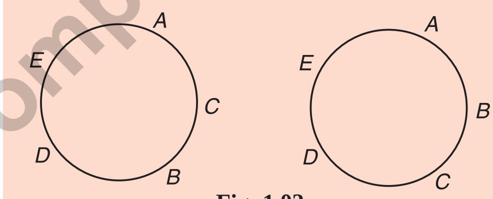


Fig. 1.02

As the number of circular permutations depends on the relative positions of objects, we *fix the position of one object and then arrange the remaining* $(n-1)$ *objects in all possible ways. This can be done in* $(n-1)$! *ways.*

Method II. Let the 5 persons be denoted by the letters *A,B,C, D, E* and one of the ways in which they can form a ring be as shown in Fig. 1.01.

Starting with different letters and reading them in the clockwise direction the various arrangements of the letters thus obtained are *ABCDE, BCDEA, CDEAB, DEABC, EABCD.*

These are all *different linear* arrangements but have no essential difference when regarded as circular arrangements. This shows that a single circular arrangement of the 5 letters gives rise to 5 different linear arrangements.

Hence if the required number of circular arrangements of the 5 persons be x, the total number of linear arrangements of the same persons will be $5x$.

But we know that the total number of linear arrangements of 5 persons is 5 !.

$\therefore \quad 5x = 5! \qquad \therefore \; x = \frac{5!}{5} = 4!$

Thus, the number of ways in which *five* persons form a ring $= 4! = (5-1)!$.

Similarly the number of ways in which n persons can form a ring $= (n-1)!$.

Ex. 32. 20 persons were invited for a party. In how many ways can they and the host be seated around a circular table ? In how many of these ways will two particular persons be seated on either side of the host ?

Sol. There is 1 host and 20 guests, they are to be seated around a circular table.

(*i*) Let us fix the seat of one person, say the host, the 20 guests will be seated around the circular table in 20 ! ways, [or, $(n-1)! = (21-1)! = 20!$]

(*ii*) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated around the circular table in 18 ! ways.

Hence the number of ways of seating two particular persons on either side of the host = 2 × 18 !. [or, 2! × (19 – l)!, considering the host and two particular persons as one entity.]

Ex. 33. In how many ways can a party of 4 boys and 4 girls be seated at a circular table so that no 2 boys are adjacent ? **(*ISC BM 2004*)**

Sol. Let the girls first take up their seats. They can sit in 3 ! ways. When they have been seated, then there remain 4 places for the boys each between two girls. Therefore the boys can sit in 4 ! ways. Therefore there are 3 ! × 4 !, *i.e.*, 144 ways of seating the party.

Ex. 34. A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates are

(*i*) always together, (*ii*) never together ?

Sol. (*i*) Let D_1 and D_2 be the two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all . 19 delegates can be seated round a circular table in (19 – 1) ! = 18 ! ways. But two particular delegates can seat themselves in 2! ($D_1 D_2$ or $D_2 D_1$) ways.

Hence, the total number of ways = 18! × 2! = 2 (18!)

(*ii*) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of ways of seating 20 persons, *i.e.*, (20 – 1) ! = 19! ways.

Hence, the total number of ways in this case = 19! – 2 (18!) = 19 (18!) – 2 (18!) = 17 (18!).

1.12. Clockwise and counter-clockwise permutations

We have two types of circular permutations:

(*i*) Those in which counter-clockwise and clockwise are distinguishable. Thus while seating 4 persons *A*, *B*, *C*, *D* around a table, the following permutations are considered different. (Fig. 1.03)

(*ii*) Those in which counter-clockwise and anti-clockwise are not distinguishable. (Fig. 1.04)

Thus, (*a*) while forming a garland of roses or jasmine, the following arrangements are not disturbed if we turn the garland over.

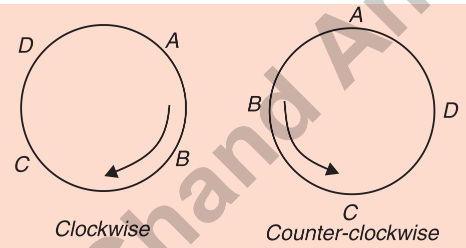


Fig. 1.03

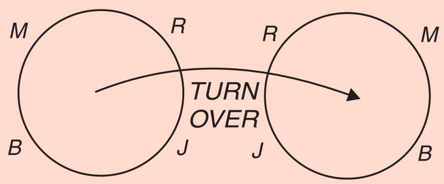


Fig. 1.04

(*b*) The distinction between clockwise and anti-clockwise is ignored when a number of people have to be seated around a table so as not to have the same neighbours.

Ex. 35. Find the number of ways in which (*i*) *n* different beads, (*ii*) 10 different beads can be arranged to form a necklace.

Sol. Fixing the position of one bead, the remaining beads can be arranged in (*n* – 1)! ways. As there is no distinction between the clockwise and anti-clockwise arrangements, the required number of ways = $\frac{(n-1)!}{2}$.

Number of ways in which 10 different beads can be arranged = $\frac{(10-1)!}{2} = \frac{1}{2}$ (9!).

Ex. 36. Find the number of ways in which 10 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Sol. Consider 4 particular flowers as one flower. Then, we have 7 flowers which can be strung to form a garland in (7 – 1)! = 6! ways. But 4 particular flowers can be arranged in 4! ways.

Hence, the required number of ways = $\frac{1}{2}$ (6! × 4!) = **8640.**

Ex. 37. In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements.

Sol. 7 persons can sit around a table in 6 ! ways but as each person will have the same neighbours in clockwise and anti-clockwise arrangements, the required number = $\frac{1}{2}$ 6! = 360.

EXERCISE 1 (*c*)

1. Of 12 different books a shelf will hold 5; how many different arrangements may be made on the shelf ?
2. In how many ways can the letters of the following words be arranged:

   (*a*) *RADIO* (*b*) *FOREIGN* ?
3. In how many other ways can the letters of the word '*SIMPLETON*' be arranged ?
4. How many different words beginning and ending with a consonant can be made out of the letters of the word '*EQUATION*'?
5. How many permutations can be made out of the letters of the word '*TRIANGLE*' ? How many of these will begin with *T* and end with *E* ?
6. How many different words can be formed of the letters of the word '*MALENKOV*' so that

   (*i*) no two vowels are together,

   (*ii*) the vowels may occupy odd places ?
7. In how many ways can the letters of the word '*COMBINE*' be arranged so that;

   (*i*) the vowels are never separated ;

   (*ii*) all the vowels never come together ;

   (*iii*) vowels occupy only the odd places ?
8. Three persons have 4 coats, 5 waistcoats, and 6 hats. Find in how many ways can they put on the clothes.
9. If out of 6 flags any number of flags can be shown at a time, find how many different signals can be made out of them.
10. In how many ways can 9 things be arranged taken 4 at a time, and in how many of these arrangements will a particular thing be always included ?
11. How many different numbers of 4 digits each can be formed with the ten digits, 0, 1, 2,...9 when digits are not repeated ?
12. From the digits 1, 2, 3, 4, 5, 6, how many three-digit odd numbers can be formed when the repetition of the digits is not allowed ? **(*ISC*)**
13. (*i*) How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5?

    (*ii*) How many of them are divisible by 10?

    (*iii*) How many of them will have zero in the ten's place?
14. How many 5-digit telephone numbers can be formed with the digits 0, 1, 2,..., 8, 9 if each number starts with 35 and no digit appears more than once ? **(*ISC*)**
15. There are 5 boys and 3 girls. In how many ways can they stand in a row so that no two girls are together?
16. There are 5 red, 4 white and 3 blue marbles in a bag. They are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements. **(*ISC*)**

17. How many 7-digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4? ***(ISC)***

[**Hint.** Reqd. number of numbers = Total possible arrangements – those arrangements have 0 in the extreme left position. $= \dfrac{7!}{3!2!} - \dfrac{6!}{3!2!}$].

18. (*i*) How many different words can be formed with the letters of the word '*BANGCHUNG*'?

(*ii*) In how many of these *B* and *H* are never together?

(*iii*) How many of these begin with *B* and end with *A*?

19. (*i*) Find how many arrangements can be made with the letters of the word "*MATHEMATICS*"?

(*ii*) In how many of them the vowels occur together? (***ISC***)

[**Hint.** (*ii*) $\boxed{A, A, E, I,}$ *M, T, H, M, T, C, S*

Imagine 4 vowels written together, Then these 8 letters can be permuted in $\dfrac{8!}{2!2!}$ = 10080 ways.

Corresponding to each of these permutations, the 4 vowels can be arranged among themselves in

$\dfrac{4!}{2!}$ = 12 ways.

∴ Reqd. number of words in which vowels occur together = 10080 × 12 = 120960].

20. Ten different books are arranged on a shelf. Find the number of different ways in which this can be done, if two specified books are (*a*) to be together, (*b*) not to be together.

21. In how many ways can 20 books be arranged on a shelf so that a particular pair of books shall not come together?

22. Find the number of permutations of the letters of the words

(*i*) *INDIA* (*ii*) *BHUTAN* (*iii*) *MALDIVES* (*iv*) *PAKISTAN.*

23. Find the number of ways in which five identical balls can be distributed among ten identical boxes, if not more than one can go into a box.

24. How many numbers are there in all which consist of 5 digits?

25. In how many ways can 5 prizes be distributed among 4 students, when each student may receive any number of prizes?

26. In how many ways can 3 letters be posted in four letter boxes in a village ? If all the three letters are not posted in the same letter box, find the corresponding number of ways of posting.

27. In how many ways can 8 people sit around a table?

28. In how many ways can 10 people sit around a table so that all shall not have the same neighbours in any two arrangements?

29. A committee of 11 members sits at a round table. In how many ways can they be seated if the 'President' and the 'Secretary' choose to sit together?

30. In how many ways can 30 different pearls be arranged to form a necklace?

31. In how many ways 6 gentlemen and 3 ladies can be seated round a table so that every gentleman may have a lady by his side.

[**Hint.** Arrange the seats for 6 gentlemen and 3 ladies as shown. This can be done in 5! (Gentlemen) × 3! (Ladies) = 720 ways.

If arrangements are made in the opposite direction, then the number of arrangements = 5! × 3! = 720.

∴ Total number of required arrangements = 720 + 720 = 1440.]

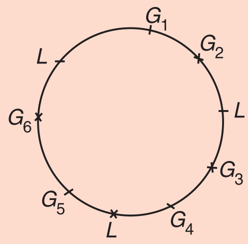


Fig. 1.05

ANSWERS

1. $^{12}P_5$	**2.** (*a*) 120 (*b*) 5040	**3.** (9!) – 1	**4.** 4320	**5.** 40320, 720
6. (*i*) 14400 (*ii*) 2880	**7.** 720 ; 4320 ; 576	**8.** 172800	**9.** 1956	**10.** 3024, 1344
11. 4536	**12.** 60	**13.** (*i*) 600 (*ii*) 120 (*iii*) 120		**14.** 336
15. 14400	**16.** 27720	**17.** 360	**18.** (*i*) 90720 (*ii*) 80640 (*iii*) 1260	
19. (*i*) $\frac{11!}{2!2!2!} = 4989600$ (*ii*) 120960		**20.** (*a*) 725760 (*b*) 2903040		**21.** 18(19!)
22. (*i*) 60 (*ii*) 720 (*iii*) 40320 (*iv*) 2520		**23.** $\frac{10!}{5!5!}$	**24.** 90,000	**25.** $2^4 = 1024$
26. 64, 60	**27.** 5040	**28.** $\frac{1}{2}$ (9 !)	**29.** 2 (9 !)	**30.** $\frac{1}{2}$ (29 !)

1.13. Combinations

Def. ***Each of the different groups or selections which can be made by taking some or all of a number of things at a time (irrespective of the order) is called a combination.***

By the number of combinations of n things taken r at a time is meant the number of groups of r things which can be formed from the n things. The same is denoted by the symbol $\boldsymbol{^nC_r}$ or $\boldsymbol{C(n, r)}$ or $\boldsymbol{\binom{n}{r}}$.

1.14. Value of ^nC_r

Each combination consists of r different things which can be arranged among themselves in r ! ways. Therefore, the number of arrangements for all the ^nC_r combinations is $^nC_r \times r$!. This is equal to the permutations of n different things taken r at a time.

$$\therefore \quad {}^nC_r \times \underline{|r} = {}^nP_r$$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{\underline{|r}} = \frac{n(n-1)(n-2)........(n-r+1)}{1.2.3........r}$$

Cor. 1. $^nP_r = \dfrac{n!}{\underline{|n-r}}$ $\therefore {}^nC_r = \dfrac{n!}{\underline{|r} \,.\, \underline{|n-r}}$

Cor. 2. To prove that $\boldsymbol{^nC_n = 1}$.

Putting $r = n$ in $^nC_r = \dfrac{n!}{(n-r)!} \cdot \dfrac{1}{r!}$, we have

$$^nC_n = \frac{n!}{(n-n)!} \cdot \frac{1}{n!} = \frac{n!}{0!} \cdot \frac{1}{n!} = \frac{n!}{1} \cdot \frac{1}{n!} = 1. \qquad (\because 0! = 1)$$

Cor. 3. The number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken $(n – r)$ at a time, *i.e.*, $\boldsymbol{^nC_r = {}^nC_{n-r}}$.

Proof.

Method I. Every time we select a group of r things we leave behind another group of $(n – r)$ things. Thus for every combination of $(n – r)$ things there corresponds a combination of r things. $\therefore \; {}^nC_r = {}^nC_{n-r}$.

Method II. $^nC_r = \dfrac{n!}{\lfloor r \times \lfloor n-r}$ and $^nC_{n-r} = \dfrac{n!}{\lfloor n-r \times \lfloor n-(n-r)} = \dfrac{n!}{\lfloor (n-r) \cdot \lfloor r}$

$^nC_r = {^nC_{n-r}}$.

Cor. 4. If $^nC_x = {^nC_y}$, then, either $x = y$ or $x + y = n$.

Since $^nC_x = {^nC_y} = {^nC_{n-y}}$ $\quad \therefore\ x = y$ or $x = n - y$, *i.e.*, $x + y = n$.

Cor. 5. An important formula

$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$ **($1 \le r \le n$) (*Pascal's Rule*)**

$$^nC_r + {^nC_{r-1}} = \frac{\lfloor n}{\lfloor r \,.\lfloor n-r} + \frac{\lfloor n}{\lfloor (r-1)\,.\,\lfloor (n-r+1)}$$

$$= \lfloor n \left\{ \frac{(n-r+1)+r}{\lfloor r\,.\,\lfloor (n-r+1)} \right\} = \frac{(n+1)\lfloor n}{\lfloor r \cdot \lfloor (n-r+1)} = \frac{\lfloor (n+1)}{\lfloor r \cdot \lfloor (n-r+1)} = {^{n+1}C_r}.$$

Another form. $C(n, r) + C(n, r-1) = C(n+1, r)$,

i.e., $C(n, r-1) = C(n+1, r) - C(n, r)$

Alternative Proof. The total number of combinations of $n + 1$ things taken r at a time = combinations that contain a particular thing + combinations that do not contain a particular thing.

$\therefore\ {^{n+1}C_r} = {^nC_{r-1}} + {^nC_r}$.

Note. ^nC_r is greatest if (*i*) $r = \dfrac{n}{2}$, when n is even, (*ii*) $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ when n is odd.

Ex. 38. Find the values of $^6C_3$ and $^{30}C_{28}$.

Sol. (*i*) $^6C_3 = \dfrac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20.$

(*ii*) $^{30}C_{28} = {^{30}C_{30-28}} = {^{30}C_2} = \dfrac{30 \times 29}{1 \times 2} = 435.$

Ex. 39. If $^{18}C_r = {^{18}C_{r+2}}$, find the value of $^rC_5$.

Sol. As $r \ne r+2$, so $r + (r+2) = 18$ $\quad \therefore\ r = 8\ \therefore\ {^rC_5} = {^8C_5} = {^8C_3} = \dfrac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$

Ex. 40. Find the value of $^{47}C_4 + \sum\limits_{r=1}^{5} {^{52-r}C_3}$.

Sol. $^{47}C_4 + \sum\limits_{r=1}^{5} {^{52-r}C_3} = {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{48}C_3} + {^{47}C_3} + {^{47}C_4}$

$= {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{48}C_3} + {^{48}C_4}$ $\quad [\because {^nC_r} + {^nC_{r-1}} = {^{n+1}C_r}]$

$= {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{49}C_4} = {^{51}C_3} + {^{50}C_3} + {^{50}C_4}$

$= {^{51}C_3} + {^{51}C_4} = \mathbf{^{52}C_4}$.

Ex. 41. If ^nC_r denotes the number of combinations of n things taken r at a time, then the expression $^nC_{r+1} + {^nC_{r-1}} + 2 \times {^nC_r}$ equals

(*a*) $^{n+1}C_{r+1}$ (*b*) $^{n+2}C_r$ (*c*) $^{n+2}C_{r+1}$ (*d*) $^{n+1}C_r$

Sol. (*c*) Given exp. $= {}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r = {}^nC_{r+1} + {}^nC_{r-1} + {}^nC_r + {}^nC_r$ (write $2 \times {}^nC_r = {}^nC_r + {}^nC_r$)

$$= ({}^nC_{r+1} + {}^nC_r) + ({}^nC_{r-1} + {}^nC_r)$$

$$= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1}$$

Ex. 42. If $^nC_{r-1} = 36$, $^nC_r = 84$ and $^nC_{r+1} = 126$, then find the value of r.

Sol. $$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36} \Rightarrow \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{84}{36} \Rightarrow \frac{n-r+1}{r} = \frac{7}{3} \quad .....(i)$$

Also, $$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{126}{84} \Rightarrow \frac{n!}{(r+1)!(n-r-1)!} \times \frac{r!(n-r)!}{n!} = \frac{3}{2} \Rightarrow \frac{n-r}{r+1} = \frac{3}{2} \quad .....(ii)$$

From (*ii*) $2n - 2r = 3r + 3 \Rightarrow n = \dfrac{5r+3}{2}$

Substituting in (*i*), we get $\dfrac{\frac{5r+3}{2} - r + 1}{r} = \dfrac{7}{3}$ $\Rightarrow$ **$r = 3$.**

EXERCISE 1 (*d*)

1. Find the value of:

   (*a*) $^5C_2$ (*b*) $^{10}C_4$ (*c*) $^{50}C_{47}$

2. Evaluate :

   (*i*) $C(15, 14)$ (*ii*) $C(8, 5)$ (*iii*) $^{11}C_2$.

3. Evaluate :

   (*i*) $C(19, 17) + C(19, 18)$ (*ii*) $C(31, 26) - C(30, 26)$.

   [**Hint.** Using $C(n, r) + C(n, r-1) = C(n+1, r)$, we have $C(30, 26) + C(30, 25) = C(31, 26)$

   $\Rightarrow C(31, 26) - C(30, 26) = C(30, 25)$]

4. If $^4P_2 = n.{}^4C_2$, find n.

5. If $^nC_4 = {}^nC_6$, find n. (*SC*)

6. If $C(2n, 3) : C(n, 2) = 12 : 1$, find n.

7. If $^nC_r : {}^nC_{r+1} = 1 : 2$ and $^nC_{r+1} : {}^nC_{r+2} = 2 : 3$, determine the values of n and r. (*ISC*)

8. If $C(n, 10) = C(n, 12)$, determine $C(n, 5)$.

9. If $C(2n, r) = C(2n, r+2)$, find r in term of n.

10. The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is (*a*) $^{55}C_4$ (*b*) $^{55}C_3$ (*c*) $^{56}C_3$ (*d*) $^{56}C_4$

11. $^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$ if (*a*) $n > 7$ (*b*) $n \geq 7$ (*c*) $n > 6$ (*d*) $n \geq 6$ (*IIT*)

ANSWERS

1. (*a*) 10 (*b*) 210 (*c*) 19600	**2.** (*i*) 15 (*ii*) 56 (*iii*) 55	**3.** (*i*) 190 (*ii*) 142506	**4.** 2
5. 10	**6.** 5	**7.** $n = 14$, $r = 4$	**8.** 26334
9. $r = n - 1$	**10.** (*d*)	**11.** (*a*)	

1.15. Practical problems on combination

Ex. 43. In how many ways can 4 persons be selected from amongst 9 persons ? How many times will a particular person be always selected?

Sol. The number of ways in which 4 persons can be selected from amongst 9 persons

$$= {}^9C_4 = \frac{9\times8\times7\times6}{1\times2\times3\times4} = 126.$$

The number of ways in which a particular person is always to be selected

$$= {}^8C_3 = \frac{8\times7\times6}{1\times2\times3} = 56.$$

Ex. 44. Find the number of diagonals that can be drawn by joining the angular points of a heptagon.

Sol. A heptagon has seven angular points and seven sides.

The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points $= {}^7C_2 = \dfrac{7\times6}{1\times2} = 21.$

But the number of sides = 7 $\quad\therefore$ The number of diagonals = 21 – 7 = 14.

Ex. 45. A committee of 4 is to be selected from amongst 5 boys and 6 girls. In how many ways can this be done so as to include (*i*) exactly one girl, (*ii*) at least one girl?

Sol. (*i*) In this case we have to select one girl out of 6 and 3 boys out of 5.

The number of ways of selecting 3 boys $= {}^5C_3 = {}^5C_2 = 10$.

The number of ways of selecting one girl $= {}^6C_1 = 6$.

$\therefore$ The required committee can be formed in 6 × 10 = 60 ways.

(*ii*) The committee can be formed with

(*a*) one boy and three girls, or (*b*) 2 boys and 2 girls,

or (*c*) 3 boys and one girl, or (*d*) 4 girls alone.

The committee can be formed in (*a*) ${}^5C_1 \times {}^6C_3$ ways ;

The committee can be formed in (*b*) ${}^5C_2 \times {}^6C_2$ ways ;

The committee can be formed in (*c*) ${}^5C_3 \times {}^6C_1$ ways ;

The committee can be formed in (*d*) ${}^6C_4$ ways.

Hence the required number of ways of forming the committee

$= {}^5C_1 \times {}^6C_3 + {}^5C_2 \times {}^6C_2 + {}^5C_3 \times {}^6C_1 + {}^6C_4 = 100 + 150 + 60 + 15 = 325$ ways.

(*ii*) **Method II.** Required ways = (Committees of 4 out of 11 without any restriction) – (Committees in which no girl is included) $= {}^{11}C_4 - {}^5C_4 = 325$.

Ex. 46. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. Find the number of choices available to him.

Sol. Two cases are possible :

(*i*) Selecting 4 out of first five questions and 6 out of remaining 8 questions

$\therefore$ Number of choices in this case $= {}^5C_4 \times {}^8C_6 = {}^5C_1 \times {}^8C_2 = \dfrac{5\times8\times7}{1\times2} = 140$

(*ii*) Selecting 5 out of first five questions and 5 out of remaining 8 questions.

⇒ Number of choices = $^5C_5 \times {}^8C_5 = 1 \times {}^8C_3 = \dfrac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$

∴ Total number of choices = 140 + 56 = **196.**

1.16. Miscellaneous types

Type I. Total number of combinations. ***To find the total number of combinations of n dissimilar things taking*** **any number** ***of them at a time.***

Case I. When all things are different.

Each thing may be disposed of in two ways. It may either be included or rejected.

∴ The total number of ways of disposing of all the things = 2 × 2 × 2 × ......n times = 2^n

But this includes the case in which all the things are rejected.

Hence the total number of ways in which *one* or *more* things are taken = $2^n - 1$.

Cor. $2^n - 1$ is also the total number of the combinations of n things taken 1, 2, 3, ... or n at a time. Hence, $^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$.

Ex. 47. There are 5 questions in a question paper. In how many ways can a boy solve one or more questions?

Sol. The boy can dispose of each question in two ways. He may either solve it or leave it. Thus the number of ways of disposing of all the questions = 2^5.

But this includes the case in which he has left all the questions unsolved.

Hence the total number of ways of solving the paper = $2^5 - 1 = 31$.

Case II. When all things are not different.

Suppose, out of ($p + q + r + \ldots$) *things, p are alike of one kind, q are alike of a second kind, r alike of a third kind, and the rest different.*

Out of p things we may take 0, 1, 2, 3...... or p. Hence they may be disposed of in (p + 1) ways. Similarly, q alike things may be disposed of in (q + 1) and r alike things in (r + 1) ways. The t different things may be disposed of in 2^t ways.

This includes that case in which all are rejected.

∴ The total number of selections = $(p + 1)(q + 1)(r + 1)\, 2^t - 1$.

Ex. 48. Prove that from the letters of the sentence, 'Daddy did a deadly deed', one or more letters can be selected in 1919 ways.

Sol. In the given sentence, there are 9 *d*'s ; 3 *a*'s ; 3 *e*'s ; 2 *y*'s ; 1 *i*; and 1 *l*.

∴ The total number of selections. = (9 + 1) (3 + 1) (3 + 1) (2 + 1) (1 + 1) (1 + 1) – 1

= 10 × 4 × 4 × 3 × 2 × 2 – 1 = 1919.

Type II. Division into groups.

To find the number of ways in which p + q things can be divided into two groups containing p and q things respectively.

Every time when a set of p things is taken, a second set of q things is left behind. Hence the required number of ways = the number of combinations of ($p + q$) things taking p at a time

$$= {}^{p+q}C_p = \frac{(p+q)!}{p!\,q!}$$

Cor. 1. Generalisation. The number of ways in which $p + q + r$ things can be divided into three groups containing p, q and r things respectively

$$= {}^{p+q+r}C_p \times {}^{q+r}C_q \times {}^{r}C_r = \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!r!} \times 1 = \frac{(p+q+r)!}{p!q!r!}$$

Similarly the result can be extended to the case of dividing a given number of things into more than three groups.

Cor. 2. The number of ways in which $3p$ things can be divided equally into three *distinct* groups is $\frac{(3p)!}{(p!)^3} \cdot (q = p, r = p)$

Cor. 3. The number of ways in which $3p$ things can be divided into three *identical* groups is $\frac{(3p)!}{3!(p!)^3}$.

Ex. 49. In how many ways can 15 things be divided into 3 groups containing 8, 4 and 3 things respectively ?

Sol. The number of ways = $\frac{(15)!}{8!4!3!} = 225225.$

Ex. 50. In how many ways can 18 different books be divided equally among 3 students?

Sol. The required number of ways = $\frac{(18)!}{(6!)^3}$

Ex. 51. In how many ways can 52 playing cards be placed in 4 heaps of 13 cards each ? In how many ways can they be dealt out to four players giving 13 cards each ?

Sol. (*i*) The number of ways = $\frac{52!}{4!(13!)^4}$.

(*ii*) The number of ways = $\frac{52!}{(13!)^4}$.

Type III. Permutations and Combinations occurring simultaneously.

The method is illustrated by the following examples.

Ex. 52. How many different words, each containing 2 vowels and 3 consonants, can be formed with 5 vowels and 17 consonants ? **(*ISC 1996 Type*)**

Sol. Two vowels can be selected in ${}^5C_2$ ways.

Three consonants can be selected in ${}^{17}C_3$ ways.

∴ 2 vowels and 3 consonants can be selected in ${}^5C_2 \times {}^{17}C_3$ ways.

Now, each group of 2 vowels and 3 consonants can be arranged in 5 ! ways.

∴ The total number of words = ${}^5C_2 \times {}^{17}C_3 \times 5! =$ **816000.**

Ex. 53. Find the number of (*i*) combinations, (*ii*) permutations of four letters taken from the word EXAMINATION. **(*ISC 1992 Type*)**

Sol. There are 11 letters not all different. They are as: (*AA*) ; (*II*) ; (*N*, *N*) ; *E* ; *X* ; *M* ; *T* ; 0.

The following combinations are possible:

(*a*) 2 alike, 2 alike $= {}^3C_2 = 3$;

(*b*) 2 alike, 2 different $= {}^3C_1 \times {}^7C_2 = 63$;

(*c*) All 4 different $= {}^8C_4 = 70.$

∴ The total number of combinations = 3 + 63 + 70 = 136.

∴ The number of permutations in (*a*) to (*c*)

$$= 3 \times \frac{4!}{2!2!} + 63 \times \frac{4!}{2!} + 70 \times 4! = 18 + 756 + 1680 = \mathbf{2454.}$$

EXERCISE 1 (*e*)

1. In how many ways can a committee of 8 be chosen from 10 individuals?

2. In how many ways can a committee of five persons be formed out of 8 members when a particular member is taken every time ?

3. In how many ways can a committee of 4 be selected out of 12 persons so that a particular person may (*i*) always be taken, (*ii*) never be taken ?

4. In how many ways can a team of 11 players be selected from 14 players when two of them can play as goalkeepers only ?

5. A person has got 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests such that 5 of them may be relatives ?

6. How many diagonals are there in a polygon of (*i*) 8 sides, (*ii*) 10 sides ? **(*ISC*)**

7. In how many ways can a committee, consisting of a chairman, secretary, treasurer and four other members be chosen from eight persons ?

(Committees with different chairmen, secretaries, treasurers count as different committees.) **(*SC*)**

8. (*a*) In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student? **(*ISC*)**

(*b*) In how many ways can we select a cricket eleven from 17 players in which 5 players can bowl? Each cricket team must include 2 bowlers. **(*SC*)**

9. How many committees of 5 members each can be formed with 8 officials and 4 non-official members in the following cases :

(*a*) each consists of 3 officials and 2 non-official members ;

(*b*) each contains at least two non-official members ;

(*c*) a particular official member is never included ;

(*d*) a particular non-official member is always included ?

10. In a college team there are 15 players of whom 3 are teachers. In how many ways can a team of 11 players be selected so as to include (*i*) only one teacher, (*ii*) at least one teacher?

11. How many different groups can be selected for playing tennis out of 4 ladies and 3 gentlemen, there being one lady and one gentleman on each side ? **(*ISC*)**

12. If ${}^nC_{10} = {}^nC_{14}$, find the value of ${}^nC_{20}$ and ${}^{25}C_n$.

13. In how many ways can I invite one or more of six friends to a dinner ?

14. In how many ways can 10 marbles be divided between two boys so that one of them may get 2 and the other 8 ?

15. In how many ways can a selection be made out of 5 oranges, 8 mangoes and 7 apples ?

16. In how many ways can 20 articles be packed in the three parcels so that the first contains 8 articles, the second 7 and the third 5 ?

17. In how many ways can 28 different things be formed into 4 heaps so that each may contain 7?

18. In how many ways can 20 students be divided into four equal groups ? In how many ways can these be sent to four different schools ?

19. Find the number of four letter arrangements of the letters of the word '*SHOOT*'. How many of them begin with *O* ? **(*SC*)**

20. To go on a journey 8 persons are to be divided into 2 groups, one group to go by car and the other by train. In how many ways can this be done if there must be at least 3 persons in each group ? **(*ISC*)**

21. A table has 7 seats, 4 being on one side facing the window and 3 being on the opposite side. In how many ways can 7 people be seated at the table.

(*a*) if 2 people, *X* and *Y*, must sit on the same side; (*b*) *X* and *Y* must sit on opposite sides ;

(*c*) if 3 people, *X*, *Y* and *Z*, must sit on the side facing the window ? **(*ISC*)**

22. Seven cards, each bearing a letter, can be arranged to spell the word "*DOUBLES*". How many three-letter code-words can be formed from these cards ?

How many of these words

(*a*) contain the letter *S* ; (*b*) do not contain the letter *O* ;

(*c*) consist of a vowel between two consonants ? **(*GCE*)**

23. A committee of 5 is to be formed from a group of 12 students consisting of 8 boys and 4 girls. In how many ways can the committee be formed if it

(*i*) consists of exactly 3 boys and 2 girls ; (*ii*) contains at least 3 girls ? **(*ISC*)**

24. There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?

[Hint : 5 gentlemen can seat in (5 – 1) ! = 4 ! = 24 ways. The four ladies will occupy the places marked *L* so that no two ladies sit together. They can do so in $^5C_4 \times 4! = 5 \times 4! = 5! = 120$ ways.

$\therefore$ Reqd. number of ways = 24 × 120].

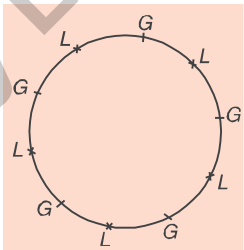


Fig. 1.06

25. There are 12 points in a plane, of which 5 are collinear. Find **(*ISC 1999*)**

(*i*) the number of triangles that can be formed with vertices at these points ;

(*ii*) the number of straight lines obtained by joining these points in pairs. **(*ISC*)**

26. A committee of 5 is to be formed from a group of 10 people, consisting of 4 single men, 4 single women and a married couple. The committee is to consist of a chairman, who must be a single man, 2 other men and 2 women,

(*i*) Find the total number of committees possible,

(*ii*) How many of these would include the married couple ? **(*ISC*)**

27. A committee of 5 persons is to be formed from a group of 6 gentlemen and 4 ladies. In how many ways can this be done if the committee is to include at least one lady ? **(*ISC*)**

[Sol. The no. of committees is equal to

$$({}^4C_1 \times {}^6C_4) + ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1)$$

$$= (4 \times 15) + (6 \times 20) + (4 \times 15) + (1 \times 6)$$

$$= 60 + 120 + 60 + 6 = 246.]$$

Method II. $^{10}C_5 - {}^6C_5 = 246$. See solved Ex. 48.]

28. Out of 3 books on Economics, 4 books on Political Science and 5 books on Geography, how many collections can be made, if each collection consists of

(*i*) exactly one book on each subject, (*ii*) at least one book on each subject ? **(*ISC 1990*)**

[Sol. (*i*) The no. of collections = $^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$.

(*ii*) The no. of collections $= ({}^3C_1 + {}^3C_2 + {}^3C_3)({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4)({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$

$= (3 + 3 + 1)(4 + 6 + 4 + 1)(5 + 10 + 10 + 5 + 1) = 7 \times 15 \times 31 = 3255.]$

29. Find the number of words which can be formed by taking two alike and two different letters from the word "*COMBINATION*". **(*ISC 1992*)**

[Sol. There are 11 letters, not all different. They are as (*O, O*), (*I, I*), (*N, N*) ; *C, M, B, A, T*.

The number of combinations taking two alike and two different = $^3C_1 \times {}^7C_2 = 3 \times \dfrac{7\times6}{2\times1} = 63$.

Each of these combinations gives rise to $\dfrac{\lfloor 4}{\lfloor 2}$, *i.e.*, 12 words.

Therefore, 63 combinations give rise to 63 × 12, *i.e.* 756 words. (See also solved Ex. 53)]

ANSWERS

1. 45 **2.** 35 **3.** (*i*) 165 (*ii*) 330

4. 132 **5.** 336 **6.** (*i*) 20 (*ii*) 35

7. 1680 **8.** (*a*) 35 , (*b*) 2200 **9.** (*a*) 336 (*b*) 456 (*c*) 462 (*d*) 330

10. 198 ; 1353 **11.** 36 **12.** 10626. 25

13. 63 **14.** 90 **15.** 431

16. $\dfrac{20!}{8!7!5!}$ **17.** $\dfrac{28!}{4!(7!)^4}$ **18.** $\dfrac{20!}{4!(5!)^4}, \dfrac{20!}{(5!)^4}$

19. 60 ; 24 **20.** 182 **21.** (*a*) 2160 (*b*) 2880 (*c*) 576

22. 210 (*a*) 90 (*b*) 120 (*c*) 36 **23.** (*i*) 336 (*ii*) 120 **24.** 2880

25. (*i*) 210 (*ii*) 57 **26.** (*i*) 240 (*ii*) 48 **27.** 246

REVISION EXERCISE

1. The set $S = \{1, 2, 3, ... , 12\}$ is to be partitioned into three sets *A, B, C* of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition *S* is

(*a*) $\dfrac{12!}{(4!)^3}$ (*b*) $\dfrac{12!}{(4!)^4}$ (*c*) $\dfrac{12!}{3!\,(4!)^3}$ (*d*) $\dfrac{12!}{3!\,(4!)^4}$.

[Hint. Number of ways = $^{12}C_4 \times {}^8C_4 \times {}^4C_4 = \dfrac{12!}{(4!)^3}$]

2. The number of permutations of the letters of the word '*CONSEQUENCE*' in which all the three *E*'s are together is

(*a*) 9! 3! (*b*) $\dfrac{9!}{2!}$ (*c*) $\dfrac{9!}{2!\,2!\,3!}$ (*d*) $\dfrac{9!}{2!\,2!}$

3. In how many ways can the letters of the word '*CABLE*' be arranged so that the vowels should always accupy odd positions ?

(*a*) 12 (*b*) 18 (*c*) 24 (*d*) 36

4. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be selected. If a voter votes for at least one candidate, then the number of ways in which he can vote is

(*a*) 5040 (*b*) 6210 (*c*) 1110 (*d*) 385

[Hint. The no. of ways in which a voter can vote = $^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$.]

5. If the letters of the word *SACHIN* are arranged in all possible ways and these words are written out as in dictionary, then *SACHIN* appears at serial number

(*a*) 601 (*b*) 600 (*c*) 603 (*d*) 602

6. If $^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of r is

(*a*) 12 (*b*) 8 (*c*) 6 (*d*) 10

7. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(*a*) 6! × 5! (*b*) 30 (*c*) 5! × 4! (*d*) 7! × 5!

[Hint. Same type as Q. 24 in Exercise 1(*e*)]

8. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, Then n =

(*a*) 5 (*b*) 7 (*c*) 6 (*d*) 4

[Hint. To form a triangle, 3 out of n vertices of a regular polygon should be chosen. Hence, the number of triangles that can be formed = $T_n = {}^nC_3$

∴ It is given that $T_{n+1} - T_n = 21$. So $^{n+1}C_3 - {}^nC_3 = 21$.

$\Rightarrow {}^nC_3 + {}^nC_2 - {}^nC_3 = 21 \Rightarrow {}^nC_2 = {}^7C_2 \Rightarrow n = 7$]

9. If $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, then r =

(*a*) 40 (*b*) 41 (*c*) 42 (*d*) 39

[Hint. $1 \times \frac{56!}{(50-r)!} = 30800 \times \frac{54!}{(51-r)!} \Rightarrow 51 - r = \frac{30800}{56 \times 55}$]

10. If a polygon has 35 diagonals, then the number of its sides is

(*a*) 8 (*b*) 9 (*c*) 10 (*d*) 11

[Hint. $^nC_2 - n = 35$]

11. The number of words that can be formed out of the letters of the word *ARTICLE* so that the vowels occupy even places is

(*a*) 36 (*b*) 574 (*c*) 144 (*d*) 754

[Hint. There are 4 odd places and 3 even places and 4 consonents and 3 vowels]

12. There are 10 lamps in a hall. Each one of them can be switched on independently. The number of ways in which the hall can be illuminated is

(*a*) 10^2 (*b*) 1023 (*c*) 2^{10} (*d*) 10!

13. In an examination there are three multiple choice questions and each question has 4 choices. Numbers of ways in which a student can fail to get all answers correct is

(*a*) 11 (*b*) 12 (*c*) 27 (*d*) 63.

[Hint. Each question can be answered in 4 ways and all questions can be answered correctly in only one way.].

14. The number of triangles that can be formed by choosing the vertices from a set of 12 points, seven of which lie on the same straight line, is

(*a*) 185 (*b*) 175 (*c*) 115 (*d*) 105

15. The number of divisors of 9600 including 1 and 9600 are

(*a*) 60 (*b*) 58 (*c*) 48 (*d*) 46

[Hint. Given $9600 = 2^7 \times 3 \times 5^2$, therefore, number of divisors = (7 + 1) (1 + 1) (2 + 1) = 48.]

16. How many different words can be formed by jumbling the letters in the word *MISSISSIPPI* in which no two *S* are adjacent ?

(*a*) $8.{}^{6}C_4.{}^{7}C_4$ (*b*) $6.7.{}^{8}C_4$ (*c*) $6.8.{}^{7}C_4$ (*d*) $7.{}^{6}C_4\,{}^{8}C_4$

[Hint. First let us arrange *MIIII PP*

This can be done in $\frac{7!}{4!2!}$ ways.

$M \times I \times I \times I \times I \times P \times P$

From the above it is clear that the four *S* can be put in any of the 8 places marked '×' so that no two *S* are adjacent in ${}^{8}C_4$ ways.

∴ Total number of required ways $= \frac{7!}{4!2!}.{}^{8}C_4 = 7 \times {}^{6}C_4 \times {}^{8}C_4$]

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*d*)	**4.** (*d*)
5. (*a*)	**6.** (*a*)	**7.** (*a*)	**8.** (*b*)
9. (*b*)	**10.** (*c*)	**11.** (*c*)	**12.** (*b*)
13. $4^3 - 1 = 63$	**14.** (*a*) ${}^{12}C_3 - {}^{7}C_3 = 185$	**15.** (*c*)	**16.** (*d*)

DETERMINANTS

Syllabus

- of order 2 and 3
- Minors and co-factors of a determinant
- Expansion of a determinant
- Properties of a determinant and their use in the evaluation of a determinant
- Product of determinants (without proof)
- Solution of simultaneous equations in 2 or 3 variables using Cramer's rule
- Conditions for consistency of 3 equations in two variables

HISTORICAL NOTE

The Japanese mathematician ***Seki Kowa*** (1683) systematized an old Chinese method of solving simultaneous linear equations whose coefficients were represented by calculating sticks, bamboo rods, placed in squares on a table with the positions of the different squares corresponding to the coefficients. In the process of working out his system, Kowa rearranged the rods in a way similar to that used in our simplification of determinants; thus it is thought that he had the idea of determinant.

Then years later in Europe ***Gottfied Wilhelm Von Leibnitz*** formally originated determinants and gave a written notation for them. The now-standard "vertical line notation" was given in 1841 by ***Arthur Cayley.***

Determinants were invented independently by ***Gabriel Cramer,*** whose now well-known rule for solving linear systems was published in 1750, although not in present day notation.

Many other mathematicians also made contributions to determinant theory—among them ***Alexandre Theophile Vandermonde, Pierre Simon Laplace. Josef Maria Wronski*** **and** ***Augustin Louis Cauchy*** were prominent. It was Cauchy who applied the word "determinant" to the subject.

2

Determinants

2.01. What is a determinant?

Any four members a, b, c and d are arranged in two rows and two columns between two vertical bars, as shown below, form what is called a determinant of the second order or second order determinant.

Thus the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ represents a second order determinant and its meaning or value is defined to be $ad - bc$.

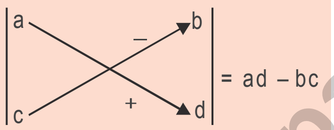


Fig. 2.01

For example, $\begin{vmatrix} 5 & -3 \\ 2 & 1 \end{vmatrix} = 5 \times 1 - (2 \times -3) = 5 - (-6) = 11.$

The numbers a, b, c and d are called the **elements** of the determinant and the expression $ad - bc$ on R.H.S. is called the **expansion** of the determinant. The elements a and b are said to be in the *first row*, the elements c and d in the *second row*, the elements a and c in the *first column* and the elements b and d in the *second column*. Thus a determinant of second order contains 2 rows and 2 columns. The first element, *i.e.*, a is called the leading element and the diagonal ad the leading diagonal.

If we consider the two simultaneous equations :

$$\left.\begin{aligned} a_1 x + b_1 y &= 0 \\ a_2 x + b_2 y &= 0 \end{aligned}\right\}$$

Elimination of x and y gives

$$a_1 b_2 - a_2 b_1 = 0$$

We may write the expression $(a_1 b_2 - a_2 b_1)$ in the determinant form as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Thus $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

If we denote an element in the first row and first column by a_{11}, an element in the first row and second column by a_{12} and an element in the second row and first column by a_{21} and an element in the second row and second column by a_{22} (the first subscripts refer to rows while the second subscripts refer to columns) then we may denote a second order determinant by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

Definition : *For any numbers a_1, b_1, a_2, b_2, the* **determinant**

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ has the value } \mathbf{a_1 b_2 - a_2 b_1}.$$

Caution. The determinant | – 3 | should not be confused with the absolute value | – 3 | which is equal to 3.

Note. We may name a given determinant by Δ or D.

Ex. 1. Evaluate the determinant $\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix}$.

Sol. $\Delta = \begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} = 1 \times 2 - (4 \times -3) = 2 + 12 = 14.$

Ex. 2. Find the value of the determinant

$$\begin{vmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{vmatrix}$$

Sol. $\Delta = \cos A \times \cos A - (-\sin A \times \sin A) = \cos^2 A + \sin^2 A = 1.$

EXERCISE 2 (*a*)

1. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}$ (*iii*) $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$

(*iv*) $\begin{vmatrix} x+2 & 2x+5 \\ 3x-1 & x-3 \end{vmatrix}$ (*v*) $\begin{vmatrix} y-x & -x^2+xy-y^2 \\ x+y & x^2+xy+y^2 \end{vmatrix}$

2. Prove that $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1.$

3. If $\begin{vmatrix} 3 & m \\ 4 & 5 \end{vmatrix} = 3$, find the value of m.

4. If $\begin{vmatrix} x-1 & x-2 \\ x & x-3 \end{vmatrix} = 0$, find the value of x.

5. Determine the value of k for which $\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 0.$

ANSWERS

1. (*i*) –18 (*ii*) 1 (*iii*) $a^2 + b^2$ (*iv*) $-5x^2 - 14x - 1$ (*v*) $2y^3$

3. $m = 3$ **4.** $x = \frac{3}{2}$ **5.** $k = 0, 2$

2.02. Determinant of order 3

An expression of the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a determinant of the third order. It contains 3 rows and 3 columns and 3^2, *i.e.*, 9 elements. A determinant of third order may be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}.$$

2.03. Value of a determinant of order 3

Consider the three simultaneous equations :

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

From the second and third of these equations, proportional values of x, y, z can be easily found by the method of cross-multiplication.

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - a_2c_3} = \frac{z}{a_2b_3 - a_3b_2} = K, \text{ (say)}$$

$$\Rightarrow \quad x = K\,(b_2c_3 - b_3c_2),\ y = K\,(c_2a_3 - a_2c_3),\ z = K\,(a_2b_3 - a_3b_2)$$

Substituting in the first equation, we get

$$a_1\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2) = 0 \qquad \text{...(1)}$$

The expression on left-hand side of (1), which is a function of nine elements $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ is represented by the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{...(2)}$$

Thus $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \text{...(3)}$$

$$= a_1\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2) \qquad \text{...(4)}$$

The above gives you an insight into the origin of determinants. By definition, the expression on the R.H.S. of (4) is called the value of expansion of the determinant given in (2).

This may be remembered by the following rule.

2.04. The rule of Sarrus

Sarrus gave a rule for a determinant of order 3. It should be noted that such a schematic representation does not exist for determinants of order greater than 3.

Rule. Write down the three columns of the determinant, and repeat the first two. The positive terms in the expansion are obtained from the elements on the lines, running diagonally from the top to bottom (dotted lines), and the negative terms from the elements running diagonally from bottom to top (full lines), beginning being made, in each case, from an element in the first column.

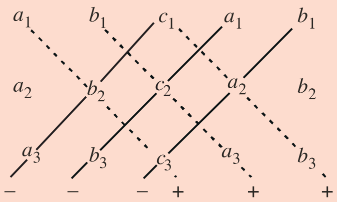


Fig. 2.02

Ex. 3. Evaluate $\begin{vmatrix} 3 & -2 & 2 \\ 6 & 1 & -1 \\ -2 & -3 & 2 \end{vmatrix}$.

Sol. By Sarrus diagram

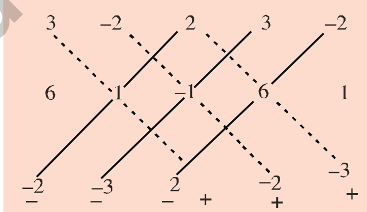


Fig. 2.03

The value of the determinant

$= \{3 \times 1 \times 2\} + \{(-2) \times (-1) \times (-2)\} + \{2 \times 6 \times (-3)\} - \{2 \times 1 \times (-2)\} - \{3 \times (-1) \times (-3)\} - \{(-2) \times 6 \times 2\}$

$= 6 - 4 - 36 + 4 - 9 + 24 = -15$

Aid to memory

We have defined the value of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

as $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ which can be put in the form

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

We start with the element a_1 in the first row and first column. We omit the other elements in that row and column as indicated below :

$$\begin{vmatrix} a_1 \cdots\cdots & b_1 \cdots\cdots & c_1 \cdots \\ \vdots & & \\ a_2 & b_2 & c_2 \\ \vdots & & \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The second order determinant that remains, is the multiplier of a_1 in the expression. The same procedure is followed for obtaining the sub-determinants corresponding to b_1 and c_1.

$$\begin{matrix} & \vdots & \\ a_1 \cdots\cdots & b_1 \cdots\cdots & c_1 \\ & \vdots & \\ a_2 & b_2 & c_2 \\ & \vdots & \\ a_3 & b_3 & c_3 \end{matrix} \qquad\qquad \begin{matrix} & & \vdots \\ a_1 \cdots\cdots & b_1 \cdots\cdots & c_1 \\ & & \vdots \\ a_2 & b_2 & c_2 \\ & & \vdots \\ a_3 & b_3 & c_3 \end{matrix}$$

The signs of the multipliers are taken to be alternately positive and negative. This is called the expansion of the determinant by *the elements of the first row.*

Important note. We can expand a given determinant by the elements of any row or column and prefix signs as under :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The signs, *alternate*, starting with (+) in the left upper hand corner.

Aid to memory. The sign corresponding to i^{th} row and j^{th} column will be $(-1)^{i+j}$. Thus the sign corresponding to element of the third row and second column, *i.e.*, a_{32} will be $(-1)^{3+2}$ or '–', similarly the sign that will go with element of the third row and third column, *i.e.*, a_{33} is $(-1)^{3+3}$ or '+'.

Thus, if we expand the determinant by the element of the first column, we have

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

The expansion by using the second row will be

$$-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Working rule : 1. *You may multiply by the elements of any row or column.*

2. *Multiply each element of a row or column you have chosen by the sub-determinant obtained by deleting the row and the column in which that element lies. Take signs of products as per rule given above.*

Ex. 4. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{4} & \mathbf{1} & \mathbf{-1} \\ \mathbf{5} & \mathbf{-3} & \mathbf{2} \end{vmatrix}$

Sol. Expanding, using the first row, we have

$$\Delta = 1\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix}$$
$$= 1(2-3) - 3(8+5) - 2(-12-5) = -6.$$

Ex. 5. Find the value of $\begin{vmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-8} \\ \mathbf{0} & \mathbf{-5} & \mathbf{7} \end{vmatrix}$

Sol. Since there are zeros in the first column, it is more convenient to expand by using the first column.

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -8 \\ 0 & -5 & 7 \end{vmatrix} = 3\begin{vmatrix} 1 & -8 \\ -5 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ -5 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & -8 \end{vmatrix} = 3(7-40) = 3(-33) = -99.$$

Ex. 6. Prove that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$

Sol. Expanding by using first row, we have :

$$\Delta = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix} = a(bc - f^2) - h(ch - fg) + g(hf - bg)$$
$$= abc - af^2 - ch^2 + fgh + fgh - bg^2 = abc + 2fgh - af^2 - bg^2 - ch^2.$$

2.05. Minors and co-factors

Definition. The minor of an element *in a determinant is a determinant that is left after removing the row and the column which intersect at the element, and is of order one less than that of the given determinant.*

Consider the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$. Similarly, the minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, and that of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$, and so on.

In general, the minor M_{ij} of an element a_{ij} is the value of the determinant obtained by deleting the i-th row and j-th column of the given determinant.

Co-factors. If we apply the appropriate sign to the minor of an element, we have its **co-factor**. Thus the minor of b_1 is

$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and the co-factor of b_1 denoted by

B_1 is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, *i.e.*, $B_1 = -(a_2 c_3 - a_3 c_2)$.

$$\boldsymbol{A_{ij} = (-1)^{i+j} M_{ij}}$$

Co-factor of a_{ij} Minor of a_{ij}

It is clear that the minor and the co-factor of an element have the same value when the sum of the number of the row and the number of the column of the element is an even number.

In general, *the* **co-factor** A_{ij} *of* a_{ij} *is defined as* $\mathbf{A_{ij} = (-1)^{i+j} M_{ij}}$.

Obviously, the co-factor of a_{ij} is M_{ij} or $-M_{ij}$ according as $(i + j)$ is even or odd.

Remark. It is easy to see that

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 = b_1 B_1 + b_2 B_2 + b_3 B_3 = a_1 A_1 + b_1 B_1 + c_1 C_1 \text{ etc.,}$$

i.e., the value of a determinant is obtained by multiplying the elements with co-factors and adding the resulting products.

Also, it can be easily verified that if we multiply the elements of any row (or column) with the corresponding co-factors of any other row (or column) and add them, the result is zero, *i.e.*, $a_1B_1 + a_2 B_2 + a_3 B_3 = 0$, $a_1 C_1 + a_2 C_2 + a_3 C_3 = 0$ etc.

Ex. 7. Find the minors and co-factors of the elements of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Sol. The element a_1 occurs in the first row and first column.

Therefore, to find the minor of a_1, we delete the first row and first column of the given determinant. The minor M_{11} of a_1 is given by

$$M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)$$

The co-factor of a_1 is $A_{11} = (-1)^{1+1} M_{11} = M_{11} = (b_2c_3 - b_3c_2)$

Similarly, minor of b_1 is $M_{12} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = (a_2c_3 - a_3c_2)$

Co-factor of b_1 is $A_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_2 c_3 - a_3 c_2) = (a_3c_2 - a_2c_3)$

Minor of c_1 is $M_{13} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)$

Co-factor of c_1 is $A_{13} = (-1)^{1+3} M_{13} = M_{13} = (a_2b_3 - a_3b_2)$.

In a similar manner, we can find the minor and co-factor of each of the remaining elements.

Ex. 8. Find the minor and co-factor of each element of the determinant

$$\begin{vmatrix} 2 & -2 & 3 \\ 1 & 4 & 5 \\ 2 & 1 & -3 \end{vmatrix}.$$

Sol. The minors are

$$M_{11} = \begin{vmatrix} 4 & 5 \\ 1 & -3 \end{vmatrix} = -17, \quad M_{12} = \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = -13, \quad M_{13} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7$$

$$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 3, \quad M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12, \quad M_{23} = \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = 6$$

$$M_{31} = \begin{vmatrix} -2 & 3 \\ 4 & 5 \end{vmatrix} = -22, \quad M_{32} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7, \quad M_{33} = \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} = 10$$

The co-factors are

$$A_{11} = (-1)^{1+1} M_{11} = M_{11} = -17, \quad A_{12} = (-1)^{1+2} M_{12} = -M_{12} = 13,$$

$$A_{13} = (-1)^{1+3} M_{13} = -M_{13} = -7$$

$$A_{21} = (-1)^{2+1} M_{21} = -M_{21} = -3, \quad A_{22} = (-1)^{2+2} M_{22} = M_{22} = -12,$$

$$A_{23} = (-1)^{2+3} M_{23} = -M_{23} = -6 \quad A_{31} = (-1)^{3+1} M_{31} = M_{31} = -22,$$

$$A_{32} = (-1)^{3+2} M_{32} = -M_{32} = -7, \quad A_{33} = (-1)^{3+3} M_{33} = M_{33} = 10.$$

2.06. Value of a determinant in terms of minors and co-factors

Now we can define the value of a determinant of order 3 as follows :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

$$= a_1 \text{ (its minor)} - b_1 \text{ (its minor)} + c_1 \text{ (its minor)}.$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

This is called the expansion of the determinant by minors of the elements of the first row.

Note 1. Also from the above, we have

$$\boldsymbol{\Delta = a_1 \text{ . (its co-factor)} + b_1 \text{ . (its co-factor)} + c_1 \text{ . (its co-factor)}}$$

($\because$ Co-factor of b_1 = negative of minor of b_1)

Note 2. Since it is possible, however, to evaluate a determinant by expanding by minors, using *any* row or column, not just the first row, therefore, there are six distinct ways of expanding by minors a determinant of order 3.

Note 3. We may point out that each element of the determinant is paired with a particular minor; thus, **a function is established.** The ordered pairs are (an element, its minor).

Note 4. While expanding a determinant by any row or column using minors, we may keep in mind the following scheme of signs for a third order determinant.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Note 5. If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.

Ex. 9. Express the value of the determinant $\begin{vmatrix} -2 & 4 & 2 \\ 1 & 5 & -3 \\ 5 & -2 & 2 \end{vmatrix}$ in terms of the minors of the elements of the third row.

Sol. Value of the determinant

= sum of the elements each multiplied by the associated co-factor

$$= (5)\left\{+\begin{vmatrix} 4 & 2 \\ 5 & -3 \end{vmatrix}\right\} + (-2)\left\{-\begin{vmatrix} -2 & 2 \\ 1 & -3 \end{vmatrix}\right\} + (2)\left\{+\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix}\right\}$$

Upon evaluating each of the 2nd order determinants the value – 130 is obtained.

EXERCISE 2 (*b*)

1. Write the minors and co-factors of each element of the first column of the following determinants and evaluate the determinant in each case.

(*i*) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (*iii*) $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 5 & 1 & 0 \\ 2 & 3 & -1 \\ -3 & 2 & 0 \end{vmatrix}$. ***(ISC)***

3. Find the value of Δ, where $\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$.

4. Find the value of the determinants

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$ (ii) $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

(iii) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (iv) $\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix}$.

5. Expand the determinants by minors of the given row or column.

(i) $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 6 & 1 & 1 \end{vmatrix}$; column 1 (ii) $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$; column 2

(iii) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & 3 & -1 \\ 4 & 2 & 3 \end{vmatrix}$; row 2 (iv) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix}$; row 1

6. Solve for x :

(i) $\begin{vmatrix} x & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix} = 3$ (ii) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

7. Show that

$$\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

8. Expand and simplify the following :

(i) $\begin{vmatrix} 1 & x & y \\ 0 & \cos x & \sin y \\ 0 & \sin x & \cos y \end{vmatrix}$ (ii) $\begin{vmatrix} 0 & \tan\theta & 1 \\ 1 & -\sec\theta & 0 \\ \sec\theta & \tan\theta & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} \sin\theta & 1 & 0 \\ 0 & \cos\phi & -\cos\theta \\ \sin\phi & 0 & 1 \end{vmatrix}$

9. If one root of $\begin{vmatrix} 7 & 6 & x \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$ is $x = -9$, find the other roots.

ANSWERS

1. (i) $M_{11} = -1$, $M_{21} = 20$, $A_{11} = -1$, $A_{21} = -20$, value of determinant = – 5

(ii) $M_{11} = \begin{vmatrix} b & ca \\ c & ab \end{vmatrix}$, $M_{21} = \begin{vmatrix} b & bc \\ c & ab \end{vmatrix}$, $M_{31} = \begin{vmatrix} a & bc \\ b & ca \end{vmatrix}$

$A_{11} = \begin{vmatrix} b & ca \\ c & ab \end{vmatrix}$, $A_{21} = -\begin{vmatrix} a & bc \\ c & ab \end{vmatrix}$, $A_{31} = \begin{vmatrix} a & bc \\ b & ca \end{vmatrix}$

Value of the determinant = $ab\,(b-a) + bc\,(c-b) + ac\,(a-c)$

(iii) $M_{11} = \begin{vmatrix} 5 & 0 \\ 7 & 1 \end{vmatrix}$, $M_{21} = \begin{vmatrix} 2 & 6 \\ 7 & 1 \end{vmatrix}$, $M_{31} = \begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix}$.

$A_{11} = \begin{vmatrix} 5 & 0 \\ 7 & 1 \end{vmatrix}$, $A_{21} = -\begin{vmatrix} 2 & 6 \\ 7 & 1 \end{vmatrix}$, $A_{31} = \begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix}$.

Value of the determinant = – 50.

2. 13 **3.** 0 **4.** (i) 0 (ii) $\lambda^2(3x + \lambda)$ (iii) – 8 (iv) 0

5. (i) 67 (ii) 5 (iii) 53 (iv) 4 **6.** (i) $x = 3$ (ii) $x = 2, \dfrac{-17}{7}$

8. (i) $\cos(x + y)$ (ii) $\sec^2\theta$ (iii) $\sin(\theta - \phi)$ **9.** 2, 7

2.07. Applications of determinants

Area of a triangle

You have learnt in class XI that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by the expression

$$\frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$= \frac{1}{2}\,[x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2y_3 - x_3y_2]$$

This expression is the expansion of the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Remark. Since the area has to be a positive quantity, we always take the absolute value of the determinant for the area.

Hence, area of triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Note : The area of the quadrilateral *ABCD*, with vertices at (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) respectively, is the sum of the areas of the two triangles *ABC* and *ACD*. Hence

Area of quadrilateral *ABCD* = Area of $\Delta\,ABC$ + Area of $\Delta\,ACD$

$$= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| + \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\right|.$$

Ex. 10. Find the area of the triangle whose vertices are

(–2, – 3), (3, 2), (–1, – 8).

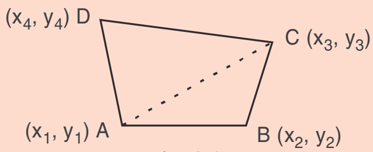


Fig. 2.04

Sol. The area of the triangle is given by

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}[1(-24 + 2) - 1(16 - 3) + 1(-4 + 9)]\right|$$

(by expanding along the third column).

$$= \left|\frac{1}{2}[-22 - 13 + 5]\right| = 15$$ Hence, required area = 15 sq units.

Ex. 11. If the points (a, b) (a', b') and $(a - a', b - b')$ are collinear, show that $ab' = a'b$.

Sol. The given points are collinear if the area of the triangle formed by them is zero,

i.e., if $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a - a' & b - b' & 1 \end{vmatrix} = 0$

Expanding along the third column, we get,

$$\Rightarrow\quad 1\begin{vmatrix} a' & b' \\ a - a' & b - b' \end{vmatrix} - 1\begin{vmatrix} a & b \\ a - a' & b - b' \end{vmatrix} + 1\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 0$$

$$\Rightarrow\quad a'(b - b') - b'(a - a') - a(b - b') + b(a - a') + ab' - a'b = 0$$

$$\Rightarrow\quad a'b - a'b' - ab' + a'b' - ab + ab' + ab - a'b + ab' - a'b = 0$$

$$\Rightarrow\quad ab' - a'b = 0 \Rightarrow ab' = a'b.$$

Note. After doing properties of determinants you can simplify more easily by applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, and expanding along C_3. $\Rightarrow \frac{1}{2}\begin{vmatrix} a & b & 1 \\ a'-a & b'-b & 0 \\ -a' & -b' & 0 \end{vmatrix} = 0$

Ex. 12. Find the value of *x*, if the area of the triangle with vertices (*x*, 4) (2, – 6) and (5, 4) be 35 sq. cm.

Sol. Let the vertices be A $(x, 4)$, B $(2, -6)$ and C $(5, 4)$. As, shown, there are two different possible positions of the vertex A $(x, 4)$.

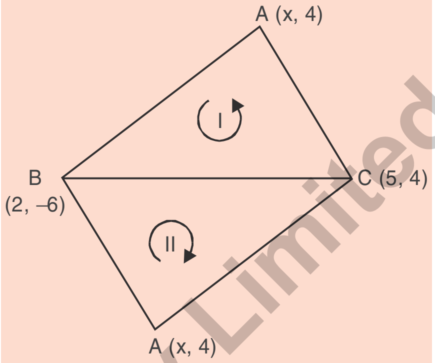


Fig. 2.05

For triangle I, the vertices A, B, C are anticlockwise.

$\therefore$ Area $\Delta ABC = 35$

$$\Rightarrow \left| \frac{1}{2}\begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} \right| = 35$$

$$\Rightarrow \quad x(-6-4) - 4(2-5) + 1(8+30) = 70 \text{ or } -70$$

Case (*i*) $\Rightarrow -10x + 12 + 38 = 70 \Rightarrow 10x = -20$ *i.e.* $x = -2$

Case (*ii*) $\Rightarrow x(-6-4) - 4(2-5) + 1(8+30) = -70$

$$\therefore \quad -10x + 12 + 38 = -70 \quad \Rightarrow \quad 10x = 120 \text{ i.e. } x = 12$$

$$\therefore \quad x = -2, 12$$

EXERCISE 2 (*c*)

1. Find the area of the triangle whose vertices are

(*i*) (– 8, – 2), (– 4, – 6), (– 1, 5) (*ii*) (3, 8), (– 4, 2), (5, – 1)

(*iii*) (– 3, 5), (3, – 6), (7, 2).

2. Using determinants, prove that the following points are collinear

(*i*) (11, 7), (5, 5), (– 1, 3) (*ii*) (0, 3), (4, 6) and (– 8, – 3) (*iii*) (– 2, 5), (– 6, – 7), (– 5, – 4).

3. Using determinants, find the area of the triangle whose vertices are (4, – 2), (– 2, 3) and (3, 5). Are the given points collinear ?

4. Find the area of the quadrilateral whose vertices are A (1, – 1), B (3, 1), C (– 2, 3) and D (– 1, – 2).

5. Show that the points $(b, c + a)$, $(c, a + b)$ and $(a, b + c)$ are collinear.

6. (*i*) Find x so that the points, (3, – 2), (x, 2) and (8, 8) be on a line.

(*ii*) For what value of k the points (5, 5), (k, 1) and (11, 7) are collinear.

7. If (x, y), $(a, 0)$, $(0, b)$ are collinear, then using determinants prove that $\frac{x}{a} + \frac{y}{b} = 1$.

ANSWERS

1. (*i*) 28 sq units (*ii*) 37.5 sq units (*iii*) 46 sq units

3. 18.5 sq units, not collinear. **4.** $\frac{25}{2}$ sq. units **6.** (*i*) $x = 5$ (*ii*) $k = -7$

2.08. Properties of determinants

Determinants have some properties that are useful by virtue of the fact that they permit to generate equal determinants with different and simpler configurations of entries. This, in turn, helps us find values of determinants. In other words, they help us in their *transformations*. We shall list these properties as shown below, and illustrate them by using third order determinants, but the properties are valid for determinants of any order.

Property 1. ***If each entry in any row, or each entry in any column, of a determinant is 0, then the determinants is equal to 0.***

For example, $\begin{vmatrix} 3 & 1 & 5 \\ 0 & 0 & 0 \\ 2 & 4 & -2 \end{vmatrix} = -0\begin{vmatrix} 1 & 5 \\ 4 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 5 \\ 2 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 0$.

Property 2. ***If rows be changed into columns and columns into rows, the determinant remains unaltered.***

For example, $\begin{vmatrix} 5 & 2 & 11 \\ 3 & 4 & 0 \\ -7 & 8 & -6 \end{vmatrix} = \begin{vmatrix} 5 & 3 & -7 \\ 2 & 4 & 8 \\ 11 & 0 & -6 \end{vmatrix}$ You may evaluate the values of the two determinants and verify.

Property 3. ***If any two rows (or columns) of a determinant are interchanged, the resulting determinant is the negative of the original determinant.***

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$

For example, if $\Delta_1 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix}$ and $\Delta_2 = -\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix}$ then, $\Delta_1 = -\Delta_2$.

(Formed by interchanging C_1 & C_2 of Δ_1)

We have, $\Delta_1 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix}$

$= 1(1 \times 1 - 0 \times 2) - 2(2 \times 1 - (-1) \times 2) + 1(2 \times 0 - (-1) \times 1)$ expanding by 1st column

$= 1 - 8 + 1 = -6$

$\Delta_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix}$

$= 2(2 \times 1 - (-1) \times 2) - 1(1 \times 1 - 0 \times 2) + 1(1 \times (-1) + 0 \times 2)$ expanding by 1st row

$= 8 - 1 - 1 = 6$

$\therefore$ Since $\Delta_1 = -6$ and $\Delta_2 = 6$, therefore, $\Delta_1 = -\Delta_2$ or $\Delta_2 = -\Delta_1$.

Cor. 1. *If any line of a determinant Δ be passed over m parallel lines, the resulting determinant* $\Delta' = (-1)^m \Delta$. For example,

if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$ then $\Delta' = (-1)^2 \Delta = \Delta$.

Cor. 2. *The sign of any term will be positive or negative according as that term can be made the principal term in Δ' by an even or odd number of movements of lines of Δ.*

For example, $\begin{vmatrix} -2 & 1 & -4 \\ 5 & 7 & 1 \\ 0 & 2 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 & -4 \\ 7 & 5 & 1 \\ 2 & 0 & 0 \end{vmatrix}$ (Number of jumped columns = 1)

$= (-1)^2\begin{vmatrix} 1 & -4 & -2 \\ 7 & 1 & 5 \\ 2 & 0 & 0 \end{vmatrix}$ (Number of jumped columns = 2)

Property 4. ***If two rows (or two columns) in a determinant are identical, the determinant is equal to zero.***

i.e., $\boldsymbol{\Delta} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \mathbf{0}$

Proof.

$$\begin{aligned}\Delta &= a_1(b_1c_2 - b_2c_1) - b_1(a_1c_2 - a_2c_1) + c_1(a_1b_2 - a_2b_1)\\ &= a_1b_1c_2 - a_1b_2c_1 - a_1b_1c_2 + a_2b_1c_1 + a_1b_2c_1 - a_2b_1c_1 = 0\end{aligned}$$

Hence proved.

Illustration:

$$\begin{vmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{3} & \mathbf{-2} \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-4-12) - 3(-6-4) - 2(9-2) = -16 + 30 - 14 = 0.$$

Property 5. ***If all the elements of any row (or column) be multiplied by a non zero real number k, then the value of the new determinant is k times the value of the original determinant.***

Thus, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **and** $\Delta' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **then** $\Delta' = k\Delta$.

For example, $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3(2) & 3(-1) & 3(5) \end{vmatrix} = 3\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 2 & (-1) & 5 \end{vmatrix}.$

$$\text{L.H.S.} = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 6 & -3 & 15 \end{vmatrix} = 1(1\times 15 + 3\times 4) - 3(2\times 15 - 6\times 4) - 1(2\times -3 - 6\times 1)$$

$$= 1\times 27 - (3\times 6) - 1(1\times -12) = 27 - 18 + 12 = 21.$$

$$\text{R.H.S.} = 3\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 2 & -1 & 5 \end{vmatrix} = 3[1(1\times 5 + 1\times 4) - 3(2\times 5 - 2\times 4) - 1(2\times -1 - 2\times 1)]$$

$$= 3[1\times 9 - 3\times 2 - (1\times -4)]$$
$$= 3[9 - 6 + 4) = 3\times 7 = 21.$$

$\Rightarrow$ L.H.S. = R.H.S.

Cor. *If two parallel lines (rows or columns) be such that the elements of one are the equi-multiplies of the elements of the other, the determinant is equal to 0.*

Property 6. ***If each entry in a row (or column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants,***

i.e., $\Delta = \begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}.$

Illustration:

$$\begin{vmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2+3 & 3 \\ 2 & 1+5 & 2 \\ 3 & 4+3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 3 \\ 2 & 5 & 2 \\ 3 & 3 & 1 \end{vmatrix}$$

L.H.S. $= 1(6-14) - 2(5-21) + 3(10-18) = -8 + 32 - 24 = 0$

R.H.S. $= [1(1-8) - 2(2-12) + 3(4-3)] + [1(5-6) - 2(3-9) + 3(6-15)]$

$= (-7 + 20 + 3) + (-1 + 12 - 27) = 16 - 16 = 0.$

Thus, L.H.S. = R.H.S.

Property 7. ***If each entry of one row (or column) of a determinant is multiplied by a real number k and the resulting product is added to the corresponding entry in another row (or column respectively) in the determinant, then the resulting determinant is equal to the original determinant.***

i.e., if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **and** $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 + ka_1 & b_3 + kb_1 & c_3 + kc_1 \end{vmatrix}$ **then** $\Delta = \Delta'$.

Proof : $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$

$= \Delta + k \,.\, 0 = \Delta$ (Since 2 rows are identical in the 2nd determinant)

For example, $\begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 1+3(1) & -2+3(3) & 5+3(-2) \end{vmatrix}$

L.H.S. $= 1(20+6) - 0(15-4) + 1(9+8) = 26 + 17 = 43$

R.H.S. $= \begin{vmatrix} 1 & 3 & -2 \\ 0 & 4 & 3 \\ 4 & 7 & -1 \end{vmatrix} = 1(-4-21) - 0(-3+14) + 4(9+8) = -25 + 68 = 43.$

Thus, L.H.S. = R.H.S.

Property 8. ***If to each element of a line (row or column) of a determinant be added the equi-multiples of the corresponding elements of one or more parallel lines, the determinant remains unaltered, i.e.,***

$$\begin{vmatrix} a_1 + la_2 + ma_3 & a_2 & a_3 \\ b_1 + lb_2 + mb_3 & b_2 & b_3 \\ c_1 + lc_2 + mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

L.H.S. $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} la_2 & a_2 & a_3 \\ lb_2 & b_2 & b_3 \\ lc_2 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ma_3 & a_2 & a_3 \\ mb_3 & b_2 & b_3 \\ mc_3 & c_2 & c_3 \end{vmatrix}$ (By Property 6)

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + l \begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} + m \begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix}$ (By Property 5)

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + l\,(0) + m\,(0)$ (By Property 4)

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$ R.H.S.

A useful notation. Suppose to the 5th row we add m times the 7th row and n times the ninth row, we can denote this operations by $R_5 \to R_5 + mR_7 + nR_9$. Similarly $C_3 \to C_3 + qC_5 + rC_7$ will stand for the statement "to the third column add q times the fifth column and r times the seventh column."

Note. These properties, specially property 8, provide very powerful methods for calculating the values of determinants. We try to get as many zeros as possible in a row or a column.

Property 9. ***If the elements of a determinant that involve x are polynomials in x, and if the determinant is equal to 0 when a is substituted for x, then x – a is a factor of the determinant.***

Since the element of the determinant involving x are polynomials in x, the expansion of the determinant will also be a polynomial in x. As the determinant is 0, for $x = a$, $x - a$ is a factor of the determinant.

For example take the following determinant

$$\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

If a be substituted for b, first two columns become identical and, therefore, $\Delta = 0$. Thus, $a - b$ is a factor, similarly $b - c$ and $c - a$ are factors.

Note. A transformation of any of the following type is called an **elementary transformation :**

(*i*) Interchange of two rows or two columns.

(*ii*) Multiplication of a row or column of a determinant by a constant different from zero.

(*iii*) Addition to the entries of a line of the determinant the constant multiples of a parallel line.

If two determinants are such that each can be obtained from the other by a finite number of elementary transformations, they are said to be **equivalent.**

Property 10. Product of two determinants :

The product of two determinants is obtained by multiplying

(1) row by row (2) column by column (3) column by row and (4) row by column

Thus, $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

Note 1. The above is **row** *by* **row** multiplication. As stated above you can also multiply rows by columns or columns by columns.

Note 2. We are not stating the proof here. This is not required at this level.

Illustrations:

1. Let $\Delta_1 = \begin{vmatrix} 3 & -6 \\ 4 & 8 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 2 & 3 \\ -5 & 7 \end{vmatrix}$, then

$$\Delta_1\Delta_2 = \begin{vmatrix} 3 & -6 \\ 4 & 8 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ -5 & 7 \end{vmatrix} = \begin{vmatrix} 3\times 2 + (-6)\times 3 & 3\times(-5) + (-6)\times 7 \\ 4\times 2 + 8\times 3 & 4\times(-5) + 8\times 7 \end{vmatrix}$$

$$= \begin{vmatrix} 6-18 & -15-42 \\ 8+24 & -20+56 \end{vmatrix} = \begin{vmatrix} -12 & -57 \\ 32 & 36 \end{vmatrix}$$

$$= -12\times 36 - (32\times -57) = -432 + 1824 = \mathbf{1392.}$$

2. Let $\Delta_1 = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} -3 & -2 & 0 \\ 0 & 5 & -1 \\ 4 & 6 & 0 \end{vmatrix}$, then

$$\Delta_1\Delta_2 = \begin{vmatrix} 1\times(-3)+3\times(-2)+5\times 0 & 1\times 0+3\times 5+5\times(-1) & 1\times 4+3\times 6+5\times 0 \\ 2\times(-3)+4\times(-2)+6\times 0 & 2\times 0+4\times 5+6\times(-1) & 2\times 4+4\times 6+6\times 0 \\ 3\times(-3)+5\times(-2)+7\times 0 & 3\times 0+5\times 5+7\times(-1) & 3\times 4+5\times 6+7\times 0 \end{vmatrix}$$

$$= \begin{vmatrix} -3-6+0 & 0+15-5 & 4+18+0 \\ -6-8+0 & 0+20-6 & 8+24+0 \\ -9-10+0 & 0+25-7 & 12+30+0 \end{vmatrix} = \begin{vmatrix} -9 & 10 & 22 \\ -14 & 14 & 32 \\ -19 & 18 & 42 \end{vmatrix}$$

$$= -9\,(14\times 42 - 18\times 32) - (-14)\,(10\times 42 - 18\times 22) - 19\,(10\times 32 - 14\times 22)$$

$$= -9\,(588 - 576) + 14\,(420 - 396) - 19\,(320 - 308)$$

$$= -9\times 12 + 14\times 24 - 19\times(-12) = -108 + 336 + 228 = \mathbf{456.}$$

Property 11. ***If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding co-factors, then $\Delta' = \Delta^2$.***

It suffices to illustrate the theorem for a determinant of third order.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$, then

$$A_1 = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 2,\ A_2 = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = -18,$$

$$A_3 = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -8,\ B_1 = (-1)^{1+2}\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = 7,\ B_2 = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3,$$

$$B_3 = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = -8,\ C_1 = (-1)^{1+3}\begin{vmatrix} -3 & 2 \\ 2 & 4 \end{vmatrix} = 8,\ C_2 = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = 8,$$

$$C_3 = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 8$$

$$\therefore \quad \Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 8 \\ -18 & -3 & 8 \\ -8 & -8 & 8 \end{vmatrix}$$

$$= 2\begin{vmatrix} -3 & 8 \\ -8 & 8 \end{vmatrix} - 7\begin{vmatrix} -18 & 8 \\ -8 & 8 \end{vmatrix} + 8\begin{vmatrix} -18 & -3 \\ -8 & -8 \end{vmatrix}$$

$$= 2(-24 + 64) - 7(-144 + 64) + 8(144 - 24)$$

$$= 2 \times 40 - 7 \times (-80) + 8 \times 120 = 80 + 560 + 960$$

$$= 1600 \qquad \text{... (1)}$$

$$\Delta^2 = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1\times1+2\times2+3\times3 & 1\times-3+2\times2+3\times(-1) & 1\times2+2\times(-4)+3\times3 \\ -3\times1+2\times2+(-1)\times3 & (-3)\times(-3)+2\times2+(-1)\times(-1) & (-3)\times2+2\times(-4)+(-1)\times3 \\ 2\times1+(-4)\times2+3\times3 & 2\times(-3)+(-4)\times2+(3)\times(-1) & 2\times2+(-4)\times(-4)+3\times3 \end{vmatrix}$$

$$= \begin{vmatrix} 14 & -2 & 3 \\ -2 & 14 & -17 \\ 3 & -17 & 29 \end{vmatrix} = 14\begin{vmatrix} 14 & -17 \\ -17 & 29 \end{vmatrix} + 2\begin{vmatrix} -2 & -17 \\ 3 & 29 \end{vmatrix} + 3\begin{vmatrix} -2 & 14 \\ 3 & -17 \end{vmatrix}$$

$$= 14(406 - 289) + 2(-58 + 51) + 3(34 - 42)$$

$$= 14 \times 117 + 2\times(-7) + 3\times(-8) = 1638 - 14 - 24 = 1600 \qquad \text{... (2)}$$

(1) and (2) $\Rightarrow \Delta' = \Delta^2$.

Ex. 13. Without expanding, *i.e.*, using properties of determinants, show that

(*i*) $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix} = 0$ **(*ii*)** $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix} = 0$

(*iii*) $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix} = 0$ **(*ISC*)** **(*iv*)** $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$ **(*ISC*)**

Sol. (*i*) Operating $C_1 \to C_1 - 7C_3$, we have

$$\Delta = \begin{vmatrix} 43-7\times 6 & 1 & 6 \\ 35-7\times 4 & 7 & 4 \\ 17-7\times 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} = 0,\ C_1 \text{ and } C_2 \text{ being identical} \quad \text{(Property 4)}$$

(*ii*) First take out 3 common from C_2 and then 4 common from C_3

$$\Delta = 3\begin{vmatrix} 9 & 3 & 12 \\ 1 & -1 & -4 \\ 1 & 3 & 12 \end{vmatrix} = 3\times 4\begin{vmatrix} 9 & 3 & 3 \\ 1 & -1 & -1 \\ 1 & 3 & 3 \end{vmatrix} = 12\times 0 = 0,\ (C_2 \text{ and } C_3 \text{ being identical}).$$

(*iii*) Given, $\Delta = \begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & b & ca \\ \frac{1}{c} & c & ab \end{vmatrix}$ Multiply R_1 by a, R_2 by b and R_3 by c. Then,

$$\Delta = \frac{1}{abc}\begin{vmatrix} 1 & a^2 & abc \\ 1 & b^2 & abc \\ 1 & c^2 & abc \end{vmatrix} = \frac{1}{abc}\,.\,abc\begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} = 1\times 0 = 0.$$

[Two columns being identical]

(*iv*) Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = a^2b^2c^2\begin{vmatrix} bc & 1 & \frac{1}{b}+\frac{1}{c} \\ ca & 1 & \frac{1}{c}+\frac{1}{a} \\ ab & 1 & \frac{1}{a}+\frac{1}{b} \end{vmatrix} = 0$

[Taking bc common from R_1, ca from R_2 and ab from R_3]

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Delta = a^2b^2c^2\begin{vmatrix} bc & 1 & \frac{1}{b}+\frac{1}{c} \\ c\,(a-b) & 0 & \frac{1}{a}-\frac{1}{b} \\ b\,(a-c) & 0 & \frac{1}{a}-\frac{1}{c} \end{vmatrix} = a^2b^2c^2\begin{vmatrix} bc & 1 & \frac{c+b}{bc} \\ c\,(a-b) & 0 & -\frac{(a-b)}{ab} \\ b\,(a-c) & 0 & -\frac{(a-c)}{ac} \end{vmatrix}$$

$$= a^2b^2c^2\,(a-b)\,(a-c)\begin{vmatrix} bc & 1 & \frac{c+b}{bc} \\ c & 0 & -\frac{1}{ab} \\ b & 0 & -\frac{1}{ac} \end{vmatrix}$$

$$= \frac{a^2b^2c^2\,(a-b)\,(a-c)}{ab\,.\,ac}\begin{vmatrix} bc & 1 & \frac{c+b}{bc} \\ abc & 0 & -1 \\ abc & 0 & -1 \end{vmatrix} \quad \text{(Multiplying } R_2 \text{ by } ab \text{ and } R_3 \text{ by } ac)$$

$$= bc\,(a-b)\,(a-c)\times 0 = 0 \qquad [R_2 \text{ and } R_3 \text{ being identical rows}]$$

Ex. 14. For positive number x, y and z, show that the numerical value of the determinant

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0. \qquad \textbf{\textit{(IIT)}}$$

Sol. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$, $\left(\text{Using } \log_n m = \dfrac{\log m}{\log n}\right)$

$$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

multiplying R_1, R_2, R_3 by $\log x, \log y, \log z$ respectively.

$= 0$ $\quad (\because\ R_1, R_2, R_3$ are identical)

Ex. 15. Without expanding the determinant, prove that

(*i*) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0;$ **(*ISC*)**

(*ii*) $\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3.$ **(*ISC*)**

Sol. (*i*) Operating $R_1 \to R_1 + R_2$, we have

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0, \text{ since two rows are identical.}$$

(*ii*)

$$\Delta = \begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = \begin{vmatrix} 3x & 2x & x \\ 4x & 3x & 3x \\ 5x & 4x & 6x \end{vmatrix} + \begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix}$$

$$= x^3\begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y\begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ 6 & 4 & 6 \end{vmatrix}$$

$$= x^3\begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y\,(0) \quad \text{since } C_1 \text{ and } C_3 \text{ are identical.}$$

$$= x^3[3(18-12) - 2(24-15) + 1(16-15)] = x^3(18-18+1) = x^3.$$

Ex. 16. Show without expanding the determinant that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} c & b & a \\ b & a & c \\ a & c & b \end{vmatrix}.$

Sol. Let $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ then $(-D) = \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix}$ (Interchanging C_1 and C_3)

Again, $-(-D) = \begin{vmatrix} c & b & a \\ b & a & c \\ a & c & b \end{vmatrix}$ (Interchanging R_2 and R_3)

$\Rightarrow$ $D = \begin{vmatrix} c & b & a \\ b & a & c \\ a & c & b \end{vmatrix}.$

Ex. 17. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$ **(ISC)**

Sol. Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Expanding by first column

$$\Delta = a\begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix} = a\,(7a^2 + 3ab - 6a^2 - 3ab) = a^3.$$

Ex. 18. Without expanding the determinant at any stage, prove that

$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ **= 0, where a, b, c are in A.P.** **(NMOC)**

Sol. Given a, b, c are in A.P. $\therefore$ $b - a = c - b = d$ (common difference) ...(1)

Let $$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$. Then,

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & b-a \\ 1 & 1 & c-a \end{vmatrix} = \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & d \\ 1 & 1 & d \end{vmatrix}$$ [From (1)]

= 0, since R_2 and R_3 are identical.

Ex. 19. Using properties of determinants, prove that

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2).$ **(ISC)**

Sol. $$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$$, Operating $C_1 \to C_1 + C_2 + C_3$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

[Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b+c)\{(c-b)(b-c)-(a-b)(a-c)\}$, expanding along 1st column.

$= (a+b+c)\{bc-b^2-c^2+bc-(a^2-ab-ac+bc)\}$

$= (a+b+c)(ab+bc+ca-a^2-b^2-c^2).$

Ex. 20. Prove the following identity : $\begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix} = 4\alpha^2\beta^2\gamma^2.$

Sol. $$\Delta = \begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix}$$

Taking, α, β, γ common from C_1, C_2, C_3 respectively

$$\Delta = \alpha\beta\gamma\begin{vmatrix} -\alpha & \alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & \gamma & -\gamma \end{vmatrix}$$

Now taking α, β, γ common from R_1, R_2, R_3 respectively

$$\Delta = \alpha^2\beta^2\gamma^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we have

$$\Delta = \alpha^2\beta^2\gamma^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

Now expanding by using C_1, $\Delta = \alpha^2\beta^2\gamma^2 (-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -\alpha^2\beta^2\gamma^2 (0 - 4) = 4\alpha^2\beta^2\gamma^2$.

Ex. 21. Evaluate (*i*) $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$, **(*ii*)** $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$. **(*ISC*)**

Sol. (*i*) $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = abc\begin{vmatrix} 0 & b^2 & c^2 \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix} = abc\,(a^2b^2c^2)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= (abc)\,(a^2\,b^2c^2)\,[-1\,(0-1) + 1\,(1-0)] = 2a^3b^3c^3.$

(*ii*) Expanding by using first row, we get

$$\Delta = (b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$= (b+c)\,(ca + bc + a^2 + ab - bc) - a\,(ab + b^2 - bc) + a\,(bc - c^2 - ac)$

$= 4\,abc$, all other terms cancelling out on simplification.

Ex. 22. (*a*) Solve the following equations :

(*i*) $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

(*ii*) $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$

(*b*) Use properties of determinants to solve for *x*: **(*ISC*)**

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$$

Sol. (*a*) (*i*) Given, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ $[R_1 \to R_1 + R_2 + R_3]$

$\Rightarrow (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0$ $[C_2 \Rightarrow C_2 - C_1, C_3 \to C_3 - C_1]$

$\Rightarrow \quad (x+9)\begin{vmatrix} x-2 & 0 \\ -1 & x-7 \end{vmatrix} = 0$ (expanding along R_1) $\Rightarrow (x+9)(x-2)(x-7)=0$

$\Rightarrow \quad x=-9 \quad$ or $\quad x=2 \quad$ or $\quad x=7 \quad \therefore$ The solution set is $\{-9, 2, 7\}$.

(*ii*) Given, $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -x & -6 & 3 \\ -x & 3-x & 3 \\ -x & 3 & -6-x \end{vmatrix} = 0 \quad [C_1 \to C_1 + C_2 + C_3]$

$\Rightarrow \quad -x\begin{vmatrix} 1 & -6 & 3 \\ 1 & 3-x & 3 \\ 1 & 3 & -6-x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -6 & 3 \\ 0 & 9-x & 0 \\ 0 & 9 & -9-x \end{vmatrix} = 0 \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$

$\Rightarrow \quad -x \times 1\begin{vmatrix} 9-x & 0 \\ 9 & -(9+x) \end{vmatrix} = 0$ (expanding along C_1)

$\Rightarrow \quad -x[-(9-x)(9+x)-0]=0 \quad \Rightarrow \quad x(9-x)(x+9)=0$

$\Rightarrow \quad x=0 \quad$ or $\quad x=9 \quad$ or $\quad x=-9. \quad \therefore$ The solution set is $\{-9, 0, 9\}$.

(*b*) $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix}$ (Using $C_1 \to C_1 + C_2 + C_3$)

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & a-c \\ 0 & -x & x+c-a \end{vmatrix}$

(Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$)

$= (x+a+b+c)[x(x+c-a)+x(a-c)]$

[Expanding by means of elements in C_1]

$= (x+a+b+c)x^2$

$\therefore \quad \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)=0$

$\therefore \quad x = 0, \quad$ or $\quad -(a+b-c)$.

Ex. 23. Prove that $\begin{vmatrix} \mathbf{b+c} & \mathbf{c+a} & \mathbf{a+b} \\ \mathbf{q+r} & \mathbf{r+p} & \mathbf{p+q} \\ \mathbf{y+z} & \mathbf{z+x} & \mathbf{x+y} \end{vmatrix} = 2\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{vmatrix}$.

Sol. Operating $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} 2(b+c+a) & c+a & a+b \\ 2(q+r+p) & r+p & p+q \\ 2(y+z+x) & z+x & x+y \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$\Delta = 2\begin{vmatrix} b+c+a & -b & -c \\ q+r+p & -q & -r \\ y+z+x & -y & -z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$. Operating $C_1 \to C_1 + C_2 + C_3$ and taking '–' common from C_2 and C_3.

Ex. 24. Prove that $\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2(a+b+c)^3}$.

Sol. Given determinant = $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + (C_2 + C_3)$, we obtain

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1 \Rightarrow$

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}.$$

$$= 2(a+b+c)\,.\,1\,\{(b+c+a)(c+a+b) - (0 \times 0)\}$$

$$= 2(a+b+c)^3. \qquad \text{[Expanding by using 1st column]}$$

Ex. 25. Show that $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$ **= (α – β) (β – γ) (γ – α) (αβ + βγ + γα).** ***(ISC 2006)***

Sol. Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ \alpha^3 & \beta^3-\alpha^3 & \gamma^3-\alpha^3 \end{vmatrix} = 1\,.\begin{vmatrix} \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ \beta^3-\alpha^3 & \gamma^3-\alpha^3 \end{vmatrix} \quad \text{expanding along } R_1$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \beta+\alpha & \gamma+\alpha \\ \beta^2+\alpha\beta+\alpha^2 & \gamma^2+\alpha\gamma+\alpha^2 \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \beta-\gamma & \gamma+\alpha \\ \beta^2+\alpha\beta-\gamma^2-\alpha\gamma & \gamma^2+\alpha\gamma+\alpha^2 \end{vmatrix} \quad \text{Applying } C_1 \to C_1 - C_2$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \beta-\gamma & \gamma+\alpha \\ (\beta^2-\gamma^2)+\alpha(\beta-\gamma) & \gamma^2+\alpha\gamma+\alpha^2 \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)(\beta-\gamma)\begin{vmatrix} 1 & \gamma+\alpha \\ \beta+\gamma+\alpha & \gamma^2+\alpha\gamma+\alpha^2 \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)(\beta-\gamma)\,[1\,.\,(\gamma^2+\alpha\gamma+\alpha^2) - (\gamma+\alpha)(\beta+\gamma+\alpha)]$$

$$= -(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\gamma^2+\alpha\gamma+\alpha^2-\gamma\beta-\gamma^2-\gamma\alpha-\alpha\beta-\alpha\gamma-\alpha^2)$$

$$= -(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(-\alpha\beta-\beta\gamma-\gamma\alpha)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha).$$

Ex. 26. Express the determinant | *A* | in factors, where

$$|A| = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^3} & \boldsymbol{b^3} & \boldsymbol{c^3} \end{vmatrix} = \boldsymbol{(a-b)(b-c)(c-a)(a+b+c)} \qquad \textbf{(ISC)}$$

Sol. Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expanding by using R_1

$(a-b)(b-c).1[1(b^2+bc+c^2)-1(a^2+ab+b^2)]$

$= (a-b)(b-c)(bc+c^2-a^2-ab) = (a-b)(b-c)\{b(c-a)+(c-a)(c+a)\}$

$= (a-b)(b-c)(c-a)(b+c+a) = (a-b)(b-c)(c-a)(a+b+c).$

Ex. 27. Prove that $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$ ***(ISC)***

Sol. Applying $C_2 \to C_2 - 2C_1 - 2C_3$, we get

$$\Delta = \begin{vmatrix} a^2 & -(a^2+b^2+c^2) & bc \\ b^2 & -(a^2+b^2+c^2) & ca \\ c^2 & -(a^2+b^2+c^2) & ab \end{vmatrix} = -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} \quad \text{Interchanging } C_1 \text{ and } C_2$$

Multiplying R_1, R_2 and R_3 by a, b and c respectively.

$$= (a^2+b^2+c^2) \times \frac{1}{abc}\begin{vmatrix} a & a^3 & abc \\ b & b^3 & abc \\ c & c^3 & abc \end{vmatrix}$$

$$= (a^2+b^2+c^2) \times \frac{abc}{abc}\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$$

Now prove as in Ex. 26.

Ex. 28. Prove that $\begin{vmatrix} x^2 & y^2 & z^2 \\ yz & zx & xy \\ x & y & z \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$ ***(NMOC 1994)***

Sol. Let $\Delta = \begin{vmatrix} x^2 & y^2 & z^2 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x^3 & y^3 & z^3 \\ xyz & xyz & xyz \\ x^2 & y^2 & z^2 \end{vmatrix}$$, multiplying C_1, C_2 and C_3 by x, y and z respectively.

$$= \frac{xyz}{xyz}\begin{vmatrix} x^3 & y^3 & z^3 \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ x^3 & y^3 & z^3 \\ x^2 & y^2 & z^2 \end{vmatrix} \qquad (R_1 \leftrightarrow R_2)^*$$

* *i.e.*, interchanging R_1 and R_2.

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \qquad (R_2 \leftrightarrow R_3)$$

Now, operating $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} = \begin{vmatrix} y^2-x^2 & z^2-x^2 \\ y^3-x^3 & z^3-x^3 \end{vmatrix}, \text{ expanding along } R_1$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+x^2+xy & z^2+x^2+zx \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+x^2+xy & (z^2-y^2)+x(z-y) \end{vmatrix}, \text{ operating } C_2 \to C_2 - C_1$$

$$= (y-x)(z-x)(z-y)\begin{vmatrix} y+x & 1 \\ y^2+x^2+xy & x+y+z \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)[(y+x)(x+y+z) - 1(y^2+x^2+xy)]$$

$$= (x-y)(y-z)(z-x)[xy+y^2+yz+x^2+xy+zx-y^2-x^2-xy]$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx).$$

Ex. 29. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$. **(ISC)**

$$= abc + ab + bc + ca.$$

Sol. Let the given determinant be = Δ. Then, $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ gives

$$\Delta = \begin{vmatrix} a & 0 & 1 \\ -b & b & 1 \\ 0 & -c & 1+c \end{vmatrix} = a\begin{vmatrix} b & 1 \\ -c & 1+c \end{vmatrix} + 1\begin{vmatrix} -b & b \\ 0 & -c \end{vmatrix}$$

$$= a(b+bc+c) + bc = abc + ab + ac + bc = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Ex. 30. Using properties of determinants, show that

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$$

Sol. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} \quad \text{Multiplying and dividing } R_1 \text{ by } a, R_2 \text{ by } b \text{ and } R_3 \text{ by } c.$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \quad \text{Taking } abc \text{ common from } C_3$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} \quad \text{Interchanging } C_2 \text{ and } C_3$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0.$$ Interchanging C_1 and C_2.

Ex. 31. Show that

$$\begin{vmatrix} \sin^2 A & \sin A & \cos^2 A \\ \sin^2 B & \sin B & \cos^2 B \\ \sin^2 C & \sin C & \cos^2 C \end{vmatrix} = -(\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A)$$

(*ISC*)

Sol. Let $\Delta = \begin{vmatrix} \sin^2 A & \sin A & \cos^2 A \\ \sin^2 B & \sin B & \cos^2 B \\ \sin^2 C & \sin C & \cos^2 C \end{vmatrix} = \begin{vmatrix} \sin^2 A & \sin A & 1-\sin^2 A \\ \sin^2 B & \sin B & 1-\sin^2 B \\ \sin^2 C & \sin C & 1-\sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} \sin^2 A & \sin A & 1 \\ \sin^2 B & \sin B & 1 \\ \sin^2 C & \sin C & 1 \end{vmatrix} - \begin{vmatrix} \sin^2 A & \sin A & \sin^2 A \\ \sin^2 B & \sin B & \sin^2 B \\ \sin^2 C & \sin C & \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2 A - \sin^2 B & \sin A - \sin B & 0 \\ \sin^2 B - \sin^2 C & \sin B - \sin C & 0 \\ \sin^2 C & \sin C & 1 \end{vmatrix} - 0.$$ [Two columns being identical 2nd det. = 0. In first det. apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.]

$$= 1\begin{vmatrix} \sin^2 A - \sin^2 B & \sin A - \sin B \\ \sin^2 B - \sin^2 C & \sin B - \sin C \end{vmatrix},$$ expanding along C_3.

$$= (\sin A - \sin B)(\sin B - \sin C)\begin{vmatrix} \sin A + \sin B & 1 \\ \sin B + \sin C & 1 \end{vmatrix},$$ Taking common factors out from R_1 and R_2.

$= (\sin A - \sin B)(\sin B - \sin C)\,[1.(\sin A + \sin B) - 1(\sin B + \sin C)]$

$= (\sin A - \sin B)(\sin B - \sin C)(\sin A - \sin C)$

$= -(\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A).$

Ex. 32. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

Sol. Let the given determinant be denoted by Δ.
Operating $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & (c-a-b) & (a+b)^2 \end{vmatrix}$$ Taking $(a+b+c)$ common from C_1 and C_2

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \text{Operating } R_3 \to R_3 - (R_1 + R_2)$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix} \quad (\text{Applying } C_1 \to aC_1, C_2 \to bC_2)$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad (\text{Applying } C_1 \to C_1 + C_3, C_2 \to C_2 + C_3)$$

$$= \frac{(a+b+c)^2}{ab}\, ab \,.\, 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Taking } a, b \text{ and } 2ab \text{ common from } R_1, R_2 \text{ and } R_3 \text{ respectively}$$

$$= 2\,ab\,(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} \quad \text{expanding along } R_3$$

$$= 2\,ab\,(a+b+c)^2 \{(b+c)(c+a) - ab\} = 2\,abc\,(a+b+c)^3.$$

Ex. 33. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$ ***(ISC)***

Sol. $\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$ Applying $C_1 \to C_1 - bC_3$, $C_2 \to C_2 + aC_3$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix} \quad \text{Taking out } 1+a^2+b^2 \text{ common from } C_1 \text{ and } C_2$$

Expanding along C_1, we get

$$\Delta = (1+a^2+b^2)^2 [1 \,.\, (1-a^2-b^2+2a^2) + b\,(0+2b)]$$

$$= (1+a^2+b^2)^2 (1-a^2-b^2+2a^2+2b^2)$$

$$= (1+a^2+b^2)^2 (1+a^2+b^2) = (1+a^2+b^2)^3.$$

Ex. 34. If x, y, z are different and $A = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1 + xyz = 0$. ***(ISC 1988, 1995, 2004)***

Sol.
$$A = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(Number of jumped columns = 2 in the first determinant)

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz).$$

Now $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$ (applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$= (y-x)(z-x) . 1 \{1 (z+x) - 1 (y+x)\}$ expanding by using R_1

$= (y-x)(z-x)(z-y) = (x-y)(y-z)(z-x)$

$\therefore \quad A = (1+xyz)(x-y)(y-z)(z-x).$

It is given that $A = 0 \quad \therefore \quad (1+xyz)(x-y)(y-z)(z-x) = 0$

But $\quad x \neq y \neq z \quad \Rightarrow \quad (x-y)(y-z)(z-x) \neq 0 \quad \therefore \quad 1 + xyz = 0.$

Ex. 35. Prove that $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ac & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = (a-b)(c-b)(c-a)(a^2+b^2+c^2).$

Sol. Operating, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$D = \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & (b^2-a^2) - c(b-a) & b^3-a^3 \\ 0 & (c^2-a^2) - b(c-a) & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & (b-a)(b+a-c) & (b-a)(b^2+a^2+ab) \\ 0 & (c-a)(c+a-b) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b+a-c & b^2+a^2+ab \\ 0 & c+a-b & c^2+a^2+ac \end{vmatrix}$$

$= (b-a)(c-a) \begin{vmatrix} b+a-c & b^2+a^2+ab \\ c+a-b & c^2+a^2+ac \end{vmatrix}$, expanding along C_1.

$\Rightarrow \quad D = (b-a)(c-a) \begin{vmatrix} 2(b-c) & (b-c)(a+b+c) \\ (c+a-b) & c^2+a^2+ac \end{vmatrix}$ Applying $R_1 \to R_1 - R_2$

$$= (b-a)(c-a)(b-c) \begin{vmatrix} 2 & a+b+c \\ c+a-b & c^2+a^2+ac \end{vmatrix}$$

$= (a-b)(c-b)(c-a)[2(c^2+a^2+ac) - (a+b+c)(c+a-b)]$

$= (a-b)(c-b)(c-a)(a^2+b^2+c^2).$

Ex. 36. If p, q, r are not in G.P., and $\begin{vmatrix} 1 & \frac{q}{p} & \alpha + \frac{q}{p} \\ 1 & \frac{r}{q} & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0,$

show that, $p\alpha^2 + 2q\alpha + r = 0$. **(*ISC*)**

Sol. $(C_2 \to C_2 - C_1) \Rightarrow \begin{vmatrix} 1 & \frac{q}{p} & \alpha + \frac{q}{p} \\ 1 & \frac{r}{q} & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = \begin{vmatrix} 1 & -\alpha & \alpha + \frac{q}{p} \\ 1 & -\alpha & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & \frac{q}{p} - \frac{r}{q} \\ 1 & -\alpha & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} \quad (R_1 \to R_1 - R_2)$$

$$= \left(\frac{q}{p} - \frac{r}{q}\right) \begin{vmatrix} 1 & -\alpha \\ p\alpha + q & q\alpha + r \end{vmatrix}, \text{ using first row}$$

$$= \frac{q^2 - pr}{pq} [(q\alpha + r) \times 1 - (-\alpha)(p\alpha + q)]$$

$$= \frac{q^2 - pr}{pq} (p\alpha^2 + 2q\alpha + r) = 0, \text{ if } p\alpha^2 + q\alpha + r = 0.$$

[since $q^2 - pr \neq 0$, as p, q, r are not in G.P.]

Ex. 37. If $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$**, find the value of** $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ **$(p \neq a, q \neq b, r \neq c)$.**

Sol. Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, the given equation becomes

$$\begin{vmatrix} p-a & 0 & c-r \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 \Rightarrow (p-a)\{r(q-b) - b(c-r)\} + (c-r)\{-a(q-b)\} = 0$$

$\Rightarrow \; r(p-a)(q-b) + b(p-a)(r-c) + a(q-b)(r-c) = 0$ [Expanding by R_1]

$\Rightarrow \; \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$ [Dividing throughout by $(p-a)(q-b)(r-c)$]

$\Rightarrow \; \frac{r}{r-c} + \frac{(b-q)+q}{q-b} + \frac{(a-p)+p}{p-a} = 0 \Rightarrow \frac{r}{r-c} - 1 + \frac{q}{q-b} - 1 + \frac{p}{p-a} = 0$

$\Rightarrow \; \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$

Ex. 38. Show that the value of the following determinant is negative, if a, b and c are positive and unequal $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. **(*ISC, IIT*)**

Sol. $R_1 \to R_1 + R_2 + R_3$ gives

$$D = \begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$D = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = (a+b+c)\begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix}, \text{ expanding along } R_1$$

$$= (a+b+c)\,[-b^2 - c^2 + 2\,bc - a^2 + ab + ac - bc]$$
$$= -(a+b+c)\,[b^2 + c^2 + a^2 - ab - ac - bc]$$

$$= -\frac{1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a^2)].$$

The expression $(a-b)^2 + (b-c)^2 + (c-a)^2$ being the sum of perfect squares is always positive. Therefore, the above expression is necessarily negative as a, b and c are positive and unequal.

EXERCISE 2 (*d*)

1. (*a*) **Without evaluating Problems (*i*) to (*x*) state why each statement is true.**

(*i*) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 0$ (*ii*) $\begin{vmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$ (*iii*) $\begin{vmatrix} -2 & 1 & 0 \\ 3 & 4 & 1 \\ -4 & 2 & 0 \end{vmatrix} = 0$

(*iv*) $\begin{vmatrix} 7 & 3 & 2 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{vmatrix} = 0$ (*v*) $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 0$ (*vi*) $\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -4 & 0 & 2 & 8 \end{vmatrix} = 0$

(*vii*) $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 7 \\ 5 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 5 \\ 2 & 0 & 3 \\ 1 & 7 & 4 \end{vmatrix}$ (*viii*) $\begin{vmatrix} 2 & 3 & 21 \\ 11 & 4 & 7 \\ 6 & 15 & 8 \end{vmatrix} = \begin{vmatrix} 21 & 2 & 3 \\ 7 & 11 & 4 \\ 8 & 6 & 15 \end{vmatrix}$

(*ix*) $\begin{vmatrix} 1 & 2 & 7 \\ 6 & 0 & 13 \\ 8 & 3 & 5 \end{vmatrix} = -\begin{vmatrix} 6 & 0 & 13 \\ 1 & 2 & 7 \\ 8 & 3 & 3 \end{vmatrix}$ (*x*) $\begin{vmatrix} 2+3 & -1 & 2 \\ 3+4 & 0 & 1 \\ 4+5 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 0 & 1 \\ 4 & 3 & 0 \end{vmatrix} + \begin{vmatrix} 3 & -1 & 2 \\ 4 & 0 & 1 \\ 5 & 3 & 0 \end{vmatrix}$

(*b*) If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 3 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, without expanding show that $|A| = |B|$.

[Hint. Use property 2]. **(*NMOC*)**

(*c*) Without actually expanding the determinant but stating and using the theorems on determinants, show that $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$

2. Without expanding the determinant show that :

(*i*) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix} = 0$ **[Hint.** Take out 7 common from C_1]. (*ii*) $\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix} = 0$

(*iii*) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

[Hint. Operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$. Now, expand by C_3].

(iv) $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix} = 0$

3. Using the properties of determinants, show that :

(i) $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$ **(*NMOC*)**

(ii) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$ (iv) $\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix} = 0$

(v) $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix} = \lambda^2(3x+\lambda)$.

(vi) $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \cos^2 x & \sin^2 x & 1 \\ -10 & 12 & 2 \end{vmatrix} = 0$

4. Without expanding the determinant at any stage, prove that

$\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$, where α, β, γ are in A.P. **(*NMOC*)**

Using properties of Determinants, evaluate :

5. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ **(*ISC*)**

6. $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

7. $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ **[Hint.** Apply $C_2 \to C_2 - 4C_1$, $C_3 \to C_3 - 9C_1$]. **(*ISC*)**

8. Solve the following equations :

(i) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ **(*IIT*)** (ii) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$ **(*NMOC*)**

(iii) $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ (iv) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

(v) $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

9. $\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & a+c \\ c & c^2 & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

[Hint. Operate $C_1 \to C_1 + C_3$, take out $(a+b+c)$ common. Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, expand.]

10. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

11. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2 b^2 c^2$ ***(ISC 2002)***

12. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4\,xyz$ ***(ISC 1996)***

[Hint. Use first $C_1 \to C_1 - (C_2 + C_3)$, then $R_2 \to R_2 - R_3$, and expand]

13. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2\,(a+b)(b+c)(c+a)$.

[Hint. Apply $C_1 \to C_1 + C_2$, $C_2 \to C_2 + C_3$ then apply $R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3$ and expand along C_1].

14. $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$ [See Solved Ex. 26]

15. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ ***(ISC)***

16. $\begin{vmatrix} 1 & bc & bc\,(b+c) \\ 1 & ca & ca\,(c+a) \\ 1 & ab & ab\,(a+b) \end{vmatrix} = 0$ ***(ISC)***

17. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ or $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(NMOC)

18. $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

[Hint. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

19. (*i*) $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\,(a+b+c)(ab+bc+ca-a^2-b^2-c^2)$

(*ii*) $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

[Hint. Apply $C_1 \to C_1 + C_2 + C_3$, then $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$.

Now, $C_1 \to C_1 - (C_2 + C_3)$ Finally, interchange R_1, R_2 and R_2, R_3]

20. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$ **(*ISC* 1991)**

21. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\,(x-y)(y-z)(z-x)$ **(*ISC*)**

22. (*i*) Without expanding the determinant, show that $(x+y+z)$ is a factor of the determinant and also find its value: $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$ **(*ISC* 2007)**

[Hint. $R_1 \to R_1 + R_2 + R_3$].

(*ii*) Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$ **(*ISC*)**

[Hint. Apply $R_1 \to R_1 + R_2 + R_3$. Then apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$].

23. $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4\,(a+b)(b+c)(c+a)$. **[Hint.** Apply factor property (Property 9)]

24. (*i*) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2\,(x+a+b+c)$ **(*ISC*)**

[Hint. Apply $C_1 \to C_1 + C_2 + C_3$ and take out common factor, Then $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$].

(*ii*) $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x+p+q)(x-p)(x-q)$.

(*iii*) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2$. **(*ISC*)**

(*iv*) $\begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix} = 1 + a_1 + a_2 + a_3$.

[Hint. (*ii*), (*iii*) and (*iv*). Apply same method as for (*i*)]

25. (*i*) $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$. **(*ISC*)**

[Hint. Applying $C_1 \to C_1 + C_2 + C_3$, $a+b+c$ is easily seen to be a factor].

(*ii*) $\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a-b)$ **(*ISC*)**

26. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

[Hint. (*i*) Take *a*, *b* and *c* common from C_1, C_2 and C_3 respectively,

(*ii*) Apply $R_2 \to R_2 - R_1 - R_3$, (*iii*) Apply $C_2 \to C_2 - C_3$, (*iv*) Expand along R_2].

27. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$ **(*ISC 2005*)**

28. Without expanding the determinants, show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

[Hint. 1. Operate aR_1, bR_2, cR_3 and divide the whole det. by *abc*. **2.** Interchange C_3 with C_2 **3.** Interchange C_1 and C_2]

29. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$ **(*ISC*)**

[Hint. $R_1 \to R_1 + R_2 + R_3$, Take out $(a + b + c)$ common. Then, $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$. Now, expand along R_1]

30. Using properties of determinants, prove that $\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & yzx & zxy \end{vmatrix} = xyz\,(x-y)\,(y-z)\,(z-x)\,(xy+zy+zx)$

(*ISC 2001*)

[Hint. Take *xyz* common from R_3. Then apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, etc]

31. Using properties of determinants, prove that $\begin{vmatrix} y+z & x+y & x \\ z+x & y+z & y \\ x+y & z+x & z \end{vmatrix} = x^3 + y^3 + z^3 - 3xyz$ **(*ISC 2000*)**

[Hint. $\begin{vmatrix} y+z & x+y & x \\ z+x & y+z & y \\ x+y & z+x & z \end{vmatrix} = \begin{vmatrix} x+y+z & y & x \\ z+x+y & z & y \\ x+y+z & x & z \end{vmatrix}$ $(C_1 \to C_1 + C_3)$ and $(C_2 \to C_2 - C_3)$

$= (x+y+z)\begin{vmatrix} 1 & y & x \\ 1 & z & y \\ 1 & x & z \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & y & x \\ 0 & z-y & y-x \\ 0 & x-z & z-y \end{vmatrix}$ $(R_2 \to R_2 - R_1)$ and $(R_3 \to R_3 - R_2)$

$= (x + y + z)\,[1.\,(z - y)^2 - (y - x)\,(x - z)$ etc.]

32. Using properties of determinants, find the value of the following determinant :

$\begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix}$ **(*ISC 2003*)**

33. Prove that $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2 - ac)\,(ax^2 + 2bxy + cy^2)$ **(*ISC 2008*)**

[Hint. Operate $R_3 \to R_3 - x\,R_1 - yR_2$]

ANSWERS

1. (*a*) (*i*) Each element in the second row is zero (*ii*) First and third columns are identical.
(*iii*) The elements in the third row are equi-multiples of the corresponding elements in the first row.
(*iv*) Each element in the fourth column is zero (*v*) The first and the third rows are identical.
(*vi*) The elements in the fourth row are equi-multiples of the corresponding elements in the second row.
(*vii*) The rows are changed into columns and columns into rows.
(*viii*) The third column has passed over two (*even*) columns, namely first and second.
(*ix*) When the first and the second rows are interchanged, the resulting determinant is the negative of the original determinant.
(*x*) Each element in the first column is the sum of two terms, therefore the determinant can be written as the sum of two determinants.

5. 0 **6.** 4 **7.** – 8

8. (*i*) $x = -1, 2$ (*ii*) $x = 4$ (*iii*) $x = -1, 2, 2$ (*iv*) $x = 0, -(a+b+c)$
(*v*) $\lambda = 2, 3, 6$ **22.** (*i*) $(x+y+z)^3$ **32.** $xyz(x-y)(y-z)(z-x)$

2.09. Solutions of linear equations using determinants. (Cramer's rule)

1. Consider the simultaneous equations $a_1x + b_1y = c_1,\ a_2x + b_2y = c_2$

Solving these equations, we get $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1},\ y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Writing in determinant form, we can express the solution as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D},\quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}.$$

Observe that determinant D in the denominator is really the determinant of the co-efficients. In the determinant D_x, the co-efficients of x, *i.e.*, a_1 and a_2 are replaced by the constant terms c_1 and c_2 while in the determinant D_y, the co-efficients of y, *i.e.* b_1 and b_2 are replaced by the constant terms.

2. Now consider the following system of equations :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \text{...(1)}$$

then $$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$ [Substituting for d_1, d_2, d_3 from (1)]

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$$= x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow x = \frac{|d_1b_2c_3|}{|a_1b_2c_3|}$$ Using the short notations, where $|d_1b_2c_3|$ is the determinant obtained from the determinant $|a_1b_2c_3|$ of the co-efficients, by replacing a_1, a_2, a_3 by d_1, d_2, d_3 respectively.

Hence, $$x = \frac{|d_1\ b_2\ c_3|}{|a_1\ b_2\ c_3|},\ \text{provided } |a_1 b_2 c_3| \neq 0$$

In the same way, $$y = \frac{|a_1\ d_2\ c_3|}{|a_1\ b_2\ c_3|} \text{ and } z = \frac{|a_1\ b_2\ d_3|}{|a_1\ b_2\ c_3|}$$

These equations are referred to as **Cramer's Rule** after the Swiss mathematician Gabriel Cramer (1704 – 1752).

For short, we can write the above results as

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D} \text{ or } \frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D},\ \text{provided } D \neq 0.$$

Important note : It may be noted that constant terms in the given equations are on the right side.

Aid to memory. **Note that**

(1) The denominator (D) in all the three x, y and z is the determinant of the co-efficients of x, y and z in the equations.

(2) The numerator D_x in x is the determinant obtained from the determinant D by replacing the x co-efficients a_1, a_2, a_3 by the constant terms d_1, d_2, d_3.

(3) D_y is obtained on replacing the y-coeffts. by constant terms and D_z is obtained on replacing the z-coeffts. by constant terms.

Remarks 1. Cramer's rule can be used in exactly the same way to solve the system of n equations in n unknowns.

2. Cramer's rule does not apply if $D = 0$.

Working rule. Let the system of three linear equations in x, y, z be

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \quad \ldots(i)$$

Step 1. Evaluate $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Step 2. If $D \neq 0$, then the given equations have a unique solution and $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$, where D_x is the determinant obtained on replacing the x-coeffts. a_1, a_2, a_3 by the constants d_1, d_2, d_3

i.e., $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$.

Similarly, D_y is the determinant obtained on replacing the y-coeffts. b_1, b_2, b_3 by the constants d_1, d_2, d_3 and d_z is the determinant obtained on replacing the z-coeffts. c_1, c_2, c_3 by the constants d_1, d_2, d_3.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note. If $D = 0$, then the given system does not have a unique solution. For detailed discussion of this case, read the next article.

Remark. If $d_1 = d_2 = d_3 = 0$ in (*i*), then the system of equations is said to be **homogeneous system**, otherwise it is called a **non-homogeneous system** of equations.

2.10. Consistent, inconsistent and dependent equations

Consider the following systems of equations :

(1) $\begin{aligned} x + y &= -1 \\ 8x + 3y &= 2 \end{aligned}$ (2) $\begin{aligned} x + y &= 5 \\ 4x + 4y &= 20 \end{aligned}$ (3) $\begin{aligned} x + y &= 5 \\ 4x + 4y &= 7 \end{aligned}$

On solving these systems of equations, you will find that

(*i*) the first system has only ***one solution***, namely, $x = 1, y = -2$. We say that the system is consistent.

(*ii*) the second system ***has infinitely many solutions***, namely $x = k, y = 5 - k$.

(*iii*) the third system has *no* solution, *i.e.*, we cannot find values of x and y that may satisfy both the equations simultaneously. In this case, we say that the system is *inconsistent.*

Remark. If a system of linear equations has an infinite number of solutions, then the equations are said to be **dependent.**

A system of equations is said to be consistent if its solution exists whether unique or not, otherwise it is said to be inconsistent.

2.11. Conditions of inconsistency

(A) ***For a system of simultaneous linear equations with 2 unknowns.***

Rule.

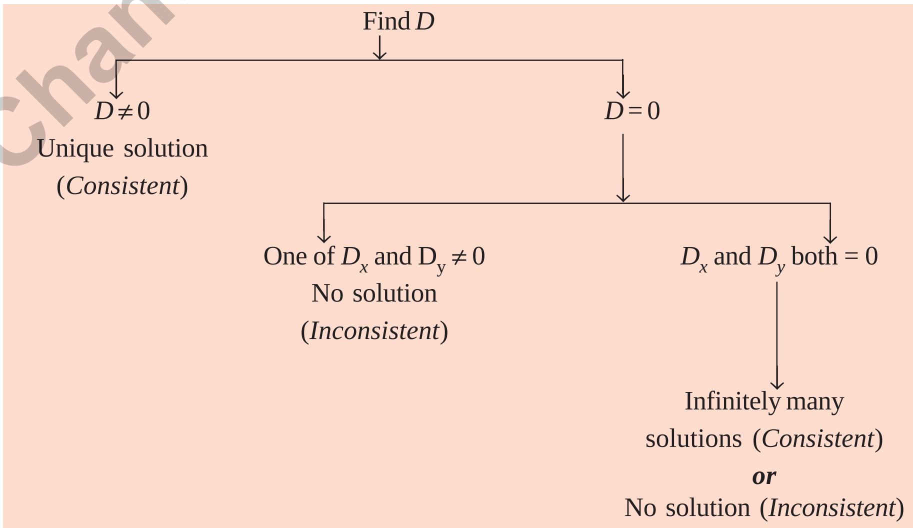
(*i*) If $D \neq 0$, then the given system of equations is consistent and has a unique solution, namely.

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \Rightarrow \frac{x}{D_x} = \frac{y}{D_y} = \frac{1}{D}$$ **[Solved Ex. 39]**

(*ii*) If $D = 0$ and $D_x = D_y = 0$, then the system may be consistent with infinitely many solutions or inconsistent. **[Solved Ex. 40 (*ii*)]**

(*iii*) If $D = 0$ and at least one of D_x and D_y is non-zero then the system has no solution, *i.e.*, the system is inconsistent. **[Solved Ex. 40 (*i*)]**

The above rule is diagrammatically displayed below :



(B) ***For a system of 3 simultaneous linear equations involving 3 unknowns.***

Rule.

(*i*) If $D \neq 0$, then the given system of equations is consistent and has a unique solution namely,

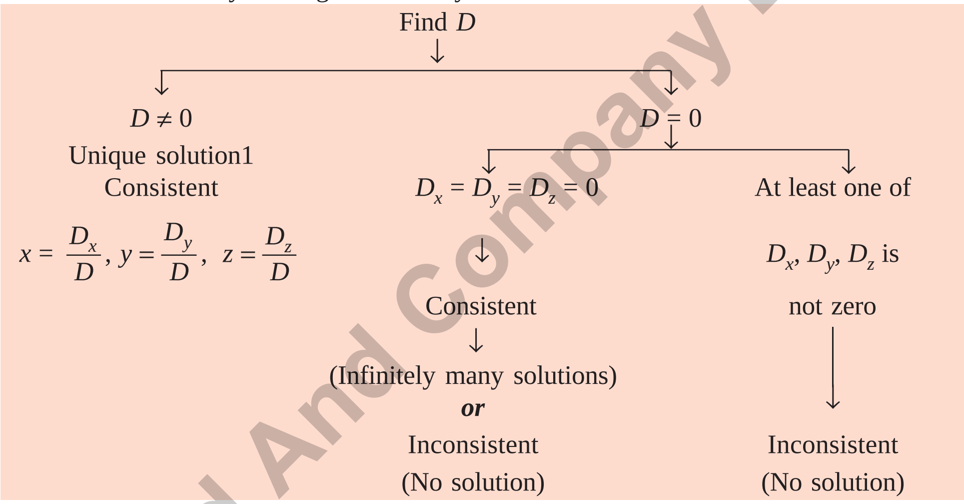
$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D} \Rightarrow \frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D}$$. **[Solved Ex. 41, 42, 43, 44]**

(*ii*) If $D = 0$ and $D_x = D_y = D_z = 0$, then the given system may be consistent with infinitely many solutions [Ex 46, 47, 48] or inconsistent having no solution [Ex. 49]. If consistent, then take any two equations out of the three given equations and shift one of the variables say z on the right hand side to obtain two equations in x, y. Solve these two equations by cramer's rule to obtain x, y in terms of z. **[Solved Ex. 46, 47, 48]**

(*iii*) If $D = 0$ and at least one of the determinants D_x, D_y, D_z is not zero, than the given system of equations is inconsistent. **[Solved Ex. 45, 50]**

(*iv*) If $d_1 = d_2 = d_3 = 0$, *i.e.*, the given system consists of homogeneous linear equations, then system has only the **trivial solution,** *i.e.* $x = y = z = 0$. If $D = 0$, the system of equations **[Solved Ex. 51, 52, 53]** is consistent and will have infinite solutions.

The above discussion may be diagrammatically summarised as under :



2.12. Consistency of three equations in two variables (Concurrency test)

Consider the equations

$$a_1 x + b_1 y + c_1 = 0 \quad \text{...(1)}$$
$$a_2 x + b_2 y + c_2 = 0 \quad \text{...(2)}$$
$$a_3 x + b_3 y + c_3 = 0 \quad \text{...(3)}$$

The system of three equations is consistent, if there exists an ordered pair (h, k) of rational numbers, which is a solution of all the three equations. A discussion of the condition of consistency in the general case being beyond the scope of the book, we assume that two of the three equations admit of a unique solution. We suppose that the first two equations have a unique solution and then find the condition that the system is consistent.

Thus assuming that $(a_1 b_2 - a_2 b_1) \neq 0$, the unique solution of (1) and (2) is

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right)$$

It will also be a solution of (3) provided

$$a_3 \times \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} + b_3 \times \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} + c_3 = 0$$

or equivalently,

$$a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - b_2 a_3) = 0$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad ...(4)$$

Thus, (4) is the condition of consistency of the given system under the assumption $a_1b_2 - a_2 b_1 \neq 0$. This condition (4) is also referred to as the **eliminant** of the given system of equations and the process of finding the condition of consistency is called **elimination.**

Ex. 39. Use Cramer's rule to solve

(*i*) $7x + 2y - 25 = 0$

$2x - y - 4 = 0,$

(*ii*) $\frac{2}{x} + \frac{3}{y} = 0$

$\frac{1}{x} - \frac{2}{y} = 7$

Sol. (*i*) The given equations are

$$7x + 2y = 25 \quad ...(1)$$

$$2x - y = 4 \quad ...(2)$$

$$D = \begin{vmatrix} 7 & 2 \\ 2 & -1 \end{vmatrix} = -7 - 4 = -11. \text{ Since } D \neq 0, \text{ therefore, the solution exists.}$$

$$D_x = \begin{vmatrix} 25 & 2 \\ 4 & -1 \end{vmatrix} = -25 - 8 = -33, \; D_y = \begin{vmatrix} 7 & 25 \\ 2 & 4 \end{vmatrix} = 28 - 50 = -22$$

$$\therefore \quad x = \frac{D_x}{D} = \frac{-33}{-11} = 3 \quad y = \frac{D_y}{D} = \frac{-22}{-11} = 2. \text{ Hence, } x = 3, y = 2.$$

(*ii*) Given system of equations are $\frac{2}{x} + \frac{3}{y} = 0$

$\frac{1}{x} - \frac{2}{y} = 7$

Let $\frac{1}{x} = u$ and $\frac{1}{y} = v$. Then the given equations become

$$2u + 3v = 0$$

$$u - 2v = 7$$

$$D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7, D_u = \begin{vmatrix} 0 & 3 \\ 7 & -2 \end{vmatrix} = -21, D_v = \begin{vmatrix} 2 & 0 \\ 1 & 7 \end{vmatrix} = 14$$

$$\therefore \text{ By Cramer's rule } u = \frac{D_u}{D} = \frac{-21}{-7} = 3 \Rightarrow x = \frac{1}{3}, v = \frac{D_v}{D} = \frac{14}{-7} = -2 \Rightarrow y = -\frac{1}{2}$$

Hence, $x = \frac{1}{3}$ and $y = -\frac{1}{2}$

Ex. 40. Solve the following system of equations

(*i*) $x + 3y = 2$

$2x + 6y = 7$

(*ii*) $2x + 7y = 9$

$4x + 14y = 18$

Sol. (*i*)

$$x + 3y = 2 \quad \text{...(1)}$$
$$2x + 6y = 7 \quad \text{...(2)}$$

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0, \; D_x = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = 12 - 21 = -9$$

Since at least one of the determinants $D_x \neq 0$, therefore, the given system is inconsistent *i.e.*, it has no solution

(*ii*)

$$2x + 7y = 9 \quad \text{... (1)}$$
$$4x + 14y = 18 \quad \text{...(2)}$$

$$D = \begin{vmatrix} 2 & 7 \\ 4 & 14 \end{vmatrix} = 28 - 28 = 0$$

$$D_x = \begin{vmatrix} 9 & 7 \\ 18 & 14 \end{vmatrix} = 126 - 126 = 0, \; D_y = \begin{vmatrix} 2 & 9 \\ 4 & 18 \end{vmatrix} = 36 - 36 = 0$$

Since $D = 0$ and $D_x = D_y = 0$, therefore, the system has infinitely many solutions or is inconsistent.

Let $x = k$. Then, from (1), $2k + 7y = 9 \Rightarrow y = \dfrac{9 - 2k}{7}$

Substituting this value in (2), we get

$$4\,(k) + 14\left(\frac{9 - 2k}{7}\right) = 18 \quad \Rightarrow \quad 4k + 18 - 4k = 18 \quad \Rightarrow \quad 18 = 18, \text{ which is true.}$$

$\therefore$ The given system has infinitely many solutions given by $x = k,\; y = \dfrac{9 - 2k}{7}$, where k is any real number. The system is dependent.

Ex. 41. Using determinants, solve the equations

$$\mathbf{-4x + 2y - 9z = 2, \quad 3x + 4y + z = 5, \quad x - 3y + 2z = 8}$$

Sol. Let $D = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$

$$= -4\,(8 + 3) - 2\,(6 - 1) - 9\,(-9 - 4) = -44 - 10 + 117 = 63$$

$\because \quad D \neq 0 \qquad \therefore$ The solution exists.

$$D_x = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix} = 2\,(8 + 3) - 2\,(10 - 8) - 9\,(-15 - 32) = 22 - 4 + 423 = 441$$

$$D_y = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix} = -4\,(10 - 8) - 2\,(6 - 1) - 9\,(24 - 5) = -8 - 10 - 171 = -189$$

$$D_z = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix} = -4\,(32 + 15) - 2(24 - 5) + 2\,(-9 - 4) = -188 - 38 - 26 = -252$$

$$\therefore \quad x = \frac{D_x}{D} = \frac{441}{63} = 7, \;\; y = \frac{D_y}{D} = \frac{-189}{63} = -3, \;\; z = \frac{D_z}{D} = \frac{-252}{63} = -4$$

$\therefore$ The solution is $x = 7,\; y = -3,\; z = -4$.

Ex. 42. Solve the following system of equations by Cramer's rule

$$\mathbf{\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \; \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \text{ and } \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.}$$

Sol. Let $\frac{1}{x} = u$, $\frac{1}{y} = v$ and $\frac{1}{z} = w$. Then, the above set of equations becomes

$$\begin{aligned} 2u + 3v + 10w &= 4 \\ 4u - 6v + 5w &= 1 \\ 6u + 9v - 20w &= 2 \end{aligned}$$

$$\because \quad D = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} \begin{aligned} &= 2\,(120 - 45) - 3\,(-80 - 30) + 10\,(36 + 36) \\ &= 150 + 330 + 720 = 1200 \neq 0 \ \therefore \ \text{solution exists.} \end{aligned}$$

$$D_u = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix} \begin{aligned} &= 4\,(120 - 45) - 3\,(-20 - 10) + 10\,(9 + 12) \\ &= 300 + 90 + 210 = 600 \end{aligned}$$

$$D_v = \begin{vmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 12 & -20 \end{vmatrix} \begin{aligned} &= 2\,(-20 - 10) - 4\,(-80 - 30) + 10\,(8 - 6) \\ &= -60 + 440 + 20 = 400 \end{aligned}$$

$$D_w = \begin{vmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 9 & 2 \end{vmatrix} \begin{aligned} &= 2\,(-12 - 9) - 3\,(8 - 6) + 4\,(36 + 36) \\ &= -42 - 6 + 288 = 240 \end{aligned}$$

$$\therefore \quad u = \frac{D_u}{D} = \frac{600}{1200} = \frac{1}{2}, \quad v = \frac{D_v}{D} = \frac{400}{1200} = \frac{1}{3}, \quad w = \frac{D_w}{D} = \frac{240}{1200} = \frac{1}{5}$$

i.e., $\frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{4} \Rightarrow x = 2,\ y = 3,\ z = 5.$

Ex. 43. Use Cramer's rule to solve

$$\mathbf{x + y + z = 1, \ ax + by + cz = k, \ a^2x + b^2y + c^2z = k^2} \qquad \textit{(ISC)}$$

Sol. Let
$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)\,(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)\,(c-a)\,1 \,.\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)\,(c-a)\,(c+a-b-a)$$

$$= (b-a)\,(c-a)\,(c-b) = (a-b)\,(b-c)\,(c-a) \qquad \text{...(1)}$$

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (k-b)\,(b-c)\,(c-k) \text{ replacing } a \text{ by } k \text{ in (1)}$$

Similarly,

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (a-k)\,(k-c)\,(c-a), \quad D_z = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)\,(b-k)\,(k-a)$$

Then $$x = \frac{D_x}{D} = \frac{(k-b)\,(c-k)}{(a-b)\,(c-a)}, \quad y = \frac{D_y}{D} = \frac{(a-k)\,(k-c)}{(a-b)\,(b-c)}, \quad z = \frac{D_z}{D} = \frac{(b-k)\,(k-a)}{(b-c)\,(c-a)}.$$

Ex. 44. Find a, b, c when $f(x) = ax^2 + bx + c$, $f(0) = 6$, $f(2) = 11$ and $f(-3) = 6$. Determine the quadratic function $f(x)$ and find its value when $x = 1$.

Sol.
$$\begin{aligned} f(x) &= ax^2 + bx + c \ ; \quad \text{Putting} \quad x = 0,\ 2,\ -3 \text{ we have} \\ f(0) &= a \cdot 0 + b \cdot 0 + c = 6, \ f(2) = a \cdot 4 + b \cdot 2 + c = 11, \\ f(-3) &= a \cdot 9 + b \cdot (-3) + c = 6 \end{aligned}$$

$$\Rightarrow \quad 0 \cdot a + 0 \cdot b + c = 6, \ 4a + 2b + c = 11, \ 9a - 3b + c = 6$$

$$\therefore \quad D = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{vmatrix} = 1\cdot(-12-18) = -30 \neq 0 \quad \therefore \text{ The solution exists.}$$

$$D_a = \begin{vmatrix} 6 & 0 & 1 \\ 11 & 2 & 1 \\ 6 & -3 & 1 \end{vmatrix} = 6(2+3)+1(-33-12) = 30-45 = -15, \text{expanding along 1st row.}$$

$$D_b = \begin{vmatrix} 0 & 6 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix} = -6(4-9)+1(24-99) = 30-75 = -45$$

$$D_c = \begin{vmatrix} 0 & 0 & 6 \\ 4 & 2 & 11 \\ 9 & -3 & 6 \end{vmatrix} = 6(-12-18) = 6(-30) = -180$$

$$\therefore \quad a = \frac{D_a}{D} = \frac{-15}{-30} = \frac{1}{2},\ b = \frac{D_b}{D} = \frac{-45}{-30} = \frac{3}{2},\ c = \frac{D_c}{D} = \frac{-180}{-30} = 6$$

$$\therefore \quad f(x) = ax^2 + bx + c = \frac{1}{2}x^2 + \frac{3}{2}x + 6.$$

when $x = 1,\ f(1) = \frac{1}{2}\times(1)^2 + \frac{3}{2}\times 1 + 6 = 8.$ Hence, the value of (x) at $x = 1$ is 8.

Ex. 45. Determine whether the system

$$\mathbf{x - 3y + 2z = 4,\ 2x + y - 3z = -2,\ 4x - 5y + z = 5 \text{ is consistent.}}$$

Sol. $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -5 & 1 \end{vmatrix} = 0$. However, $\Delta_x = \begin{vmatrix} 4 & -3 & 2 \\ -2 & 1 & -3 \\ 5 & -5 & 1 \end{vmatrix} = -7$

Since at least one of the determinants $\Delta_x, \Delta_y, \Delta_z \neq 0$ so the equations are inconsistent. This could be seen in another way by multiplying the first equation by 2 and adding to the second equation to obtain $4x - 5y + z = 6$ which is not consistent with the last equation.

Ex. 46. Determine whether the system

$$\mathbf{4x - 2y + 6z = 8,\ 2x - y + 3z = 5,\ 2x - y + 3z = 4 \text{ is consistent.}}$$

Sol. The given equations are

$$4x - 2y + 6z = 8 \quad ...(1)$$
$$2x - y + 3z = 5 \quad ...(2)$$
$$2x - y + 3z = 4 \quad ...(3)$$

$$D = \begin{vmatrix} 4 & -2 & 6 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{vmatrix} = 0,\quad D_x = \begin{vmatrix} 8 & -2 & 6 \\ 5 & -1 & 3 \\ 4 & -1 & 3 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 4 & 8 & 6 \\ 2 & 5 & 3 \\ 2 & 4 & 3 \end{vmatrix} = 0,\quad D_z = \begin{vmatrix} 4 & -2 & 8 \\ 2 & -1 & 5 \\ 2 & -1 & 4 \end{vmatrix} = 0$$

Since $D = D_x = D_y = D_z = 0$, therefore, nothing definite can be concluded from these facts. The system may be consistent or inconsistent.

On closer examination of the system, we notice that the second and third equations are inconsistent. Hence the system is inconsistent.

Ex. 47. Solve the following system of equations by using determinants :

$$\mathbf{x - y + 3z = 6}$$
$$\mathbf{x + 3y - 3z = -4}$$
$$\mathbf{5x + 3y + 3z = 10} \quad \textbf{\textit{(ISC)}}$$

Sol. The given equations are

$$x - y + 3z = 6$$
$$x + 3y - 3z = -4$$
$$5x + 3y + 3z = 10$$

$$\therefore \quad D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 1(9+9) + 1(3+15) + 3(3-15) = 18 + 18 - 36 = 0$$

Further,
$$D_x = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} = 6(18) + 1(-12+30) + 3(-12-30) = 0$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} = 1(-12+30) - 6(3+15) + 3(10+20) = 0$$

$$D_z = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} = 1(30+12) + 1(10+20) + 6(3-15) = 0$$

$\therefore$ The system has either infinitely many solutions or no solution.

Putting $z = k$ in the first two equations, we get

$$x - y = 6 - 3k$$
$$x + 3y = -4 + 3k$$

This represents a system of two equations in two variables x, y.

$$\therefore \quad D = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 3 + 1 = 4 \neq 0.$$

$$D_x = \begin{vmatrix} 6-3k & -1 \\ -4+3k & 3 \end{vmatrix} = 18 - 9k - 4 + 3k = 14 - 6k$$

$$D_y = \begin{vmatrix} 1 & 6-3k \\ 1 & -4+3k \end{vmatrix} = -4 + 3k - 6 + 3k = 6k - 10.$$

$$\therefore \quad x = \frac{D_x}{D} = \frac{14-6k}{4} = \frac{7-3k}{2}, \quad y = \frac{D_y}{D} = \frac{6k-10}{4} = \frac{3k-5}{2}$$

Putting the values of x, y and z in the last equation, we get

$$5\left(\frac{7-3k}{2}\right) + 3\left(\frac{3k-5}{2}\right) + 3k = 10.$$

$$\Rightarrow \quad 35 - 15k + 9k - 15 + 6k = 20 \quad \Rightarrow \quad 20 = 20, \text{ which is true.}$$

$\therefore$ The solution is $x = \dfrac{7-3k}{2}$, $y = \dfrac{3k-5}{2}$, $z = k$, where k is any real number.

These are infinitely many solutions.

Ex. 48. Determine whether the system

$$\mathbf{2x + y - 2z = 4, \quad x - 2y + z = -2, \quad 5x - 5y + z = -2 \text{ is consistent.}}$$

Sol. $D = D_x = D_y = D_z = 0$. Hence nothing can be concluded from these facts.

Solving the first two equations for x and y (in terms of z), $x = \dfrac{3}{5}(z+2)$, $y = \dfrac{4}{5}(z+2)$.

These values are found by substitution to satisfy the third equation. (If they did not satisfy the third equation the system would be inconsistent).

Hence the values $x = \dfrac{3}{5}(z+2)$, $y = \dfrac{4}{5}(z+2)$ satisfy the system and there are infinite sets of solutions, obtained by assigning various values to z. Thus if $z = 3$, then $x = 3$, $y = 4$; if $z = -2$, then $x = 0$, $y = 0$; etc. Hence the system of equations is ***dependent.***

Ex. 49. Solve : $\boldsymbol{x + y + z = 1, 2x + 2y + 2z = 2, 3x + 3y + 3z = 4.}$

Sol. The given equations are

$$x + y + z = 1 \quad ...(1)$$
$$2x + 2y + 2z = 2 \quad ...(2)$$
$$3x + 3y + 3z = 4 \quad ...(3)$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 0$$

Since $D = 0$, Cramer's rule does not apply. However, we observe that $D_x = D_y = D_z = 0$. Therefore, the system may have infinite solutions or no solution.

To check, consider equations (1) and (3) and let $z = k$, we have

$$x + y = 1 - k \quad ...(4)$$
$$3x + 3y = 4 - 3k \quad ...(5)$$

Since $D = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 0$ and $D_x = \begin{vmatrix} 1-k & 1 \\ 4-3k & 3 \end{vmatrix} = 3 - 3k - 4 + 3k = -1 \neq 0.$

Therefore, the system of equations given by equations (4) and (5) has no solution. [Remark (*iii*), Art. 2.11].

Hence, the system of given equations will have no solution, *i.e.*, it is inconsistent.

Ex. 50. By using determinants, prove that there is no solution for the equations:

$$\boldsymbol{x + 4y - 2z = 3,\ 3x + y + 5z = 7,\ 2x + 3y + z = 5.} \qquad \textbf{(ISC)}$$

Sol. The given equations are

$$x + 4y - 2z = 3$$
$$3x + y + 5z = 7$$
$$2z + 3y + z = 5$$

Remark. In order to show that the given system of equations is inconsistent, it is also sufficient to show that $D = 0$ and at least one of D_x, D_y, D_z is non–zero.

Here, $D = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix} = 1(1-15) - 4(3-10) - 2(9-2) = -14 + 28 - 14 = 0$

$\therefore$ The system has either infinitely many solutions or no solution.

Further, $D_x = \begin{vmatrix} 3 & 4 & -2 \\ 7 & 1 & 5 \\ 5 & 3 & 1 \end{vmatrix} = 3(1-15) - 4(7-25) - 2(21-5) = -2$

Since, at least one of the determinants D_x, D_y, $D_z \neq 0$ therefore the given system is inconsistent.

Homogeneous equations

Ex. 51. Prove that the following system of equations are consistent.

(*i*) $\boldsymbol{4x + 5y - 9 = 0}$	**(*ii*)** $\boldsymbol{x + 3y + z = 0}$
$\boldsymbol{11x - 4y - 7 = 0}$	$\boldsymbol{55x + 5y + 7z = 0}$
$\boldsymbol{x + 2y - 3 = 0}$	$\boldsymbol{9x + 7y + 3z = 0}$

Sol. (*i*) The equations are consistent

if $\begin{vmatrix} 4 & 5 & -9 \\ 11 & -4 & -7 \\ 1 & 2 & -3 \end{vmatrix} = 0,$

i.e., if $4(12+14)-5(-33+7)-9(22+4)=0$, *i.e.*, if $104+130-234=0$, which is true. Hence, the equations are consistent.

(*ii*) The equations are consistent if

$$\begin{vmatrix} 1 & 3 & 1 \\ 55 & 5 & 7 \\ 9 & 7 & 3 \end{vmatrix} = 0$$

L.H.S. $= 1(15-49)-55(9-7)+9(21-5) = -34-110+144 = 0$

Hence, the equations are consistent.

Ex. 52. Solve the following system of homogeneous equations :

$$3x - 4y + 5z = 0$$
$$x + y - 2z = 0$$
$$2x + 3y + z = 0$$

Sol. $D = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3(1+6)+4(1+4)+5(3-2) = 21+20+5 = 46 \neq 0$

Since $D \neq 0$, by remark (*iv*), Art. 2.11 (B), the system of equations has only trivial solution, *i.e.*, $x = y = z = 0$.

Ex. 53. Solve the following system of homogeneous equations

$$x + y - 2z = 0 \quad ...(1)$$
$$2x + y - 3z = 0 \quad ...(2)$$
$$5x + 4y - 9z = 0 \quad ...(3)$$

Sol. $D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix} = 1(-9+12)-1(-18+15)-2(8-5) = 3+3-6 = 0$

Since $D = 0$, by remark (*iv*), Art 2.11 (B) in Cramer's rule the system has infinite solutions. Consider first two equations and take $z = k$. Then,

$$x + y = 2k,$$
$$2x + y = 3k$$

Solving these equations by Cramer's rule, we have

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0,\ D_x = \begin{vmatrix} 2k & 1 \\ 3k & 1 \end{vmatrix} = -k,\ D_y = \begin{vmatrix} 1 & 2k \\ 2 & 3k \end{vmatrix} = -k$$

Hence by Cramer's rule $x = \dfrac{D_x}{D_y} = \dfrac{-k}{-1} = k,\ y = \dfrac{D_y}{D} = \dfrac{-k}{-1} = k$

By substituting the values of x, y, z in equation (3), we have

$$\text{L.H.S.} = 5x + 4y - 9z = 5k + 4k - 9k = 0 = \text{R.H.S.}$$

Since equation (3) is satisfied, therefore, $x = k, y = k, z = k$ is the solution for each $k \in R$.

EXERCISE 2 (*e*)

Solve the following systems of equations, using determinants :

1. $7x - 2y = -7$
   $2x - y = 1$
2. $5x + 2y = 3$
   $3x + 2y = 5$
3. $2x + y = 1$
   $x - 2y = 8$
4. $3x + ay = 4$
   $2x + ay = 2,\ a \neq 0$
5. (*i*) $\dfrac{1}{x} + \dfrac{2}{y} = 8$
   $\dfrac{3}{x} + \dfrac{1}{y} = -1$

   (*ii*) $\dfrac{2}{x} - \dfrac{1}{y} = 8$
   $\dfrac{3}{x} + \dfrac{2}{y} = 5$

6. $2x + 3y = 0$
$2x - 3y = 0$

[Hint. $D \neq 0, D_x = 0, D_y = 0$

$\therefore$ The system has only the trivial solution $x = y = 0$]

7. $3x + 4y = 1$
$6x + 8y = 2$

8. $5x - y = 7$
$15x - 3y = 21$

9. $x - y = 1$
$x + z = -6$
$x + y - 2z = 3$ ***(ISC 2002)***

10. $5x - 7y + z = 11$
$6x - 8y - z = 15$
$3x + 2y - 6z = 7$

11. $x + 3y + 5z = 22$
$5x - 3y + 2z = 5$
$9x + 8y - 3z = 16$

12. $x - y + z = 4$
$2x + y - 3z = 0$
$3x + y + z = 6$

13. $2x - y + 3z = 1$
$x + 2y - z = 2$
$5y - 5z = 3$

14. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12. Use determinants to find the numbers.

[Hint. The equations are $x + y + z = 6$, $2z + x = 7$, $y + z + 3x = 12$. Solve these by Cramer's rule.]

15. The perimeter of a triangle is 45 cm. The longest side exceeds the shortest side by 8 cm and the sum of the lengths of the longest and the shortest side is twice the length of the other side. Find the lengths of sides of the triangle.

Determine whether each system is consistent.

16. $2x - y = 5$
$4x - 2y = 7$ ***(ISC)***

17. $2x - 3y + z = 1$
$x + 2y - z = 1$
$3x - y + 2z = 6$

18. $x + 3y - 2z = 2$
$3x - y - z = 1$
$2x + 6y - 4z = 3$

19. $2x - y + z = 4$
$x + 3y + 2z = 12$
$3x + 2y + 3z = 10$

20. $x + y = 2$
$2x - z = 1$
$2y - 3z = 1$ ***(ISC 1998)***

21. (*i*) Show that the following equations are consistent :
$4x - 3y + 1 = 0$, $7x - 8y + 10 = 0$, $x + y - 5 = 0$

(*ii*) Find whether the following equations are consistent :
$2x + 3y - 17 = 0$, $x - 2y + 16 = 0$, $3x + y - 1 = 0$

22. Find k so that the following equations are consistent : ***(ISC)***

(*a*) $2x - y + 3 = 0$
$kx - y + 1 = 0$
$5x - y - 3 = 0$

(*b*) $2x + 3y + 4 = 0$
$3x + 4y + 6 = 0$
$4x + 5y - k = 0$ ***(ISC)***

[Hint. (*a*) Solving 1st and 3rd equations by Cramer's rule, we get $x = 2$ and $y = 7$. For consistency the third equation must be satisfied by these values].

23. Obtain the condition of consistency in the form of a determinant for the following three equations :

$2x + 3y - 8 = 0$; $7x - 5y + 3 = 0$; $4x - 6y + \lambda = 0$

and hence find the value of λ. ***(ISC)***

24. Find the values of k if the following equations are consistent,

$x + y - 3 = 0$; $(1 + k)\, x + (2 + k)\, y - 8 = 0$; $kx - (1 + k)\, y + 2 + k = 0$ ***(ISC)***

25. If the following equations are consistent and have more than one solution, find the values of λ :

$u + v = -(\lambda v + 1)$; $u + 2v = -\lambda(v - 1) + 1$; $3u + 8v = \lambda + 2$ ***(ISC 1990)***

26. Given $x = cy + bz$, $y = az + cx$, $z = bx + ay$, where x, y, z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$

[$x - cy - bz = 0$, $cx - y + az = 0$, $bx + ay - z = 0$, x, y, z are not zero, it has non-zero solution if, $D = 0$, *i.e.* $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$.

ANSWERS

1. $x = -3, y = -7$

2. $x = -1, y = 4$

3. $x = 2, y = -3$

4. $x = 2, y = -\dfrac{2}{a}, a \neq 0$

5. (*i*) $x = -\dfrac{1}{2}, y = \dfrac{1}{5}$, (*ii*) $x = \dfrac{1}{3}, y = \dfrac{-1}{2}$

6. $x = 0, y = 0$

7. $x = k, y = \dfrac{1 - 3k}{4}$, where k is any number

8. $x = k, y = 5k - 7$, where k is any number

9. $x = -2, y = -3, z = -4$

10. $x = 1, y = -1, z = -1$

11. $x = 1, y = 2, z = 3$

12. $x = 2, y = -1, z = 1$

13. $x = \dfrac{7 - 5k}{5}, y = k, z = \dfrac{5k - 3}{5}$, where k is any number

14. 3, 1, 2

15. 19 cm, 15 cm, 11 cm

16. Inconsistent

17. Consistent

18. Inconsistent

19. Inconsistent

20. Consistent

21. (*ii*) Consistent

22. (*a*) $k = 3$, (*b*) $k = -8$

23. $\begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0, \Rightarrow \lambda = 8$

24. $k = 1$ or $-\dfrac{1}{2}$

25. $\lambda = 1$ or $\dfrac{-5}{3}$

MATRICES

Syllabus

- **Of order $m \times n$, when $m, n \leq 3$, including case $m = n$; Types of matrices.**
- **Operations : Addition / Subtraction (compatibility); Multiplication by a scalar; Multiplication of two matrices (compatibility).**

  Application of matrix multiplication
- **Adjoint and inverse of a matrix**
- **Use of matrices to solve simultaneous linear equations in 2 or 3 unknowns.**

3

Matrices

3.01. The concept of a matrix

It is often desirable to present a set of numbers (or other elements) in a rectangular array of rows and columns. The table of values of trigonometric functions is an example of such an arrangement; in it the columns have the headings sine, cosine, tangent, and cotangent, and the rows are designated by angles, expressed in degrees. It is conventional to call the vertical lines ***columns*** and the horizontal lines ***rows.***

1. Here is a Bowling Analysis in cricket.

	Overs	*Maidens*	*Runs*	*Wickets*
Yogesh	15	7	70	5
John	18	6	55	4
Aslam	10	3	21	1

2. Here is an example of simultaneous linear equations :

$$2x - 3y = 7$$

$$\frac{1}{2}x + 5y = 9$$

These may be set down as below :

Coefficient of x	*Coefficient of y*	*Constant term*
2	– 3	7
$\frac{1}{2}$	5	9

Tables are a concise method of presenting a mass of information. When we construct a table from a collection of data, we generally arrange the data in rows and columns. We extract the information from the table by reading the entry corresponding to a row and column intersection. Any table is a matrix.

Definition. ***A Matrix (plural is matrices) is an array of real numbers (or other suitable entities), arranged in rows and columns.***

Entries. The entities are called **entries,** or **elements,** of a matrix. In this book we shall consider only real numbers as entries. A matrix is customarily displayed in a pair of brackets or parentheses. If a matrix appears in an understood context, we may omit the row and column headings. Thus the matrices of illustration Nos. 1 and 2 above can be represented as

$$\begin{bmatrix} 15 & 7 & 70 & 5 \\ 18 & 6 & 55 & 4 \\ 10 & 3 & 21 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -3 & 7 \\ \frac{1}{2} & 5 & 9 \end{bmatrix} \text{ respectively.}$$

The following are all matrices :

$$\begin{bmatrix} 3 & 0 & 2 \\ 5 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

3.02. The order of a matrix

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. Thus,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are 2×3 (read "two by three"), 3×1 (read "three by one") and 4×4 (read "four by four") matrices, respectively. Note that the **number of *rows* is given first, and then the number of *columns*.**

Generally, a matrix that has m rows and n columns is called an $m \times n$ (read "m by n") matrix, or a matrix of order $m \times n$.

A matrix is also denoted by using double subscripts, where a single letter, say a is used to denote an entry in a matrix and then *two* subscripts are appended, the first subscript telling in which *row* the entry occurs, and the second telling which *column*. Thus, we write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{ij}
Elements in ith row and jth column.

where a_{21} is the element in the *second row* and *first column*, a_{32} is the element in **the *third row* and *second column*. Generalising, a_{ij} is the element in ith row and jth column.**

The general form of a matrix with m rows and n columns is

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

This $(m \times n)$ matrix can be denoted compactly as $(a_{ij})_{m \times n}$.

Ex. 1. Let $A = \begin{bmatrix} \mathbf{5} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{3} & \mathbf{7} \\ \mathbf{6} & \mathbf{1} & \mathbf{4} \end{bmatrix}$.

Sol. Then A is a 3×3 matrix. The element 7 occurs in the second row and third column. We may write $a_{23} = 7$. Similarly $a_{31} = 6$.

Ex. 2. If a matrix has 12 elements, what are the possible orders it can have ? What will be the possible orders if it has 7 elements ?

Sol. We know that a matrix of order $m \times n$ has mn elements. Hence, to find all possible orders of a matrix having 12 elements, we will find all ordered pairs the product of whose components is 12.

The possible ordered pairs having the property stated above are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3). Hence possible orders are

$$1 \times 12,\ 12 \times 1,\ 2 \times 6,\ 6 \times 2,\ 3 \times 4 \text{ and } 4 \times 3.$$

Remark. If the matrix has 7 elements, then the possible orders will be 1×7 and 7×1.

Ex. 3. (*a*) Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+2j)^2}{2}$.

(*b*) Construct a 2×3 matrix A, whose elements are given by $a_{ij} = \frac{(i-2j)^2}{2}$.

Sol. (*a*) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Putting $i = 1,\ j = 1,\ a_{11} = \frac{(1+2)^2}{2} = \frac{9}{2}$.

Similarly, $a_{12} = \frac{(1+2\times 2)^2}{2} = \frac{25}{2},\ a_{21} = \frac{(2+2\times 1)^2}{2} = 8,\ a_{22} = \frac{(2+2\times 2)^2}{2} = 18$

$\therefore\ A = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$.

(*b*) Given, $a_{ij} = \frac{(i-2j)^2}{2}$ where $1 \le i \le 2,\ 1 \le j \le 3$.

Let the reqd. 2×3 matrix A be $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Then, $a_{11} = \frac{(1-2)^2}{2} = \frac{1}{2},\ a_{12} = \frac{(1-4)^2}{2} = \frac{9}{2},\ a_{13} = \frac{(1-6)^2}{2} = \frac{25}{2}$

$a_{21} = \frac{(2-2)^2}{2} = 0,\ a_{22} = \frac{(2-4)^2}{2} = 2,\ a_{23} = \frac{(2-6)^2}{2} = 8$

$\therefore$ The required matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}.$$

Ex. 4. Construct a 3×4 matrix whose elements are $a_{ij} = i + j$.

Sol. Since $a_{ij} = i + j$, therefore,

$a_{11} = 1 + 1 = 2,\quad a_{12} = 1 + 2 = 3,\quad a_{13} = 1 + 3 = 4,\quad a_{14} = 1 + 4 = 5$

$a_{21} = 2 + 1 = 3,\quad a_{22} = 2 + 2 = 4,\quad a_{23} = 2 + 3 = 5,\quad a_{24} = 2 + 4 = 6$

$a_{31} = 3 + 1 = 4,\quad a_{32} = 3 + 2 = 5,\quad a_{33} = 3 + 3 = 6,\quad a_{34} = 3 + 4 = 7$

Hence, the required matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ or $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$.

Ex. 5. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by

(*i*) $a_{ij} = \frac{1}{2}|2i - 3j|$

(*ii*) $a_{ij} = \begin{cases} i - j \text{ if } i \ge j \\ i + j \text{ if } i < j \end{cases}$

Sol. (*i*) Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given, $a_{ij} = \frac{1}{2}|2i - 3j|,\ a_{11} = \frac{1}{2}|2-3| = \frac{1}{2}|-1| = \frac{1}{2}\times 1 = \frac{1}{2}$

$a_{12} = \frac{1}{2}|2 - 3(2)| = \frac{1}{2}|-4| = \frac{1}{2}\times 4 = 2,\ a_{21} = \frac{1}{2}|2(2) - 3| = \frac{1}{2}\times 1 = \frac{1}{2}$

$a_{22} = \frac{1}{2}|2(2) - 3(2)| = \frac{1}{2}|-2| = \frac{1}{2}\times 2 = 1$

Hence, the required matrix is $A = \begin{bmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$.

(*ii*) Here, $a_{11} = 1 - 1 = 0, a_{12} = 1 + 2 = 3, a_{21} = 2 - 1 = 1, a_{22} = 2 - 2 = 0$

$\therefore \quad A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$.

3.03. Types of matrices

(*i*) Rectangular matrix. ***Any $m \times n$ matrix, where $m \neq n$, is called a rectangular matrix.***

For example, $\begin{bmatrix} 1 & -4 \\ 0 & 2 \\ 6 & 3 \end{bmatrix}$ is a rectangular matrix.

(*ii*) Row matrix. ***A matrix having only one row is called a row matrix.***

e.g., $[3 \quad 7 \quad 1 \quad -2]_{1\times 4}$, $[0 \quad -3 \quad 6]_{1\times 3}$ are row matrices.

(*iii*) Column matrix. *A matrix which has only one column is called a column matrix.*

e.g., $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} 0 \\ -3 \\ 4 \\ 5 \end{bmatrix}_{4\times 1}$, etc.

(*iv*) Square matrix. *A matrix in which the number of rows is equal to the number of columns is called a square matrix. An $m \times m$ matrix is termed as a square matrix of order m.* A 2×2 matrix is a square matrix of order 2, a 3×3 matrix is a square matrix of order 3.

For example, the matrices $\begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}$, $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ are square matrices.

The fact that matrices may be interpreted in various ways makes them useful in coordinate geometry. For example, the point 5, – 2 can be denoted by the matrix $[5 \quad -2]$. Such a matrix having only one row is called a **row matrix** or a **row vector.** It may also be denoted by the matrix $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$. A matrix of this type having one column is called a **column matrix** or a **column vector.**

The coordinates (1, 2, 3) of a point in space may be designated as $[1 \quad 2 \quad 3]$ by a row matrix or as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ by a column matrix.

It is no accident that the same notation is used for a column matrix and a column vector : you have seen in Book 1 in the chapters on vectors that they have the same properties.

A polygon is determined by its vertices, and each vertex is a point (x, y). We can use matrices to denote polygons. For example quadrilateral *ABCD* in Fig. 2.01 has vertices $\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Putting these column vertices into single 2 by 4 matrix.

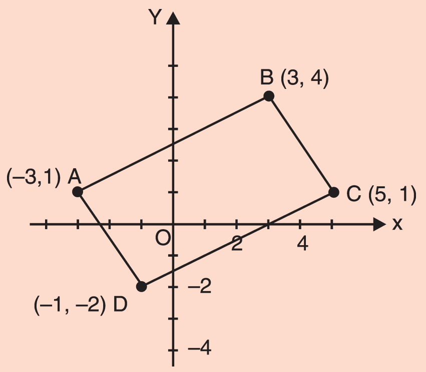


Fig. 3.01

$$\begin{bmatrix} -3 & 3 & 5 & -1 \\ 1 & 4 & 1 & -2 \end{bmatrix}$$

We have a matrix representing quad. *ABCD*. Each column refers to one vertex of the quadrilateral.

Similarly, the diagonal *DB* of *ABCD* is represented by $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$. What could $\begin{bmatrix} -3 & 3 & -1 \\ 1 & 4 & -2 \end{bmatrix}$ represent?

Note. If $A = [a_{ij}]$ be a square matrix of order *m*, then the elements a_{ij} for which $i = j$, are called the diagonal elements of *A*.

Thus, the diagonal elements of $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$.

The line along which the diagonal elements of a square matrix lie, is known as the **main** or **leading diagonal** of the matrix.

e.g., The matrix $\begin{bmatrix} 7 & -2 & 1 \\ 3 & 0 & 5 \\ -1 & 4 & 8 \end{bmatrix}$ is a 3-rowed square matrix in which the diagonal elements are 7, 0, and 8.

(*v*) Diagonal matrix. *It is a square matrix all of whose elements except those in the leading diagonal, are zero.*

For example, $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Definition. ***A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if all the elements, except those in the leading diagonal are zero, i.e., $a_{ij} = 0$ for all $i \neq j$.***

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots\ldots, d_n$ as diagonal elements may be denoted by diag. $[d_1, d_2, \ldots\ldots, d_n]$.

Thus, the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ may be denoted by diag [3, 4, – 7].

(*vi*) Scalar matrix. *A square matrix in which the diagonal elements are all equal, all other elements being zeros, is called a scalar matrix.*

e.g., $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a third order scalar matrix.

Definition. ***A square matrix $A = [a_{ij}]_{n \times n}$ is called a scalar matrix if***

$$a_{ij} = \begin{cases} 0, & \text{for all } i \neq j \\ \alpha, & \text{if } i = j, \text{ where } \alpha \neq 0 \end{cases}$$

(*vii*) Unit matrix or Identity matrix. *A square matrix in which each diagonal element is unity, all other elements being zeros, is called a unit matrix or an identity matrix.*

Unit matrix of order *n* is denoted by I_n.

e.g., $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. ***A square matrix $A = [a_{ij}]_{n \times n}$ is called an identity or unit matrix if***

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

(*viii*) Sub-matrix. *A matrix obtained by deleting the rows or columns or both of a matrix is called sub-matrix.*

For example, $A = \begin{bmatrix} 5 & 7 \\ -1 & 2 \end{bmatrix}$ is sub-matrix of matrix $B = \begin{bmatrix} 5 & 7 & 3 \\ -1 & 2 & 0 \\ 4 & 1 & 0 \end{bmatrix}$, obtained by deleting third row and third column of matrix B.

(*ix*) **Comparable matrices.** Two matrices A and B are said to be comparable if they are of the same order, *i.e.*, they have the same number of rows and the same number of columns.

e.g., $\begin{bmatrix} 1 & 0 & -3 \\ 2 & 7 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$ are comparable matrices, each being of order 2×3.

3.04. Equality of matrices

Two matrices A and B are equal if and only if both matrices are of the same order and each element of one is equal to the corresponding element of the other,

i.e., $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if $a_{ij} = b_{ij}\ \forall\ i, j$.

Thus $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$, but $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Ex. 6. Find the values of *x, y, z* and *t* which satisfy the matrix equation

$$\begin{bmatrix} x+3 & x+2y \\ z-1 & 4t-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2t \end{bmatrix}.$$

Sol. By the principle of equality of matrices,

$$\begin{bmatrix} x+3 & x+2y \\ z-1 & 4t-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2t \end{bmatrix} \Rightarrow x+3=0 \Rightarrow x=-3$$

$z - 1 = 3 \Rightarrow z = 4,\quad x + 2y = -7 \Rightarrow -3 + 2y = -7$

$\Rightarrow y = -2,\ 4t - 6 = 2t \Rightarrow t = 3$

Ex. 7. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, **why ?**

Sol. Because the given matrices are not comparable *i.e.*, they are not of the same order.

Ex. 8. Find the values of *x* and *y* so that the matrices

$$A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix},\ B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

may be equal.

Sol. $A = B \Rightarrow 2x+1 = x+3,\quad 3y = y^2+2,\quad y^2 - 5y = -6$

(*i*) Now, $2x + 1 = x + 3 \Rightarrow x = 2$

(*ii*) $3y = y^2 + 2 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow (y-2)(y-1) = 0 \Rightarrow y = 1$ or 2.

(*iii*) $y^2 - 5y = -6 \Rightarrow y^2 - 5y + 6 = 0 \Rightarrow (y-3)(y-2) = 0 \Rightarrow y = 3$ or 2

Since $3y = y^2 + 2$ and $y^2 - 5y = -6$ must hold simultaneously.

we take the common solution of these two equations, *i.e.*, $y = 2$.

Hence, $A = B$ if $x = 2, y = 2$.

EXERCISE 3 (*a*)

1. If a matrix has 8 elements, what are the possible orders it can have ? What if it has 5 elements ?

2. How many entries are there in (*i*) a 3×3 matrix, (*ii*) a 3×4 matrix, (*iii*) an $m \times n$ matrix, (*iv*) a square matrix of order n ?

3. Write out the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ given that $a_{ij} = 4i - 3j$.

4. Construct a 2×2 matrix $B = [b_{ij}]$ whose elements are given by

(*i*) $b_{ij} = \dfrac{(i-2j)^2}{2}$ (*ii*) $\dfrac{1}{2}(-3i+j)$

5. Construct a 3×4 matrix whose elements are :

(*i*) $a_{ij} = i - j$ (*ii*) $a_{ij} = ij$ (*iii*) $a_{ij} = \dfrac{i}{j}$.

6. (*a*) Construct a 2×3 matrix whose elements are given by

(*i*) $a_{ij} = \dfrac{3i - j}{2}$. (*ii*) $a_{ij} = \dfrac{i+3j}{2}$

(*b*) Construct a 3×2 matrix whose elements in the *i*th row and *j*th column are given by

(*i*) $a_{ij} = \dfrac{i+3j}{2}$. (*ii*) $a_{ij} = \dfrac{(i+2j)^2}{2}$.

7. If $A = \begin{bmatrix} 5 & -2 & 1 & 0 & 3 \\ 7 & 6 & 4 & 2 & -1 \\ 0 & 8 & 3 & 5 & 6 \end{bmatrix}$, then

(*i*) State the order of A; (*ii*) Write down the entries of the second row of A ; (*iii*) Write down the entries of the third column of A ; (*iv*) State the entries a_{12}, a_{23}, a_{34}, a_{15} of the above matrix; (*v*) If $a_{ij} = 4$, find i, j.

8. Find x and y such that

(*i*) $\begin{bmatrix} x & y \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -1 & 5 \end{bmatrix}$ (*ii*) $[x \;\; 3] = [-1 \;\; y]$ (*iii*) $\begin{bmatrix} x+1 \\ -3+y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

9. If $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, find x, y, z.

10. (*i*) If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$, find x, y, z, w.

(*ii*) If matrix $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, find the values of a and b.

ANSWERS

1. 1×8, 8×1, 2×4, 4×2. When a matrix has 5 elements, the possible orders are 1×5 and 5×1.

2. (*i*) 9 (*ii*) 12 (*iii*) mn (*iv*) n^2 **3.** $\begin{bmatrix} 1 & -2 & -5 \\ 5 & 2 & -1 \\ 9 & 6 & 3 \end{bmatrix}$ **4.** (*i*) $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$

5. (*i*) $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 2 & 1 & 2/3 & 1/2 \\ 3 & 3/2 & 1 & 3/4 \end{bmatrix}$

6. (*a*) (*i*) $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$, (*ii*) $\begin{bmatrix} 2 & \frac{7}{2} & 5 \\ \frac{5}{2} & 4 & \frac{11}{2} \end{bmatrix}$ (*b*) (*i*) $\begin{bmatrix} 2 & \frac{7}{2} \\ \frac{5}{2} & 4 \\ 3 & \frac{9}{2} \end{bmatrix}$ (*ii*) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \\ \frac{25}{2} & \frac{49}{2} \end{bmatrix}$

7. (*i*) 3×5 (*ii*) 7, 6, 4, 2, – 1 (*iii*) 1, 4, 3 (*iv*) – 2, 4, 5, 3 (*v*) $i = 2, j = 3$

8. (*i*) $x = -2, y = 0$ (*ii*) $x = -1, y = 3$ (*iii*) $x = -3, y = 3$

9. $x = 1, y = 2, z = 3$ **10.** (*i*) $x = 1, y = 2, z = 3, w = 4$, (*ii*) $a = 2, b = 4$ or $a = 4, b = 2$

OPERATIONS ON MATRICES

3.05. The sum or addition of matrices

Consider the following example :

Rakesh and Anil are close competitors in the mathematics class. They compare their marks at the end of the second term, the scores for the two terms being as given below :

	First term		*Second term*		*Total*	
	Rakesh	*Anil*	*Rakesh*	*Anil*	*Rakesh*	*Anil*
Algebra	95	90	90	92	185	182
Geometry	85	87	88	89	173	176

If we set out this information in matrix form, we can write

$$\begin{bmatrix} 95 & 90 \\ 85 & 87 \end{bmatrix} + \begin{bmatrix} 90 & 92 \\ 88 & 89 \end{bmatrix} = \begin{bmatrix} 185 & 182 \\ 173 & 176 \end{bmatrix}$$

This method of combining matrix is called the sum or addition of matrices.

Definition. ***The sum of two matrices of the same order, $A_{m\times n}$ and $B_{m\times n}$, is the matrix $(A + B)_{m\times n}$ in which the entry in the ith row and jth column is $a_{ij} + b_{ij}$, for i = 1, 2, 3, ...m and j = 1, 2, 3, ...n.***

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then

$$A + B = [a_{ij} + b_{ij}]_{m\times n}$$

In other words, the sum of two matrices is a matrix of the same order, whose entries are the sums of the corresponding entries of $A_{m\times n}$ and $B_{m\times n}$.

Ex. 9. $$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3+1 & 1+0 & 2+2 \\ 2+(-1) & 1+3 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 4 & 4 \end{bmatrix}$$

Note. Two matrices of the same order are said to be **compatible** or **conformable** for addition. The sum of two matrices of different orders is not defined.

Ex. 10. Is it possible to define the matrix $A + B$, when

(*i*) A has 3 rows and B has 2 rows

(*ii*) A has 2 columns and B has 4 columns

(*iii*) A has 3 rows and B has 2 columns

(*iv*) Both A and B are square matrices of the same order ?

Sol. (*i*) No, because $A + B$ is defined only if A and B are of the same order.

(*ii*) No. As above.

(*iii*) Yes, only when A has 2 columns and B has 3 rows for in that case both will be of the same order.

(*iv*) Yes. Always.

3.06. Zero matrix or null matrix

In the algebra of real numbers R, the equation $a + 0 = a$ is satisfied for all $a \in R$. Accordingly we say that 0 is the identity element for addition in R. In the algebra of matrices, the matrices all of whose entries are 0 play a corresponding role. Thus

$$\begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5+0 & 2+0 \\ -2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix}$$

Such a matrix is called a **zero matrix** and is denoted by **0.** It may be of any order. An $m \times n$ zero matrix may also be denoted by $\mathbf{0}_{m\times n}$, or if the matrix is square, we might write $\mathbf{0}_n$, where n indicates the order of the matrix. Thus,

$$\mathbf{0}_{1\times 2} = [0\ 0],\ \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{0}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.07. The negative of the matrix

The negative of the matrix $A_{m \times n}$ *denoted by* $-A_{m \times n}$, *is the matrix formed by replacing each entry in the matrix* $A_{m \times n}$ *with the additive inverse.*

For example if
$$A_{3\times 2} = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}, \text{ then } -A_{3\times 2} = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$$

The sum $B_{m \times n} + (-A_{m \times n})$ is called the **difference** of $B_{m \times n}$ and $A_{m \times n}$ and is denoted by $B_{m\times n} - A_{m\times n}$.

Definition. *If* $A = (a_{ij})_{m\times n}$, *and X is any matrix of the same order such that* $\mathbf{A + X = 0}$, *the zero matrix, then X is called the* **additive inverse of A.** It is clear that $X = -A$. *For example*

$$\text{If } A = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -a \\ -b \end{bmatrix} \text{ and } A + (-A) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} a-a \\ b-b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Ex. 11. Find the additive inverse of $A = \begin{bmatrix} 5 & -7 & 1 \\ 3 & 0 & -2 \end{bmatrix}$.

Sol. We know that additive inverse of A is a matrix of same order each of whose elements is the negative of corresponding element of A.

$$\therefore \quad -A = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 0 & 2 \end{bmatrix}.$$

3.08. Subtraction of matrices

For the set of real numbers, subtraction was defined as follows :

$$\forall\, a, b \in R,\; a - b = a + (-b).$$

We will define the subtraction of matrices in a similar way.

Definition. *If A and B are matrices of the same order, then the sum* $B + (-A)$ *is called the* **difference** or **subtraction** *of B and A is denoted by* $B - A$.

Ex. 12. If $L = \begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix}$ **and** $M = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$, **find** $L - M$.

Sol.
$$L - M = \begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 2-1 & 0+2 \\ -3+0 & 6-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$$

We can obtain the difference directly by subtracting their corresponding entries. Thus

$$L - M = \begin{bmatrix} 2-1 & 0-(-2) \\ -3+0 & 6-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$$

3.09. Properties of sums of matrices

At this point, we are able to establish the following facts concerning the set $S_{m\times n}$ $(\forall\, m, n \in N)$ of all $m \times n$ matrices with real number entries.

Theorem: *If A, B and C are members of the set $S_{m\times n}$ of all $m \times n$ matrices with real number entries, then*

I	$A + B \in S_{m\times n}$	***Closure law for addition***
II	$A + B = B + A$	***Commutative law for addition***
III	$(A + B) + C = A + (B + C)$	***Associative law for addition***
IV	*The matrix* $\mathbf{0}_{m\times n}$ *has the property that for every matrix* $A_{m\times n}$. $A + \mathbf{0} = A$ *and* $\mathbf{0} + A = A$	***Additive-identity law***
V	*For every matrix* $A_{m\times n}$ *the matrix* $-A_{m\times n}$ *has the property that* $A + (-A) = \mathbf{0}$ and $(-A) + (A) = \mathbf{0}$	***Additive-inverse law***
VI	If A, B, C are three matrices of the same order, then	
	$A + B = A + C \Rightarrow B = C$	***Left cancellation law***
	$B + A = C + A \Rightarrow B = C$	***Right cancellation law***

Compare the properties listed above with the axioms of addition in R.

3.10. Solving matrix equations

By using the substitution principle and the properties of equality, we can solve certain matrix equations. Suppose that we have to solve the equation $X + A = B$ for the unknown matrix X. The answer is easy. We do exactly what we learnt to do with numbers. Add the matrix $-A$ to both sides. This gives

$$X + A + (-A) = B + (-A)$$

$$\Rightarrow \quad X + \mathbf{0} = B - A, \text{ since } A + (-A) = \mathbf{0}$$

$$\Rightarrow \quad X = B - A,$$

which is the required solution.

Ex. 13. **Solve** $X + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}$ **for the 2×2 matrix X.**

Sol.

$$X + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}$$

$$\Rightarrow \left[X + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right] + \begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow X + \left[\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}\right] = \begin{bmatrix} 4 + (-1) & -1 + (-3) \\ 6 + 1 & 2 + (-2) \end{bmatrix}$$

$$\Rightarrow X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 7 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 3 & -4 \\ 7 & 0 \end{bmatrix}$$

Checking, we find that

$$\begin{bmatrix} 3 & -4 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3+1 & -4+3 \\ 7+(-1) & 0+2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}$$

Therefore, the solution set is $\left\{\begin{bmatrix} 3 & -4 \\ 7 & 0 \end{bmatrix}\right\}$.

3.11. Multiplication of a matrix by a scalar

While dealing with matrices, real numbers are often referred to as *scalars*. We know that $\forall x \in R, x + x = 2x$ and $x + x + x = 2x + x = 3x$. Similarly, we have repeated addition of the same matrix.

Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. By definition of the addition of matrices

$$A + A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & 2c \\ 2b & 2d \end{bmatrix}$$

and
$$A + A + A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & 2c \\ 2b & 2d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3a & 3c \\ 3b & 3d \end{bmatrix}$$

The above examples suggest that we may write $A + A$ as $2A$ and $A + A + A$ as $3A$.

We define the product of a real number or scalar k and a matrix A, denoted by kA, as the matrix whose entries are the products of k and the corresponding entries of A. Thus

$$k\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ka & kc \\ kb & kd \end{bmatrix} \text{ where } k \in R.$$

Definition. ***The product of a real number c and an m × n matrix A with entries a_{ij}, is the matrix cA with corresponding entries ca_{ij}, for i = 1, 2, 3, ..., m and j = 1, 2, 3, ... n.***

Ex. 14. If $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, find 3A and – 2A.

Sol.
$$A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \Rightarrow 3A = 3\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3\times 1 & 3\times -3 \\ 3\times 0 & 3\times 2 \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ 0 & 6 \end{bmatrix}$$

Similarly,
$$-2A = -2\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2\times 1 & -2\times -3 \\ -2\times 0 & -2\times 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 0 & -4 \end{bmatrix}$$

Notice that the product of a real number and a matrix is a matrix.

3.12. Properties of products of matrices and real numbers

Products of scalars and matrices have a number of basic properties which follow from the definition given above and the properties of real numbers. These basic properties are given in the following theorem.

Theorem. *If $A \in S_{m \times n}$ and $B \in S_{m \times n}$, where m, n are any given natural numbers and $c \in R$, $d \in R$, then*

I	cA	$\in$	$S_{m \times n}$	V	$1A$	=	A
II	$c(A + B)$	=	$cA + cB$	VI	$(-1)A$	=	$-A$
III	$(c + d)A$	=	$cA + dA$	VII	$\mathbf{0}A$	=	$\mathbf{0}$
IV	$(cd)A$	=	$c(dA) = d(cA)$	VIII	$c\mathbf{0}$	=	$\mathbf{0}$

Ex. 15. Express in a single matrix :

$$4\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}.$$

Sol.
$$4\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 4 & -16 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 2 & -20 \end{bmatrix}.$$

Ex. 16. Find a matrix X such that $X + A = 0$ and $A = \begin{bmatrix} 2 & 5 & 7 \\ -3 & 0 & -4 \\ 3 & 4 & 5 \end{bmatrix}$.

Sol.

$$X + A = 0 \Rightarrow X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 5 & 7 \\ -3 & 0 & -4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0-2 & 0-5 & 0-7 \\ 0-(-3) & 0-0 & 0-(-4) \\ 0-3 & 0-4 & 0-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 & -7 \\ 3 & 0 & 4 \\ -3 & -4 & -5 \end{bmatrix}.$$

Ex. 17. Find a matrix X such that $3A - 2B + X = 0$, where

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

Sol. Given,

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

Now,

$$3A - 2B + X = \mathbf{0} \Leftrightarrow X = -3A + 2B = -3\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\Leftrightarrow \quad X = \begin{bmatrix} -12 & -6 \\ -3 & -9 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -12-4 & -6+2 \\ -3+6 & -9+4 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$$

Ex. 18. Solve the equation

$$-2\left[X + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right] = 3X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ over } S_{3\times 3}.$$

Sol. We first perform the indicated multiplication by – 2 in accordance with part IV of the above theorem to get

$$-2X + \begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 3X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we add $2X$ to both sides of the equation to obtain

$$\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 3X + 2X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we use part III of the above theorem to find that $3X + 2X = 5X$, so that

$$\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 5X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding $-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to both sides, we get

$$\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5X \Rightarrow \begin{bmatrix} -3 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{bmatrix} = 5X$$

Multiplying both sides of this equation by $\frac{1}{5}$, we get by part II of the theorem

$$\begin{bmatrix} \frac{-3}{5} & \frac{-4}{5} & \frac{-6}{5} \\ 0 & \frac{-2}{5} & \frac{-4}{5} \\ 0 & 0 & \frac{-3}{5} \end{bmatrix} = X \Rightarrow X = \begin{bmatrix} \frac{-3}{5} & \frac{-4}{5} & \frac{-6}{5} \\ 0 & \frac{-2}{5} & \frac{-4}{5} \\ 0 & 0 & \frac{-3}{5} \end{bmatrix}$$

This is the required solution.

Ex. 19. If $A = \text{diag}\,[3 \quad -2 \quad 1]$ and $B = \text{diag}\,[1 \quad 3 \quad -2]$, find $2A - 3B$.

Sol. Given

$$A = \text{diag}\,[3\ -2\ 1] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \text{diag}\,[1\ 3\ -2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore \quad 2A - 3B = 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & 0 & 0 \\ 0 & -4+9 & 0 \\ 0 & 0 & 2-6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \text{dig.}\,[3 \quad 5 \quad -4]$$

Ex. 20. Simplify : $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$.

Sol. $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\cos\theta\sin\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

EXERCISE 3 (*b*)

1. Write each sum or difference as a single matrix.

(*i*) $[2 \quad 3] + [5 \quad 1]$

(*ii*) $\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(*iii*) $\begin{bmatrix} -3 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$

(*iv*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(*v*) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \\ 3 & 1 & -6 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 1 \\ -3 & 0 & -4 \\ -2 & -1 & 6 \end{bmatrix}$

(*vi*) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ -\cos x & -\sin^2 x \end{bmatrix}$

2. (*i*) Given $A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ 3 & -2\\ 0 & 1\end{bmatrix}$, and $C=\begin{bmatrix}4 & 2\\ 1 & 0\\ -2 & -4\end{bmatrix}$ compute the following :

(*a*) $A + B$, (*b*) $(A + B) + C$, (*c*) $A + (B + C)$, (*d*) $A - B$, (*e*) $(A - B) + C$, (*f*) $B - A$.

(*ii*) Consider the answers to part (*b*) and (*c*), what law is illustrated ?

(*iii*) Consider the parts (*d*) and (*f*), what conclusion can be drawn ?

3. Solve the matrix equation $X+\begin{bmatrix}2 & 1\\ 6 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ for the 2×2 matrix.

4. Solve the equation $X+\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}2 & 1 & 2\\ 3 & 2 & 3\\ 4 & 3 & 4\end{bmatrix}$ for the 3×3 matrix X.

5. If $\begin{bmatrix}2 & -3\\ 4 & 0\end{bmatrix}-\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}=\begin{bmatrix}-3 & 4\\ 5 & -1\end{bmatrix}$ determine x_1, x_2, y_1, and y_2.

6. If $A=\begin{bmatrix}1 & 2\\ -1 & 8\\ 4 & 9\end{bmatrix}$, construct a matrix X such that $X + A = 0$.

7. If $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0 & 3\\ 0 & 0 & 5\end{bmatrix}$, verify the commutative law of addition.

8. Is the equation $\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 3\\ 1 & 4 & 4\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 3\\ 1 & 4 & 4\end{bmatrix}=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ valid ?

9. If $\begin{bmatrix}x^2 & 3 & 4\\ 1 & 9 & 8\end{bmatrix}+\begin{bmatrix}-3x & 1 & -5\\ -3 & -2 & -6\end{bmatrix}=\begin{bmatrix}4 & 4 & -1\\ -2 & 7 & 2\end{bmatrix}$, find the values of x. ***(ISC 2003)***

10. For $A=\begin{bmatrix}2 & 2 & 2\\ 2 & 1 & -3\\ 1 & 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}3 & 3 & 3\\ 3 & 0 & 5\\ 6 & 9 & -1\end{bmatrix}$, $C=\begin{bmatrix}4 & 4 & 4\\ 5 & -1 & 4\\ 7 & 8 & -1\end{bmatrix}$, compute (*a*) $3A - 6B + 9C$, (*b*) $7A - 2B - C$.

11. If $A=\begin{bmatrix}1 & -2\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}$, solve each over $S_{2\times 2}$:

(*i*) $X + 2A = B$ (*ii*) $X - A = 3B$ (*iii*) $2X - 3A = 2B - X$

12. If $A=\begin{bmatrix}1 & 2 & 3\\ -2 & 5 & 7\end{bmatrix}$ and $2A-3B=\begin{bmatrix}4 & 5 & -9\\ 1 & 2 & 3\end{bmatrix}$, find B. ***(NMOC)***

13. If $A=\begin{bmatrix}1 & -3 & 2\\ 2 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\end{bmatrix}$, find the matrix C such that $A + B + C$ is a zero matrix.

14. Find X and Y if $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$. ***(ISC 2008)***

15. If $\begin{bmatrix}x & 1\\ -1 & -y\end{bmatrix}+\begin{bmatrix}y & 1\\ 3 & x\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ show that $x = 1$ and $y = 0$.

16. If A = diag (1 –4 8), B = diag (–2 3 5), C = diag. (–3 7 10) find

(*i*) $2A + 3B$ (*ii*) $B + 2C - A$ (*iii*) $3A - B + 4C$.

17. Solve the matrix equation : $2\begin{bmatrix}x & y\\ z & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\ 4 & 6\end{bmatrix}$.

18. Find x, y, z and w if $3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$.

19. Solve for x and y. $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2\begin{bmatrix} 2x \\ 3y \end{bmatrix} = 3\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ **(*NMOC*)**

20. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$, then find a matrix $C = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$ such that $A + B - C = 0$.

21. Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, find a matrix B such that $A + B - 4I = 0$.

ANSWERS

1. (*i*) $[7 \quad 4]$ (*ii*) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 0 & 2b \\ -2b & 0 \end{bmatrix}$ (*v*) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (*vi*) $\begin{bmatrix} \dfrac{1}{\sin x - \cos x} & \dfrac{1}{\cos 2x} \end{bmatrix}$

2. (*i*) (*a*) $\begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 5 & 7 \end{bmatrix}$ (*b*) $\begin{bmatrix} 7 & 3 \\ 7 & 2 \\ 3 & 3 \end{bmatrix}$ (*c*) $\begin{bmatrix} 7 & 3 \\ 7 & 2 \\ 3 & 3 \end{bmatrix}$

(*d*) $\begin{bmatrix} -1 & 3 \\ 0 & 6 \\ 5 & 5 \end{bmatrix}$ (*e*) $\begin{bmatrix} 3 & 5 \\ 1 & 6 \\ 3 & 1 \end{bmatrix}$ (*f*) $\begin{bmatrix} 1 & -3 \\ 0 & -6 \\ -5 & -5 \end{bmatrix}$

(*ii*) The associative law for addition (*iii*) $A - B = -(B - A)$.

3. $\begin{bmatrix} -1 & 0 \\ -6 & 0 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \\ 3 & 3 & 4 \end{bmatrix}$

5. $x_1 = 5, x_2 = -7, y_1 = -1, y_2 = 1$ **6.** $\begin{bmatrix} -1 & -2 \\ 1 & -8 \\ -4 & -9 \end{bmatrix}$

8. No, although both members of the equation are equal to zero matrices, the orders are not the same.

9. $x = -1, 4$ **10.** (*a*) $\begin{bmatrix} 24 & 24 & 24 \\ 33 & -6 & -3 \\ 30 & 18 & 9 \end{bmatrix}$ (*b*) $\begin{bmatrix} 4 & 4 & 4 \\ 3 & 8 & -35 \\ -12 & -26 & 31 \end{bmatrix}$

11. (*i*) $\begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 7 & 1 \\ -2 & 3 \end{bmatrix}$ (*iii*) $\dfrac{1}{3}\begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}$ **12.** $B = -\dfrac{1}{3}\begin{bmatrix} 2 & 1 & -15 \\ 5 & -8 & -11 \end{bmatrix}$

13. $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$ **14.** $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

16. (*i*) diag (–4 1 31) (*ii*) diag (–9 21 17) (*iii*) diag (–7 13 59)

17. $x = 3, y = 9, z = 6, t = 6$ **18.** $x = 2, y = 4, z = 1, w = 3$ **19.** $x = -7$ or 3; $y = -3$.

20. $\begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$ **21.** $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & -2 \\ -3 & -4 & -1 \end{bmatrix}$

3.13. Multiplication of matrices

(1) Let us consider the two matrices

$$A_{1\times 3} = [a \quad b \quad c] \text{ and } B_{3\times 2} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$$

Suppose that the numbers a, b, c represent the number of bags of wheat flour, rice, and sugar purchased each week by a shopkeeper, while x_1, y_1, z_1 represent the cost per bag of each respectively the first week, and x_2, y_2, z_2 the costs the second week. How much would the shopkeeper have to pay for these purchases ?

The total amount payable for the first week is clearly

$$ax_1 + by_1 + cz_1$$

while the total cost payable for the second week is

$$ax_2 + by_2 + cz_2$$

Here we have added the products obtained by multiplying the elements of a *row* in one matrix by the corresponding elements of a *column* in the other matrix.

(2) The number of tubes and the number of speakers used in assembling TV sets of three different models were specified by the following table :

	Model A	*Model B*	*Model C*
Number of tubes	13	18	20
Number of speakers	2	3	4

This array may be called the parts per set matrix. Suppose orders were received in January for 12 sets of Model A, 24 sets of Model B and 12 sets of Model C; and in February for 6 sets of Model A, 12 of models B, and 9 of Model C. We can arrange the information in the form of the following matrix :

	January	February
Model A	12	6
Model B	24	12
Model C	12	9

We may call this array as the sets per month matrix. To determine the number of tubes and speakers required in each of the months for these orders, it is clear that we must use both sets of information. For instance, to compute the number of tubes needed in January, we multiply each entry in the first *row* of the parts per set matrix by the corresponding entry in the first *column* of the sets per month matrix, and then add the three products. Thus the number of tubes required in January is

$$13\times 12 + 18\times 24 + 20\times 12 = 828$$

To compute the number of speakers needed in January we multiply each entry in the second row of the parts per set matrix by the corresponding entry in the first column of the sets per month matrix and then add the products. Thus, the number of speakers for January is

$$2\times 12 + 3\times 24 + 4\times 12 = 144$$

Similarly, the number of tubes and speakers for February are respectively.

$$13\times 6 + 18\times 12 + 20\times 9 = 474$$

and $$2\times 6 + 3\times 12 + 4\times 9 = 84$$

We can arrange the four sums in an array, which we shall call the parts per month matrix.

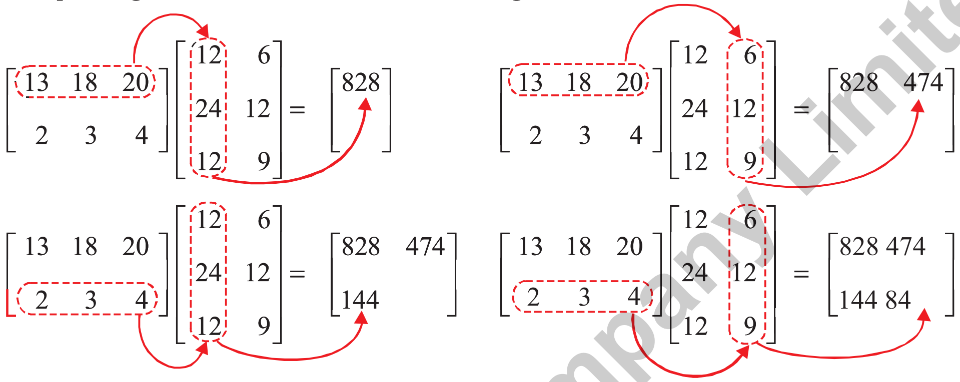
	January	February
Number of tubes	828	474
Number of speakers	144	84

We can represent our "operations" in equation form as under :

$$\begin{bmatrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix}$$

We have *multiplied* the parts per set matrix by the set per month matrix to get just what should be expected, the parts per month matrix.

Note that, in the foregoing equation, 828 equals the sum of the products of the entries in the first row of the left-hand factor by the corresponding entries in the first column of the right-hand factor. Likewise 474 equals the sum of the products of the entries in the first row of the left-hand factor by the corresponding entries in second column of the right-hand factor, and so on.



A method for remembering how to write down the elements in multiplication of one matrix by another is shown below:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

$$= \begin{bmatrix} \textbf{1st row} \times \textbf{1st column} & \textbf{1st row} \times \textbf{2nd column} \\ \textbf{2nd row} \times \textbf{1st column} & \textbf{2nd row} \times \textbf{2nd column} \end{bmatrix}$$

This is a **"Multiply row by column"** process. *We multiply the entries of a row by the corresponding entries of a column and then add the products.*

Thus $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \end{bmatrix}_{3\times 2}$$

Note. The definition of the product of two matrices, A and B, requires that the **matrix A has the same number of columns, as B has rows ;** the result, AB, then has the same number of rows as A and the same number of columns as B.

$$A_{m\times p} \times B_{p\times n} = C_{m\times n}$$

Such matrices A and B are said to be **conformable** or **compatible** for multiplication. The fact that two matrices are conformable in the order AB, however, does not mean that they necessarily are conformable in the order BA.

Ex. 21. Calculate : $\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$.

Sol. $$[1 \quad 3 \quad -2]\begin{bmatrix}4\\6\\5\end{bmatrix} = [1\times4+3\times6+(-2)(5)] = [4+18-10] = \mathbf{[12]}.$$

Ex. 22. **If** $A = \begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}$ **and** $B = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$, **find** AB.

Sol. $$AB = \begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1\times2+2\times1 & 1\times1+2\times1\\-1\times2+3\times1 & -1\times1+3\times1\end{bmatrix} = \begin{bmatrix}2+2 & 1+2\\-2+3 & -1+3\end{bmatrix} = \begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}$$

$$BA = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix} = \begin{bmatrix}2\times1+1\times(-1) & 2\times2+1\times3\\1\times1+1\times(-1) & 1\times2+1\times3\end{bmatrix} = \begin{bmatrix}2-1 & 4+3\\1-1 & 2+3\end{bmatrix} = \begin{bmatrix}1 & 7\\0 & 5\end{bmatrix}$$

This example shows that the multiplication of matrices, in general, is *not commutative*. Thus, when discussing products of matrices we must specify the *order* in which the matrices are to be considered as factors. *For the product AB, we say that* **A is right-multiplied by B**, or *that* **B is left-multiplied by A.** This is also expressed as **"post-multiplication of A by B"** or **"pre-multiplication of B by A."**

Ex. 23. **If** $A = \begin{bmatrix}3 & 1 & 2\\1 & 0 & 1\end{bmatrix}$ **and** $B = \begin{bmatrix}1 & -1\\2 & 1\\3 & 1\end{bmatrix}$, **find** AB.

Sol. Since A is a 2×3 matrix, and B is a 3×2 matrix *i.e.*, A has the same number of columns as B has rows, therefore, they are conformable for multiplication. We have

$$AB = \begin{bmatrix}3 & 1 & 2\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 1\\3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}3\times1+1\times2+2\times3 & 3\times(-1)+1\times1+2\times1\\1\times1+0\times2+1\times3 & 1\times(-1)+0\times1+1\times1\end{bmatrix} = \begin{bmatrix}3+2+6 & -3+1+2\\1+0+3 & -1+0+1\end{bmatrix} = \begin{bmatrix}11 & 0\\4 & 0\end{bmatrix}.$$

In much of the matrix work in this book, we shall focus our attention on matrices having the same number of rows as columns. For brevity, a matrix of order $n\times n$ is often called a **square matrix of order *n*.** Although many of the ideas we shall discuss are applicable to matrices of any order, we shall apply the notions only to square matrices. If A is a square matrix, then A^2, A^3, etc., denote AA, $(AA)A$, etc.

Ex. 24. **Compute :** (*i*) $\begin{bmatrix}1 & 6\\-3 & 5\end{bmatrix}\begin{bmatrix}4 & 0\\2 & -1\end{bmatrix}$ (*ii*) $\begin{bmatrix}1 & 6\\-3 & 5\end{bmatrix}\begin{bmatrix}2\\-7\end{bmatrix}$

(*iii*) $\begin{bmatrix}1\\-6\end{bmatrix}\begin{bmatrix}1 & 6\\-3 & 5\end{bmatrix}$ (*iv*) $\begin{bmatrix}1\\6\end{bmatrix}[3 \quad 2]$ (*v*) $[2 \quad -1]\begin{bmatrix}1\\-6\end{bmatrix}$.

Sol. (*i*) The first factor is a 2×2 matrix and the second factor is also a 2×2 matrix, so that product is defined and is a 2×2 matrix.

$$\begin{bmatrix}1 & 6\\-3 & 5\end{bmatrix}\begin{bmatrix}4 & 0\\2 & -1\end{bmatrix} = \begin{bmatrix}1\times4+6\times2 & 1\times0+6\times(-1)\\-3\times4+5\times2 & -3\times0+5\times(-1)\end{bmatrix} = \begin{bmatrix}16 & -6\\-2 & -5\end{bmatrix}.$$

(*ii*) The first factor is a 2×2 matrix and the second factor is a 2×1 matrix, so that product is defined and is a 2×1 matrix.

$$\begin{bmatrix}1 & 6\\-3 & 5\end{bmatrix}\begin{bmatrix}2\\-7\end{bmatrix} = \begin{bmatrix}1\times2+6\times(-7)\\-3\times2+5\times(-7)\end{bmatrix} = \begin{bmatrix}-40\\-41\end{bmatrix}.$$

(*iii*) The first factor is a 2×1 matrix and the second is a 2×2 matrix. Since the number of columns in the first matrix is not equal to the number of rows in the second matrix, therefore, the product is not defined.

(*iv*) The first factor is a 2×1 matrix and the second is a 1×2 matrix, so the product is defined and is a 2×2 matrix. Thus

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\times 3 & 1\times 2 \\ 6\times 3 & 6\times 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 18 & 12 \end{bmatrix}$$

(*v*) The first factor is a 1×2 matrix and the second is a 2×1 matrix, so the product is defined and is a 1×1 matrix which we frequently write as a scalar. Thus

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = [2\times 1 + (-1)(-6)] = [8] = 8.$$

3.14. Definitions

(1) The **principal diagonal** *of a square matrix is the ordered set of entries* a_{ij}, *where* $i = j$, *extending from the upper left-hand corner to the lower right-hand corner of the matrix.*

For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of 1, 2, and 0, in that order.

(2) A **diagonal matrix** *is a square matrix in which all entries, but not in the principal diagonal, are 0.*

Thus $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are diagonal matrices.

(3) $I_{n\times n}$ *denotes the diagonal matrix having* 1's *for entries on the principal diagonal.*

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The importance of $I_{n\times n}$ to the operation of multiplication of $n \times n$ matrices is apparent in Art. 3.15.

3.15. Identity element or unity element or unit matrix

Theorem. ***For each matrix*** $A_{n\times n}$, ***we have*** $A_{n\times n}\, I_{n\times n} = I_{n\times n}\, A_{n\times n} = A_{n\times n}$

Further, if for the matrix $B_{n\times n}$, ***we have*** $A_{n\times n}\, B_{n\times n} = B_{n\times n}\, A_{n\times n} = A_{n\times n}$ **for all matrices** $A_{n\times n}$, **then** $B_{n\times n} = I_{n\times n}$.

Accordingly, $I_{n\times n}$ is the **identity element** for multiplication in the set $n\times n$ square matrices, and $I_{n \times n}$ is unique. The proof of this theorem, for the illustrative case $n = 2$, is left as an exercise to the student.

Note. Since $I_{n\times n}$ is a square matrix of order, n, it may be denoted by I_n. Thus

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

Ex. 25. If $A = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}$, **show that** $AI_2 = I_2 A$.

Sol. $AI_2 = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5\times 1 + 2\times 0 & 5\times 0 + 2\times 1 \\ -3\times 1 + 4\times 0 & -3\times 0 + 4\times 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} = A.$

$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1\times 5 + 0\times(-3) & 1\times 2 + 0\times 4 \\ 0\times 5 + 1\times(-3) & 0\times 2 + 1\times 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} = A \Rightarrow AI_2 = I_2 A.$

Ex. 26. **If** $A = \begin{bmatrix} 3 & 3 & 5 \\ 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix}$, **show that** $AI_3 = I_3A$.

Sol.

$$AI_3 = \begin{bmatrix} 3 & 3 & 5 \\ 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0 & 0+3+0 & 0+0+5 \\ 2+0+0 & 0+3+0 & 0+0+4 \\ 5+0+0 & 0+2+0 & 0+0+3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \\ 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix} = A$$

Similarly, it may be verified that $I_3A = A \quad \Rightarrow \quad AI_3 = I_3A$.

3.16. Now we state the last theorem in a more general form as below:

Theorem. ***If A is an n × n matrix, then*** $AI_n = A$ ***and*** $I_nA = A$.

3.17. The scalar matrix and the diagonal matrix

The matrix $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is called a **scalar matrix** and is obviously obtained by the scalar multiplication of matrix I by k as $kI = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$.

The scalar matrix is a special case of a **diagonal matrix.** In a diagonal matrix all the elements except those on the leading diagonal are zeros.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a simple example of a diagonal matrix.

3.18. Properties of matrix multiplication

We have learnt that in so far as only addition and subtraction are involved, the algebra of matrices is exactly like the ordinary algebra of numbers. Now we shall show that it is not so for the operation of multiplication. There are some major differences in the two algebras when multiplication is involved.

1. ***The product of matrices is not commutative.***

(*a*) Whenever AB exists, BA is not always defined. For example if A be a 5×4 matrix and B be a 4×3 matrix, then AB is defined while BA is not defined.

(*b*) If AB and BA are both defined, it is not necessary that they should be equal. For example if A be a 4×3 matrix and B a 3×4 matrix, then AB is defined and is a 4×4 matrix. BA is also defined but is a 3×3 matrix. AB and BA being of different orders, $AB \neq BA$.

(*c*) Even if AB and BA are both defined and are of the same order, it is not necessary that $AB = BA$.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1\times0+2\times1 & 1\times3+2\times4 \\ 0\times0+3\times1 & 0\times3+3\times4 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 3 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0\times1+3\times0 & 0\times2+3\times3 \\ 1\times1+4\times0 & 1\times2+4\times3 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 1 & 14 \end{bmatrix}$$

Thus $AB \neq BA$

(*d*) However, it is not always that AB is not equal to BA.

Let $$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 8 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\times5+0\times2-4\times1 & 1\times8+0\times3-4\times2 & 1\times4+0\times2-4\times1 \\ 0\times5-1\times2+2\times1 & 0\times8-1\times3+2\times2 & 0\times4-1\times2+2\times1 \\ -1\times5+2\times2+1\times1 & -1\times8+2\times3+1\times2 & -1\times4+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5\times1+8\times0+4(-1) & 5\times0+8(-1)+4\times2 & 5(-4)+8\times2+4\times1 \\ 2\times1+3\times0+2(-1) & 2\times0+3(-1)+2\times2 & 2(-4)+3\times2+2\times1 \\ 1\times1+2\times0+1(-1) & 1\times0+2(-1)+1\times2 & 1(-4)+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ Hence, $AB = BA$.

2. ***The product of two matrices can be zero without either factor being a zero matrix.***

This fact is borne out by the following example.

Let $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ where a, b, c, d are all different from zero.

Here $A \neq 0, B \neq 0$. Also, $AB = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$

Ex. 27. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$, **and** $C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$, **show that** AB **and** CA **are null matrices but** $BA \neq 0, AC \neq 0$.

Sol. Since A, B are square matrices, therefore, AB and BA are both defined.

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-5 \\ -2-18+20 & -4-36+40 & -2-18+20 \\ -3-12+15 & -6-24+30 & -3-12+15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{null matrix}$$

$$BA = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4-3 & -1+6+2 & 1-8-3 \\ 6+24+18 & 6-36+12 & -6+48+18 \\ 5+20+15 & 5-30-10 & -5+40+15 \end{bmatrix} = \begin{bmatrix} -8 & 7 & -10 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix}$$

Similarly, you can show that

$CA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = null matrix, and $AC = \begin{bmatrix} 4 & 4 & -4 \\ -20 & -20 & 20 \\ -16 & -16 & 16 \end{bmatrix}$, which is not a null matrix.

3. ***Cancellation law for the multiplication of real numbers is not valid for the multiplication of matrices.***

The breakdown for matrix algebra of the law that $xy = yx$ and of the law that $xy = 0$ only if either x or y is zero causes additional differences. For instance, for real numbers we know that if $ab = ac$, and $a \neq 0$, then $b = c$. This property is called the *cancellation law of multiplication.* This does not hold for the multiplication of matrices, that is AB can be equal to AC with the conditions $A \neq 0$, and $B \neq C$. Let us consider the following example :

If $$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

then $$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & -7 \end{bmatrix} = AC \quad \text{and} \quad A \neq 0, \quad \text{and} \quad B \neq C.$$

Thus, $AB = AC$ *does not imply that* $B = C$.

Note. The invalidity of the commutative law and the concellation law for the multiplication of matrices should not lead us to conclude a total collapse of all the other familiar laws. Except for these two laws and the fact that, as we shall see later, many matrices do not have multiplicative inverses (reciprocals), the other basic laws of ordinary algebra generally remain valid for matrices. The associative law holds for the multiplication of matrices and there are distributive laws that unite addition and multiplication.

4. ***Matrix multiplication is associative if conformability is assured, i.e., A (BC) = (AB) C.***

5. ***Matrix multiplication is distributive with respect to matrix addition, i.e.,***

$$\boldsymbol{A(B + C) = AB + AC.}$$

Note. It can be proved that

(*i*) $(B + C)A = BA + CA$ **(*ii*)** $A(B - C) = AB - AC$ **(*iii*)** $(B - C)A = BA - CA$

Ex. 28. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, **and** $C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, **show that**

(*i*) $(AB)C = A(BC)$; **(*ii*)** $A(B + C) = AB + AC$; **(*iii*)** $(B + C)A = BA + CA$

Sol. $$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

(*i*) $$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+0 \\ 2+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 7 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 7 \end{bmatrix} \Rightarrow (AB)C = A(BC)$$

(*ii*) $$B + C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+2 \\ 1+3 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 \\ 1+4 & 2+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \quad AC = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1+0 & 2+0 \\ -1+3 & 2+1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+2 \\ 3+2 & 1+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A(B+C) = AB + AC$$

(*iii*)
$$B + C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

$$(B+C)A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+2 \\ 4+2 & 0+2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+0 \\ 1+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 0+2 \\ 3+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$BA + CA = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2+1 & 0+2 \\ 2+4 & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (B+C)A = BA + CA.$$

It may be noted that in the above example $A(B+C) \neq (B+C)A$.

Ex. 29. If A, B, C are three matrices such that

$$\boldsymbol{A = [x \quad y \quad z], \quad B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \quad C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}} \text{ evaluate } \boldsymbol{ABC}.$$

(*NMOC*)

Sol.
$$AB = [x \quad y \quad z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$= [xa + yh + zg \quad xh + yb + zf \quad xg + yf + zc]$$

$$\therefore \quad ABC = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$$

6. Zero matrix. Earlier we defined the zero matrix of order $m \times n$ and showed that it is the identity element for matrix addition, *i.e.*, $\boldsymbol{A + 0 = A}$, where A is any matrix of order $m \times n$. This zero matrix plays the same role in the multiplication of matrices as the number zero does in the multiplication of real numbers. For example, we have

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}_2$$

thus, for any matrix $A_{p \times n}$, we have

$$\mathbf{0}_{m \times p}\, \boldsymbol{A}_{p \times n} = \mathbf{0}_{m \times n} \text{ and } \boldsymbol{A}_{p \times n}\, \mathbf{0}_{n \times q} = \mathbf{0}_{p \times q}$$

Some of the above results may be concisely put in the form of a theorem as under :

Theorem. *If $S_{n \times n}$ is the set of $n \times n$ square matrices for n, a fixed positive integer $a \in R$; A, B, $C \in S_{n \times n}$, then*

I	$AB \in S_{n \times n}$	*Closure law for multiplication.*
II	$(AB)C = A(BC)$	*Associative law for multiplication.*
III	$A(B + C) = AB + AC$ and $(B + C)A = BA + CA$	*Distributive laws.*
IV	$AI_{n \times n} = A$ and $I_{n \times n} A = A$	*Multiplicative-identity law.*

3.19. Positive integral powers of matrices

If A is any matrix, the product AA is defined only when A is a square matrix. Let us denote this product AA as A^2.

Now by the Associative Law,

$$A^2A = (AA)A = A(AA) = AA^2 = AAA.$$

If we denote this product by A^3, then $A^3 = A^2A = AA^2 = AAA$.

Since the Associative law is true for any number of matrices, we can denote

$(AAA...m \text{ times})$ by A^m.

Note 1. If I is a unit matrix of any order, then $I = I^2 = I^3 = I^4 = I^n$.

Note 2. We can also form polynomials in A that is for any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + ...+ a_nx^n$$

we define $f(A)$ to be the matrix

$$f(A) = a_0I + a_1A + a_2A^2 + ...+ a_nA^n$$

In case $f(A)$ is the zero matrix, then A is said to be *zero* or *root* of the polynomial $f(x)$.

Ex. 30. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, find

(*i*) A^2, (*ii*) A^3, (*iii*) $f(A)$, where $f(x) = 2x^3 - 4x + 5$; (*iv*) Show that A is a zero of the polynomial $g(x) = x^2 + 2x - 11$.

Sol. (*i*) $$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$$

(*ii*) $$A^3 = AA^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$$

(*iii*) To find $f(A)$, first substitute A for x and $5I$ for the constant 5 in the given polynomial $2x^3 - 4x + 5$.

$$f(A) = 2A^3 - 4A + 5I = 2\begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 60 \\ 120 & -134 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -16 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14-4+5 & 60-8+0 \\ 120-16+0 & -134+12+5 \end{bmatrix} = \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}$$

(*iv*) Now A is zero of $g(x)$ if the matrix $g(A)$ is the zero matrix. Compute $g(A)$ as was done for $f(A)$, that is first substitute A for x and $11I$ for the constant 11 in $g(x) = x^2 + 2x - 11$.

$$g(A) = A^2 + 2A - 11I = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} + \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2-11 & -4+4+0 \\ -8+8+0 & 17-6-11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $g(A) = \mathbf{0}$, therefore A is a zero of the polynomial $g(x)$.

Ex. 31. Show that if $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ **then** $(A+B)^2 \neq A^2 + 2AB + B^2$.

Sol. $$(A+B)^2 = \left\{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}\right\}^2 = \begin{bmatrix} 0 & 7 \\ 4 & 7 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0 & 7 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 0 & 7 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 28 & 49 \\ 28 & 77 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 21 \\ 14 & 42 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 10 & 14 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 5 & 7 \\ 10 & 14 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 20 & 28 \end{bmatrix}, \quad B^2 = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 7 & 21 \\ 14 & 42 \end{bmatrix} + \begin{bmatrix} 10 & 14 \\ 20 & 28 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 26 & 35 \\ 34 & 79 \end{bmatrix}$$

Since $\begin{bmatrix} 28 & 49 \\ 28 & 77 \end{bmatrix} \neq \begin{bmatrix} 26 & 35 \\ 34 & 79 \end{bmatrix}$, therefore, $(A+B)^2 \neq A^2 + 2AB + B^2$

Note. If we expand $(A+B)^2$ in the form

$$(A+B) \times (A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2$$

We can see why the two expressions in the foregoing example are not equal. $(A+B)^2$ is equal to $A^2 + AB + BA + B^2$ and, therefore, cannot be equal to $A^2 + 2AB + B^2$ unless $AB = BA$, which, here, it is not.

$$BA = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 21 \\ 4 & 12 \end{bmatrix}$$

Ex. 32. If $T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, **find** T^2, T^3.

Sol. $$T^2 = T \times T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ the null matrix.}$$

$\Rightarrow$ $T^2 = \mathbf{0} \;\Rightarrow\; T^3 = T^2 \times T = 0 \times T = \mathbf{0}$

Thus T is a matrix which has the special property, namely that powers of itself are zero.

Note. In number algebra the only number whose square is zero is zero itself. This is another difference between matrix algebra and number algebra. Such matrices as T, *whose integral power is zero*, are said to be **NIL POTENT** matrices.

Ex. 33. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, **show that** $A^2 - 5A + 7I_2 = 0$

Sol. Given $$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$\Rightarrow$ $$A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$-5A = -5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (-5)(3) & (-5)\times 1 \\ (-5)(-1) & (-5)\times 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix}$$

$$7I_2 = 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 7I_2 = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} + \begin{bmatrix}-15 & -5\\5 & -10\end{bmatrix} + \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$$

$$= \begin{bmatrix}8-15+7 & 5-5+0\\-5+5+0 & 3-10+7\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Ex. 34. If $A = \begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix}$ and $f(x) = x^2 - 4x + 7$, show that $f(A) = 0$. Use this result to find A^5.

Sol. Given, $f(x) = x^2 - 4x + 7 \Rightarrow f(A) = A^2 - 4A + 7I_2$

$$A^2 = AA = \begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix} = \begin{bmatrix}2\times2+3\times-1 & 2\times3+3\times2\\-1\times2+2\times-1 & -1\times3+2\times2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 12\\-4 & 1\end{bmatrix}$$

$$4A = 4\begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix} = \begin{bmatrix}8 & 12\\-4 & 8\end{bmatrix}, \quad 7I_2 = 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$$

$$\therefore \quad f(A) = A^2 - 4A + 7I_2 = \begin{bmatrix}1 & 12\\-4 & 1\end{bmatrix} - \begin{bmatrix}8 & 12\\-4 & 8\end{bmatrix} + \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$$

$$= \begin{bmatrix}1-8+7 & 12-12+0\\-4+4+0 & 1-8+7\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \mathbf{0}$$

Now, $f(A) = 0 \Rightarrow A^2 - 4A + 7I_2 = 0 \Rightarrow A^2 = 4A - 7I_2$ ...(1)

$A^3 = A^2A = (4A - 7I_2)A = 4A^2 - 7I_2A = 4A^2 - 7A$ $(\because I_2A = A)$

$\Rightarrow A^3 = 4(4A - 7I_2) - 7A$ [From (1)]

$\Rightarrow A^3 = 9A - 28I_2$ ... (2)

$A^4 = A^3A = (9A - 28I_2)A = 9A^2 - 28I_2A = 9A^2 - 28A$ [Using (2)]

$\Rightarrow A^4 = 9(4A - 7I_2) - 28A$ [Putting $A^2 = 4A - 7I_2$ from (1)]

$\Rightarrow A^4 = 36A - 63I_2 - 28A = 8A - 63I_2$

$\therefore A^5 = A^4A = (8A - 63I_2)A = 8A^2 - 63I_2A = 8A^2 - 63A$

$= 8(4A - 7I_2) - 63A$ [Putting $A^2 = 4A - 7I_2$]

$= 32A - 56I_2 - 63A = -31A - 56I_2$

$$= -31\begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix} - 56\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-62 & -93\\31 & -62\end{bmatrix} - \begin{bmatrix}56 & 0\\0 & 56\end{bmatrix}$$

Hence, $$A^5 = \begin{bmatrix}-62-56 & -93-0\\31-0 & -62-56\end{bmatrix} = \begin{bmatrix}-118 & -93\\31 & -118\end{bmatrix}.$$

EXERCISE 3 (*c*)

1. Calculate : (*i*) $\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -10 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 5 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 6 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (*v*) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

2. Find the value of x in the following :

(*i*) $\begin{bmatrix} x & 7 \end{bmatrix}\begin{bmatrix} 4 \\ x \end{bmatrix} = [22]$ (*ii*) $\begin{bmatrix} -2 & x & 4 \end{bmatrix}\begin{bmatrix} x \\ 3 \\ 5 \end{bmatrix} = [15]$

3. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ in each of the problems through (*i*) to (*xii*), find a 2×2 matrix equal to the given product.

(*i*) AB (*ii*) BA (*iii*) AC (*iv*) CA (*v*) BC (*vi*) CB

(*vii*) A^2 (*viii*) B^2 (*ix*) $(A + B)C$ (*x*) $C(A + B)$ (*xi*) $(A + B)^2$ (*xii*) $(C - A)^2$

Do you find that $AB \neq BA$, $AC \neq CA$, $BC \neq CB$, $(A + B)C \neq C(A + B)$?

4. (*a*) If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, show that $AB \neq BA$, where $i^2 = -1$.

(*b*) If $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, show that $AB \neq 0$ but $BA = 0$.

5. If $A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 11 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -3 & -3 \end{bmatrix}$, state the order of each of the following matrices :

(*a*) AB (*b*) DA (*c*) AD (*d*) CB

(*e*) BD (*f*) $D(AB)$ (*g*) $(CB)(DA)$ (*h*) $B(DA)$.

6. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, obtain the product AB and explain why BA is not defined.

7. (*i*) If $A = \begin{bmatrix} 4 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, find AB and BA.

(*ii*) AB and BA if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

8. Evaluate :

(*i*) $\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -5 & 7 \end{bmatrix}$.

9. (*i*) If $A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$, find $-A^2 + 6A$.

(*ii*) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 7 \end{bmatrix}$, find $3A^2 - 2B$.

10. (*i*) If $A = \begin{bmatrix} 0 & 3 \\ -7 & 5 \end{bmatrix}$, find k so that $kA^2 = 5A - 21\,I$.

(*ii*) If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k such that $A^2 = kA - I_2$.

(*iii*) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, show that $A^2 - 3\,I = 2A$. ***(ISC 2007)***

11. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, by computing each expression determine whether or not the given statement is true.

(*a*) $(AB)\,C = A\,(BC)$ (*b*) $AC = 0$ (*c*) $AC = CA$

(*d*) $A\,(B + C) = (B + C)\,A$ (*e*) $(-A)\,C = A\,(-C)$ (*f*) $B^2 = (-B)^2$

12. (*i*) If $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, verify whether

$$(A + B)\,(A + B) = A^2 + 2\,AB + B^2.$$

Explain your result with proper reasoning. ***(ISC)***

(*ii*) If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ where $i^2 = -1$, verify that $(A + B)^2 = A^2 + B^2$.

13. If $A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, find $(A - 2I)\,(A - 3I)$

where I is the unit matrix, and express the above product in a matrix form. ***(ISC)***

14. (*i*) $f(x) = x^2 - 5x + 7$, find $f(A)$ when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

[Hint. Type solved Ex. 30]

(*ii*) $f(x) = x^2 - 5x + 6$, find $f(A)$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

15. (*i*) Without using the concept of inverse of a matrix, find the matrix

$\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ ***(ISC)***

(*ii*) Without using the concept of inverse of a matrix, find the matrix X so that $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, where X is a 2×2 matrix.

[Hint. Let $X = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$]

16. If $A = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ show that $A^2B^2 = A^2$.

17. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, find $A^2 - 5A - 14\,I$ ***(ISC)***

18. Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -2 \end{pmatrix}$

Is it possible to compute AB ? If it is possible to compute AB, then write

(*i*) the order of the product matrix AB, and

(*ii*) the value of each of the elements a_{11}, a_{12} and a_{31} of the matrix AB. ***(ISC 1990)***

[Note. Element ajk means in the jth row and kth column of the matrix.]

19. Given $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, find the matrix X, such that $AX = 3B + 2C$. ***(ISC 1991)***

20. A is a $5 \times p$ matrix. B is a $2 \times q$ matrix. A is conformable to B, for pre-multiplication (*i.e.*, AB can be worked out). AB works out to be a 5×4 matrix. Write the values of p and q. **(*ISC 1992*)**

21. (*i*) Verify that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 + A = 0$.

(*ii*) Show that the matrix $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$ satisfies the equation $A^3 - 7A^2 - 5A + 13\,I = 0$.

(*iii*) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$ and I is the identify matrix of order 3, show that $A^3 = pI + qA + rA^2$.

22. (*i*) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that $A^2 = I$.

(*ii*) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, show that $A^2 = 3A$.

23. Solve the following matrix equation for x :

(*i*) $\begin{bmatrix} 1 & 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$

[Hint. $\begin{bmatrix} 1+0+2x & 0+2+x & 2+1+0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2x+1 & 2+x & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$

$\Rightarrow [2x+1+2+x+3] = 0 \Rightarrow 3x+6=0 \Rightarrow x = -2]$

(*ii*) $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$

24. (*i*) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$.

(*ii*) If $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{bmatrix}$.

25. Matrix $R(t)$ is given by $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, show that $R(s)\,R(t) = R(s+t)$.

26. (*a*) Give examples of two matrices A and B such that

(*i*) $A \neq O, B \neq O, AB \neq O$ and $BA = O$; **(*Pb 1994*)**

(*ii*) $A \neq O, B \neq O, AB = BA = O$;

(*iii*) $AB \neq BA$.

(*b*) A, B and C such that $AB = AC$ but $B \neq C, A \neq O$.

ANSWERS

1. (*i*) [4] (*ii*) [26] (*iii*) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$ (*v*) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

2. (*i*) $x = 2$ (*ii*) $x = -5$ **3.** (*i*) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ (*iii*) $\begin{bmatrix} -1 & 7 \\ 3 & 4 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 4 & 7 \\ 3 & -1 \end{bmatrix}$ (*v*) $\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$ (*vi*) $\begin{bmatrix} -1 & 3 \\ 3 & -4 \end{bmatrix}$ (*vii*) $\begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix}$ (*viii*) $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

(*ix*) $\begin{bmatrix} -5 & 10 \\ 6 & 3 \end{bmatrix}$ (*x*) $\begin{bmatrix} 3 & 10 \\ 6 & -5 \end{bmatrix}$ (*xi*) $\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$ (*xii*) $\begin{bmatrix} 7 & -4 \\ -6 & 7 \end{bmatrix}$

5. (*a*) 2×3 (*b*) 3×3 (*c*) 2×2 (*d*) 4×3 (*e*) 3×2
(*f*) 3×3 (*g*) 4×3 (*h*) 3×3

6. $\begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$, since *B* has 2 columns and *A* has 3 rows, *i.e.*, no. of columns in *B* are not equal to no. of rows in *A*, therefore, *BA* is not defined.

7. (*i*) $AB = [13]$, $BA = \begin{bmatrix} 8 & -4 & 10 \\ 0 & 0 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ (*ii*) $AB = [30]$, $BA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$

8. (*i*) $[ac + bd + a^2 + b^2 + c^2 + d^2]$ (*ii*) $[-21 \quad 15 \quad -10]$

9. (*i*) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -20 \\ 38 & -11 \end{bmatrix}$ **10.** (*i*) $k = 1$ (*ii*) $k = 1$

11. (*a*) Yes (*b*) No (*c*) No (*d*) No (*e*) Yes
(*f*) Yes

12. No **13.** $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$ **14.** (*i*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

15. (*i*) $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$ **17.** (*i*) 0 **18.** Yes

(*i*) 3×2 (*ii*) $a_{11} = 0, a_{22} = -4, a_{31} = 0$ **19.** $X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

20. $p = 2, q = 4$ **23.** (*i*) $x = -2$ (*ii*) $x = -2$ or $x = -14$

26. (*a*) (*i*) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(*iii*) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (*b*) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

3.20. Application of matrix multiplication

Ex. 35. A trust fund has Rs 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of (*a*) Rs 1800, (*b*) 2000, (*c*) Rs 1600.

Sol. Let Rs x be invested in first type of bonds and Rs $(30000 - x)$ in second type of bonds.

The value of these bonds can be written in the form of a row matrix A given by

$$A = [x \quad 30000 - x] \text{ which is a } 1 \times 2 \text{ matrix.}$$

And the amounts received as interest per rupee annually from these two types of bonds can be written in the form of a column matrix B given by

$$B = \begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}, \text{ which is a } 2 \times 1 \text{ matrix.}$$

The interest to be annually is a single number, *i.e.*, a matrix of order 1×1 which can be obtained by the product matrix AB, since the product matrix would be a 1×1 matrix

$$AB = [x \quad 30000 - x] \begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}$$

$$= \left[x . \frac{5}{100} + (30000 - x) . \frac{7}{100} \right] = \left[2100 - \frac{2x}{100} \right]$$

(*a*) Since the total annual interest is given to be Rs 1800, therefore,

$$\left[2100 - \frac{2x}{100} \right] = [1800] \Rightarrow 2100 - \frac{2x}{100} = 1800$$

$$\Rightarrow \qquad x = 15000 \quad \Rightarrow \quad 30000 - x = 15000$$

Hence the investments in the two types of bond are Rs 15,000 each.

(*b*) Put $\left[2100 - \frac{2x}{100} \right] = [2000]$ **Ans.** Rs 5000, Rs 25000.

(*c*) Put $\left[2100 - \frac{2x}{100} \right] = 1600$ **Ans.** Rs 25000, Rs 5000.

Ex. 36. The amounts of carbohydrates, fats, and proteins in bread, butter, and cheese are as in the following table :

	Carbohydrates	***Fats***	***Proteins***
Bread	**0.52**	**0.02**	**0.09**
Butter	**0.00**	**0.81**	**0.01**
Cheese	**0.00**	**0.25**	**0.20**

Now suppose a cheese sandwich and roll contain the following amounts of bread, butter and cheese.

	Bread	***Butter***	***Cheese***
Sandwich	**80**	**20**	**50**
Roll	**50**	**10**	**0**

Use matrix multiplication to show the dietary values of the sandwich and roll in terms of carbohydrates, fats, and proteins.

If now the breakfast matrix of John and Kamla is given as follows :

$$\begin{array}{l} \\ \textbf{John} \\ \textbf{Kamla} \end{array} \begin{array}{c} \textit{Sandwich} \quad \textit{Roll} \\ \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \end{array}$$

Find the dietary composition of John and Kamla in terms of carbohydrates, fats, and proteins. ***(ISC)***

Sol. The dietary values of the sandwich and roll are given by the following matrix multiplication :

$$\begin{array}{c} \text{Sandwich} \\ \text{Roll} \end{array} \begin{bmatrix} 80 & 20 & 50 \\ 50 & 10 & 0 \end{bmatrix} \begin{bmatrix} 0.52 & 0.02 & 0.09 \\ 0.00 & 0.81 & 0.01 \\ 0.00 & 0.25 & 0.20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \times 0.52 + 20 \times 0 + 50 \times 0 & 80 \times 0.02 + 20 \times 0.81 + 50 \times 0.25 & 80 \times 0.09 + 20 \times 0.01 + 50 \times 0.20 \\ 50 \times 0.52 + 10 \times 0 + 0 & 50 \times 0.02 + 10 \times 0.81 + 0 & 50 \times 0.09 + 10 \times 0.01 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 41.6 & 1.6+16.2+12.5 & 7.2+0.2+10 \\ 26 & 1+8.1 & 4.5+0.1 \end{bmatrix} = \begin{matrix} \textit{Carbohydrates} & \textit{Fats} & \textit{Proteins} \\ \end{matrix} \begin{bmatrix} 41.6 & 30.3 & 17.4 \\ 26.0 & 9.1 & 4.6 \end{bmatrix}$$

The dietary composition of John and Kamla is given by the following matrix multiplication :

$$\begin{matrix} \text{John} \\ \text{Kamla} \end{matrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 41.6 & 30.3 & 17.4 \\ 26.0 & 9.1 & 4.6 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 41.6 + 2\times 26.0 & 3\times 30.3 + 2\times 9.1 & 3\times 17.4 + 2\times 4.6 \\ 1\times 41.6 + 1\times 26.0 & 1\times 30.3 + 1\times 9.1 & 1\times 17.4 + 1\times 4.6 \end{bmatrix}$$

$$= \begin{bmatrix} 124.8+52 & 90.9+18.2 & 52.2+9.2 \\ 41.6+26.0 & 30.3+9.1 & 17.4+4.6 \end{bmatrix} = \begin{matrix} \textit{Carbohydrates} & \textit{Fats} & \textit{Proteins} \end{matrix} \begin{bmatrix} 176.8 & 109.1 & 61.4 \\ 67.6 & 39.4 & 22.0 \end{bmatrix}$$

Thus, the breakfast of John consists of 176.8 carbohydrates, 109.1 fats, and 61.4 proteins. The corresponding amounts for Kamla are 67.6, 39.4, 22.0 respectively.

Ex. 37. The number of gadgets *R* and *S* produced per day by each of two factories *P* and *Q* and the number of days per week that each factory operates, are shown in the following table :

		FACTORY *P*	FACTORY *Q*
Gadgets per day	***R***	**2**	**1**
	S	**4**	**3**
No. of days operating per week		**5**	**6**

Determine the product of the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \text{ and } N = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

and explain what each element of the product represents in the context of the above.

Given that : $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$**, find the values of** x_1 **and** x_2 **(*ISC*)**

Sol. Product of A and N

$$= \begin{matrix} R \\ S \end{matrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 10+6 \\ 20+18 \end{pmatrix} = \begin{pmatrix} 16 \\ 38 \end{pmatrix}$$

This means that both the factories P and Q produce in a week 16 gadgets R and 38 gadgets S.

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 2x_1 + x_2 = 17 \\ 4x_1 + 3x_2 = 39 \end{matrix} \quad \Rightarrow \begin{matrix} 4x_1 + 2x_2 = 34 \\ 4x_1 + 3x_2 = 39 \end{matrix}$$

Subtracting, we get $x_2 = 5$

Substituting this value of x_2, we get $2x_1 + 5 = 17 \Rightarrow x_1 = 6$

Hence $x_1 = 6, x_2 = 5.$

Ex. 38. There are two families *A* and *B*. There are 4 men, 6 women and 2 children in family *A*, with 2 men, 2 women and 4 children in family *B*. The recommended daily allowance for calories are: man : 2400, woman : 1900, child : 1800 and for proteins are : man : 55 gm, woman : 45 gm and child : 33 gm.

Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

Sol. Let F denote the family matrix and R the recommended daily allowance matrix. Then F a 2×3 matrix and R a 3×2 matrix can be represented as under :

$$F = \begin{array}{c} \\ A \\ B \end{array}\begin{bmatrix} M & W & C \\ 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \qquad R = \begin{array}{c} \\ M \\ W \\ C \end{array}\begin{bmatrix} \text{Calories} & \text{Proteins} \\ 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication as shown below :

$$FR = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \times 2400 + 6 \times 1900 \times 2 \times 1800 & 4 \times 55 + 6 \times 45 + 2 \times 33 \\ 2 \times 2400 + 2 \times 1900 + 4 \times 1800 & 2 \times 55 + 2 \times 45 + 4 \times 33 \end{bmatrix} = \begin{array}{c} A \\ B \end{array}\begin{bmatrix} 24600 & 556 \\ 15800 & 332 \end{bmatrix}$$

Thus, family A requires 24,600 calories and 556 gm protein and family B requires 15,800 calories and 332 gm proteins.

Ex. 39. Farm *A* and farm *B* are classified on the basis of the number of oxen, camels and tractors which are given in the table as follows :

	Oxen	**Camels**	**Tractors**
Farm *A* :	**20**	**5**	**4**
Farm *B* :	**10**	**6**	**6**

Now the cost (in thousand Rs) average expenditure (in Rs) and average daily return (in Rs) on each ox, camel and tractor is given in the following table :

	Cost (in thousand Rs)	**Average daily expenses (in Rs)**	**Average daily income (in Rs)**
Ox : 2.5	**20**	**25**	
Camel :	**3.0**	**20**	**30**
Tractor :	**52.5**	**80**	**150**

If now the number of Farm *A* and Farm *B* in States *X* and *Y* are as follows :

	Farm *A*	**Farm *B***
State *X* :	**2**	**4**
State *Y* :	**6**	**1**

Find by using matrix multiplication the total cost expenditure, and income on oxen, camels and tractors in State *X* and State *Y*.

Comparing the net income with the total money invested in cost by each State, find which state is getting a comparatively better advantage. **(*ISC*)**

Sol. Let $P = \begin{pmatrix} 20 & 5 & 4 \\ 10 & 6 & 6 \end{pmatrix}$; $Q = \begin{pmatrix} 2.5 & 20 & 25 \\ 3.0 & 20 & 30 \\ 52.0 & 80 & 150 \end{pmatrix}$ and $R = \begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix}$

$$RP = \begin{array}{c} X \\ Y \end{array}\begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} \text{Oxen} & \text{Camels} & \text{Tractors} \\ 20 & 5 & 4 \\ 10 & 6 & 6 \end{pmatrix} = \begin{array}{c} X \\ Y \end{array}\begin{pmatrix} \text{Oxen} & \text{Camels} & \text{Tractors} \\ 80 & 34 & 32 \\ 130 & 36 & 30 \end{pmatrix}$$

$$(RP)\,Q = \begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} \text{Oxen} & \text{Camels} & \text{Tractors} \\ 80 & 34 & 32 \\ 130 & 36 & 30 \end{pmatrix}\begin{pmatrix} \text{Cost} & \text{Av. daily expenses} & \text{Av. daily income} \\ 2.5 & 20 & 25 \\ 3.0 & 20 & 30 \\ 52.5 & 80 & 150 \end{pmatrix}$$

$$= \begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} \text{Cost} & \text{Av. daily expenses} & \text{Av. daily income} \\ 1982 & 4840 & 7820 \\ 2008 & 5720 & 8830 \end{pmatrix}$$

Thus

	Cost (*in thousand Rs*)	*Av. daily expenses* (*in Rs*)	*Av. daily income* (*in Rs*)
State X :	1982	4840	7820
State Y :	2008	5720	8830

Net daily incomes of X and Y are Rs 2980 and Rs 3110 respectively.

$$\text{State } X: \frac{\text{Net income}}{\text{Investment}} \times 100 = \frac{2980}{1982000} \times 100 = \frac{298}{1982} = 0.150\ \%$$

$$\text{State } Y: \frac{\text{Net income}}{\text{Investment}} \times 100 = \frac{3110}{2008000} \times 100 = \frac{311}{2008} = 0.154\ \%$$

Therefore the State Y is getting a comparatively better advantage.

EXERCISE 3 (*d*)

1. A man buys 8 dozen mangoes, 10 dozen apples and 4 dozen bananas. Mangoes cost Rs 18 per dozen, apples Rs 9 per dozen and bananas Rs 6 per dozen. Represent the quantities bought by a row matrix and the prices by a column matrix and hence obtain the total cost.

2. A store has in stock 20 dozen shirts, 15 dozen trousers and 25 dozen pairs of socks. If the selling prices are Rs 50 per shirt, Rs 90 per trousers and Rs 12 per pair of socks, then find the total amount the store owner will get after selling all the items in the stock.

3. A shopkeeper has 10 dozen Physics books, 8 dozen Chemistry books and 5 dozen Mathematics books. If their selling prices are Rs 65.70, Rs 43.20 and Rs 76.50 each respectively, find by matrix method the total amount of the sale if all the books are sold.

4. A manufacturer produces three products A, B, C which he sells in the market. Annual sale volumes are indicated as follows :

Markets	*Products*		
	A	*B*	*C*
I	8,000	10,000	15,000
II	10,000	2,000	20,000

If unit sale prices of A, B and C are Rs 2.25, Rs 1.50 and Rs 1.25 respectively, find the total revenue in each market with the help of matrices.

5. A man invests Rs 50,000 into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix multiplication, determine how to divide Rs 50,000 among the two types of bonds so as to obtain an annual total interest of Rs 2780.

6. In a development plan of a city, a contractor has taken a contract to construct certain houses for which he needs building materials like stones, sand etc. There are three firms A, B, C that can supply him these materials. At one time these firms A, B, C supplied him 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively. If the cost of one truck load of stone and sand is Rs 1200 and Rs 500 respectively, then find the total amount paid by the contractor to each of these firms A, B, C respectively.

[Hint : $P = \begin{matrix} \\ \text{Stone} \\ \text{Sand} \end{matrix}\begin{bmatrix} A & B & C \\ 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$ $\quad Q = \begin{matrix} \text{Stones} & \text{Sand} \\ [1200 & 500] \end{matrix}$

The reqd. total amount paid to each of the firms is given by $[1200 \quad 500]\begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$

Note that here PQ cannot be calculated.]

ANSWERS

1. Rs 258 **2.** Rs 31800 **3.** Rs 16621.20

4. From market I, Rs 51,750, From market II, Rs 50,500 **5.** Rs 22,000; Rs 28,000

6. Rs 53,000; Rs 44,500; Rs 34,000.

3.21. Transpose matrix

Definition. ***The matrix obtained from any given matrix A by interchanging its rows and the columns is called the transpose of the given matrix and is denoted by A^T or A'.***

Thus (*i*) if the order of A is $m \times n$, then the order of A' is $n \times m$.

(*ii*) ($i - j$th) element of A = ($j - i$th) element of A'.

For example, $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $A' = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, and if $A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$, then $A' = \begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$.

If $A = \begin{bmatrix} 3 & 5 & 7 \\ -1 & 0 & 2 \\ 4 & 6 & 1 \end{bmatrix}$, then $A' = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 0 & 6 \\ 7 & 2 & 1 \end{bmatrix}$.

3.22. Properties of transpose matrix

Property 1. If A is any matrix then $(A')' = A$.

Property 2. If A and B are two matrices of the same order then $(A + B)' = A' + B'$.

Property 3. If A is $m \times p$ matrix and B is $p \times n$ matrix then $(AB)' = B'A'$.

Note. $(ABCD \ldots Z)' = Z' \ldots D' . C' . B' . A'$ a scalar.

Property 4. If A is a matrix and k is a scalar then $(kA)' = kA'$.

Summary

1. $(A')' = A$ 2. $(A + B)' = A' + B'$
3. $(AB)' = B'A'$ 4. $(kA)' = kA'$

Remember the above results.

Ex. 40. For matrix $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$, **find** $\frac{1}{2}(A - A')$, **where A' is the transpose of matrix A.**

Sol. Given, $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} -2 & 5 & 7 \\ 3 & -4 & 2 \\ 4 & -3 & 9 \end{bmatrix}$

$$\therefore \quad \frac{1}{2}(A - A') = \frac{1}{2}\left\{\begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 5 & 7 \\ 3 & -4 & 2 \\ 4 & -3 & 9 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix}.$$

Ex. 41. **If** $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$, **find** $(AB)'$.

Sol. Given $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6-4-2 & 8+8-1 \\ -3+0+4 & -4+0+2 \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ 1 & -2 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}.$$

Ex. 42. **If** $A = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$, $B = [3 \quad 1 - 2]$, **verify that** $(AB)' = B'A'$. **(ISC)**

Sol.
$$AB = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}[3 \quad 1 \quad -2] = \begin{bmatrix} 3 & 1 & -2 \\ -15 & -5 & 10 \\ 21 & 7 & -14 \end{bmatrix} \quad \therefore \text{ L.H.S.} = (AB)' = \begin{bmatrix} 3 & -15 & 21 \\ 1 & -5 & 7 \\ -2 & 10 & -14 \end{bmatrix}$$

$$A' = [1 \quad -5 \quad 7], \quad B' = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\text{R.H.S.} = B'A' = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}[1 \quad -5 \quad 7] = \begin{bmatrix} 3 & -15 & 21 \\ 1 & -5 & 7 \\ -2 & 10 & -14 \end{bmatrix}.$$

Thus, L.H.S. = R.H.S.

Ex. 43. **If** $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, **verify that** $(AB)' = B'A'$.

Sol. Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 8+3+0 & 0+6+12 & 10+0+4 \\ 20+7+0 & 0+14+27 & 25+0+9 \\ -8+1+0 & 0+2+3 & -10+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 18 & 14 \\ 27 & 41 & 34 \\ -7 & 5 & -9 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 11 & 27 & -7 \\ 18 & 41 & 5 \\ 14 & 34 & -9 \end{bmatrix} \qquad ...(1)$$

Also
$$B'A' = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 3 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & -2 \\ 3 & 7 & 1 \\ 4 & 9 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3+0 & 20+7+0 & -8+1+0 \\ 0+6+12 & 0+14+27 & 0+2+3 \\ 10+0+4 & 25+0+9 & -10+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 27 & -7 \\ 18 & 41 & 5 \\ 14 & 34 & -9 \end{bmatrix} \qquad ...(2)$$

From (1) and (2) it follows that $(AB)' = B'A'$.

EXERCISE 3 (*e*)

1. (*i*) If $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 0 \end{bmatrix}$, find $A + A^T$. **(NMOC)**

(*ii*) For matrix $A = \begin{bmatrix} -3 & 6 & 0 \\ 4 & -5 & 8 \\ 0 & -7 & -2 \end{bmatrix}$, find $\frac{1}{2}(A - A')$.

(*iii*) Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, where $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

2. If $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, find $(BC)'$.

3. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, find (*i*) AA', (*ii*) $A'A$.

4. If $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$, find (*i*) $A'B$, (*ii*) AB'.

5. Verify that $(AB)' = B'A'$ if (*i*) $A = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ and $B = [1 \quad -5 \quad 7]$;

(*ii*) $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \quad -1 \quad -4]$;

(*iii*) If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$;

(*iv*) $A = \begin{bmatrix} -1 & 3 & 0 \\ -7 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 0 \\ 0 & 3 \\ 1 & -8 \end{bmatrix}$.

6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, verify that $(A + B)' = A' + B'$.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$, find each of the following :

(*i*) $2A' - B'$ (*ii*) $(BC)'$. Is $(BC)' = C'B'$?

(*iii*) $(A + B + C)'$. Is $(A + B + C)' = A' + B' + C'$?

8. If $A = [2 \;\; 5 \;\; 7 \;\; 9]$ and $B = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 4 \end{bmatrix}$, then prove that $(A + B') = (A' + B)'$.

9. Find x and y if the matrix

$$A = \frac{1}{3}\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{vmatrix}$$ may satisfy the condition $AA' = A'A = I_3$.

10. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$, then verify that $(A^2)' = (A')^2$.

11. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A'A = I$.

ANSWERS

1. (i) $\begin{bmatrix} 2 & 6 & 5 \\ 6 & 10 & 2 \\ 5 & 2 & 0 \end{bmatrix}$ (ii) $\frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 15 \\ 0 & -15 & 0 \end{bmatrix}$ (iii) $\frac{1}{2}(A + A') = 0, \frac{1}{2}(A - A') = A$

2. $\begin{bmatrix} 7 & 19 \\ 17 & 10 \end{bmatrix}$ **3.** (i) $\begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix}$ (iii) $\begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix}$

4. (i) $\begin{bmatrix} -2 & 3 \\ 2 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} -5 & -3 \\ -1 & -1 \end{bmatrix}$ **7.** (i) $\begin{bmatrix} -2 & -1 & 0 \\ -1 & 0 & -7 \\ 0 & 3 & -4 \end{bmatrix}$

(ii) $\begin{bmatrix} -15 & 0 & -5 \\ -10 & 0 & -6 \\ 31 & 1 & 9 \end{bmatrix}$, yes (iii) $\begin{bmatrix} 4 & -3 & 2 \\ 5 & 2 & -4 \\ 10 & 6 & 3 \end{bmatrix}$, yes **9.** $x = -2,\ y = -1$

ADJOINT AND INVERSE OF A MATRIX

3.23. The determinant function (Determinant of a square matrix)

Associated with each square matrix A having real-number entries is a real number called the **determinant** of A and denoted by δA or ΔA or $|A|$ or det (A). Thus δ (*delta*) *is a function or mapping with domain the set of all square matrices having real number entries and with the range the set of all real-numbers.* We say that $\delta(A_{n \times n})$ is a determinant of **order** n.

Let us begin by examining δ over the set $S_{2 \times 2}$ of 2×2 matrices.

Definition. *The determinant of the matrix*

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{ is the number } a_1b_2 - a_2b_1.$$

We indicate this as follows :

$$\delta : \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \to a_1b_2 - a_2b_1 \quad \text{that is } \delta : A \to \delta(A)$$

3.24. Determinant notation

The determinant of a matrix is customarily displayed in the same form as a matrix, but with vertical bars in lieu of parentheses.

Thus,

$$\delta\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

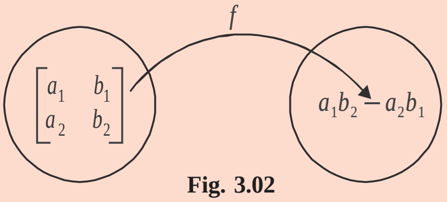


Fig. 3.02

Note 1. It may be emphasised that while a matrix being just an arrangement of numbers has no value, its determinant det A or $|A|$ is not an array of numbers enclosed between two vertical lines. It is a scalar quantity having definite value. For example, if

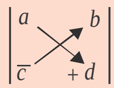


Fig. 3.03

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then det } A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ has the value } ad - bc.$$

Note 2. Determinant of order one. The determinant of a matrix $[a]$ is the number 'a' itself. Thus, det $[a] = |a| = a$, or det $[-3] = |-3| = -3$.

Caution. The determinant | –3 | should not be confused with the absolute value | –3 | of –3.

Ex. 44. If $A = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$, find det (A).

Sol. $\det(A) = \begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 3 \times 2 - (-1)\,5 = 11.$

3.25. Determinants of order 3

Let $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a 3 × 3 matrix. Then $\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ...(1)

$= a_1b_2c_3 + a_2b_3c_1 - a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$ ...(2)

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Ex. 45. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 1 & -1 \\ 5 & -3 & 2 \end{bmatrix}$$

Sol. $|A|$ or $\det A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & 1 & -1 \\ 5 & -3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix}$

$= 1(2-3) - 3(8+5) - 2(-12-5) = -1 - 39 + 34 = -6$

3.26. Singular and non-singular matrices

A *square matrix A is said to be* singular *if det* [A] = *0, otherwise it is said to be* non-singular.

3.27. The Inverse of a square matrix of order 2

You have seen that the unit matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the zero matrix $\boldsymbol{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ of order 2 have the special properties that $\boldsymbol{AI = IA = A}$ and $\boldsymbol{AO = OA = O}$. Hence the matrices I and 0 are analogous to the number 1 and 0 respectively in ordinary algebra. Furthermore, in the set of all real numbers, we know that for each non-zero real number $\boldsymbol{a}$, there exists another real number $1/\boldsymbol{a}$ or $\boldsymbol{a}^{-1}$ such that $a\,.\,(1/a) = 1$ or $aa^{-1} = a^{-1}\ a = 1$. This number, you know, is called the multiplicative inverse. The question arises whether every square matrix A also has a multiplicative inverses A^{-1}. That is, given a non-zero matrix A of order 2, does there exist a square matrix B such that $AB = I = BA$?

As will be seen from the following examples, it may or may not exist.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then for any matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $AB = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq I$ and $BA = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \neq I$.

Thus, no such matrix B exists such that $AB = BA = I$ which means in other words that A^{-1} does not exist in this case.

Now consider another example. Let $A = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ since $AB = BA = I$, therefore, each of the matrices $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ is the inverse of the other.

Definitions.

1. If A and B are square matrices such that AB = BA = I, then B is called the inverse of A and written as $A^{-1} = B$ and similarly, A is called the inverse of B written as $B^{-1} = A$.

Thus $AA^{-1} = A^{-1}A = I.$

2. If inverse of A exists, *i.e.*, if A is non-singular, then A is called an **invertible matrix.**

A non-zero square matrix of order n is said to be **invertible** ***if there exists a square matrix B of order n such that*** $AB = BA = I_n$.

For the inverse of a matrix to exist, the following requirements are necessary:

(1) ***The matrix must be a square matrix.*** This requirement is essential because in equation $AB = BA = I$ if A is, say, of order 2×3 and B is of order 3×2, then AB and BA are both defined but have different orders, namely, 2×2 and 3×3 and hence cannot be equal.

(2) ***The equation AB = BA = I must be satisfied.*** For instance, the matrices $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 5 & 10 \\ 20 & 5 \end{bmatrix} = BA$, but AB is not equal to I. This leads us to the fact that as 0 has no inverse in the set of real numbers, so the zero matrix $\boldsymbol{O}$ also has no inverse, because $\boldsymbol{OB} = \boldsymbol{O}$ and not $\boldsymbol{I}$.

3.28. The uniqueness theorem–an invertible matrix has a unique inverse

Proof. Let A be an invertible square matrix of order n. If possible, let B and C be two inverses of A.

Then $AB = BA = I_n$ (by definition of inverse matrix)

$AC = CA = I_n$

Now $B = BI_n = B(AC)$

$= (BA)C$ [Matrix multiplication is associative.]

$= I_nC = C$

$\therefore$ $B = C$, *i.e.*, any two inverses of A are equal matrices.

Hence the inverse of A is unique.

3.29. Method of finding the inverse of a square matrix of order 2

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have to see whether there exists a 2×2 matrix B such that $AB = BA = I$. Let the matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be the inverse of A. Then, we have,

$$AB = I \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = I \Rightarrow \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is true if and only if (iff)

$$ax + bz = 1, \quad ay + bw = 0$$

$$cx + dz = 0, \quad cy + dw = 1 \quad \text{[rule of equality of matrices]}$$

Solving simultaneously, we get

$$x = \frac{d}{ad-bc}, y = \frac{-b}{ad-bc}, z = \frac{-c}{ad-bc}, w = \frac{a}{ad-bc}$$

Therefore, $$B = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

B will not exist if $ad - bc = 0$

Conclusion. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then the* **inverse** *matrix of A denoted by* A^{-1} *exists if and only if* $ad - bc \neq 0$ *and in this case* $\mathbf{A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$.

> **Procedure.**
>
> **Step 1.** *Check whether the inverse exists by finding det A or* $|A|$. *The inverse will exist iff* $|A| \neq 0$.
>
> **Step 2.** *Interchange the entries on the leading diagonal and change the signs of the entries on the other diagonal.*
>
> **Step 3.** *Divide the result of step 2 by* $|A|$.

Ex. 46. If $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, **find** A^{-1}.

Sol. $$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -1 - 6 = -7,$$ Since $|A| \neq 0$, therefore, A has an inverse.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{bmatrix}$$

It is a good idea to check the result when finding A^{-1}, because there is much room for blundering in the process of determining the inverse. In the above example, we have

$$A^{-1}A = -\frac{1}{7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Ex. 47. Show that $\begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$ **is singular, and hence no inverse exists.**

Sol. Let $$A = \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}.$$ Then

$$|A| = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = 30 - 30 = 0,$$ therefore, no inverse exists.

Ex. 48. If $C = \begin{bmatrix} 2 & x \\ 4 & 2 \end{bmatrix}$, **find** C^2. **Under what conditions is** C^{-2} **(inverse of** C^2**) defined ? Find** C^{-2} **under that condition.**

Sol. $$C = \begin{bmatrix} 2 & x \\ 4 & 2 \end{bmatrix},\ C^2 = \begin{bmatrix} 2 & x \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & x \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 4+4x & 4x \\ 16 & 4x+4 \end{bmatrix}$$

C^{-2} exists if the determinant $\begin{vmatrix} 4+4x & 4x \\ 16 & 4x+4 \end{vmatrix} \neq 0$

i.e. if $(4+4x)^2 - 64x \neq 0$ or $(x-1)^2 \neq 0$ or $x \neq 1$ which is the required condition.

$$\therefore \quad C^{-2} = \frac{1}{16(1-x)^2}\begin{bmatrix} 4+4x & -4x \\ -16 & 4+4x \end{bmatrix}.$$

3.30. Cofactors of a matrix

Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactor of any element a_{ij} of the matrix is denoted by A_{ij} and is equal to

$$\mathbf{(-1)^{i+j} \times \text{minor of } a_{ij} \text{ in det } A.}$$

Thus, for the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of $a_{12} = A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of $a_{31} = A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

Cofactor of $a_{32} = A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Ex. 49. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$, find the cofactors of elements 7 and 12.

Sol. The cofactor of the element a_{22} *i.e.*, 7 is

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 4 \\ 9 & 12 \end{vmatrix} = +(12-36) = -20.$$

The cofactor of the element a_{33} *i.e.*, 12 is

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} = (7-10) = -3.$$

3.31. Adjoint of a square matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

be a 3 × 3 square matrix, and A_{ij} be the cofactor of a_{ij}, the adjoint of A denoted by **adj.** $\boldsymbol{A}$ is defined by

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Note carefully that **cofactors of the row entries in *A* are column entries in adj *A*.**

Definition. ***The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its cofactor in | A |.***

Note. If A is of order 3 × 3 and k is any number, then **adj (kA) = k^2 (adj A).**

Ex. 50. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. First we find all the co-factors.

$A_{11} = (-1)^{1+1}|-1| = -1$ *Missing out entries in the 1st row and 1st column*

$A_{12} = (-1)^{1+2}|4| = -4,\quad A_{21} = (-1)^{2+1}|1| = -1,\ A_{22} = (-1)^{2+2}|2| = 2$

$$\therefore \quad \text{adj.}\, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ -4 & 2 \end{bmatrix}.$$

Ex. 51. Find adjoint of A **where** $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

We know that adj A is the transpose of the matrix obtained by replacing the elements of A by their corresponding cofactors.

A_{11} = cofactor of a_{11}, *i.e.* $1 = (-1)^{1+1}\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

A_{12} = cofactor of a_{12}, *i.e.* $-1 = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12$

A_{13} = cofactor of a_{13}, *i.e.* $2 = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$

A_{21} = cofactor of a_{21}, *i.e.* $2 = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$

A_{22} = cofactor of a_{22}, *i.e.* $3 = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$

A_{23} = cofactor of a_{23}, *i.e.* $5 = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$

A_{31} = cofactor of a_{31}, *i.e.* $-2 = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$

A_{32} = cofactor of a_{32}, *i.e.* $0 = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -5 - (-4) = -1$

A_{33} = cofactor of a_{33}, *i.e.* $1 = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5.$

$$\text{adj.}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Ex. 52. Find the adjoint of the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{4} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{2} \end{pmatrix}$$

Sol. A_{11} = cofactor of a_{11} in $|A| = 1\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2;\ A_{12} = -\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = -5.$

$$A_{13} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2,\ A_{21} = -\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = -2\ ;\ A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7.$$

$$A_{23} = -\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = 10,\ A_{31} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -2$$

$$A_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 11,\ A_{33} = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14$$

$$\text{Adj } A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 2 & -2 & -2 \\ -5 & -7 & 11 \\ 2 & 10 & -14 \end{pmatrix}$$

3.32. An important relation between a square matrix and adj *A*

Theorem. ***If A be any n-rowed square matrix, then (adj A) A = A (adj A) = |A| I_n, where I_n is the n-rowed unit matrix.***

According to this theorem, the product of the matrices A and adj A is commutative and is a scalar matrix every diagonal element of which is $|A|$.

Let us prove this result when A is a 3×3 matrix.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\quad\therefore$ adj $A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

where A_1, B_1, C_1, ... are cofactors of a_1, b_1, c_1 ....

$$\therefore A(\text{adj } A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I_3 \text{ [Remark Art. 2.05 in Chapter 2]}$$

Since $a_1A_2 + b_1B_2 + c_1C_2 = a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - c_1a_3) + c_1(a_1b_3 - a_3b_1) = 0$, etc.

Similarly we can prove that (adj A) $A = |A|. I_3$

$\therefore$ $\boldsymbol{A}$ **(adj** $\boldsymbol{A}$**)** $=$ **(adj** $\boldsymbol{A}$**)** $\boldsymbol{A} = |\boldsymbol{A}|.\boldsymbol{I}_3$

Note. (1) Adj $I = I$, (2) Adj $O = O$.

Ex. 53. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then verify that $A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I_2$.

Sol. We have $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Here A_{11} = cofactor of $a_{11}\,(=1) = (-1)^{1+1}|4| = 4,$

A_{12} = cofactor of $a_{12}\,(=2) = (-1)^{1+2}|3| = -3$

A_{21} = cofactor of a_{21}, *i.e.* $3 = (-1)^{2+1}|2| = -2$

A_{22} = cofactor of a_{22}, *i.e.* $4 = (-1)^{2+2}|1| = 1$

$$\therefore \quad (\text{adj}\,A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \quad A\,(\text{adj}\,A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4-6 & -2+2 \\ 12-12 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \quad (\text{adj}\,A)\,A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-6 & 8-8 \\ -3+3 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$|A|\,I_2 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (4-6).\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \quad \mathbf{A\,(adj\,A) = (adj\,A)\,A = |A|\,I_2.}$$

Ex. 54. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find adj A and verify that

$A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I_3$. ***(NMOC)***

Sol. Given $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\alpha\,(\cos\alpha - 0) + \sin\alpha\,(\sin\alpha - 0) = \cos^2\alpha + \sin^2\alpha = 1$$

$A_{11} = (-1)^{1+1}(\cos\alpha - 0) = \cos\alpha\,;\ A_{12} = (-1)^{1+2}(\sin\alpha - 0) = -\sin\alpha\,;$

$A_{13} = (-1)^{1+3}(0-0) = 0$

$A_{21} = (-1)^{2+1}(-\sin\alpha - 0) = \sin\alpha\,;\ A_{22} = (-1)^{2+2}(\cos\alpha - 0) = \cos\alpha\,;$

$A_{23} = (-1)^{2+3}(0-0) = 0$

$A_{31} = (-1)^{3+1}(0-0) = 0;\ A_{32} = (-1)^{3+2}(0-0) = 0;\ A_{33} = (-1)^{3+3}(\cos^2\alpha + \sin^2\alpha) = 1$

$$\therefore \quad \text{Adj.}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{21} & A_{22} & A_{32} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now
$$A\,(\text{Adj.}\,A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha + 0 & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & 0+0+0 \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 = |A| I_3. \qquad [\because |A| = 1]$$

Ex. 55. If A is a 3×3 non-singular matrix, then $|\text{adj } A| = |A|^2$.

Sol. We know that $A\,(\text{adj } A) = |A| I_3$

$$\therefore \; A\,(\text{adj } A) = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \Rightarrow |A\,(\text{adj } A)| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow \quad |A\,(\text{adj } A)| = |A|^3 \Rightarrow |A|\,|\text{adj } A| = |A|^3 \Rightarrow |\text{adj } A| = |A|^2 \qquad (\because |A| \neq 0)$$

Note. If A is a $n \times n$ non-singular matrix, then $|\text{adj } A| = |A|^{n-1}$.

Ex. 56. If A is a 3×3 singular matrix, prove that $A\,(\text{adj } A) = O$.

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where $A_{11}, A_{12}, A_{13}, \ldots$ are the cofactors of $a_{11}, a_{12}, a_{13}, \ldots$ respectively.

$$\therefore \quad A\,(\text{adj } A) = |A| I_3 = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O, \qquad (A \text{ being a singular matrix}, |A| = 0)$$

Ex. 57. If $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix}$, find the value of (adj. A) A without finding Adj. A. (*NMOC*)

Sol. Given $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{vmatrix} = 1(3-0) - 3(6-0) = -14$

We know that (Adj. A) $A = |A| I$

$$\therefore \text{ Here (adj } A)\,A = -14I = -14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}.$$

3.33. Theorem

If A and B are two non-singular matrices of the same type, then

$$\textbf{adj }(AB) = (\textbf{adj } B)\,(\textbf{adj } A)$$

Proof. We have $(AB)\,(\text{adj } AB) = |AB|\, I = (\text{adj } AB)\,(AB)$ — Theorem (Art. 3.32)

Now $(AB)\,(\text{adj } B)\,(\text{adj } A) = A\,(B \text{ adj } B)\,(\text{adj } A)$

$$= A\,(|B| I)\,(\text{adj } A) = |B|\, A\,(\text{adj } A) = |B|\,|A|\, I = |A|\,|B| I = |AB|\, I$$

$$= (AB)\,(\text{adj } AB)$$

$$\Rightarrow \quad (\text{adj } B)\,(\text{adj } A) = \text{adj }(AB)$$

$AB = AC$ does not necessarily imply $B = C$. However, if A, B, C are square matrices of the same type and if A is non-singular then $AB = AC \Rightarrow B = C$

Ex. 58. Find the adjoint of the matrix

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{0} & \mathbf{-6} & \mathbf{-7} \end{bmatrix}$$ **and verify the theorem** $A\,(\text{adj}\,A) = |A|\,I_3 = (\text{adj}\,A)\,A.$ ***(ISC 2006)***

Sol. Given
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = 1\,(-28+30) - 1\,(-18-0), = 20, \text{ expanding along } R_1$$

$$A_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2,\; A_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21,\; A_{13} = \begin{bmatrix} 3 & 4 \\ 0 & -6 \end{bmatrix} = -18$$

$$A_{21} = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = 6,\; A_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7,\; A_{23} = -\begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix} = 6$$

$$A_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 4,\; A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -8,\; A_{33} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = 4$$

$$\text{Adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Now
$$A\,(\text{adj}\,A) = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+18 & 6+0-6 & 4+0-4 \\ 6+84-90 & 18-28+30 & 12-32+20 \\ 0-126+126 & 0+42-42 & 0+48-28 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I_3.$$

Also
$$(\text{adj}\,A)\,A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2+18+0 & 0+24-24 & -2+30-28 \\ 21-21+0 & 0-28+48 & -21-35+56 \\ -18+18+0 & 0+24-24 & 18+30-28 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I_3$$

Hence, $A\,(\text{adj}\,A) = |A|\,I_3 = (\text{adj}\,A)\,A.$

EXERCISE 3 (*g*)

1. Find the adjoint of the following matrices

(*a*) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, (*b*) $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, (*c*) $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$, (*d*) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

2. Find the adjoint of the matrix A and verify A (adj. A) = (adj A) $A = |A|\, I_2$.

(*i*) $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix}$.

3. For the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, prove that A (adj A) = O.

4. Find A (adj A) for the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$.

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$, find the value of A (Adj. A) without finding Adj A. **(*NMOC*)**

[Hint. A (adj A) = $|A|\, I$.]

6. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj $A = A$.

7. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, verify the theorem A (adj A) = (adj A) $A = |A|\, I$.

8. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, prove that adj AB = (adj B) (adj A).

9. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj $A = 3\,A'$.

10. If $A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, find a non-zero unit matrix B such that $AB = BA$.

[Hint. A (adj A) = (adj A) A ∴ B = adj A. Hence calculate adj A.] [Art. 3.32]

11. Prove that | adj AB | = | adj A | | adj B |

[Hint. By Art 3.34, we know that adj AB = (adj B) (adj A)

⇒ | adj AB | = | adj B | | adj A | = | adj A | | adj B |].

12. Given that $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ and A(adj. A) = $K \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the value of K. **(*ISC 2005*)**

[Hint. $K = |A|$, by Art. 3.32**]**

ANSWERS

1. (*a*) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$ (*c*) $\begin{bmatrix} 7 & -11 & -5 \\ 0 & 10 & -5 \\ 14 & 3 & -5 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$ **5.** $\begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$ **10.** $\begin{bmatrix} -3 & 6 & 6 \\ -6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ **12.** $K = 1$

3.34. Inverse of an $n \times n$ matrix

If A is an n × n non-singular matrix of order n, then $A^{-1} = \dfrac{1}{|A|}$ **(adj A).**

By definition of an inverse matrix, we know that if $AB = BA = I$, then B is called the inverse of A, *i.e.*, $B = A^{-1}$

In Art. 3.32 we have seen that, $A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\,I$ ...(1)

Since A is non-singular, $|A| \neq 0$, therefore,

(1) $\Rightarrow$ $A\left(\dfrac{1}{|A|}\text{adj } A\right) = \left(\dfrac{1}{|A|}\text{adj } A\right)A = I \Rightarrow \mathbf{A^{-1} = \dfrac{1}{|A|}(adj\ A)}$

Note. 1. $\text{adj } A = |A|\,A^{-1}$, if A is non-singular **2.** $(\text{adj } A)^{-1} = \dfrac{1}{|A|}A$

3. $\text{adj } I = I$ **4.** $\text{adj } O = O$ **5.** The inverse of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is itself.

6. If A is a non-singular matrix of order 3 and k is any non-zero number, then

$$(kA)^{-1} = \frac{1}{|kA|}(\text{adj } kA) = \frac{1}{k^3|A|}k^2\,(\text{adj } A) \qquad \text{[See note Art. 3.31]}$$

$$= \frac{1}{k}\cdot\frac{1}{|A|}\text{adj } A = \frac{1}{k}A^{-1}.$$

3.35. Theorem

The necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$, i.e. A is non-singular.

Proof. **(*a*) The condition is necessary,** *i.e.*, given that A has inverse, to show that $|A| \neq 0$.

Let B be the inverse of A, then $AB = BA = I$

$\Rightarrow$ $|AB| = |BA| = |I| \Rightarrow |A|\,|B| = |B|\,|A| = 1 \Rightarrow |A| \neq 0.$

(*b*) The condition is sufficient, *i.e.*, given that $|A| \neq 0$, to show that A has inverse.

Since $|A| \neq 0$, consider $B = \dfrac{\text{adj } A}{|A|}$

$$AB = A\left(\frac{\text{adj}}{|A|}\right) = \frac{I}{|A|}A\,(\text{adj } A) = \frac{I}{|A|}(|A|\,I) = I \qquad \because\ A\,(\text{adj } A) = |A|$$

$$BA = \left(\frac{\text{adj}}{|A|}\right)A = \frac{I}{|A|}(\text{adj } A)\,A = \frac{I}{|A|}(|A|\,I) = I \qquad \therefore\ (\text{adj } A)\,A = |A|\,I$$

Thus $AB = BA = I$

$\therefore$ The inverse of A exists and $A^{-1} = B = \dfrac{\text{adj } A}{|A|}$.

From the above theorem, we conclude that **A has inverse if and only if $|A| \neq 0$. Then $A^{-1} = \dfrac{\text{adj } A}{|A|}$.**

Ex. 59. Find the inverse of the matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, if $ps - rq \neq 0$

Sol. Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Then $|A| = \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - rq \neq 0$ (given). Hence A^{-1} exists.

Now $A_{11} = s,\ A_{12} = -r,\ A_{21} = -q,\ A_{22} = p$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{ps - rq} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}.$$

Method II. By Art. 3.29, the inverse of $\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \frac{1}{ps - rq} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$, provided $ps - rq \neq 0$.

Note. In the case of 2×2 matrix it is easier to find inverse by the method of Art. 3.29.

Ex. 60. Find the inverse of the matrix $\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$ and verify your answer.

Sol. Let $A = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$ then $|A| = -2 \times 4 - 3 \times 5 = -23 \neq 0$

Since $|A| \neq 0$, therefore, A is non-singular and A^{-1} exists.

Now $A_{11} = (-1)^{1+1} |4| = 4, A_{12} = (-1)^{1+2} |3| = -3$

$A_{21} = (-1)^{2+1} |5| = -5, A_{22} = (-1)^{2+2} |-2| = -2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-23} \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix}.$$

Verification :

To verify we have to show that

$$AA^{-1} = A^{-1}A = I_2$$

$$AA^{-1} = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} \times -\frac{1}{23} \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix} = -\frac{1}{23} \begin{bmatrix} -8-15 & 10-10 \\ 12-12 & -15-8 \end{bmatrix}$$

$$= \frac{-1}{23} \begin{bmatrix} -23 & 0 \\ 0 & -23 \end{bmatrix} = \frac{-23}{-23} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

$$A^{-1}A = -\frac{1}{23} \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} = -\frac{1}{23} \begin{bmatrix} -8-15 & 20-20 \\ 6-6 & -15-8 \end{bmatrix}$$

$$= -\frac{1}{23} \begin{bmatrix} -23 & 0 \\ 0 & -23 \end{bmatrix} = \frac{-23}{-23} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Hence, verified.

Ex. 61. Find a 2×2 matrix B such that $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

Sol. $B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1}$

Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$. We have to find A^{-1}.

$A_{11} = 4,\ A_{12} = -1,\ A_{21} = 2, A_{22} = 1, |A| = 4 + 2 = 6, A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$$\therefore \quad B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = 6 \times \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = I. \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}.$$

Ex. 62. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$. **Verify your answer.**

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix} = 1(8-6) + 1(0+9) + 2(0-6)$

$$= 2 + 9 - 12 = -1 \neq 0$$

$\therefore$ A is invertible and $A^{-1} = \dfrac{\text{Adj. } A}{|A|}$. Now we find adj. A.

$A_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = (8-6) = 2,$ $A_{12} = (-1)^3 \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9$

$A_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = (0-6) = -6,$ $A_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4+4) = 0$

$A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4-6) = -2,$ $A_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2+3) = -1$

$A_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = (3-4) = -1,$ $A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$

$A_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = (2+0) = 2.$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}.$$

Verification :

To verify we have to show that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$$A^{-1}A = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+3 & 2+0-2 & -4+0+4 \\ 9+0-9 & -9+4+6 & 18-6-12 \\ 6+0-6 & -6+2+4 & 12-3-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, verified.

3.36. Properties of matrices and inverses

There are a number of useful properties associated with matrices and their inverses. For example, we have the following:

Theorem 1. ***If A, B be two n-rowed non-singular matrices, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$ (Reversal law for the inverse of a product)***

i.e., the inverse of a product is the product of the inverses taken in the reverse order.

Ex. 63. **If $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.**

Sol. $AB = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 7 \end{bmatrix}$, $|AB| = \begin{vmatrix} 0 & 2 \\ 2 & 7 \end{vmatrix} = 0 \times 7 - 2 \times 2 = -4 \neq 0$

Since $|AB| \neq 0$, therefore, $(AB)^{-1}$ exists and is $= \dfrac{\text{adj }(AB)}{|AB|}$

The cofactors of first row in $|AB|$ are $(-1)^{1+1}|7|$ and $(-1)^{1+2}, |2|$ *i.e.*, 7 and – 2.
The cofactors of second row are $(-1)^{2+1}|2|$ and $(-1)^{2+2}|0|$, *i.e.*, – 2 and 0.

$$\therefore \quad \text{adj }(AB) = \begin{bmatrix} 7 & -2 \\ -2 & 0 \end{bmatrix}' = \begin{bmatrix} 7 & -2 \\ -2 & 0 \end{bmatrix} \quad \therefore \ (AB)^{-1} = -\frac{1}{4}\begin{bmatrix} 7 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad ...(1)$$

Again, $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \therefore |A| = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \neq 0 \quad \therefore \ A^{-1}$ exists.

Co-factors of first row in $|A|$ are $(-1)^{1+1}|1| = 1$ and
$(-1)^{1+2}|3| = -3$. Co-factors of second row are $(-1)^{2+1}|0| = 0$ and $(-1)^{2+2}|2| = 2$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \quad \therefore \ A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} \therefore |B| = \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = -2 \neq 0 \qquad \therefore \ B^{-1} \text{ exists}$$

Co-factors of first row in $|B|$ are $(-1)^{1+1}|4| = 4$ and $(-1)^{1+2}|2| = -2$.
Co-factors of second row are $(-1)^{2+1}|1| = -1$ and $(-1)^{2+2}|0| = 0$

$$\therefore \quad \text{adj. } B = \begin{bmatrix} 4 & -2 \\ -1 & 0 \end{bmatrix}' = \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} \quad \therefore \ B^{-1} = \frac{\text{adj } B}{|B|} = -\frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}$$

Also $$B^{-1}A^{-1} = -\frac{1}{4}\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 7 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad ...(2)$$

From (1) and (2) it follows that $(AB)^{-1} = B^{-1}A^{-1}$.

Ex. 64. **If $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ find $(AB)^{-1}$.** ***(NMOC)***

Sol. We will find $(AB)^{-1}$ by using the relation $(AB)^{-1} = B^{-1}A^{-1}$. So first we find A^{-1}.

$$|A| = \begin{vmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 5(3-4) - 0(2-2) + 4(4-3) = -5 + 4 = -1 \neq 0$$

$\therefore \quad A^{-1}$ exists.

$A_{11} = (-1)^{1+1}(3-4) = -1$; $\quad A_{12} = (-1)^{1+2}(2-2) = 0$; $\quad A_{13} = (-1)^{1+3}(4-3) = 1$
$A_{21} = (-1)^{2+1}(0-8) = 8$; $\quad A_{22} = (-1)^{2+2}(5-4) = 1$; $\quad A_{23} = (-1)^{2+3}(10-0) = -10$
$A_{31} = (-1)^{3+1}(0-12) = -12$; $\quad A_{32} = (-1)^{3+2}(10-8) = -2$; $\quad A_{33} = (-1)^{3+3}(15-0) = 15$

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & -15 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{-1}\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & 10 & 15 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$\therefore \quad (AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-3 & -8-2+30 & 12+4-45 \\ 1+0-3 & -8-4+30 & 12+8-45 \\ 1+0-4 & -8-3+40 & 12+6-60 \end{bmatrix} = \begin{bmatrix} -2 & 20 & -29 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}.$$

Theorem 2. $(A')^{-1} = (A^{-1})'$.

Ex. 65. **If** $A = \begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$**, verify that** $(A')^{-1} = (A^{-1})'$.

Sol. $A = \begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, (say)

Then, $|A| = \begin{vmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -10 & -1 \\ 0 & 8 & 2 \\ 0 & -4 & -2 \end{vmatrix}$, operating $C_1 + C_3$

$$= 2\begin{vmatrix} 8 & 2 \\ -4 & -2 \end{vmatrix} = 2(-16+8) = -16 \neq 0 \quad \therefore A^{-1} \text{ exists.}$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 8 & 2 \\ -4 & -2 \end{vmatrix} = -8, \; A_{21} = -\begin{vmatrix} -10 & -1 \\ -4 & -2 \end{vmatrix} = -16, \; A_{31} = \begin{vmatrix} -10 & -1 \\ 8 & 2 \end{vmatrix} = -12$$

$$A_{12} = -\begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0, \; A_{22} = \begin{vmatrix} 3 & -1 \\ -2 & -2 \end{vmatrix} = -4, \; A_{32} = -\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} = -4$$

$$A_{13} = \begin{vmatrix} -2 & 8 \\ 2 & -4 \end{vmatrix} = -8, \; A_{23} = -\begin{vmatrix} 3 & -10 \\ 2 & -4 \end{vmatrix} = -8, \; A_{33} = \begin{vmatrix} 3 & -10 \\ -2 & 8 \end{vmatrix} = 4$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} \text{ and } A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{16}\begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad (A^{-1})' = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 \\ 4 & 1 & 2 \\ 3 & 1 & -1 \end{bmatrix} \qquad \text{...(1)}$$

Now, $A' = \begin{bmatrix} 3 & -2 & 2 \\ -10 & 8 & -4 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, (say)

$$|A'| = 3\begin{vmatrix} 8 & -4 \\ 2 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -10 & -4 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} -10 & 8 \\ -1 & 2 \end{vmatrix}$$

$$= 3(-16+8) + 2(20-4) + 2(-20+8) = -24 + 32 - 24 = -16 \neq 0$$

Hence, $(A')^{-1}$ exists.

$$\text{adj } A' = \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} = \begin{bmatrix} -8 & 0 & -8 \\ 16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}$$ [Find the values of B_{11}, B_{21},.....etc., yourself]

$$\therefore \quad (A')^{-1} = \frac{\text{adj } A'}{|A'|} = -\frac{1}{16}\begin{bmatrix} -8 & 0 & -8 \\ 16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 \\ 4 & 1 & 2 \\ 3 & 1 & -1 \end{bmatrix} \quad ...(2)$$

(1) and (2) $\Rightarrow$ $\mathbf{(A')^{-1} = (A^{-1})'}$.

Ex. 66. Show that

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}$

$$|B| = 1 \times 1 - (-\tan\theta) \times \tan\theta = 1 + \tan^2\theta = \sec^2\theta$$

$$B_{11} = 1, \quad B_{12} = \tan\theta, \quad B_{21} = -\tan\theta, \quad B_{22} = 1$$

$$\therefore \quad \text{adj } B = \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{\sec^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$ [By Art. 3.29, B^{-1} can be found more easily]

$$\therefore \quad A \times B^{-1} = \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \times \frac{1}{\sec^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$= \cos^2\theta \begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix} = \cos^2\theta \begin{bmatrix} \dfrac{\cos^2-\sin^2\theta}{\cos^2\theta} & -2\dfrac{\sin\theta}{\cos\theta} \\ 2\dfrac{\sin\theta}{\cos\theta} & \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Alternative form. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A' A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Ex. 67. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $x^2 - 3x - 7 = 0$. Then find A^{-1}.

Sol. We have to prove that $A^2 - 3A - 7I = 0$

$$A^2 - 3A - 7I = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence proved.

Now, dividing both sides of $A^2 - 3A - 7I = 0$ by A, we have

$$A - 3A.A^{-1} - 7A^{-1} = 0 \quad \Rightarrow \quad A - 3I = 7A^{-1}$$

$$\Rightarrow \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 7A^{-1} \Rightarrow 7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \left[\because I = A.A^{-1} \Rightarrow \frac{I}{A} = A^{-1}\right]$$

$$\Rightarrow \qquad 7A^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}.$$

Ex. 68. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, prove that $A^{-1} = A^2 - 6A + 11I$.

Sol. $|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 2(3-0) - 0(15-0) - 1(5-0) = 6 - 5 = 1 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = (-1)^{1+1}(3-0) = 3$; $A_{12} = (-1)^{1+2}(15-0) = -15$; $A_{13} = (-1)^{1+3}(5-0) = 5$
$A_{21} = (-1)^{2+1}(0+1) = -1$; $A_{22} = (-1)^{2+2}(6-0) = 6$; $A_{23} = (-1)^{2+3}(2-0) = -2$
$A_{31} = (-1)^{3+1}(0+1) = 1$; $A_{32} = (-1)^{3+2}(0+5) = -5$; $A_{33} = (-1)^{3+3}(2-0) = 2$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \qquad ...(i)$$

Now, $$A^2 - 6A + 11I = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+0 & 0+0-1 & -2+0-3 \\ 10+5+0 & 0+1+0 & -5+0+0 \\ 0+5+0 & 0+1+3 & 0+0+9 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 30 & 6 & 0 \\ 0 & 6 & 18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12+11 & -1+0+0 & -5+6+0 \\ 15-30+0 & 1-6+11 & -5-0+0 \\ 5-0+0 & 4-6+0 & 9-18+11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \qquad ...(ii)$$

(i) and $(ii) \Rightarrow A^{-1} = A^2 - 6A + 11I.$

Ex. 79. For the matrix $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ show that $A^2-4A+5I=0$. Hence obtain A^{-1}.

Sol.
$$A^2-4A+5I = \begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}-4\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}+5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}1-2 & -1-3\\ 2+6 & -2+9\end{bmatrix}-\begin{bmatrix}4 & -4\\ 8 & 12\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}=\begin{bmatrix}-1 & -4\\ 8 & 7\end{bmatrix}-\begin{bmatrix}4 & -4\\ 8 & 12\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$
$$= \begin{bmatrix}-1-4+5 & -4+4+0\\ 8-8+0 & 7-12+5\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

Now, premultiplying both sides of $A^2-4A+5I$ by A^{-1}, we obtain
$$A^{-1}A^2-4A^{-1}A+5A^{-1}I = 0 \Rightarrow (A^{-1}A)A-4I+5A^{-1}=0 \qquad [\because A^{-1}A=I \text{ and } A^{-1}I=A^{-1}]$$
$$\Rightarrow \quad IA-4I+5A^{-1} = 0 \Rightarrow A-4I+5A^{-1}=0 \Rightarrow 5A^{-1}=4I-A$$
$$\Rightarrow \quad 5A^{-1} = 4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}-\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$$
$$\Rightarrow \quad 5A^{-1} = \begin{bmatrix}4-1 & 0-(-1)\\ 0-2 & 4-3\end{bmatrix}=\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}$$
$$\therefore \quad A^{-1} = \frac{1}{5}\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}.$$

Ex. 70. If $f(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$, show that $[f(x)]^{-1}=f(-x)$.

Sol.
$$f(x) = \begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\Rightarrow \quad f(-x) = \begin{bmatrix}\cos(-x) & -\sin(-x) & 0\\ \sin(-x) & \cos(-x) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix} \qquad ...(i)$$

Let $f(x)=A$. Then
$$|A| = \begin{vmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{vmatrix}$$
$$= \cos x(\cos x-0)+\sin x(\sin x-0)=\cos^2 x+\sin^2 x=1.$$

$\therefore$ $[f(x)]^{-1}$ exists.

$A_{11} = (-1)^{1+1}(\cos x-0)=\cos x,$ $\qquad A_{12} = (-1)^{1+2}(\sin x-0)=\sin x$

$A_{13} = (-1)^{1+3}(0-0)=0,$ $\qquad A_{21} = (-1)^{2+1}(-\sin x-0)=\sin x$

$A_{22} = (-1)^{2+2}(\cos x-0)=\cos x,$ $\qquad A_{23} = (-1)^{2+3}(0-0)=0$

$A_{31} = (-1)^{3+1}(0-0)=0,$ $\qquad A_{32} = (-1)^{3+2}(0-0)=0$

$A_{33} = (-1)^{3+3}(\cos^2 x+\sin^2 x)=1$

$$\therefore \quad \text{Adj. } A = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}=\begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\therefore \quad [f(x)]^{-1} = A^{-1}=\frac{\text{Adj. } A}{|A|}=\begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix} \qquad ...(ii)$$

(*i*) and (*ii*) $\Rightarrow [f(x)]^{-1}=f(-x)$.

Ex. 71. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y so that $A^2 + xI = yA$. Hence obtain A^{-1}.

Sol. $A^2 + xI = yA \Rightarrow \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix} \Rightarrow \begin{bmatrix} 16+x & 8 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$\therefore \quad 16 + x = 3y = 3(8) \Rightarrow x = 24 - 16 = 8$

Putting $x = 8$ and $y = 8$ in equation $A^2 + xI = yA$, we get $A^2 - 8A + 8I = 0$ ...(i)

Now, you can obtain A^{-1} by pre-multiplying by A^{-1} of both sides of (i).

Premultiplying by A^{-1} on both sides of (i),

$$A^{-1}A^2 - 8A^{-1}A + 8A^{-1}I = 0$$

$$\Rightarrow (A^{-1}A)A - 8I + 8A^{-1} = 0 \qquad [\because A^{-1}A = I \text{ and } A^{-1}I = A^{-1}]$$

$$\Rightarrow IA - 8I + 8A^{-1} = 0 \Rightarrow A - 8I + 8A^{-1} = 0 \qquad (\because IA = A)$$

$$\Rightarrow 8A^{-1} = 8I - A = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}.$$

EXERCISE 3 (h)

1. Show that A is a singular matrix if

(i) $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 2 & -3 & 1 \end{bmatrix}$

2. Find x if

(i) $\begin{bmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$ is a singular matrix. **(*NMOC*)** (ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & x \end{bmatrix}$ is a singular matrix.

3. For each of the following matrices, determine whether the inverse exists. If it exists, find it.

(i) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ **(*ISC 2002*)**

4. Find the adjoint and inverse of the following matrices.

(i) $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -6 & 5 \\ -7 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ **(*NMOC*)**

5. Find the sum of $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ and its multiplicative inverse. **(*ISC*)**

6. Find a matrix X for which $X\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Also find the inverse of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. **(*ISC*)**

7. Find the inverse of the following matrices and verify your result.

(*i*) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -4 & 3 \\ 5 & -5 \end{bmatrix}$

8. (*i*) Find a 2×2 matrix M such that $\begin{bmatrix} -5 & 2 \\ 15 & -7 \end{bmatrix} M = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.

(*ii*) Find a 2×2 matrix N such that $N \begin{bmatrix} -7 & 4 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$.

9. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19} A$. **(*ISC*)**

10. (*a*) Find the inverse of each of the following matrices and verify your result.

(*i*) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$. **(*ISC*)**

(*b*) Verify that $AA^{-1} = A^{-1}A = I$, if $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

(*c*) If $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ -6 & 4 & 7 \end{bmatrix}$ and $AB = BA = I$, find B.

11. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ for the matrices A and B where

(*i*) $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$

12. Let A be the matrix $\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}$, find A^{-1} and verify that $A^{-1} = \frac{1}{13} A - \frac{4}{13} I$ where I is 2×2 unit matrix.

13. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. **(*ISC*)**

14. (*i*) If $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$, find $(AB)^{-1}$. **(*NMOC*)**

(*ii*) If $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, and $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, find $(AB)^{-1}$. **(*ISC*)**

(*iii*) If $A^{-1} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, find $(BA)^{-1}$.

15. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$, verify that $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$.

16. If $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$, show that $A^2 + 3A + 4I = 0$. Hence find A^{-1}.

17. If $A^2 - A + I = 0$, then show that $A^{-1} = I - A$.

[Hint. Given $A^2 - A + I = 0$

Premultiplying by A^{-1} on both sides

$A^{-1}A^2 - A^{-1}A + A^{-1} I = 0 \Rightarrow (A^{-1}A) A - I + A^{-1} = 0$ $[\because A^{-1}A = I \text{ and } A^{-1} I = A^{-1}]$

$\Rightarrow IA - I + A^{-1} = 0 \Rightarrow A - I + A^{-1} = 0 \Rightarrow A^{-1} = I - A.]$ $[\because IA = A]$

18. For the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, show that $A^2 - 4A + 7I = 0$. Hence find A^{-1}.

19. Show that $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation $A^3 - A^2 - 3A - I_3 = 0$. Hence, find A^{-1}.

20. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$. Hence obtain A^{-1}.

21. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also, show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

22. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

23. Show that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $x^2 + 4x - 42 = 0$. Hence find A^{-1}. ***(ISC)***

[Hint. See solved Ex. 67]

24. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = 0$.

25. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^2 and show that $A^2 = A^{-1}$.

26. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

ANSWERS

2. (*i*) $x = \dfrac{5}{8}$ (*ii*) $x = 3$

3. (*i*) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$

(*iv*) Singular. The inverse does not exist. (*v*) The inverse does not exist

(*vi*) $\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$

4. (*i*) $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -6 & 5 \\ -7 & 6 \end{bmatrix}$

(*iii*) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

5. $\begin{bmatrix} -5 & 6 \\ 10 & 5 \end{bmatrix}$

6. $x = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; Inverse of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

7. (*i*) $\dfrac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$

8. (*i*) $\begin{bmatrix} -7 & -2 \\ -15 & -5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -8 & -4 \\ -5 & -7 \end{bmatrix}$

10. (*a*) (*i*) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$ (*c*) $\dfrac{1}{65}\begin{bmatrix} -1 & 8 & -10 \\ -61 & 33 & -25 \\ 34 & -12 & 15 \end{bmatrix}$

14. (*i*) $-\dfrac{1}{15}\begin{bmatrix} -2 & -15 \\ -1 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$ (*iii*) $\dfrac{1}{10}\begin{bmatrix} -98 & -18 & 36 \\ 21 & 6 & -7 \\ 12 & 2 & -4 \end{bmatrix}$

16. $-\dfrac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$ **18.** $\dfrac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ **19.** $A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

20. $\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$ **21.** $A^{-1} = \dfrac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ **22.** $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

23. $\frac{1}{42}\begin{bmatrix}-4 & 5\\ 2 & 8\end{bmatrix}$ **24.** $x = 9, y = 14$ **25.** $A^2 = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}$

HINTS

8. $\begin{bmatrix}-5 & 2\\ 15 & -7\end{bmatrix} M = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} \Rightarrow \begin{bmatrix}-5 & 2\\ 15 & -7\end{bmatrix} M = 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \quad AM = 5I$, where $A = \begin{bmatrix}-5 & 2\\ 15 & -7\end{bmatrix} \Rightarrow M = 5(A^{-1}I)$

$\Rightarrow \quad M = 5A^{-1} \qquad (\because A^{-1}I = A^{-1})$

10. (c) $AB = BA = I \Rightarrow B = A^{-1}$.

16. $A^{-1}(A^2 + 3A + 4I) = A^{-1}(0) \Rightarrow (A^{-1}A)A + 3A^{-1}A + 4A^{-1} = 0$

$\Rightarrow IA + 3I + 4A^{-1} = 0 \Rightarrow A^{-1} = -\frac{1}{4}(A + 3I)$

19. $I_3 = A^3 - A^2 - 3A \Rightarrow A^{-1} = A^2 - A - 3I_3$.

3.37. Application of matrices to the solution of linear equations (Martin's rule)

Consider the two simultaneous equations in two variables x and y

$$a_1x + b_1y = c_1, \quad a_2x + b_2y = c_2$$

These can be written in matrix form as

$$\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}c_1\\ c_2\end{bmatrix} \text{ or } AX = B.$$

where A is a 2×2 matrix, and X and B are 2×1 column matrices.

Similarly the three simultaneous equations

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2, \quad a_3x + b_3y + c_3z = d_3$$

can be written in the matrix form as

$$\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix} \Rightarrow AX = B.$$

where A is a 3×3 matrix, and X and B are (3×1) column matrices.

If now A is non-singular, *i.e.*, $|A| \neq 0$, then we can left-multiply both members of the equation by A^{-1} to obtain

$$\Rightarrow \quad A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B \qquad (\because IX = X)$$

Recall that equations having one or more solutions are called **consistent equations**.

(i) If $|A| \neq \mathbf{0}$, the system is *consistent* and has a unique solution.

(ii) If $|A| = \mathbf{0}$, the system of equations has either no solution or an infinite number of solutions.

(iii) Find (adj A) B.

(*a*) If (adj A) $B \neq \mathbf{0}$, the system has no solution and is, therefore, *inconsistent*.

(*b*) If (adj A) $B = \mathbf{0}$, the system is consistent and has *infinitely many solutions*. In this case we say that the equations are **dependent** equations.

To help you to remember easily, the above can be summarised diagrammatically as under :

System : $AX = B$

↓

Find $|A|$

- $|A| \neq 0$
  Unique Solution
  consistent
  $X = A^{-1}B$
- $|A| = 0$
  No solution or infinite number of solutions
  ↓
  Find (adj A) B
  - (adj A) $B \neq 0$
    ↓
    No solution
    Inconsistent
  - (adj A) $B = 0$
    ↓
    Infinitely many solutions
    Consistent (Dependent Eqns.)

Special Case : When $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In this case, $|A| \neq 0 \Rightarrow x = 0, y = 0, z = 0$. We say that the system has **trivial solution**.

If $|A| = 0$, then the system has infinitely many solutions.

Type 1.

Ex. 72. Use matrix method to solve the system of equations

$$4x - 3y = 11, \quad 3x + 7y = -1.$$

Sol. The given equations are equivalent to

$$\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \text{ in the matrix form. It is of the form}$$

$$AX = B \text{ where } A = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 4 \times 7 - 3 \times (-3) = 37 \neq 0 \quad \therefore A \text{ is non-singular.}$$

The system has a unique solution $X = A^{-1}B$.

$$A_{11} = 7, A_{12} = -3, A_{21} = 3, A_{22} = 4, \text{ adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{37}\begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{37} & \frac{3}{37} \\ \frac{-3}{37} & \frac{4}{37} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{37} & \frac{3}{37} \\ \frac{-3}{37} & \frac{4}{37} \end{bmatrix} \begin{bmatrix} 11 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{77}{37} - \frac{3}{37} \\ -\frac{33}{37} - \frac{4}{37} \end{bmatrix} = \begin{bmatrix} \frac{74}{37} \\ \frac{-37}{37} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \therefore\ x = 2, y = -1.$$

Ex. 73. Suppose the demand curve for automobiles over some time period can be written as

$$x_1 = 15000 - 0.2\, x_2$$

where x_1 is the price of an automobile and x_2 is the corresponding quantity. Suppose that the supply curve is

$$x_1 = 600 + 0.4\, x_2.$$

Use matrix theory to obtain x_1. **(*ISC*)**

Sol. The given equations are

$$x_1 + 0.2\, x_2 = 15000$$
$$x_1 - 0.4\, x_2 = 600$$

In the matrix form we get $\begin{pmatrix} 1 & 0.2 \\ 1 & -0.4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15000 \\ 600 \end{pmatrix}$; $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.2 \\ 1 & -0.4 \end{pmatrix}^{-1} \begin{pmatrix} 15000 \\ 600 \end{pmatrix}$

Inverse of $\begin{pmatrix} 1 & 0.2 \\ 1 & -0.4 \end{pmatrix} = \dfrac{1}{|A|} \begin{pmatrix} -0.4 & -0.2 \\ -1 & 1 \end{pmatrix}$ (Art. 3.29)

$$|A| = \begin{vmatrix} 1 & 0.2 \\ 1 & -0.4 \end{vmatrix} = -0.4 - 0.2 = -0.6$$

Therefore, inverse of $\begin{pmatrix} 1 & 0.2 \\ 1 & -0.4 \end{pmatrix} = \dfrac{1}{-0.6} \begin{pmatrix} -0.4 & -0.2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{-5}{3} \end{pmatrix}$

Using this, we get $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{-5}{3} \end{pmatrix} \begin{pmatrix} 15000 \\ 600 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 10200 \\ 24000 \end{pmatrix}$

$\Rightarrow \qquad x_1 = 10200, x_2 = 24000.$

Type 2.

Ex. 74. If $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the following system of linear equations:

$3x - 2y + z = 2,\ 2x + y - 3z = -5, -x + 2y + z = 6.$ **(*ISC 2008 Type*)**

Sol. Given, $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{vmatrix}$

$= 3(1 + 6) + 2(2 - 3) + 1(4 + 1) = 21 - 2 + 5 = 24 \neq 0 \Rightarrow A^{-1}$ exists.

Now, $\text{adj.}\, A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$, where A_{ij} is the cofactor of a_{ij}.

Here, $A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7, A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} = -(-1) = 1,$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, A_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = 4, A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 4$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = -4, A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 5, A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = 11$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 7$$

$$\therefore \quad \text{adj. } A = \begin{vmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{vmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{adj. } A = \frac{1}{24}\begin{vmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{vmatrix}$$

The given system of linear equation is

$$\begin{aligned} 3x - 2y + z &= 2 \\ 2x + y - 3z &= -5 \\ -x + 2y + z &= 6 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix} \Rightarrow AX = B \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}.$$

$$\Rightarrow A^{-1}(AX) = A^{-1}B \quad \Rightarrow X = A^{-1}B = \frac{1}{24}\begin{bmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{24}\begin{bmatrix} 7\times 2 + 4\times -5 + 5\times 6 \\ 1\times 2 + 4\times -5 + 11\times 6 \\ 5\times 2 + -4\times -5 + 7\times 6 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 24 \\ 48 \\ 72 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x = 1, y = 2, z = 3.$$

Ex. 75. Solve the following system of equations by matrix method $5x + 3y + z = 16$, $2x + y + 3z = 19$, $x + 2y + 4z = 25$

Sol. Writing the given equations in the matrix form $AX = B$ as

$$\begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix},$$

$$|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 5(4-6) - 3(8-3) + 1(4-1) = -22 \neq 0.$$

$\therefore$ A is non singular.

$\therefore$ The system has the unique solution $X = A^{-1}B$; $A_{11} = -2$, $A_{12} = -5$, $A_{13} = 3$, $A_{21} = -10$, $A_{22} = 19$, $A_{23} = -7$, $A_{31} = 8$, $A_{32} = -13$, $A_{33} = -1$.

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix} = \begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}$$

Now, $AX = B \quad \Rightarrow \quad X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix} \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{32+190-200}{22} \\ \frac{80-361+325}{22} \\ \frac{-48+133+25}{22} \end{bmatrix} = \begin{bmatrix} \frac{22}{22} \\ \frac{44}{22} \\ \frac{110}{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 5.$

Ex. 76. Using matrices, solve the following system of equations :

$$\mathbf{x + 2y = 5}$$
$$\mathbf{y + 2z = 8}$$
$$\mathbf{2x + z = 5}$$

(*ISC 2003*)

Sol. Writing the given equations in matrix form

$$AX = B \text{ as}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}, \text{ we have}$$

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\,(1-0) - 2\,(0-4) + 0 = 1 + 8 = 9 \neq 0.$$

$\therefore$ A is non-singular.

$\therefore$ The system has the unique solution $X = A^{-1}B$.

$\therefore$ A is invertible and $A^{-1} = \dfrac{\text{Adj}}{|A|}$

Now we find Adj A.

$$\text{Adj } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$$

Where A_{11}, A_{12}, A_{13}...., are cofactors of a_{11}, a_{12}, a_{13}....., respectively

$A_{11} = 1, A_{12} = 4, A_{13} = -2; A_{21} = -2, A_{22} = 1, A_{23} = 4; A_{31} = 4, A_{32} = -2, A_{33} = 1$

$$\text{Adj, } A = \begin{vmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{vmatrix} \quad \therefore \quad A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{9} \begin{vmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} \frac{1}{9} & \frac{-2}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & \frac{-1}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{1}{9} \end{vmatrix}$$

Substituting in $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{-2}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{1}{9} \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{9} - \frac{16}{9} + \frac{20}{9} \\ \frac{20}{9} + \frac{8}{9} - \frac{10}{9} \\ \frac{-10}{9} + \frac{32}{9} + \frac{5}{9} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x = 1, y = 2, z = 3.$$

Ex. 77. Find the product of $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & -6 \\ 3 & -2 & -2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 20 & 2 & 34 \\ 8 & 16 & -32 \\ 22 & -13 & 7 \end{bmatrix}$ **and use it to solve the system of equation given below :**

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = -3,\ \frac{5}{x} + \frac{4}{y} - \frac{6}{z} = 4,\ \frac{3}{x} - \frac{2}{y} - \frac{2}{z} = 6.$$ **(*NMOC*)**

Sol. $$AB = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & -6 \\ 3 & -2 & -2 \end{bmatrix}\begin{bmatrix} 20 & 2 & 34 \\ 8 & 16 & -32 \\ 22 & -13 & 7 \end{bmatrix} = \begin{bmatrix} 40+24+88 & 4+48-52 & 68-96+28 \\ 100+32-132 & 10+64+78 & 170-128-42 \\ 60-16-44 & 6-32+26 & 102+64-14 \end{bmatrix}$$

$$= \begin{bmatrix} 152 & 0 & 0 \\ 0 & 152 & 0 \\ 0 & 0 & 152 \end{bmatrix} = 152\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad AB = 152\,I \ \Rightarrow \frac{1}{152}(AB) = I,\ i.e.,\ \frac{1}{152}B \Rightarrow A^{-1}I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{152}B = \frac{1}{152}\begin{bmatrix} 20 & 2 & 34 \\ 8 & 16 & -32 \\ 22 & -13 & 7 \end{bmatrix}$$

Given system of equations are

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = -3 \quad \text{or} \quad 2u + 3v + 4w = -3$$

$$\frac{5}{x} + \frac{4}{y} - \frac{6}{z} = 4 \quad \text{or} \quad 5u + 4v - 6w = 4$$

$$\frac{3}{x} - \frac{2}{y} - \frac{2}{z} = 6 \quad \text{or} \quad 3u - 2v - 2w = 6, \qquad \text{where } \frac{1}{x} = u,\ \frac{1}{y} = v,\ \frac{1}{z} = w.$$

The above system can be written in the form

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & -6 \\ 3 & -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix} \Rightarrow AX = C \Rightarrow X = A^{-1}C$$

$$\Rightarrow \quad X = \frac{1}{152}\begin{bmatrix} 20 & 2 & 34 \\ 8 & 16 & -32 \\ 22 & -13 & 7 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$$

$$= \frac{1}{152}\begin{bmatrix} -60+8+204 \\ -24+64-192 \\ -66-52+42 \end{bmatrix} = \frac{1}{152}\begin{bmatrix} 152 \\ -152 \\ -76 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1/2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1/2 \end{bmatrix} \Rightarrow u = 1,\ v = -1,\ w = -1/2$$

$$\Rightarrow \quad \frac{1}{x} = 1,\ \frac{1}{y} = -1,\ \frac{1}{z} = -\frac{1}{2} \Rightarrow x = 1,\ y = -1,\ z = -2.$$

Type 3.

Ex. 78. Use matrix method to examine the following system of equations for consistency or inconsistency $2x + 5y = 7,\ 6x + 15y = 13$

Sol. Writing the given system of equations in matrix form, we have

$$\begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix} \quad \text{or} \quad AX = B, \text{ where}$$

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 7 \\ 13 \end{bmatrix}, |A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0 \quad \therefore \text{ A is singular}$$

$\therefore$ Either the given system has no solution or an infinite number of solutions.

We find (adj A) B.

$$A_{11} = 15, A_{12} = -6, A_{21} = -5, A_{22} = 2 \quad \therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$$

$$\therefore \quad (\text{adj } A)\, B = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 105 - 65 \\ -42 + 26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq 0$$

$\Rightarrow$ The given system has no solution, and is, therefore, inconsistent.

Note: $2x + 5y = 7 \Rightarrow 3\,(2x + 5y) \Rightarrow 7 \times 3 \Rightarrow 6x + 15y = 21$ whereas the other equation is $6x + 15y = 13$.

Hence, inconsistent.

Ex. 79. Using matrix method examine the consistency or inconsistency of the system

$6x + 4y = 2, 9x + 6y = 3$

Sol. Representing the system by the matrix equation $AX = B$

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 36 - 36 = 0 \quad \therefore \quad A \text{ is singular}$$

$\therefore$ Either the system has no solution or has infinite number of solutions.

To check, we find (adj. A) B.

$$A_{11} = 6, \; A_{12} = -9, \; A_{21} = -4, \; A_{22} = 6$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}, \; (\text{adj } A)\, B = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 - 12 \\ 18 - 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Since (adj A) $B = 0$, therefore, the given system is consistent and has an infinite number of solutions. Let $y = k$. Putting $y = k$ in 1st equation, we get

$$6x + 4k = 2 \;\Rightarrow\; 6x = 2 - 4k \;\Rightarrow\; x = \frac{1}{6}(2 - 4k) \;\Rightarrow\; x = \frac{1}{3}(1 - 2k)$$

Substituting the values of x and y in the 2nd equation, *e.g.* $9x + 6y = 3$, we have

$$9.\frac{1}{3}(1 - 2k) + 6k = 3 \;\Rightarrow\; 3 - 6k + 6k = 3 \;\Rightarrow\; 3 = 3, \text{ which is true.}$$

Hence the given system has infinite number of solutions, given by $x = \frac{1}{3}(1 - 2k), y = k$.

Ex. 80. Test for consistency the system of equations

$4x - 5y - 2z = 2, 5x - 4y + 2z = -2, 2x + 2y + 8z = -1.$ ***(ISC 2007)***

Sol. The system can be represented in the form $AX = B$,

where
$$A = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix} \begin{aligned} &= 4\,(-32 - 4) + 5\,(40 - 4) - 2\,(10 + 8) \\ &= -144 + 180 - 36 = -180 + 180 = 0 \end{aligned}$$

Since $|A| = 0$, $\therefore$ A^{-1} does not exist. Either the system has no solution or has infinite number of solutions.

To check, we find (adj A) B. $\left(\because X = \frac{B}{A} = A^{-1}B = \frac{Adj\ A}{|A|}.B\right)$

$A_{11} = -36, A_{12} = -36, A_{13} = 18, A_{21} = 36, A_{22} = 36, A_{23} = -18, A_{31} = -18, A_{32} = -18, A_{33} = 9.$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -36 & 36 & -18 \\ -36 & 36 & -18 \\ 18 & -18 & 9 \end{bmatrix},$$

$$\therefore \quad (\text{adj } A)\, B = \begin{bmatrix} -36 & 36 & -18 \\ -36 & 36 & -18 \\ 18 & -18 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -126 \\ -126 \\ 63 \end{bmatrix} \neq 0$$

Since $|A| = 0$ and (adj A)$B \neq 0$, the given system has no solution and is, therefore inconsistent.

Ex. 81. Test for consistency and solve the equations

$5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5.$

Sol. The given system can be written in the matrix form $AX = B$, *i.e.*,

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \text{ where } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Now $$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = 5\,(260 - 4) - 3\,(30 - 14) + 7\,(6 - 182) = 0$$

Since $|A| = 0$, therefore, A^{-1} does not exist. $\therefore$ The system has either no solution or has infinite number of solutions.

To check, we find (adj A)B

$A_{11} = 256, A_{12} = -16, A_{13} = -176, A_{21} = -16, A_{22} = 1, A_{23} = 11, A_{31} = -176, A_{32} = 11, A_{33} = 121$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$(\text{adj. } A)B = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 1024 - 144 - 880 \\ -64 + 9 + 55 \\ -704 + 99 + 605 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Since $|A| = 0$ and (adj. A)$B = 0$, the system is consistent and has infinite number of solutions.

Let $z = k$ and solve any two of the given equations as follows :

$$5x + 3y + 7y = 4 \quad \text{or} \quad 5x + 3y = 4 - 7k \qquad ...(1)$$

$$3x + 26y + 2k = 9 \quad \text{or} \quad 3x + 26y = 9 - 2k \qquad ...(2)$$

Solving (1) and (2), we have $x = \frac{1}{11}(7 - 16k)$, $y = \frac{1}{11}(k + 3)$. Also $z = k$

Substituting in the third equation, *e.g.*, $7x + 2y + 10z = 5$, we have

$$7.\frac{1}{11}(7 - 16k) + 2.\frac{1}{11}(k + 3) + 10k = 5,$$

or $\quad 49 - 112k + 2k + 6 + 110k = 55 \quad$ or $\quad 55 = 55$ which is true.

Thus, the given system of equations has infinite number of solutions given by

$$x = \frac{1}{11}(7 - 16k),\ y = \frac{1}{11}(k + 3),\ z = k \text{ where } k \text{ is any real number.}$$

EXERCISE 3 (*i*)

Solve the following systems of equations by matrix method.

1. $2x - 3y = 1$
$3x - 2y = 4$ **(*NMOC*)**

2. $2x + 3y = 23$
$3x + 4y = 32$ **(*ISC 1991*)**

3. $3x + 7y = 4$
$x + 2y = 1$

4. $2x + y = 5$
$5x - 2y = 8$

5. $3x + y + z = 3$
$2x - y - z = 2$
$-x - y + z = 1$

6. $x - y + z = 2$
$2x - y = 0$
$2y - z = 1$

7. $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$
$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$
$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

[Hint. Put $\frac{1}{x} = u$, $\frac{1}{y} = v$, $\frac{1}{z} = w$ and solve for u,v,w]

8. $x + y = 5$
$z + y = 7$
$z + x = 6$

9. $5x - y = -7$
$2x + 3z = 1$
$3y - z = 7$

10. $x_1 - 2x_2 + 3x_3 = 4$
$2x_1 + x_2 - 3x_3 = 5$
$-x_1 + x_2 + 2x_3 = 3$ **(*ISC 2002*)**
[Hint. See solved Ex. 75]

11. $x - 2y + 3z = 6,$
$x + 4y + z = 12,$
$x - 3y + 2z = 1$ **(*ISC 2001*)**

12. $x + y + z = 6$
$x - y + z = 2$
$2x + y - z = 1$ **(*ISC 2007*)**

13. (*i*) Find the inverse of the matrix $\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$ and use it in solving the equations
$0.8x_1 - 0.6x_2 = 10$
$0.6x_2 + 0.8x_2 = 20.$ **(*ISC*)**

(*ii*) Find the inverse of the matrix $\begin{pmatrix} 6 & 7 \\ 4 & 5 \end{pmatrix}$, and use it to solve the simultaneous equations
$6x + 7y = 2$
$4x + 5y = 3.$ **(*ISC*)**

14. (*i*) If $A = \begin{bmatrix} 4 & -5 & -11 \\ 1 & -3 & 1 \\ 2 & 3 & -7 \end{bmatrix}$, find A^{-1} and using A^{-1} solve the system of equations $4x - 5y - 11z = 12$, $x - 3y + z = 1$, $2x + 3y - 7z = 2$.

(*ii*) If $A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the following system of linear equations :

$8x - 4y + z = 5$, $10x + 6z = 4$, $8x + y + 6z = \frac{5}{2}$.

(*iii*) If $A = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix}$, find A^{-1} and hence solve the following system of linear equations.

$x + 2y - 3z = -4$, $2x + 3y + 2z = 2$, $3x - 3y - 4z = 11$. **(*ISC 2008*)**

[Hint. See Solved Ex. 74]

15. Find the product of two matrices A and B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

and use it to solve the system of equations

$x + y + 2z = 1$, $3x + 2y + z = 7$, $2x + y + 3z = 2$. **(*NMOC, Roorkee*)**

Use matrix method to examine the following systems of equations of consistency or inconsistency.

16. $3x - 2y = 5$
$6x - 4y = 9$

17. $4x - 2y = 3$
$6x - 3y = 5$

18. $x + 5y = 3$
$2x + 10y = 6$

19. $3x - y + 2z = 3$
$2x + y + 3z = 5$
$x - 2y - z = 1$

20. $x + y + z = 6$
$x + 2y + 3z = 14$
$x + 4y + 7z = 30$

21. $2x - y + 3z = 1$
$x + 2y - z = 2$
$5y - 5z = 3$

22. Show that the following system of equations is consistent.
$x - 2y + z = 0,\ y - z = 3,\ 2x - 3z = 10$
Also find the solution using matrix method. ***(ISC BM 2003)***

23. Find k so that the equations
$3x - 2y + 2z = 1,\ 2x + y + 3z = -1,\ x - 3y + kz = 0$ may have a unique solution.

24. For what value of k, do the equation

$$2x - 3y + 2z = a$$
$$5x + 4y - 2z = -3$$
$$x - 13]y + kz = 9$$

not have a unique solution ?

25. Suppose the demand curve for automobiles over some time period can be written as $x_1 = 15000 - 0.2\, x_2$ where x_1 is the price of an automobile and x_2 is the correponding quantity. Suppose the supply curve is $x_1 = 600 + 0.4\, x_2$. Use matrix theory to obtains x_1. ***(ISC)***

26. Gaurav purchases 3 pens, 2 bags and 1 instrument box and pays Rs 41. From the same shop, Dheeraj purchases 2 pens, 1 bag and 2 instrument boxes and pays Rs 29, while Ankur purchases 2 pens, 2 bags and 2 instrument boxes and pays Rs 44. Translate the problem into a system of equations. Solve the system of equation by matrix method and hence find the cost of one pen, one bag and one instrument box.

[Hint. Let the cost of 1 pen be Rs x, of one bag be Rs y and of one instrument box be Rs z. Then $3x + 2y + z = 41,\ 2x + y + 2z = 29,\ 2x + 2y + 2z = 44$. Solve by matrix method.]

ANSWERS

1. $x = 2, y = 1$ **2.** $x = 4, y = 5$ **3.** $x = -1, y = 1$

4. $x = 2, y = 1$ **5.** $x = 1, y = -1, z = 1$ **6.** $x = 1, y = 2, z = 3$

7. $x = 2, y = 3, z = 5$ **8.** $x = 2, y = 3, z = 4$ **9.** $x = -1, y = 2, z = 1$

10. $x_1 = 4, x_2 = 3, x_3 = 2$ **11.** $x = 1, y = 2, z = 3$ **12.** $x = 1, y = 2, z = 3$

13. (*i*) $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$; $x_1 = 20, x_2 = 10$ (*ii*) $\begin{bmatrix} \frac{5}{2} & \frac{-7}{2} \\ -2 & 3 \end{bmatrix}$; $x = \frac{-11}{2}, y = 5$

14. (*i*) $A^{-1} = \frac{-1}{72}\begin{bmatrix} 18 & -68 & -38 \\ 9 & -6 & -15 \\ 9 & -22 & -7 \end{bmatrix}$, $x = -1, y = -1, z = -1$

(*ii*) $A^{-1} = \frac{1}{10}\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$, $x = 1, y = \frac{1}{2}, z = -1$

(*iii*) $A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$, $x = 3,\ y = -2,\ z = 1$ **15.** $AB = 4I$; $x = 2, y = 1, z = -1$

16. Inconsistent **17.** Inconsistent

18. Consistent, infinitely many solutions, $x = 3 - 5k, y = k$ where $k \in R$. **19.** Inconsistent.

20. Consistent, has infinitely many solutions $z = k, x = k - 2, y = 8 - 2k$ for all $k \in R$.

21. Consistent, infinitely many solution $z = k$, $x = \frac{4 - 5k}{5}$, $y = \frac{5k + 3}{5}$, where $k \in R$.

22. $x = 8, y = 5, z = 2$ **23.** $k \neq -1$ **24.** $k \neq 8$

25. $x_1 = 10{,}200$; $x_2 = 24{,}000$. **26.** Pen → Rs 2, bag → Rs 15, instrument box → Rs 5.

HINTS

8. Write the given equation as $x + y + 0 \cdot z = 5,\ 0.x + y + z = 7,\ x + 0.y + z = 6$

14. First show that $AB = 4I$, or $\frac{A}{4} B = I \Rightarrow B^{-1} = \frac{A}{4}$ · Now $BX = D$

$$\Rightarrow B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow B^{-1}B\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{A}{4}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \text{ etc.}$$

22. The system will have a unique solution only, if the coefft, matrix is non-singular, *i.e.*, $|A| \neq 0$.

REVISION EXERCISE ON CHAPTERS 2 AND 3

1. (*i*) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ then find det A.

(*ii*) If $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$, then find the values of x, y, z, w.

(*iii*) If $A = \begin{bmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{bmatrix}$, then find the cofactors of the elements of 2nd row

2. Find θ if the matrix $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is singular.

3. Find the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

4. Find λ if the system of equations $3x - 2y + z = 0,\ \lambda x - 14y + 15z = 0,\ x + 2y - 3z = 0$ has non-zero solution.
[**Hint** Refer to Art 3.37, special cases, Page 3-62]

5. The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ is

(*a*) non-singular (*b*) singular (*c*) skew-symmetric (*d*) symmetric

6. (*i*) Show that $\begin{vmatrix} x & 1 & y+z \\ y & 1 & z+x \\ z & 1 & x+y \end{vmatrix} = 0$ (*ii*) $\begin{vmatrix} 1/a & 1 & bc \\ 1/b & 1 & ca \\ 1/c & 1 & ab \end{vmatrix} = 0$

7. If $A = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then find $(B^{-1} A^{-1})^{-1}$.

8. (*i*) If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then find $f(100)$.

(*ii*) Show that one root of the equation $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ is $-(a+b+c)$.

9. If the system of equations $x + 2y - 3z = 1,\ (\lambda + 3)\, z = 3,\ (2\lambda + 1)\, x + z = 0$ is inconsistent, then find the value of λ.

10. Evaluate $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^6 & 5^7 \end{vmatrix}$

11. If $A = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 3 & 1 \\ -5 & 7 & 2 \end{bmatrix}$ is expressed as a sum of a symmetric and skew symmetric matrix, then find the skew symmetric matrix.

12. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$ then find the value of $\begin{vmatrix} b_2c_3 - b_3c_2 & a_3c_2 - a_2c_3 & a_2b_3 - a_3b_2 \\ b_3c_1 - b_1c_3 & a_1c_3 - a_3c_1 & a_3b_1 - a_1b_3 \\ b_1c_2 - b_2c_1 & a_2c_1 - a_1c_2 & a_1b_2 - a_2b_1 \end{vmatrix}$

[Hint. See solved Ex. 55]

13. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then find the possible values of k.

14. If $\begin{vmatrix} 1+x^3 & x^2 & x \\ 1+y^3 & y^2 & y \\ 1+z^3 & z^2 & z \end{vmatrix} = 0$ and x, y, z and all different, then find the value of xyz.

15. (*i*) If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ whenever $A^2 = B$ then the value of α is

(*a*) 1 (*b*) −1 (*c*) 4 (*d*) no real value of α.

(*ii*) $x + ay = 0$, $y + az = 0$, $z + ax = 0$. The value of a for which the system of equation has infinitely many solutions is

(*a*) $a = 1$ (*b*) $a = 0$ (*c*) $a = -1$ (*d*) no value.

16. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(*i*) $\alpha = a^2 + b^2$, $\beta = ab$ (*ii*) $\alpha = a^2 + b^2$, $\beta = 2ab$,
(*iii*) $\alpha = a^2 + b^2$, $\beta = a^2 - b^2$ (*iv*) $\alpha = 2ab$, $\beta = a^2 + b^2$

17. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. The only correct statement about the matrix A is

(*a*) A^{-1} does not exist (*b*) $A = (-1)\, I$, when I is a unit matrix
(*c*) A is a zero matrix (*d*) $A^2 = I$

18. In a $\Delta\, ABC$, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then prove that $\sin^2 A + \sin^2 B + \sin^2 C = \dfrac{9}{4}$.

19. The number of values of k for which the system of equations $(k + 1)\, x + 8y = 4k$, $kx + (k + 30)y = 3k - 1$ has infinitely many solutions is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) Infinite

20. The value of the determinant $\begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$ is

(*a*) 2 (10! 11!) (*b*) 2 (10! 13!) (*c*) 2 (10! 11! 12!) (*d*) 2 (11! 12! 13!)

21. If $A^2 - A + I = 0$, then the inverse of A is

(*a*) A (*b*) $A + I$ (*c*) $I - A$ (*d*) $A - I$

22. If A is an invertible matrix, then what is det (A^{-1}) equal to

(*a*) 0 (*b*) det (A) (*c*) $\dfrac{1}{\det(A)}$ (*d*) 1

23. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if U_1, U_2 and U_3 are column matrices satisfying $A\,U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\,U_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $A\,U_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and U is a 3×3 matrix when columns are U_1, U_2, U_3 then find

(*a*) the value of $|U|$

(*b*) the sum of the elements of U^{-1}

(*c*) the value of $[3 \quad 2 \quad 0]\; U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.

24. (*i*) Evaluate $\begin{vmatrix} \log e & \log e^2 & \log e^3 \\ \log e^2 & \log e^3 & \log e^4 \\ \log e^3 & \log e^4 & \log e^5 \end{vmatrix}$

(*ii*) If A is an invertible matrix of order n, then $Adj.\,A =$

(*a*) $|A|^n$ (*b*) $|A|^{n+1}$ (*c*) $|A|^{n-1}$ (*d*) $|A|^{n+2}$.

25. Let $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ be a square matrix of order 3, then for any positive integer n, what is A^n equal to ?

(*a*) A (*b*) $3^n A$ (*c*) $(3^{n-1})\,A$ (*d*) $3A$

26. Let $A = (a_{ij})_{n \times n}$ and $adj\,A = (\alpha_{ij})$. If $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 3 & -1 \end{vmatrix}$ then find α_{23}.

27. If $A = \begin{bmatrix} x & x^2 & 1+x^2 \\ y & y^2 & 1+y^2 \\ z & z^2 & 1+z^2 \end{bmatrix}$, where x, y, and z are distinct, what is |A| ?

(*i*) 0 (*ii*) $x^2y - y^2x + xyz$ (*iii*) $(x-y)(y-z)(z-x)$ (*iv*) xyz.

28. The determinant $\begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta \\ \sin\alpha & \cos\alpha & \sin\alpha \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is independent of

(*a*) β (*b*) α (*c*) α and β both (*d*) To become rational

ANSWERS

1. (*i*) 2 (*ii*) 2, 2, 3, 4 (*iii*) 3, 3, 11

2. $\frac{\pi}{4}$ **3.** $\frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ **4.** 5

5. (*b*) **7.** $\begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$ **8.** (*i*) 0

9. $-\frac{1}{2}$ **10.** 0 **11.** $\begin{vmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{vmatrix}$

12. 25 **13.** – 1, 1 **14.** – 1

15. (*i*) *d* (*ii*) *c* **16.** (*ii*)

17. (*d*) **19.** (*b*) **20.** (*c*)

21. (*c*) **22.** (*c*)

23. $U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

(*i*) $|U| = 3$, (*ii*) 0 (*iii*) 5

24. (*i*) (0) (*ii*) (*c*) **25.** (*c*) **26.** 8 **27.** (*iii*) **28.** (*c*)

HINTS

12. Reqd. determinant = |adj *A*| where

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = |A|^{3-1} = |A|^2 = 5^2 = 25. \quad \text{(See note, solved Ex. 55)}$$

15. (*i*) $A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \therefore \alpha^2 = 1 \Rightarrow \alpha = \pm 1$

Also $\alpha + 1 = 5 \Rightarrow \alpha = 4$ $\therefore$ No real value of α.

(*ii*) $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 + a^3 = 0 \Rightarrow a = -1$

16. $A^2 = A.\,A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Equate the value of A^2 so obtained to the given value of A^2.

18. $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, on simplification $\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

It is possible only if and only if $a = b = c$, *i.e.*, ΔABC is equilateral $\Rightarrow \angle A = \angle B = \angle C = 60°$.

19. For infinitely many solutions, the two equations must be identical

$\Rightarrow \frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k^2 - 4k + 3 = 0$ and $k^2 - 3k + 2 = 0$

$$\Rightarrow \frac{k^2}{-8+9} = \frac{k}{3-2} = \frac{1}{-3+4} \Rightarrow k^2 = 1 \text{ and } k = 1$$

20. $\Delta = 10!\ 11!\ 12! \begin{vmatrix} 1 & 11 & 11\times 12 \\ 1 & 12 & 12\times 13 \\ 1 & 13 & 13\times 14 \end{vmatrix}$, Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

21. $A^2 - A + I = 0 \Rightarrow I = A - A \cdot A \Rightarrow IA^{-1} = AA^{-1} - A(AA^{-1}) \cdot A^{-1} = I - A.$

22. Since $|A| \neq 0$, therefore A^{-1} exists such that $AA^{-1} = I = A^{-1}A$

$$\Rightarrow |AA^{-1}| = |I| \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

23. Let $U_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ Similarly you may obtain $U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$, $U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ Now find U and then *adj* U and then U^{-1}

$$[3 \quad 2 \quad 0]\, U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = [3 \quad 2 \quad 0] \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

28. Expand $\cos(\alpha + \beta)$, $-\sin(\alpha + \beta)$, $\cos 2\beta$ (putting $= \cos^2\beta - \sin^2\beta$).
Now apply $R_1 \to R_1 + \sin\beta\, R_2 + \cos\beta\, R_3$. You will obtain the value of the determinant as 0.

UNIT 2
TRIGONOMETRY

- Inverse Trigonometric Functions

Syllabus

- Meaning of inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$)
- Principal values (use of graphs in explanation)
- Properties of inverse trigonometric functions (without proof)

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4

Inverse Trigonometric Functions

4.01. Inverse circular or trigonometrical functions

If $\sin\theta = x$, then $\theta = \sin^{-1} x$ (Read as sine inverse x)

Thus we see that '$\sin^{-1}x$' is a symbol which denotes an angle or a number the value of whose sine is x. Similarly, '$\cos^{-1}x$' denotes an angle whose cosine is x and so on. Thus

$$\because \quad \sin\frac{\pi}{6} = \frac{1}{2} \quad \therefore \quad \frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right); \qquad \tan\frac{\pi}{4} = 1 \quad \therefore \quad \frac{\pi}{4} = \tan^{-1}(1)$$

The expressions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ... are called ***inverse trigonometric functions.***

Note. The symbol **'arc sin *x*'** is also used for **$\sin^{-1}x$.**

Caution. It should be clearly noted that $\sin^{-1}x$ is merely a symbol to indicate the angle and should not be confused with $(\sin x)^{-1}$ or $1/\sin x$. The (–1) is not an exponent here. It is the inverse notation used in $f^{-1}(x)$.

4.02. The inverse of a *t*-function is not a function

The equation $y = \sin x$ defines the sine function and in this defining equation, x is the independent variable and y is the dependent variable. The domain is the set of all real numbers or angles, and the range is the set of all real numbers between –1 and +1 inclusive. We know that there is unique value of y for each given angle or number x. Suppose we are interested in the converse process that is, we want to find the value of x for a given value of y which, in other words, means that we wish to know a number or an angle corresponding to a particular value of the sine of the angle.

Suppose it is given that $y = \sin x = \frac{1}{2}$, then $x = \frac{\pi}{6}$ (or 30°) is a number or angle whose sine is $\frac{1}{2}$ but $\frac{5\pi}{6}$ (or 150°) is also a number or angle whose sine is $\frac{1}{2}$. Besides these, other possible values are $\frac{13\pi}{6}$ (or 390°), $\frac{17\pi}{6}$ (or 510°), $\frac{-7\pi}{6}$ (or – 210°), and $\frac{-11\pi}{6}$ (or – 330°) ...In fact, we have an infinitely many values of x, both positive and negative, for which $\sin x = \frac{1}{2}$. More concisely, if $\sin x = \frac{1}{2}$, then $x = \frac{\pi}{6} + 2n\pi$ and $\frac{5\pi}{6} + 2n\pi$, $n \in I$.

The function $\{(x, y) \mid y = \sin x\}$ will thus be an infinite set of ordered pairs given below:

$$\left\{\left(\frac{\pi}{6} + 2n\pi,\ \frac{1}{2}\right),\ \left(\frac{5\pi}{6} + 2n\pi,\ \frac{1}{2}\right),\ \ n \in I\right\}$$

when y is replaced by $\frac{1}{2}$.

The inverse of this function is the set of ordered pairs

$$\left\{\left(\frac{1}{2},\ \frac{\pi}{6} + 2n\pi\right),\ \left(\frac{1}{2},\ \frac{5\pi}{6} + 2n\pi\right),\ n \in I\right\}$$

which is obviously not a function, because corresponding to a value of the independent variable there are more than one value of the dependent variable.

4.03. Inverses of trigonometric functions

Consider the sine function $\{(x, y) : y = \sin x\}$ ... (1)

The inverse of this function is found by interchanging the x and the y in the defining equation. It is

$$\{(x, y) :\ x = \sin y\} \quad \text{... (2)}$$

and as discussed in Art. 4.01 is denoted by the symbolism

$$\mathbf{\{(x, y) : y = \sin^{-1} x\} \text{ or } \{(x, y) : y = \text{are} \sin x\}}$$

Other trigonometric functions also have inverse relations which are defined similarly

$\{(x, y) : y = \cos^{-1} x\}$ or $\{(x, y) : y = \text{arc} \cos x\}$
$\{(x, y) : y = \tan^{-1} x\}$ or $\{(x, y) : y = \text{arc} \tan x\}$
$\{(x, y) : y = \cot^{-1} x\}$ or $\{(x, y) : y = \text{arc} \cot x\}$
$\{(x, y) : y = \sec^{-1} x\}$ or $\{(x, y) : y = \text{arc} \sec x\}$
$\{(x, y) : y = \text{cosec}^{-1} x\}$ or $\{(x, y) : y = \text{arc cosec}\, x\}$

Note. As we have explained in Art. 4.02, all inverses of t-functions are not functions. We shall now find in the next section the conditions under which the inverse of a t-function may be made a function.

4.04. Defining inverse *t*-function

We have seen that the inverses of t-functions are all relations. They are not functions. In order to make these relations inverse functions certain restrictions must be placed on either the domain or the range. Since the domain is already restricted to the interval $-1 \le x \le 1$; therefore, we consider restricting the range.

Consider the sine function

$$\text{sine} = \left\{\theta : \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\} \to \{x : -1 \le x \le 1\} \text{ denoted by } \sin\theta = x.$$

We then, define the inverse sine function, as

$$\sin^{-1} : \{x : -1 \le x \le 1\} \to \left\{\theta : \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\} \text{ denoted by } \sin^{-1} x = \theta.$$

We see in the above that all values of x in the interval of the domain, *i.e.*, $(-1 \le x \le 1)$ are associated with one and only one value in the restricted range, *i.e.*, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

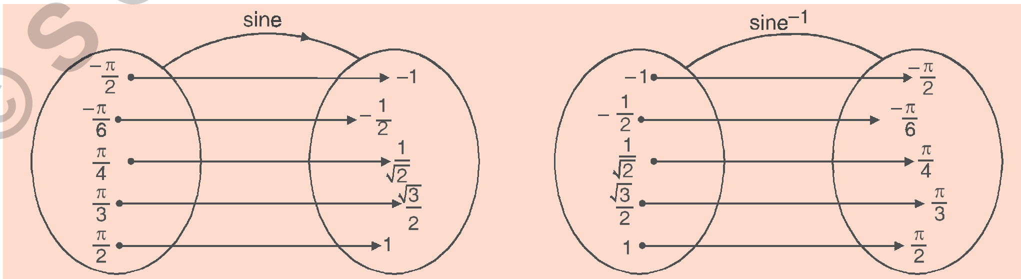


Fig. 4.01

Thus, $\sin^{-1} x$ is an angle, whose sine is x, *i.e.*, $\sin^{-1} x \Leftrightarrow x = \sin\theta$, where $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

Since all values of x in the interval of the domain $(-1 \leq x \leq 1)$ are associated with one and only one value in the restricted range $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$, the inverse function of sine function exists and is defined in the restricted range.

Similarly, in case of cosine function, its inverse function will exist, if we restrict the domain of cosine function to $0 \leq x \leq \pi$ which means that the domain of $\cos^{-1}x$ would be $-1 \leq x \leq 1$ and range $0 \leq y \leq \pi$.

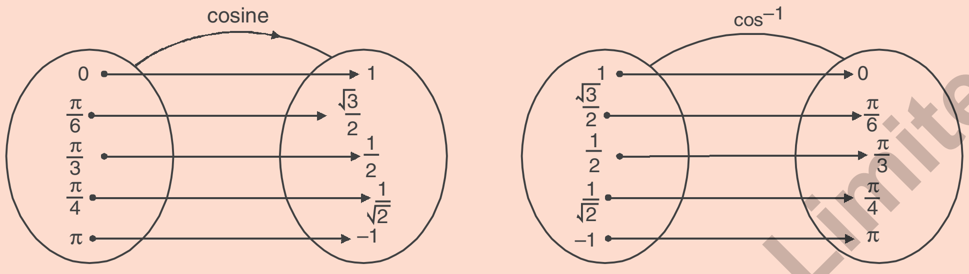


Fig. 4.02

In the table given below, we have listed the definition of the inverse trigonometric functions with *restricted ranges.*

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

4.05. $y = \sin^{-1}x$

Consider the function equation $x = \sin y$. We know that as y increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, then x increases monotonically, taking up every real value in $[-1, 1]$, so that to each value of x in this interval there corresponds *one and only one* value of y in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Thus, there is one and only one angle with a given sine.

According we define $\sin^{-1}x$ as follows:

$\sin^{-1}x$ is the angle in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, *whose sine is x.*

In others word, the domain of the $\sin^{-1}x$ is $[-1, 1]$ and the range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

To draw the graph of $y = \sin^{-1}x$.

Table for $\sin^{-1}x$

x	-1	$\frac{-\sqrt{3}}{2} = -0.87$	$\frac{-1}{\sqrt{2}} = -0.70$	$\frac{-1}{2} = -0.5$	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} = 0.70$	$\frac{\sqrt{3}}{2} = 0.87$	1
$\sin^{-1}x$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

We know that

(*i*) y increases monotonically from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ as x increases from -1 to 1.

(*ii*) $\sin^{-1}x = 0$, $\sin^{-1}1 = \frac{\pi}{2}$, $\sin^{-1}(-1) = -\frac{\pi}{2}$.

(*iii*) $\sin^{-1}x$ is defined in the interval [–1, 1] only.

Take 10 small divisions = 1 along the x-axis and 5 small division = $\frac{\pi}{6}$ along the y-axis. Plot the points and join.

We thus have the graph as drawn in Fig. 4.03.

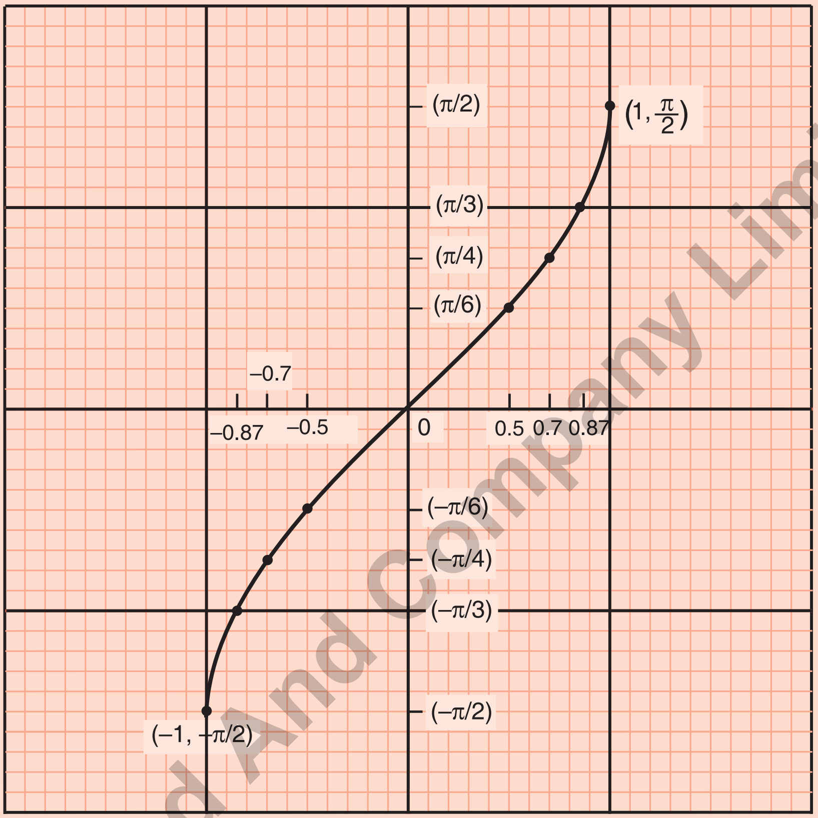


Fig. 4.03. Graph of $y = \sin^{-1}x$

Note. A portion of the graph of $y = \sin x$ is shown in Fig. 4.04. It is easily seen that the graph of $y = \sin^{-1}x$ is the reflection of $y = \sin x$, in the line $y = x$, as by interchanging x and y in $y = \sin x$ we get $x = \sin y$, *i.e.*, $y = \sin^{-1}x$.

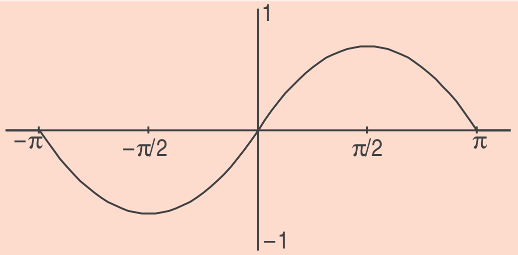


Fig. 4.04

4.06. Graph of $y = \cos^{-1}x$

Consider the functional equation $x = \cos y$.

We know that as y increases from 0 to π, then x decreases monotonically taking up every real value between 1 and –1. Thus, there is *one and only one* angle, lying between 0 to π, with a given cosine.

Accordingly we define $\cos^{-1} x$ as follows :

$\cos^{-1}x$ *is the angle in the interval* **$[0, \pi]$,** *whose cosine is* x.

In other words, the domain of $\cos^{-1} x$ is $[-1, 1]$ and the range is $[0, \pi]$.

To draw the graph of $y = \cos^{-1} x$

Table for $\cos^{-1}x$

x	-1	$\frac{-\sqrt{3}}{2}$ $= -0.87$	$\frac{-1}{\sqrt{2}}$ $= -0.70$	$\frac{-1}{2}$ $= -0.5$	0	$\frac{1}{2}$ $= 0.5$	$\frac{1}{\sqrt{2}}$ $= 0.70$	$\frac{\sqrt{3}}{2}$ $= 0.87$	1
$\cos^{-1}x$	π	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

We note that

(*i*) y decreases monotonically from π to 0 as x increases from –1 to 1.

(*ii*) $\cos^{-1}(-1) = \pi$, $\cos^{-1} 0 = \frac{\pi}{2}$, $\cos^{-1} 1 = 0$.

(*iii*) $\cos^{-1} x$ is defined in the interval $[-1, 1]$ only.

We thus have the graph as is drawn in Fig. 4.05.

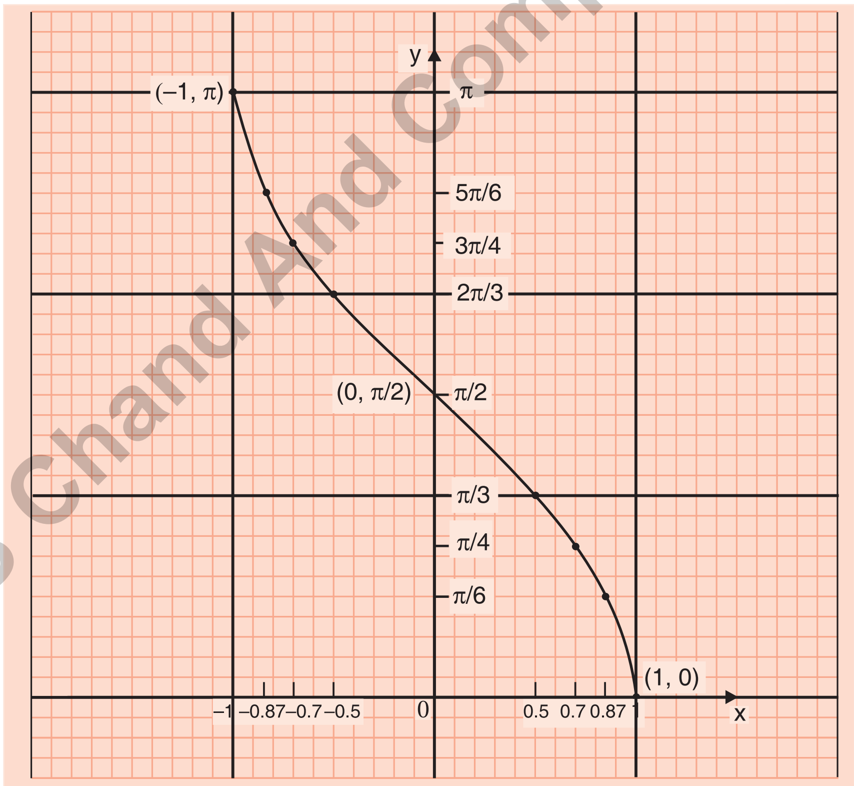
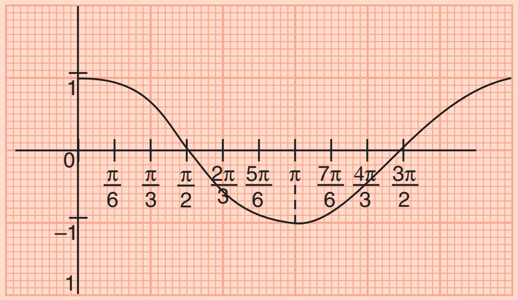


Fig. 4.05. Graph of $y = \cos^{-1}x$

Note. A portion of the graph of $y = \cos x$ is shown below. The graph of $y = \cos^{-1}x$ is reflection of $y = \cos x$ in the line $y = x$.



Graph of $y = \cos x$

4.07. $y = \tan^{-1} x$

Consider the functional equation $x = \tan y$.

We know that as y increases from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, then x increases monotonically taking up every value in $(-\infty, \infty)$ so that to each value of x in this interval there corresponds *one and only one* value of y in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accordingly we have the following definition of $\tan^{-1}x$ ·

$\tan^{-1} x$ is the angle in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, *whose tangent is x.* In other words, the domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

To draw the graph of $y = \tan^{-1} x$

Table for $\tan^{-1}x$

x	$-\sqrt{3}$ $= -1.73$	-1	$\frac{-1}{\sqrt{3}}$ $= -0.58$	0	$\frac{1}{\sqrt{3}}$ $= 0.58$	1	$\sqrt{3}$ $= 1.73$
$\tan^{-1}x$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

When $x \to \infty$, then $\tan^{-1} x \to \frac{\pi}{2}$

When $x \to -\infty$, then $\tan^{-1} x \to \frac{-\pi}{2}$

We note that

(*i*) y increases monotonically from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ as x increses from $-\infty$ to ∞.

(*ii*) $\tan^{-1} 0$ is 0.

(*iii*) $\tan^{-1} x$ is defined in the interval $(-\infty, \infty)$.

Plot the points $(-1.73, -\pi/3)$, $(-1, -\pi/4)$, $(-0.58, -\pi/6)$, $(0, 0)$, $(0.58, \pi/6)$, $(1, \pi/4)$, $(1.73, \pi/3)$ and join. Keep in mind that as y, *i.e.*, $\tan^{-1} x \to \pi/2$, $x \to \infty$ and as y, *i.e.*, $\tan^{-1} x \to -\pi/2$, $x \to \infty$.

We then have the graph as is drawn in Fig. 4.06.

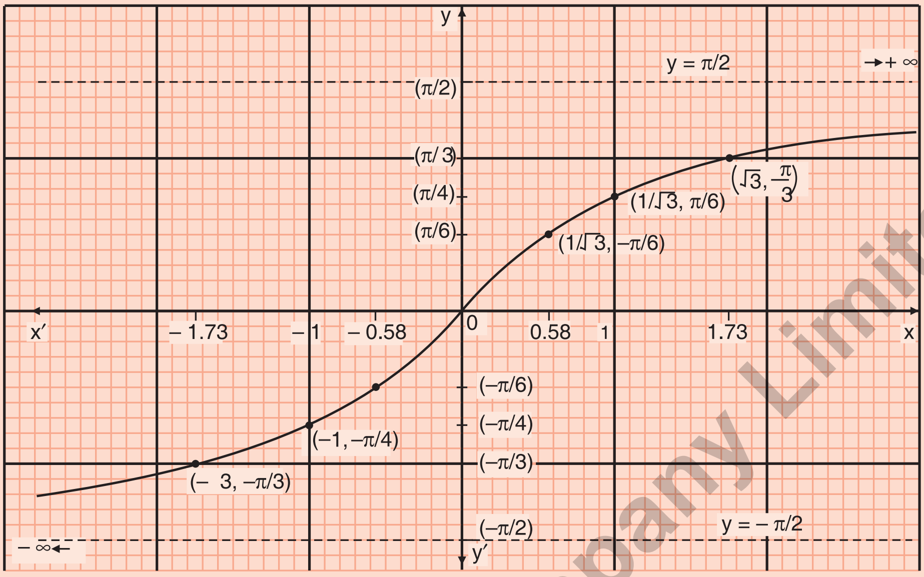
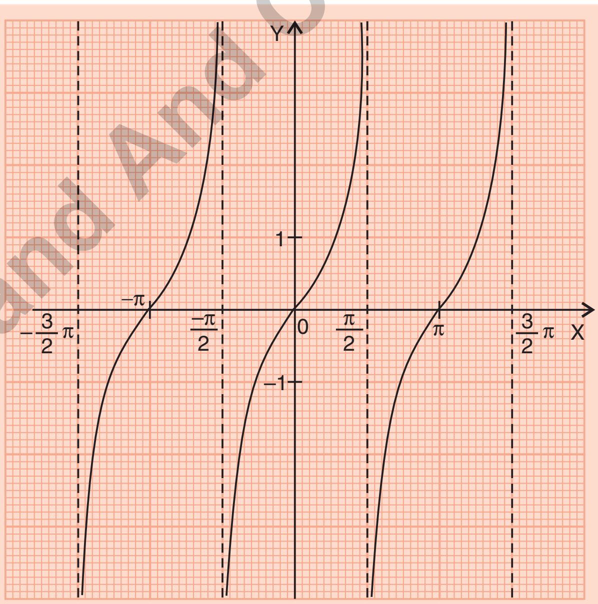


Fig. 4.06. Graph of $y = \tan^{-1} x$

Note. To help you to compare the graphs of $y = \tan^{-1} x$ and $y = \tan x$, a portion of the graph of $y = \tan x$ is shown below.



Graph of $y = \tan x$

4.08 $y = \cot^{-1} x$

Consider the functional equation $x = \cot y$. We know that as y increases from 0 to π, then x decreases monotonically from $+\infty$ to $-\infty$, taking up every real value between $-\infty$ and $+\infty$.

Thus, there is one and only one angle, lying between 0 and π, with a given cotangent. Accordingly we define $\cot^{-1} x$ as follows :

$(\cot^{-1} x)$ *is the angle in the interval* ***(0, π)*** *whose cotangent is x.*

In other words the domain of $\cot^{-1} x$ is $(-\infty, \infty)$ and the range is $(0, \pi)$.

To draw the graph of y = cot⁻¹ x

Table for $\cot^{-1}x$

Using $\cot(\pi - x) = -\cot x$, we have $\cot\frac{5\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6}$, $\cot\frac{3\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right)$ $= -\cot\frac{\pi}{4}$, $\cot\frac{2\pi}{3} = \cot\left(\pi - \frac{\pi}{3}\right) = -\cot\frac{\pi}{3}$.

Hence, we have the following table.

x	$-\sqrt{3}$ $=-1.73$	-1	$\frac{-1}{\sqrt{3}}$ $=-0.58$	0	$\frac{1}{\sqrt{3}}$ $=0.58$	1	$\sqrt{3}$ $=1.73$
$\cot^{-1}x$	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$

We note that

(*i*) y decreases monotonically from π to 0 as x increases from $-\infty$ to ∞.

(*ii*) $\cot^{-1} 0 = \frac{\pi}{2}$.

(*iii*) $\cot^{-1} x$ is defined in the interval $(-\infty, \infty)$.

Plot the points $(-1.73, \frac{5\pi}{6})$, $(-1, \frac{3\pi}{4})$, $(-0.58, \frac{2\pi}{3})$, $(0, \frac{\pi}{2})$, $(0.58, \frac{\pi}{3})$, $(1, \frac{\pi}{4})$, $(1.73, \frac{\pi}{6})$ and join.

Observe that as $x \to \infty$, $\cot^{-1} x \to 0$ and as $x \to -\infty$, $\cot^{-1} x \to \pi$.

We then have the graph as is drawn in Fig. 4.07.

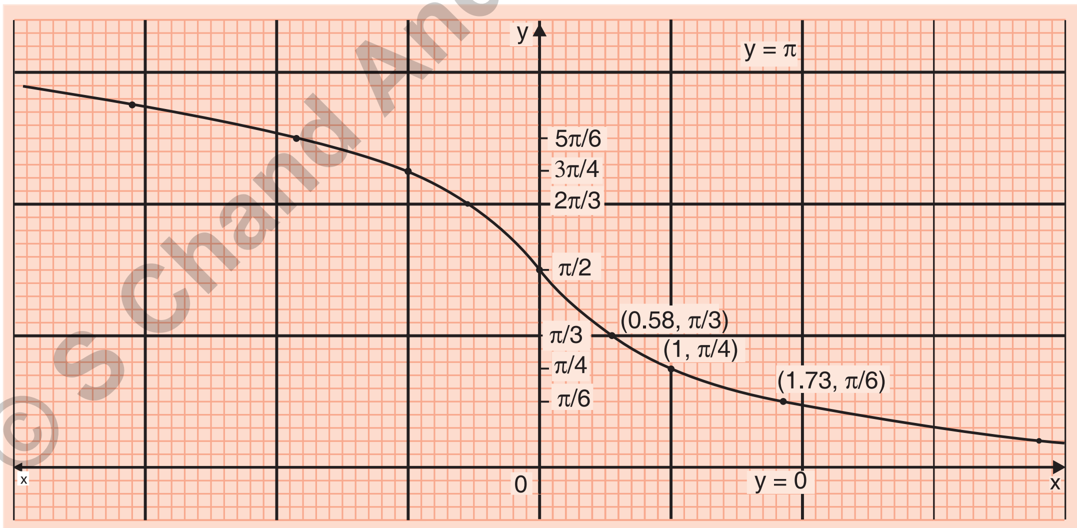
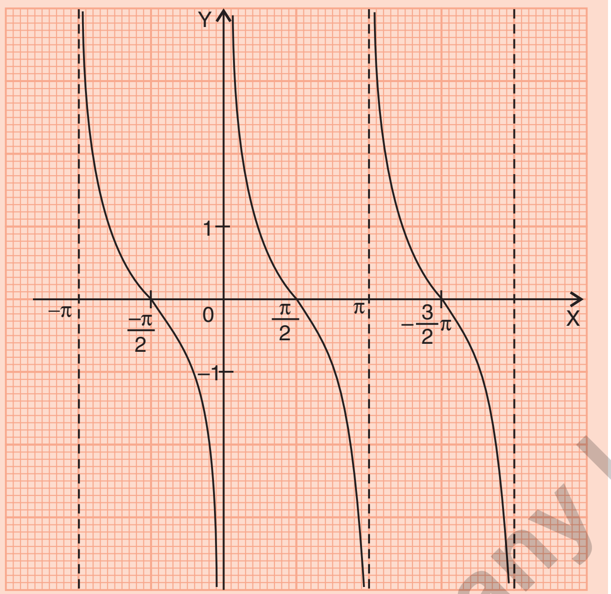


Fig. 4.07. Graph of $y = \cot^{-1}x$

Note. The graph of $y = \cot x$ is shown below.



Graph of $y = \cot x$

4.09. $y = \sec^{-1} x$

Consider the functional equation $x = \sec y$. We know that as y increases from 0 to $\frac{\pi}{2}$, then x increases monotonically from 1 to $+\infty$. Also as y increases from $\frac{\pi}{2}$ to π, then x increases monotonically from $-\infty$ to -1. Thus, there is *one and only one value* of the angle, lying between 0 and π, whose secant is any given number, *not* lying between -1 and 1.

Accordingly we define $\sec^{-1} x$ as follows :

*$\sec^{-1} x$ is the angle, lying **between 0 and π,** whose secant is x, excluding $\frac{\pi}{2}$, but including 0 and π.*

In other words, the domain is $x \le -1$ and $x \ge 1$, *i.e.*, $R - [-1, 1]$ and the range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

To draw the graph of $\sec^{-1}$ x.

Table for $\sec^{-1} x$

x	1	1.15	1.41	2	–2	–1.41	–1.15	–1
$\sec^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

We note that

(*i*) y increases from 0 to $\frac{\pi}{2}$ as x increases from 1 to $+\infty$, and y increases from $\frac{\pi}{2}$ to π as x increases from $-\infty$ to -1.

(*ii*) $\sec^{-1}(-1) = \pi$; $\sec^{-1} 1 = 0$.

(*iii*) $\sec^{-1} x$ is defined for $x \le -1$ and $x \ge 1$.

Plotting the points (1, 0), (1.15, $\frac{\pi}{6}$), ……, (–1, π) as obtained from the table.

We have the graph of the form drawn in Fig. 4.08.

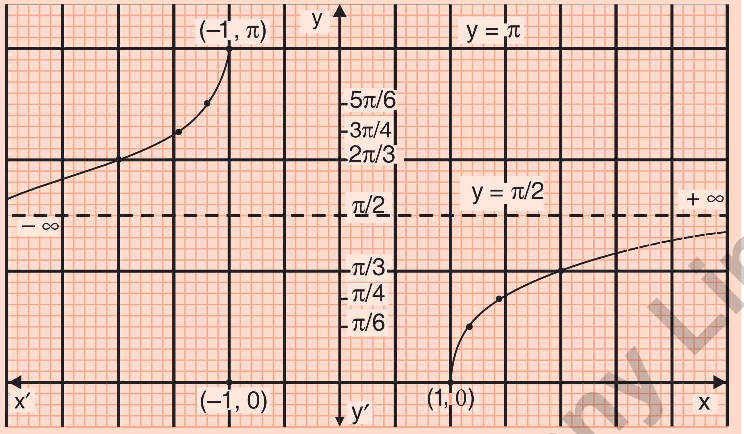
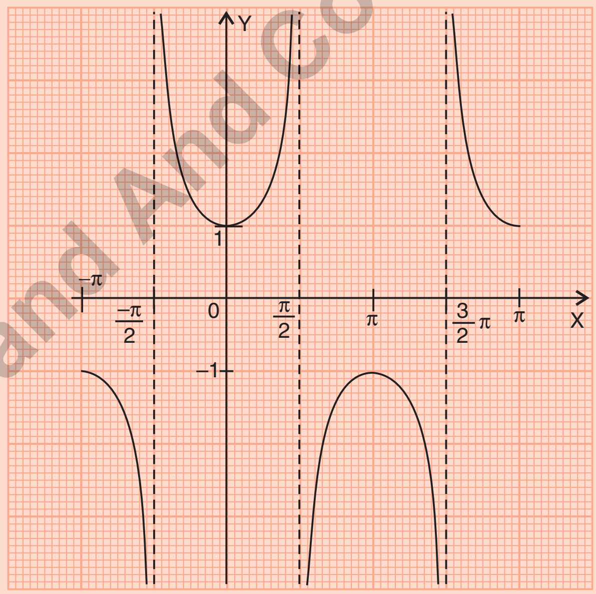


Fig. 4.08. Graph of $\sec^{-1}x$

Note. The graph of $y = \sec x$ is given below to help you to observe the difference between the graph of $y = \sec x$ and $y = \sec^{-1}x$.



Graph of $y = \sec x$

4.10. $y = \text{cosec}^{-1} x$

Consider the functional equation $x = \text{cosec } y$. We know that as y increases from $-\frac{\pi}{2}$ to 0, then x decreases monotonically from –1 to $-\infty$, and as y increases from 0 to $\frac{\pi}{2}$, x decreases monotonically from $+\infty$ to 1.

Thus, there is *one and only one* value of the angle lying between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, whose cosecent is any given number, *not* lying between –1 and 1.

Accordingly we define $\operatorname{cosec}^{-1} x$ as follows :

Cosec^{-1} x is the angle, ***lying between*** $\boldsymbol{\frac{-\pi}{2}}$ ***and*** $\boldsymbol{\frac{\pi}{2}}$, *whose cosecant is x, excluding 0, but including* $\frac{-\pi}{2}$ *and* $\frac{\pi}{2}$.

In other words the domain of $\operatorname{cosec}^{-1} x$ is $x \le -1$ and $x \ge 1$, and the range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

To draw the graph of $y = \operatorname{cosec}^{-1} x$.

Table for $\operatorname{cosec}^{-1} x$

x	2	1.41	1.15	1	–2	–1.41	–1.15	–1
$\operatorname{cosec}^{-1} x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{-\pi}{6}$	$\frac{-\pi}{4}$	$\frac{-\pi}{3}$	$\frac{-\pi}{2}$

We note that

(*i*) y decreases from 0 to $\frac{-\pi}{2}$ as x increases from $-\infty$ to –1, and y decreases from $\frac{\pi}{2}$ to 0 as x increases from 1 to $+\infty$.

(*ii*) $\operatorname{cosec}^{-1}\left(\frac{-\pi}{2}\right) = -1$; $\operatorname{cosec}^{-1}\left(\frac{\pi}{2}\right) = 1$.

(*iii*) $\operatorname{cosec}^{-1} x$ is defined for $x \le -1$ and $x \ge 1$.

Plotting the points $(2, \frac{\pi}{6})$, $(1.41, \frac{\pi}{4})$, ……, $(-1, \frac{-\pi}{2})$

We have the graph as is drawn in Fig. 4.09.

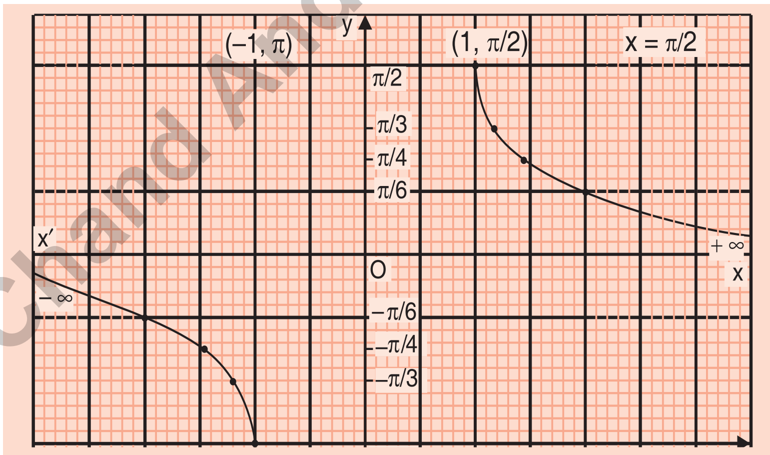
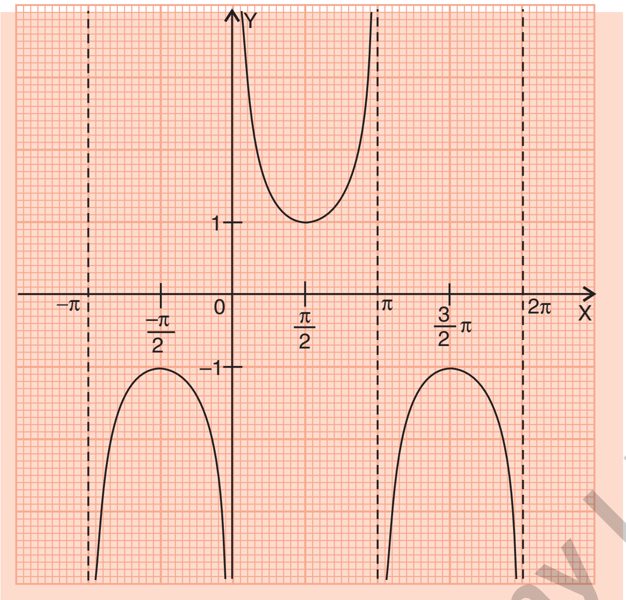


Fig. 4.09. Graph of $\operatorname{cosec}^{-1} x$

Graph of $y = \operatorname{cosec} x$ is given to help you compare the graphs of $y = \operatorname{cosec}^{-1} x$ and $y = \operatorname{cosec} x$.



Graph of $y = \operatorname{cosec} x$

Note 1. The graphs of $y = \sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$ are the reflections respectively of the graphs of $y = \sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ in the line $y = x$.

Note 2. The symbol $\sin^{-1} x$ is also written as "arc sin x" in certain books. Similarly, $\cos^{-1} x$ is written as "arc cos x" while $\tan^{-1} x$ is written as "arc tan x", and so on.

4.11. Table

Function	*Domain*	*Range*
$y = \sin^{-1}x$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, *i.e.,* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$, *i.e.,* $[0, \pi]$
$y = \tan^{-1}x$	All real numbers $(-\infty < x < \infty)$	$\frac{-\pi}{2} < y < \frac{\pi}{2}$, *i.e.,* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	All real numbers $(-\infty < x < \infty)$	$0 < y < \pi$, *i.e.,* $(0, \pi)$
$y = \sec^{-1}x$	$1 \le x < \infty$ and $-\infty < x \le -1$ } *i.e.,* $x \le -1$ or $x \ge -1$	$0 \le y < \frac{\pi}{2}$, $\frac{\pi}{2} < y \le \pi$ } Note that $y \ne \frac{\pi}{2}$, *i.e.,* $[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \operatorname{cosec}^{-1}x$	$-\infty < x \le -1$ and $1 \le x < \infty$ } *i.e.,* $x \le -1$ or $x \ge -1$	$-\frac{\pi}{2} \le y < 0$, $0 < y \le \frac{\pi}{2}$ } Note that $y \ne 0$, *i.e.,* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Remark. The choice of particular ranges for various principal-valued inverse trigonometric functions is arbitrary. The ranges that we have given are the ones customarily chosen and ordinarily used in all branches of mathematics. We have generally chosen ranges involving small values of y, to have small graphs and also to effect a one-to-one correspondence between the domain and the range. There is no universal agreement on the remaining portions of their ranges. For example, we could have the restricted range for Arc cosine function as $-\pi \le y \le 0$ or $-2\pi \le y \le -\pi$.

4.12. Principal values of inverse trigonometric functions

In our discussion in Art. 4.10, we have considered the sine function having the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ with restricted range $-1 \le x \le 1$ and then defined its inverse.

Apart from the domain of sine function taken above other domains are possible with the same range. For example, $\left\{\theta : \frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}\right\}$, $\left\{\theta : \frac{-5\pi}{2} \le \theta \le \frac{-3\pi}{2}\right\}$ etc.

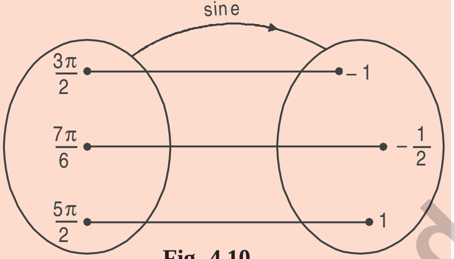


Fig. 4.10

But we consider only the **least numerical value** among all the values of the angle whose sine is x, and call it the ***principal value of*** $\boldsymbol{\sin^{-1}}$.

Thus, ***principal valued function for inverse sine function for real numbers is defined as***

$$\left\{(x, y) : y = \sin^{-1} x, -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}.$$

The smallest numerical value, either positive or negative, of an inverse trigonometric function is called the ***principal value of the function****.*

Thus, the principal values of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ *are the angles that lie between* $\frac{-\pi}{2}$ *and* $\frac{\pi}{2}$ *and the principal values of* $\cos^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ *are the angles that lie between 0 and* π.

Remark. To denote general values we use capital letters. Thus $\sin^{-1}x$ denotes all angles whose sine is x and $\sin^{-1}x$ denotes the principal value, *i.e.*, an acute angle (positive or negative) whose sine is x (depending upon whether x is positive or negative. Thus, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is 45° and that of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is – 60°.

Similarly, $\tan^{-1}x$ denotes all angles whose tangent is x and $\tan^{-1}x$ denotes the principal value, *i.e.*, an acute angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.

Note. Unless otherwise stated, by any inverse value of a t-function is meant its principal value.

4.13. Table of principal values

Function	*Domain*	*Principal Value*	*For x ≥ 0*	*For x < 0*
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$0 \le y \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le y < 0$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$	$0 \le y \le \frac{\pi}{2}$	$\frac{\pi}{2} < y \le \pi$
$y = \tan^{-1}x$	All real numbers	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$0 \le y < \frac{\pi}{2}$	$-\frac{\pi}{2} < y < 0$

$y = \cot^{-1}x$	All real numbers	$0 < y < \pi$	$0 < y \le \frac{\pi}{2}$	$\frac{\pi}{2} < y < \pi$
$y = \sec^{-1}x$	$x \ge 1$ or $x \le -1$	$0 < y \le \pi,\ y \ne \frac{\pi}{2}$	$0 < y < \frac{\pi}{2}$	$\frac{\pi}{2} < y \le \pi$
$y = \operatorname{cosec}^{-1}x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$	$0 < y \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le y < 0$

Ex. 1. Write down the values of

(*i*) $\sin^{-1}\frac{1}{2}$ **(*ii*)** $\cos^{-1}\left(-\frac{1}{2}\right)$ **(*iii*)** $\tan^{-1}(-1)$

(*iv*) $\sec^{-1}2$ **(*v*)** $\cot^{-1}\sqrt{3}$ **(*vi*)** $\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Sol. (*i*) Let $\sin^{-1}\frac{1}{2} = x,\ \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right) \Rightarrow \sin x = \frac{1}{2} = \sin\frac{\pi}{6}\ \therefore\ x = \frac{\pi}{6}.$

(*ii*) Let $x = \cos^{-1}\left(-\frac{1}{2}\right),\ (0 \le x \le \pi) \Rightarrow \cos x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$$\therefore\ x = \frac{2\pi}{3}$$

(*iii*) Let $x = \tan^{-1}(-1),\ \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right) \Rightarrow \tan x = -1 = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)\ \therefore\ x = -\frac{\pi}{4}.$

(*iv*) Let $x = \sec^{-1}2,\ (0 < x \le \pi) \Rightarrow \sec x = 2 = \sec\frac{\pi}{3}\ \therefore\ x = \frac{\pi}{3}.$

(*v*) Let $x = \cot^{-1}\sqrt{3},\ (0 < x < \pi) \Rightarrow \cot x = \sqrt{3} = \cot\frac{\pi}{6}\ \therefore\ x = \frac{\pi}{6}.$

(*vi*) Let $\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right),\ \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right) \Rightarrow \operatorname{cosec} x = \frac{-2}{\sqrt{3}} = -\operatorname{cosec}\frac{\pi}{3} = \operatorname{cosec}\left(-\frac{\pi}{3}\right)$

$$\therefore\ x = -\frac{\pi}{3}.$$

Ex. 2. Find the values of the following:

(*i*) $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

(*ii*) $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Sol. (*i*) $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} + \left(-\frac{\pi}{6}\right)$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{3}{4}\pi$$

(*ii*) $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2\cdot\frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Ex. 3. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ **is equal to**

(i) π **(ii)** $-\frac{\pi}{3}$ **(iii)** $\frac{\pi}{3}$ **(iv)** $\frac{2\pi}{3}$

Sol. $y = \tan^{-1}\sqrt{3}, \left(0 \le y < \frac{\pi}{2}\right) = \frac{\pi}{3}$

Let $x = \sec^{-1}(-2), \left(\frac{\pi}{2} < x \le \pi\right) \Rightarrow \sec x = -2 = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}$

$\therefore \quad \tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{-\pi}{3}$. **Ans. (ii)**

Similar questions for practice :

1. **Find the principal values of**

(i) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\frac{\sqrt{3}}{2}$ (iii) $\tan^{-1}(-\sqrt{3})$

(iv) $\cot^{-1}(-1)$ (v) $\sec^{-1}\frac{2}{\sqrt{3}}$ (vi) $\sec^{-1}(-2)$

(vii) $\sin^{-1}\left(-\frac{1}{2}\right)$ (viii) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (ix) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(x) $\tan^{-1}\left(-\sqrt{3}\right)$ (xi) $\sec^{-1}\left(-\sqrt{2}\right)$ (xii) $\cot^{-1}(-\sqrt{3})$

2. **Evaluate :** $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) + \operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}$

ANSWERS

1. (i) $-\frac{\pi}{4}$ (ii) $\frac{\pi}{6}$ (iii) $-\frac{\pi}{3}$ (iv) $\frac{3\pi}{4}$
(v) $\frac{\pi}{6}$ (vi) $\frac{2\pi}{3}$ (vii) $-\frac{\pi}{6}$ (viii) $\frac{5\pi}{6}$
(ix) $\frac{\pi}{6}$ (x) $-\frac{\pi}{3}$ (xi) $\frac{3\pi}{4}$ (xii) $\frac{5\pi}{6}$

2. 0

4.14. Properties of inverse *t*-functions

Below we state the properties of inverse t-functions without proof.

Theorem 1. Self-adjusting property

(i)	$\sin^{-1}(\sin\theta) = \theta$, **for all** $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\sin(\sin^{-1}x) = x$, **for all** $x \in [-1, 1]$
(ii)	$\cos^{-1}(\cos\theta) = \theta$, **for all** $\theta \in [0, \pi]$	$\cos(\cos^{-1}x) = x$, **for all** $x \in [-1, 1]$
(iii)	$\tan^{-1}(\tan\theta) = \theta$, **for all** $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\tan(\tan^{-1}x) = x$, **for all** $x \in R$
(iv)	$\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, **for all** $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\theta \ne 0$	$\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$, **for all** $x \in [-\infty, -1] \cup [1, \infty]$, *i.e.*, $R - [-1, 1]$

(*v*)	$\sec^{-1}(\sec\theta) = \theta$, for all $\theta \in [0, \pi]$, $\theta \neq \frac{\pi}{2}$	$\sec(\sec^{-1}x) = x$, for all $x\ (-\infty, -1) \cup [1, \infty]$, *i.e.*, $R - [-1, 1]$
(*vi*)	$\cot^{-1}(\cot\theta) = \theta$ for all $\theta \in (0, \pi)$	$\cot(\cot^{-1}x) = x$, for all $x \in R$

Caution. The students should note carefully the restrictions specified against each formula. Thus $\sin^{-1}(\sin\theta) \neq \theta$, if $\theta \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\tan^{-1}(\tan\theta) \neq \theta$, if $\theta \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ etc.

For example, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

In fact,

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) \qquad \because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}$$

$$= \frac{5\pi}{6} \qquad \text{and } \cos(2\pi - \theta) = \cos\theta$$

Ex. 4. Evaluate the following:

(*i*) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ (*ii*) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ (*iii*) $\tan^{-1}\left(\tan\frac{\pi}{10}\right)$

(*iv*) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ (*v*) $\cos^{-1}\left(\cos\frac{9\pi}{8}\right)$ (*vi*) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

(*vii*) $\sin^{-1}\sin\left(\frac{2\pi}{3}\right)$ (*viii*) $\sec^{-1}\left(\sec\frac{7\pi}{6}\right)$

Sol. (*i*) $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$ (*ii*) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ (*iii*) $\tan^{-1}\left(\tan\frac{\pi}{10}\right) = \frac{\pi}{10}$

(*iv*) $\frac{5\pi}{6}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

So we write $\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6}$.

$$\therefore \quad \sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

(*v*) $\frac{9\pi}{8}$ does not lie between 0 and π.

So we write $\cos\frac{9\pi}{8} = \cos\left(2\pi - \frac{7\pi}{8}\right) = \cos\frac{7\pi}{8}$

$$\therefore \quad \cos^{-1}\left(\cos\frac{9\pi}{8}\right) = \cos^{-1}\left(\cos\frac{7\pi}{8}\right) = \frac{7\pi}{8}.$$

(*vi*) $\frac{5\pi}{4}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

So we write $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$ and use $\tan(\pi + \theta) = \tan\theta$

$$\therefore \quad \tan^{-1}\left(\tan\frac{5\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi + \frac{\pi}{4}\right)\right\} = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

(*vii*) $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

So we write $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$ and use $\sin(\pi - \theta) = \sin\theta$

$$\therefore \quad \sin^{-1}\sin\left(\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\} = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}.$$

(*viii*) $\frac{7\pi}{6}$ does not lie between 0 and π.

So we write $\frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}$ and use $\sec(2\pi - \theta) = \sec\theta$

$$\therefore \quad \sec^{-1}\left(\sec\frac{7\pi}{6}\right) = \sec^{-1}\left\{\sec\left(2\pi - \frac{5\pi}{6}\right)\right\} = \sec^{-1}\left(\sec\frac{5\pi}{6}\right) = \frac{5\pi}{6}.$$

Ex. 5. Evaluate the following:

(*i*) $\sin\left(\sin^{-1}\frac{2}{5}\right)$ **(*ii*)** $\tan\left(\tan^{-1}\frac{23}{18}\right)$ **(*iii*)** $\sin\left(\cos^{-1}\frac{3}{5}\right)$

(*iv*) $\sin\left(\sec^{-1}\frac{25}{7}\right)$ **(*v*)** $\sin(\cot^{-1}x)$ **(*vi*)** $\cos(\tan^{-1}x)$

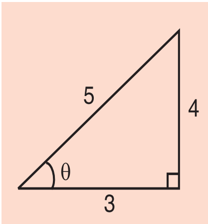
(*vii*) $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Sol. (*i*) $\sin\left(\sin^{-1}\frac{2}{5}\right) = \frac{2}{5}$ (*ii*) $\tan\left(\tan^{-1}\frac{23}{18}\right) = \frac{23}{18}$

(*iii*) Let $\cos^{-1}\frac{3}{5} = \theta \Rightarrow \cos\theta = \frac{3}{5}$

$$\Rightarrow \sin\theta = \frac{4}{5} \Rightarrow \theta = \sin^{-1}\frac{4}{5} \Rightarrow \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5}$$

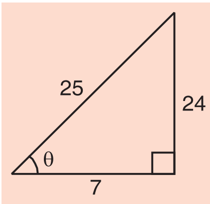
$$\therefore \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}.$$



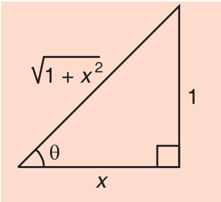
(*iv*) Let $\theta = \sec^{-1}\frac{25}{7} \Rightarrow \sec\theta = \frac{25}{7}$

$$\Rightarrow \sin\theta = \frac{24}{25} \Rightarrow \theta = \sin^{-1}\frac{24}{25}$$

$$\therefore \sin\left(\sec^{-1}\frac{25}{7}\right) = \sin\left(\sin^{-1}\frac{24}{25}\right) = \frac{24}{25}.$$



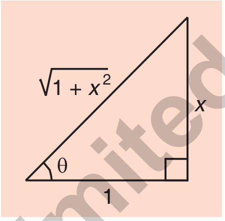
(*v*) Let $\cot^{-1} x = \theta \Rightarrow \cot\theta = x$



$$\Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \sin\left(\cot^{-1}x\right) = \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

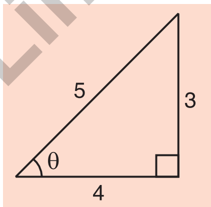
(*vi*) Let $\tan^{-1} x = \theta \Rightarrow \tan\theta = x$



$$\therefore \cos\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \cos\left(\tan^{-1}x\right) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

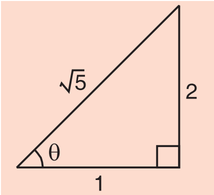
(*vii*) Let $\tan^{-1}\frac{3}{4} = \theta$. Then $\tan\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{4}{5}$

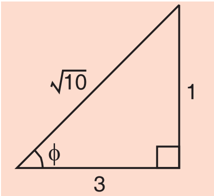


$$\Rightarrow \theta = \cos^{-1}\frac{4}{5} \Rightarrow \tan^{-1}\frac{3}{4} = \cos^{-1}\frac{4}{5}$$

$$\therefore \cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}.$$

Ex. 6. Prove that $\sec^2(\tan^{-1}2) + \mathrm{cosec}^2(\cot^{-1}3) = 15$.





Sol. $\sec^2(\tan^{-1}2) + \mathrm{cosec}^2(\cot^{-1}3)$

$$= \left\{\sec\left(\tan^{-1}2\right)\right\}^2 + \left\{\mathrm{cosec}\left(\cot^{-1}3\right)\right\}^2$$

$$= \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\mathrm{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2 + \left\{\mathrm{cosec}\left(\mathrm{cosec}^{-1}\sqrt{10}\right)\right\}^2$$

$$= \left(\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2 = 15.$$

Theorem 2. Reciprocal property

> **(*i*)** $\mathrm{cosec}^{-1}x = \sin^{-1}\frac{1}{x}, \quad x \in R - (-1, 1)$
>
> **(*ii*)** $\sec^{-1}x = \cos^{-1}\frac{1}{x}, \quad x \in R - (-1, 1)$
>
> **(*iii*)** $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x}, & x > 0 \\ \pi + \tan^{-1}\frac{1}{x}, & x < 0 \end{cases}$

Theorem 3.

> **(*i*)** $\sin^{-1}(-x) = -\sin^{-1}x, \quad x \in [-1, 1]$ **(*ii*)** $\cos^{-1}(-x) = \pi - \cos^{-1}x, \quad x \in [-1, 1]$
>
> **(*iii*)** $\tan^{-1}(-x) = -\tan^{-1}x, \quad x \in R$ **(*iv*)** $\cot^{-1}(-x) = \pi - \cot^{-1}x, \quad x \in R$

(*v*) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \; x \in R - [-1, 1]$

(*vi*) $\text{cosec}^{-1}(-x) = -\,\text{cosec}^{-1}x, \; x \in R - [-1, 1]$

Theorem 4.

(*i*) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$, for all $x \in [-1, 1]$

(*ii*) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$, for all $x \in R$

(*iii*) $\sec^{-1}x + \text{cosec}^{-1}x = \dfrac{\pi}{2}$, for all $x \in R - [-1, 1]$

Theorem 5. (Conversion property)

(*i*) $$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$
$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{cosec}^{-1}\frac{1}{x}$$

(*ii*) $$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{cosec}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$
$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

(*iii*) $$\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$
$$= \sec^{-1}\left(\sqrt{1+x^2}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

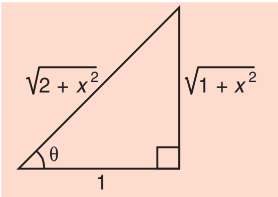
Ex. 7. Prove that

(*i*) $$\sin\left[\cot^{-1}\left\{\cos\left(\tan^{-1}x\right)\right\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$$

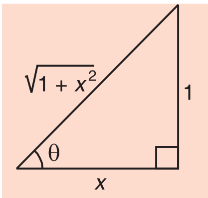
(*ii*) $$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$$

Sol. $$\cos(\tan^{-1}x) = \cos\left\{\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}}$$ (Theorem 5)

$$\Rightarrow \sin\left[\cot^{-1}\left\{\cos(\tan^{-1}x)\right\}\right] = \sin\left[\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$
$$= \sin\left\{\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \sqrt{\frac{1+x^2}{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

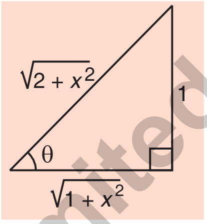


(ii) $\sin(\cot^{-1} x) = \sin\left\{\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right\} = \dfrac{1}{\sqrt{1+x^2}}$



$\therefore \quad \cos\left[\tan^{-1}\{\sin(\cot^{-1} x)\}\right] = \cos\left[\tan^{-1}\dfrac{1}{\sqrt{1+x^2}}\right]$

$= \cos\left\{\cos^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \sqrt{\dfrac{1+x^2}{2+x^2}} = \sqrt{\dfrac{x^2+1}{x^2+2}}$



Theorem 6.

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ **if** $xy < 1$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$ **if** $xy > -1$

Explanation of the restriction. Since $x > 0$, $y > 0$ and $xy < 1$, therefore $\tan(\alpha+\beta)$ is +ve $\Rightarrow \alpha+\beta$ lies in Quadrant I or Quadrant III.

In case, $\alpha+\beta$ lies in Quad. I, $\alpha+\beta < \dfrac{\pi}{2} \Rightarrow \alpha < \dfrac{\pi}{2} - \beta$

$\Rightarrow \tan\alpha < \tan\left(\dfrac{\pi}{2} - \beta\right) \Rightarrow \tan\alpha < \cot\beta \Rightarrow x < \dfrac{1}{y} \Rightarrow xy < 1.$ (Since $y > 0$)

Note. The same result holds good if $x < 0, y < 0, xy < 1$.

Remark. (i) If $x > 0, y > 0$ and $xy > 1$, then $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\dfrac{x+y}{1-xy}$

(ii) If $x < 0, y < 0$ and $xy > 1$, then $\tan^{-1}x + \tan^{-1}y = -\pi + \tan^{-1}\dfrac{x+y}{1-xy}$

The above two cases are not included in this course.

Remark. (i) If $x > 0, y < 0$ and $xy < -1$, then $\tan^{-1}x - \tan^{-1}y = \pi + \tan^{-1}\dfrac{x-y}{1+xy}$

(ii) If $x < 0, y > 0$ and $xy < -1$, then $\tan^{-1}x - \tan^{-1}y = -\pi + \tan^{-1}\dfrac{x-y}{1+xy}$

The above two cases are not included in this course.

Theorem 7.

If $x^2 < 1$,

$2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2} = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2}.$

Theorem 8.

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$, (ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

(iii) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$ (iv) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(v) $3\tan^{-1}x = \tan^{-1}\dfrac{3x - x^3}{1 - 3x^2}$

Theorem 9.

(*i*) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$, if $x, y \geq 0$ and $x^2 + y^2 \leq 1$.

(*ii*) $\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$, if $x, y \geq 0$ and $x^2 + y^2 > 1$.

(*iii*) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{1-x^2}\cdot\sqrt{1-y^2})$, if $x, y > 0$ and $x^2 + y^2 \leq 1$.

(*iv*) $\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}(xy \mp \sqrt{1-x^2}\cdot\sqrt{1-y^2})$, if $x, y > 0$ and $x^2 + y^2 > 1$.

SOLVED EXAMPLES

Ex. 8. Evaluate:

(*i*) $\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)$ (*ii*) $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$

Sol. (*i*) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then $\cos\theta = \frac{\sqrt{5}}{3}$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}$$

$$= \sqrt{\frac{(3-\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}} = \frac{3-\sqrt{5}}{\sqrt{9-5}} = \frac{1}{2}(3-\sqrt{5}).$$

(*ii*) We know, by theorem 7, that

$$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} \quad \text{if } |x| < 1$$

$$\therefore \quad 2\tan^{-1}\frac{1}{5} = \tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) \qquad \because\ x = \frac{1}{5} \text{ and } \left|\frac{1}{5}\right| < 1.$$

$$= \tan^{-1}\frac{5}{12}$$

Now $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left[\tan^{-1}\frac{5}{12} - \tan^{-1}1\right]$

We know, by theorem 6, that

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \quad \text{if } xy > -1$$

Here $x = \frac{5}{12}$, $y = 1$, therefore $xy = \frac{5}{12} > -1$,

$$\therefore \quad \tan^{-1}\frac{5}{12} - \tan^{-1}1 = \tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}\times 1}\right) = \tan^{-1}\left(\frac{-7}{17}\right)$$

$$\therefore \quad \tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left[\tan^{-1}\left(\frac{-7}{17}\right)\right] = \frac{-7}{17}.$$

Ex. 9. The value of tan $\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$ **is**

(*i*) $\frac{6}{17}$ (*ii*) $\frac{17}{6}$ (*iii*) **None of these.** (*IIT*)

Sol. If cos α = *x*, then $\tan\alpha = \frac{\sqrt{1-x^2}}{x}$

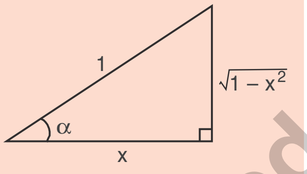


Fig. 4.11

$$\therefore \quad \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{4/5} = \tan^{-1}\frac{3}{4}$$

$$\therefore \quad \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3} \qquad \left[\text{Here } xy = \frac{3}{4}\times\frac{2}{3} = \frac{1}{2} < 1, \text{ Theorem 6}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right] = \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}.$$

Hence, (*ii*) is the answer.

Ex. 10. Evaluate : cos $\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right)$**.**

Sol. Let $\theta = \sin^{-1}\frac{3}{5}$ and $\phi = \sin^{-1}\frac{5}{13}$. Then

$$\sin\theta = \frac{3}{5} \text{ and } \sin\phi = \frac{5}{13} \text{ (Fig. 4.12)}$$

Fig. 4.12

$$\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right) = \cos(\theta+\phi) = \cos\theta.\cos\phi - \sin\theta\sin\phi$$

$$= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

Ex. 11. Evaluate : tan (2 cot^{-1} *x*).

Sol. Let $\cot^{-1} x = z$. Then $x = \cot z \Rightarrow \tan z = \frac{1}{x}$

Now $$\tan(2\cot^{-1}x) = \tan 2z = \frac{2\tan z}{1-\tan^2 z} = \frac{\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{2x}{x^2-1}.$$

Ex. 12. Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$**.**

Sol. Here $xy = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6} < 1$, therefore we can use

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\therefore \quad \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1}\frac{5/6}{5/6} = \tan^{-1}1 = \frac{\pi}{4}.$$

Ex. 13. Prove that $\tan^{-1} x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1)$. **(*ISC*)**

Sol. $\tan^{-1} x + \cot^{-1}(x+1) = \tan^{-1} x + \tan^{-1}\frac{1}{x+1}$

Since $x \times \frac{1}{x+1} = \frac{x}{x+1} < 1$, therefore we have

$$\tan^{-1} x + \tan^{-1}\frac{1}{x+1} = \tan^{-1}\left(\frac{x+\frac{1}{x+1}}{1-x\,.\,\frac{1}{x+1}}\right) = \tan^{-1}(x^2+x+1).$$

Ex. 14. Show that $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$.

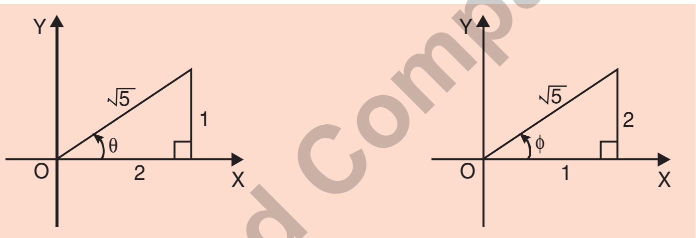


Fig. 4.13

Sol. Let $\theta = \sin^{-1}\frac{1}{\sqrt{5}}$ and $\phi = \sin^{-1}\frac{2}{\sqrt{5}}$.

Then $\sin\theta = \frac{1}{\sqrt{5}}$ and $\sin\phi = \frac{2}{\sqrt{5}}$, each angle is in the first quadrant.

We have to show that $\theta + \phi = \frac{\pi}{2}$, or $\sin(\theta+\phi) = \sin\frac{\pi}{2} = 1$.

Now $\sin(\theta+\phi) = \sin\theta\,.\cos\phi + \cos\theta\sin\phi = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\frac{2}{\sqrt{5}} = 1.$

Method II. $\sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{2} - \sin^{-1}\frac{2}{\sqrt{5}}$

$$\Rightarrow \quad \sin\left[\sin^{-1}\frac{1}{\sqrt{5}}\right] = \sin\left[\frac{\pi}{2} - \sin^{-1}\frac{2}{\sqrt{5}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{5}} = \cos\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) = \cos\left(\cos^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}.$$

Hence proved

Method III. We know that

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \text{ if } -1 \le x, y \le 1,\ x^2+y^2 \le 1.$$

Here $x = \frac{1}{\sqrt{5}}$, $y = \frac{2}{\sqrt{5}}$ and $-1 < \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} < 1$ and $x^2 + y^2 = \frac{1}{5} + \frac{4}{5} = 1$ (Theorem 9)

$$\therefore \quad \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}} + \frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}}\right] = \sin^{-1}\left(\frac{1}{5}+\frac{4}{5}\right)$$

$$= \sin^{-1} 1 = \frac{\pi}{2}.$$

Hence proved.

Ex. 15. Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Sol. $$\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} = \frac{\pi}{2} - \sin^{-1}\frac{16}{65}$$

$$\Rightarrow \quad \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} = \sin^{-1} 1 - \sin^{-1}\frac{16}{65} = \sin^{-1} 1 + \sin^{-1}\left(\frac{-16}{65}\right)$$

We know that

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \text{ if } -1 \le x, y \le 1, x^2 + y^2 \le 1.$$

On the left side, $-1 \le \frac{4}{5}, \frac{5}{13} < 1$, and or if $xy < 0, x^2 + y^2 > 1.$

$$x^2 + y^2 = \frac{16}{25} + \frac{25}{169} = \frac{2704+625}{25\times169} = \frac{3329}{4225} < 1.$$

On the right side, $$xy = \left(1 \times \frac{-16}{65}\right) = \frac{-16}{65} < 0, \text{ and}$$

$$x^2 + y^2 = 1 + \left(\frac{-16}{65}\right)^2 = 1 + \frac{256}{4225} = \frac{4481}{4225} > 1$$

Therefore, we can apply the result to both sides

$$\therefore \quad \text{L.H.S.} = \sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right]$$

$$= \sin^{-1}\left[\left(\frac{4}{5}\times\frac{12}{13}\right) + \left(\frac{5}{13}\times\frac{3}{5}\right)\right] = \sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)$$

$$= \sin^{-1}\frac{63}{65}.$$

$$\text{R.H.S.} = \sin^{-1}\left[1.\sqrt{1-\frac{256}{4225}} + \left(\frac{-16}{65}\right)\sqrt{1-1}\right] = \sin^{-1}\sqrt{\frac{3969}{4225}}$$

$$= \sin^{-1}\frac{63}{65}.$$

Hence proved.

Method II. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

$$= \sin^{-1}\left[\frac{4}{5}.\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65} \quad \text{(As done above)}$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\times\frac{63}{65}+\frac{16}{65}\times\frac{16}{65}\right]=\sin^{-1}\left[\frac{3969}{4225}+\frac{256}{4225}\right]$$

$$= \sin^{-1}\left(\frac{4225}{4225}\right)=\sin^{-1}1=\frac{\pi}{2}.$$

Hence proved.

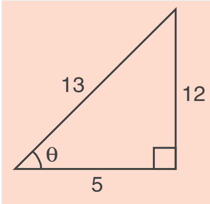
Method III. The given expression can be put as

$$\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{5}{12}=\frac{\pi}{2}-\tan^{-1}\frac{16}{63}\Rightarrow\tan^{-1}\left(\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}\right)=\frac{\pi}{2}-\tan^{-1}\frac{16}{63}$$

$$\Rightarrow \qquad \tan^{-1}\frac{63}{16} = \frac{\pi}{2}-\tan^{-1}\frac{16}{63}.$$

Now take tangent of both sides and complete it.

Ex. 16. Prove that $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.



$\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{5}{13}$

Sol. $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$

$$= \sin^{-1}\left\{\frac{5}{13}\cdot\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\cdot\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}$$

$$=\sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right\} = \sin^{-1}\frac{56}{65}.$$

Ex. 17. Prove that $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$.

Sol. L.H.S. $= 2\left(2\tan^{-1}\frac{1}{5}\right)-\left(\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right)$

We know that $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ if $|x|<1$. (Theorem 7)

Since $\frac{1}{5}<1$, therefore

$$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}=\tan^{-1}\frac{5}{12}$$

Thus, $$4\tan^{-1}\frac{1}{5} = 2\left(2\tan^{-1}\frac{1}{5}\right)=2\tan^{-1}\frac{5}{12}$$

$$= \tan^{-1}\frac{2\times\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}=\tan^{-1}\frac{120}{119}.$$

We also know that $\tan^{-1}x-\tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, if $xy>-1$. (Theorem 6)

$\therefore \quad \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99} = \tan^{-1}\left(\dfrac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\cdot\frac{1}{99}}\right) = \tan^{-1}\frac{1}{239}$ $\left[\text{Here } \frac{1}{70}\times\frac{1}{99} > -1\right]$

$\therefore \quad \text{L.H.S.} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239}$ $\left[\text{Here } \frac{120}{119}\times\frac{1}{239} > -1\right]$

$$= \tan^{-1}\left(\dfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right) = \tan^{-1}\frac{28561}{28561} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

Ex. 18. Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$.

Sol. $\text{L.H.S.} = \tan^{-1}\left(\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\cdot\frac{2}{9}}\right) = \tan^{-1}\frac{17}{34} = \tan^{-1}\frac{1}{2}$ $\left[\text{Here } \frac{1}{4}\times\frac{2}{9} = \frac{1}{18} < 1, \text{ Theorem 6}\right]$

Let $\cos^{-1}\frac{3}{5} = \theta$, so that $\cos\theta = \frac{3}{5}$.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}} = \frac{1}{2}$$

$\Rightarrow \quad \frac{\theta}{2} = \tan^{-1}\frac{1}{2} \Rightarrow \theta = 2\tan^{-1}\frac{1}{2}$, or $\cos^{-1}\frac{3}{5} = 2\tan^{-1}\frac{1}{2}$

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\times 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2}.$$

Hence L.H.S. = R.H.S.

Note. It also follows that

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \cos^{-1}\frac{2}{\sqrt{5}}, \text{ since } \tan^{-1}\frac{1}{2} = \cos^{-1}\frac{2}{\sqrt{5}}.$$

Method II. $\tan^{-1}\frac{1}{2} = \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right)$

$$= \frac{1}{2}\cos^{-1}\left(\dfrac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) = \frac{1}{2}\cos^{-1}\frac{3}{5}$$ $\left[2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, \text{ Theorem 7}\right]$

$$= \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

Ex. 19. Show that $\frac{1}{2}\tan^{-1}x = \cos^{-1}\left(\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$.

Sol. Let $\tan^{-1}x = \theta$, so that $x = \tan\theta$.

$$\text{R.H.S.} = \cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+\tan^2\theta}}{2\sqrt{1+\tan^2\theta}}}\right) = \cos^{-1}\left(\sqrt{\frac{1+\sec\theta}{2\sec\theta}}\right)$$

$$= \cos^{-1}\left(\sqrt{\frac{1+\cos\theta}{2}}\right) = \cos^{-1}\left(\sqrt{\frac{1+\left(2\cos^2\frac{\theta}{2}-1\right)}{2}}\right)$$

$$= \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x = \text{L.H.S.}$$

Hence proved.

Ex. 20. Prove that

(*i*) $\tan^{-1}\left[\frac{x}{\sqrt{a^2-x^2}}\right] = \sin^{-1}\frac{x}{a}$ **(*ii*)** $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$

Sol. (*i*) Let $\theta = \sin^{-1}\frac{x}{a}$. Then $\sin\theta = \frac{x}{a} \Rightarrow \tan\theta = \frac{x}{\sqrt{a^2-x^2}}$

$$\therefore \quad \tan^{-1}\left[\frac{x}{\sqrt{a^2-x^2}}\right] = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}.$$

(*ii*) Let $x^2 = \cos 2\theta$. Then $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}$

$$\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-(1-2\sin^2\theta)}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-(1-2\sin^2\theta)}} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \frac{1+\tan\theta}{1-\tan\theta} = \frac{1+\tan\theta}{1-1.\tan\theta}$$

$$= \frac{\tan\pi/4+\tan\theta}{1-\tan\pi/4.\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)$$

$$\therefore \quad \text{L.H.S.} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 = \text{R.H.S.}$$

Ex. 21. If $a > b > c > 0$, prove that

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right) = \pi$$

Sol. We know that if $x < 0$, $\cot^{-1}x = \pi+\tan^{-1}\frac{1}{x}$ (Theorem 2)

Here since $c < a$, therefore, in the last term $c - a < 0$.

$$\therefore \quad \cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$= \tan^{-1}\frac{a-b}{ab+1}+\tan^{-1}\frac{b-c}{bc+1}+\pi+\tan^{-1}\frac{c-a}{ca+1}$$

$= (\tan^{-1}a - \tan^{-1}b) + (\tan^{-1}b - \tan^{-1}c) + \pi + (\tan^{-1}c - \tan^{-1}a) = \pi$.

[Since a, b, c are all > 0, therefore, $ab > -1$, $bc > -1$, $ca > -1$].

Ex. 22. Solve : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$.

Sol. $$\tan^{-1}\frac{x-1}{x-2} = \frac{\pi}{4}-\tan^{-1}\frac{x+1}{x+2} = \tan^{-1}1-\tan^{-1}\frac{x+1}{x+2}$$

$$= \tan^{-1}\left(\frac{1-\dfrac{x+1}{x+2}}{1+\dfrac{x+1}{x+2}}\right) = \tan^{-1}\frac{1}{2x+3}.$$

The application of the formula is justified provided

$$1\times\frac{x+1}{x+2} > -1 \qquad \text{(Theorem 6)}$$

i.e., if $x+1 > -x-2$, or $2x > -3$, or $x > -1.5$

Now $\tan^{-1}\dfrac{x-1}{x-2} = \tan^{-1}\dfrac{1}{2x+3}$

$\Rightarrow \dfrac{x-1}{x-2} = \dfrac{1}{2x+3} \Rightarrow (x-1)(2x+3) = x-2 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$

Here, $\pm\dfrac{1}{\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2} = \pm\dfrac{1.4142}{2} = \pm .707$

Since both 0.707 and –0.707 are greater than –1, therefore both are admissible.

Hence $x = \pm\dfrac{1}{\sqrt{2}}$ is the answer.

Ex. 23. Solve the equation $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\dfrac{8}{31}$.

Sol. L.H.S. $= \tan^{-1}\left[\dfrac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right]$

$= \tan^{-1}\left(\dfrac{2x}{2-x^2}\right)$ Here $(x+1)(x-1) < 1$ (Theorem 6)

Now $\tan^{-1}\dfrac{2x}{2-x^2} = \tan^{-1}\dfrac{8}{31}$ $\Rightarrow x^2 < 2$

$\Rightarrow \dfrac{2x}{2-x^2} = \dfrac{8}{31} \Rightarrow 4x^2 + 31x - 8 = 0$ $\Rightarrow |x| < \sqrt{2}$, *i.e.*, $\sqrt{2} < x < \sqrt{2}$

$\Rightarrow (4x-1)(x+8) = 0$

$\Rightarrow x = \dfrac{1}{4}$, or –8.

Since $\left(\dfrac{1}{4}\right)^2 < 2$ and $(-8)^2 \not< 2$, therefore, $x = \dfrac{1}{4}$ is the answer.

Ex. 24. If $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$, prove that $a+b+c = abc$.

Sol. $\tan^{-1}a + \tan^{-1}b = \tan^{-1}\dfrac{a+b}{1-ab}$, provided $ab < 1$. ...(*i*)

Again, $\tan^{-1}\dfrac{a+b}{1-ab} + \tan^{-1}c = \tan^{-1}\left[\dfrac{\dfrac{a+b}{1-ab}+c}{1-\left(\dfrac{a+b}{1-ab}\right).c}\right]$, provided $\frac{a+b}{1-ab}\times c<1$, *i.e.*, $ab+bc+ca<1$. ...(*ii*)

$$\Rightarrow \quad \tan^{-1}\left[\frac{a+b+c-abc}{1-ab-ac-bc}\right] = \pi$$

$$\Rightarrow \quad \frac{a+b+c-abc}{1-ab-ac-bc} = \tan\pi = 0$$

$$\Rightarrow \quad a+b+c-abc = 0 \Rightarrow a+b+c=abc.$$

Hence proved.

Ex. 25. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$.

Sol. $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$

We know that $\cos^{-1}x+\cos^{-1}y = \cos^{-1}\left[xy-\sqrt{1-x^2}\,.\sqrt{1-y^2}, \text{if } -1\le x, y<0, x+y\ge 0\right]$

$$\therefore \quad \cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b} = \theta \qquad \text{(Theorem 9)}$$

$$\Rightarrow \quad \cos^{-1}\left[\frac{x}{a}\times\frac{x}{b}-\sqrt{1-\frac{x^2}{a^2}}\,.\sqrt{1-\frac{y^2}{b^2}}\right]=\theta, \text{ where } -1\le\frac{x}{a},\frac{y}{b}<0, \frac{x}{a}+\frac{y}{b}\ge 0$$

or $$-a \le x, y<0, \frac{x}{a}+\frac{y}{b}\ge 0$$

$$\Rightarrow \quad \frac{xy}{ab}-\sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)} = \cos\theta$$

$$\Rightarrow \quad \frac{xy}{ab}-\cos\theta = \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2}-2\frac{xy}{ab}.\cos\theta+\cos^2\theta = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2} = 1-\cos^2\theta$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2} = \sin^2\theta.$$

Ex. 26. If $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$, then find the general value of θ.

Sol. Let $\tan\theta=t$ so that $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}=\frac{2t}{1+t^2}$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-t^2}{1+t^2}$$

Then $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{1}{2}\sin^{-1}\left(\frac{3\times 2t}{5(1+t^2)+4(1-t^2)}\right)$

$$= \frac{1}{2}\sin^{-1}\frac{6t}{9+t^2} = \frac{1}{2}\sin^{-1}\left(\frac{2\times\frac{t}{3}}{1+\left(\frac{t}{3}\right)^2}\right)$$

$$= \frac{1}{2}\times 2\tan^{-1}\frac{t}{3} \qquad \left[\because \sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x\right]$$

$$= \tan^{-1}\frac{t}{3}$$

The given equation becomes

$$\theta = \tan^{-1}(2t^2) - \tan^{-1}\left(\frac{t}{3}\right)$$

$$= \tan^{-1}\frac{2t^2 - \frac{t}{3}}{1+2t^2\times\frac{t}{3}}, \text{ provided } 2t^2\times\frac{t}{3} > -1, \textit{ i.e., } t^3 > -1.5$$

(Theorem 6)

$$\Rightarrow \quad \tan\theta = \frac{t(6t-1)}{3+2t^3} \Rightarrow t = \frac{t(6t-1)}{3+2t^3}$$

$$\Rightarrow \quad t(3+2t^3) = t(6t-1) \Rightarrow t(2t^3-6t+4) = 0$$

Solving, by Remainder theorem, we get

$$t = 0, 1, -2$$

Now only $t = 0$ and $t = 1$ satisfy the condition $t^3 > -1.5$.

Therefore, $t = -2$ is rejected.

$$\therefore \quad \tan\theta = 0, 1 \Rightarrow \theta = 0, \frac{\pi}{4}.$$

Hence, the general value of θ is $n\pi$, or $n\pi + \frac{\pi}{4}$.

Ex. 27. If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\frac{\pi}{4}-\frac{u}{2}\right)$ is equal to

(*a*) $\sqrt{\tan\alpha}$ (*b*) $\sqrt{\cot\alpha}$ (*c*) $\tan^2 u$ (*d*) $\cot\alpha$.

Sol. We know that $\tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}$, or

$$\cot^{-1}\theta = \frac{\pi}{2} - \tan^{-1}\theta$$

$$\therefore \quad u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\sqrt{\tan\alpha}\right) - \tan^{-1}\sqrt{\tan\alpha} = \frac{\pi}{2} - 2\tan^{-1}\sqrt{\tan\alpha}$$

$$\Rightarrow \quad 2\tan^{-1}\sqrt{\tan\alpha} = \frac{\pi}{2} - u$$

$$\Rightarrow \quad \tan^{-1}\sqrt{\tan\alpha} = \frac{\pi}{4} - \frac{u}{2}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4} - \frac{u}{2}\right) = \sqrt{\tan\alpha}$$

Hence, (*a*) is the answer.

EXERCISE 4

1. Write down the values of :

(*i*) $\sin^{-1}\frac{\sqrt{3}}{2}$ (*ii*) $\cos^{-1}\left(\frac{1}{2}\right)$ (*iii*) $\tan^{-1} 1$ (*iv*) $\cos^{-1} 0$

(*v*) $\cot^{-1}\frac{1}{\sqrt{3}}$ (*vi*) $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ (*vii*) $\text{cosec}^{-1} 2$ (*viii*) $\cos^{-1}\left(-\frac{1}{2}\right)$.

2. Find :

(*a*) $\cos A$ if $\cos^{-1}\frac{1}{2} = A$ (*b*) $\text{cosec } A$ if $\sin^{-1}\frac{1}{3} = A$ (*c*) $\sin A$ if $\tan^{-1}\left(\frac{3}{4}\right) = A$

(*d*) θ if $\tan^{-1}\sqrt{3} = \theta$ (*e*) $\cot\theta$ if $\tan^{-1}\frac{1}{5} = \theta$ (*f*) x if $\sin^{-1}\left(\frac{1}{2}\right) = \tan^{-1} x$.

3. Find the principal value of each of the following :

(*a*) $\sin\left(\sin^{-1}\frac{1}{2}\right)$ (*b*) $\tan^{-1}\tan\left(\frac{\pi}{6}\right)$ (*c*) $\cot\left(\tan^{-1}\frac{4}{5}\right)$

(*d*) $\sin^{-1}\left(\cos\frac{\pi}{4}\right)$ (*e*) $\sin\left(\cos\frac{1}{2}\right)$ (*f*) $\cos\left(\cot^{-1}(-\sqrt{3})\right)$

(*g*) $\sin\left(2\sin^{-1}\frac{2}{3}\right)$ (*h*) $\cos^{-1}(\sin 220^\circ)$ (*i*) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

(*j*) $\tan[\sin^{-1}(-1)]$ (*k*) $\tan^{-1}\left(\cot\frac{4\pi}{3}\right)$ (*l*) $\sin(\tan^{-1} 1) + \cos\left(\cos^{-1}\frac{1}{2}\right)$

(*m*) $\tan\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) - \cot\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$ (*n*) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

(*o*) $\text{cosec}^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$ (*p*) $\cos^{-1}[\sin(\tan^{-1}(-1))]$ (*q*) $\sin(2\tan^{-1} 3)$

4. Verify the following :

(*a*) $\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}\frac{1}{2} = \frac{\pi}{12}$ (*b*) $\cos^{-1} 0 + \tan^{-1}(-1) = \tan^{-1} 1$.

5. Show that :

(*i*) $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

[Hint. Let $\sin^{-1} x = \theta$, so that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$

Now use $\sin 2\theta = 2\sin\theta\cos\theta$, etc]

(*ii*) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$ if $0 \le x \le 1$; (*iii*) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ if $-\frac{1}{2} \le x \le \frac{1}{2}$;

(*iv*) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ if $\frac{1}{2} \le x \le 1$; (*v*) $\sin^{-1}(-x) = -\sin^{-1} x$;

(*vi*) $\cos^{-1}(-x) = \pi - \cos^{-1} x$; (*vii*) $\tan^{-1}(-x) = -\tan^{-1} x$.

6. Show that :

(*a*) $\sin\left(\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{4}{5}\right)=\frac{63}{65}$

(*b*) $\sin(\tan^{-1}\sqrt{3}+\cot^{-1}\sqrt{3})=1$

(*c*) $\tan\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{3}$

(*d*) $\cos\left(\tan^{-1}\frac{15}{8}-\sin^{-1}\frac{7}{25}\right)=\frac{297}{425}$

(*e*) $\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{3}{5}\right)=\frac{3+4\sqrt{3}}{10}$

(*f*) $2\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{4}{3}$.

7. Simplify :

(*a*) $\sin(2\cos^{-1}x)$ (*b*) $\cos(2\sin^{-1}x)$ (*c*) $\tan(\sin^{-1}y)$ (*d*) $\cos\left(\frac{1}{2}\cos^{-1}x\right)$.

8. Solve the following for *x* in terms of *y* :

(*a*) $y=2\sin^{-1}3x$ (*b*) $y=3\cos^{-1}2x$ (*c*) $y=\frac{1}{2}\tan^{-1}(x+\pi)$.

9. Prove that :

(*i*) $\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$

(*ii*) $\tan^{-1}2-\tan^{-1}1=\tan^{-1}\frac{1}{3}$

(*iii*) $2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}$

(*iv*) $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

(*v*) $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{56}{65}$

(*vi*) $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

(*vii*) $\cos^{-1}\frac{4}{5}+\cot^{-1}\frac{5}{3}=\tan^{-1}\frac{27}{11}$ [**Hint.** $\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$]

(*viii*) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\cos^{-1}\frac{2}{\sqrt{5}}$.

10. Prove that

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$$ (*ISC*)

11. Prove that

$$2\left(\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right)=\pi.$$ (*ISC*)

[**Hint.** L.H.S. $=2\left[\frac{\pi}{4}+\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right]$ $\left(\text{Here } \frac{1}{2}\times\frac{1}{3}=\frac{1}{6}<1\right)$

$=2\left[\frac{\pi}{4}+\tan^{-1}1\right]$, etc.]

12. Prove that $\sin^{-1}\frac{\sqrt{3}}{2}+2\tan^{-1}\frac{1}{\sqrt{3}}=\frac{2\pi}{3}$. (*ISC*)

Prove the following :

13. $4(\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5})=\pi$.

14. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

15. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0$.

16. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, show that $xy+yz+zx=1$.

[**Hint.** Similar to solved Ex. 19.]

Solve for x the following:

17. $\cos^{-1} x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$. ***(Dhanbad)***

18. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$.

[Sol. $\sin^{-1} 2x = \frac{\pi}{3} - \sin^{-1} x$

$\Rightarrow \quad 2x = \sin\left(\frac{\pi}{3} - \sin^{-1} x\right) = \sin\frac{\pi}{3} . \cos(\sin^{-1} x) - \cos\frac{\pi}{3} \sin(\sin^{-1} x)$

$= \frac{\sqrt{3}}{2} . \cos(\cos^{-1}\sqrt{1-x^2}) - \frac{1}{2} x = \frac{\sqrt{3}}{2} . \sqrt{1-x^2} - \frac{x}{2}$

$\Rightarrow \quad 4x = \sqrt{3}\sqrt{1-x^2} - x \Rightarrow 5x = \sqrt{3}\sqrt{1-x^2}$

$\Rightarrow 25x^2 = 3 - 3x^2 \Rightarrow 28x^2 = 3 \Rightarrow x^2 = \frac{3}{28} \Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{7}} \Rightarrow x = \frac{\sqrt{3}}{2\sqrt{7}}$.

$-\frac{\sqrt{3}}{2\sqrt{7}}$ is rejected as it makes the L.H.S. negative whereas the R.H.S. is positive]

19. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$. ***(ISC)***

[Sol. $\tan^{-1} 2x + \tan^{-1} 3x = \tan^{-1}\left[\frac{2x+3x}{1-(2x)(3x)}\right]$, provided $2x \times 3x < 1 \Rightarrow x^2 < \frac{1}{6}$.

Now $\tan^{-1}\frac{5x}{1-6x^2} = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1 \Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow x = -1, \frac{1}{6}$

$x = -1$ is rejected as $(-1)^2 \not< \frac{1}{6}$; $x = \frac{1}{6}$ is accepted as $\left(\frac{1}{6}\right)^2 < \frac{1}{6}$. Hence, $x = \frac{1}{6}$ is the answer].

20. $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$.

21. $\sin\left(\frac{1}{5}\cos^{-1} x\right) = 1$.

[Hint. Let $\frac{1}{5}\cos^{-1} x = \alpha$. Since $0 \le \cos^{-1} x \le \pi$, we have

$0 \le 5\alpha \le \pi \Rightarrow 0 \le \alpha \le \frac{\pi}{5}$.

Hence $\sin\alpha \ne 1$. Therefore the given equation has no solution].

22. $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$. ***(ISC 2008)***

[Hint. Transpose $\tan^{-1} x$ to the R.H.S., and then apply equations of theorem 6]

23. $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$.

[Hint. See solved Ex. 22.]

24. $\tan^{-1}\frac{1}{2x+1} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}$.

25. **Evaluate the following :**

(*i*) $\sin\cot^{-1}\cos\tan^{-1} x$;

(*ii*) $\tan\left[\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right]$;

(*iii*) $\cos^{-1} x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right], \frac{1}{2} \le x \le 1$.

[Hint. (*i*) Let $\tan^{-1} x = \theta$. Then, $\cos\tan^{-1} x = \cos\theta = \frac{1}{\sqrt{1+x^2}} = z$, say.

Let $\cot^{-1} z = \phi$. Then $\cot \phi = z \Rightarrow \sin \phi = \dfrac{1}{\sqrt{1+z^2}} = \dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$.

(*ii*) Use $\sin^{-1} x = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}}$ and $\cot^{-1} x = \tan^{-1} \dfrac{1}{x}$. (Art. 4.07, Th. 5)

(*iii*) Let $\cos^{-1} x = \theta$. Then $\cos \theta = x$.

Now $\dfrac{1}{2} \le x \le 1 \Rightarrow \cos^{-1} 1 \le \cos^{-1} x \le \cos^{-1} \dfrac{1}{2}$ (In Quad. I cos decreases as angle increases).

$\Rightarrow 0 \le \theta \le \dfrac{\pi}{3}$

The given expression $= \cos^{-1} x + \cos^{-1}\left[x \cdot \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \cdot \sqrt{1-x^2}\right]$

$= \theta + \cos^{-1}\left[\cos\theta \cdot \cos\dfrac{\pi}{3} + \sin\dfrac{\pi}{3} \cdot \sin\theta\right]$

$= \theta + \cos^{-1} \cos\left(\dfrac{\pi}{3} - \theta\right) = \theta + \dfrac{\pi}{3} - \theta = \dfrac{\pi}{3}\Big]$.

26. Solve for *x* : $\cos(\sin^{-1} x) = \dfrac{1}{9}$. ***(ISC 1997)***

[Hint. $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$]

27. Solve the equation : $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\dfrac{2}{3}$. ***(ISC 1998, 2001)***

28. Solve the equation : $\sin^{-1}\dfrac{2a}{1+a^2} + \sin^{-1}\dfrac{2b}{1+b^2} = 2\tan^{-1} x$.

[Sol. Let $a = \tan\theta$. Then

$\sin^{-1}\dfrac{2a}{1+a^2} = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} a$.

Similarly, $\sin^{-1}\dfrac{2b}{1+b^2} = 2\tan^{-1} b$

$\therefore \quad \sin^{-1}\dfrac{2a}{1+a^2} + \sin^{-1}\dfrac{2b}{1+b^2} = 2\tan^{-1} a + 2\tan^{-1} b = 2(\tan^{-1} a + \tan^{-1} b)$

$= 2\tan^{-1}\dfrac{a+b}{1-ba}$, provided $ab < 1$. $\quad \therefore \; 2\tan^{-1}\dfrac{a+b}{1-ab} = 2\tan^{-1} x \Rightarrow x = \dfrac{a+b}{1-ab}\Big]$.

29. Solve the equation. $\sin^{-1} 6x + \sin^{-1}(6\sqrt{3}x) = -\dfrac{\pi}{2}$. ***(ISC 1999)***

[Sol. $\sin^{-1} 6x = -\dfrac{\pi}{2} - \sin^{-1}(6\sqrt{3}x)$

$\Rightarrow 6x = \sin\left[\dfrac{-\pi}{2} - \sin^{-1}(6\sqrt{3}x)\right] = -\cos\left[\sin^{-1}(6\sqrt{3}x)\right]$

$\Rightarrow 6x = -\cos\left[\cos^{-1}\sqrt{1-108x^2}\right]$ $\qquad \left[\because \sin^{-1} A = \cos^{-1}\sqrt{1-A^2}\right]$

$\Rightarrow 6x = -\sqrt{1-108x^2} \Rightarrow x = \pm\dfrac{1}{12}$.

If we put $x = \dfrac{1}{12}$ in the given equation we get $\sin^{-1}\dfrac{1}{2} + \sin^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi}{2}$, which shows that $x = \dfrac{1}{12}$ does not satisfy the given equation. If we put $x = -\dfrac{1}{12}$, we get

$\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{6}-\frac{\pi}{3}=-\frac{\pi}{2}$, which shows that $x=-\frac{1}{12}$ satisfies the given equation.

Hence, $x=-\frac{1}{12}$ is the answer].

30. Prove that $\sin^{-1}\frac{x}{\sqrt{1+x^2}}+\cos^{-1}\frac{x+1}{\sqrt{x^2+2x+2}}=\tan^{-1}(x^2+x+1)$. ***(ISC 2000, Type 2006)***

[Sol. Let $\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\theta$. Then

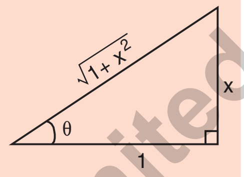


Fig. 4.14

$$\sin\theta=\frac{x}{\sqrt{1+x^2}}\Rightarrow\tan\theta=x$$

$$\Rightarrow\quad\theta=\tan^{-1}x$$

$$\Rightarrow\quad\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\tan^{-1}x$$

Let $\cos^{-1}\frac{x+1}{\sqrt{x^2+2x+2}}=\phi$. Then

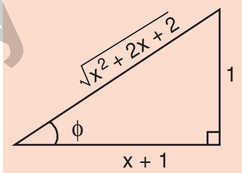


Fig. 4.15

$$\cos\phi=\frac{x+1}{\sqrt{x^2+2x+2}}\Rightarrow\tan\phi=\frac{1}{x+1}$$

$$\Rightarrow\quad\phi=\tan^{-1}\frac{1}{x+1}$$

$$\Rightarrow\quad\cos^{-1}\frac{x+1}{\sqrt{x^2+2x+2}}=\tan^{-1}\frac{1}{x+1}$$

The L.H.S. of the given equation

$$=\tan^{-1}x+\tan^{-1}\frac{1}{x+1}=\tan^{-1}\left[\frac{x+\frac{1}{x+1}}{1-x\cdot\frac{1}{x+1}}\right]$$ [Here $x\times\frac{1}{x+1}=\frac{x}{x+1}<1$, Theorem 7]

$$=\tan^{-1}(x^2+x+1)=\text{R.H.S.}$$

Hence proved].

31. Show that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right)$.

[Hint. L.H.S. $=\tan^{-1}\left[\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right]=\frac{1}{2}\left[2\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)\right]$

$$=\frac{1}{2}\cos^{-1}\left[\frac{1-\frac{\tan^2\frac{x}{2}}{3}}{1+\frac{\tan^2\frac{x}{2}}{3}}\right]$$ $\left(\because 2\tan^{-1}A=\cos^{-1}\frac{1-A^2}{1+A^2}\right.$ Theorem 7$\left.\right)$

$$=\frac{1}{2}\cos^{-1}\left(\frac{3-\tan^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}\right)=\frac{1}{2}\cos^{-1}\left[\frac{3-\frac{1-\cos x}{1+\cos x}}{3+\frac{1-\cos x}{1+\cos x}}\right]$$ $\left(\because\tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}\right)$

$$=\frac{1}{2}\cos^{-1}\left(\frac{2+4\cos x}{4+2\cos x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right).$$

32. If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$, prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$.

[Hint. $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta \quad \Rightarrow \cos^{-1}\left[\frac{x}{2}\cdot\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\cdot\sqrt{1-\frac{y^2}{9}}\right]=\theta$ (Theorem 9)

33. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that $x^2+y^2+z^2+2xyz=1$.

[Hint. $\cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$

$\Rightarrow \quad \cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}]=\pi-\cos^{-1}z$ (Theorem 9)

$\Rightarrow \quad \left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]=\cos(\pi-\cos^{-1}z) = -\cos(\cos^{-1}z)=-z$

$\Rightarrow \quad xy+z=\sqrt{1-x^2}\sqrt{1-y^2}$ etc.]

34. Show that $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{\sqrt{5}}=\cot^{-1}\frac{2}{11}$. ***(ISC 2003)***

[Hint. L.H.S. $=\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{2}{\sqrt{5}}$. Now apply Theorem 9]

35. Prove that $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$. ***(ISC 2005)***

[Hint. $\tan^{-1}\frac{1}{7}=\frac{\pi}{4}-2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}-\tan^{-1}\frac{2/3}{1-(1/3)^2}=\frac{\pi}{4}-\tan^{-1}\frac{3}{4}$, etc.] (Theorem 8)

36. Prove that $2\tan^{-1}\left(\frac{1}{3}\right)+\cot^{-1}(4) = \tan^{-1}\left(\frac{16}{13}\right)$ ***(ISC 2007)***

[Hint. Use $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $x^2<1$ and $\cot^{-1}x=\tan^{-1}\frac{1}{x}$.]

ANSWERS

1. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{\pi}{3}$ (*iii*) $\frac{\pi}{4}$ (*iv*) $\frac{\pi}{2}$ (*v*) $\frac{\pi}{3}$ (*vi*) $\frac{5\pi}{6}$ (*vii*) $\frac{\pi}{6}$ (*viii*) $\frac{2\pi}{3}$

2. (*a*) $\frac{1}{2}$ (*b*) 3 (*c*) $\frac{3}{5}$ (*d*) 60º (*e*) 5 (*f*) $\frac{1}{\sqrt{3}}$

3. (*a*) $\frac{1}{2}$ (*b*) $\frac{\pi}{6}$ (*c*) $\frac{5}{4}$ (*d*) $\frac{\pi}{4}$ (*e*) $\frac{\sqrt{3}}{2}$ (*f*) $-\frac{\sqrt{3}}{2}$ (*g*) $\frac{4\sqrt{5}}{9}$ (*h*) 130º (*i*) $\frac{1}{\sqrt{10}}$

(*j*) Undefined (*k*) $\frac{\pi}{6}$ (*l*) $\frac{\sqrt{2}+1}{\sqrt{2}}$ (*m*) 0 (*n*) $\frac{-\pi}{6}$ (*o*) $\frac{-\pi}{3}$ (*p*) $\frac{3\pi}{4}$ (*q*) $\frac{3}{5}$

7. (*a*) $2x\sqrt{1-x^2}$ (*b*) $1-2x^2$ (*c*) $\frac{y}{\sqrt{1-y^2}}$ (*d*) $\sqrt{\frac{1+x}{2}}$ **8.** (*a*) $x=\frac{1}{3}\sin\frac{y}{2}$

(*b*) $x=\frac{1}{2}\cos\frac{y}{3}$ (*c*) $x=\tan 2y-\pi$ **17.** $x=1$ **18.** $x=\frac{\sqrt{3}}{2\sqrt{7}}$ **19.** $x=\frac{1}{6}$

20. $x=\frac{1}{5}$ **21.** No solution **22.** $x=0, \pm\frac{1}{2}$ **23.** $x=\pm\frac{1}{\sqrt{2}}$ **24.** $x=3$

25. (*i*) $\sqrt{\frac{1+x^2}{2+x^2}}$ (*ii*) $\frac{17}{6}$ (*iii*) $\frac{\pi}{3}$ **26.** $\pm\frac{4\sqrt{5}}{9}$ **27.** ± 3

REVISION EXERCISE

1. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is :

(*a*) 0 (*b*) $\frac{1}{\sqrt{5}}$ (*c*) $\frac{2}{\sqrt{5}}$ (*d*) $\frac{\sqrt{3}}{2}$

2. If $4 \sin^{-1} x + \cos^{-1} x = \pi$, then x is equal to

(*a*) 0 (*b*) $\frac{1}{2}$ (*c*) $-\frac{\sqrt{3}}{2}$ (*d*) $\frac{1}{\sqrt{2}}$

3. Considering only the principal values, if $\tan(\cos^{-1} x) = \sin\left[\cot^{-1}\left(\frac{1}{2}\right)\right]$, then x is equal to

(*a*) $\frac{1}{\sqrt{5}}$ (*b*) $\frac{2}{\sqrt{5}}$ (*c*) $\frac{3}{\sqrt{5}}$ (*d*) $\frac{\sqrt{5}}{3}$

4. The value of $\tan\left[\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{\sqrt{13}}\right]$ is :

(*a*) $\frac{6}{17}$ (*b*) $\frac{6}{\sqrt{13}}$ (*c*) $\frac{\sqrt{13}}{5}$ (*d*) $\frac{17}{6}$

5. $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$, then $\sin x =$

(*a*) $\tan^2\frac{\alpha}{2}$ (*b*) $\cot^2\frac{\alpha}{2}$ (*c*) $\tan\alpha$ (*d*) $\cot\frac{\alpha}{2}$

6. If $\theta = \sin^{-1}[\sin(-600°)]$, then one of the possible values of θ is

(*a*) $\frac{\pi}{3}$ (*b*) $\frac{\pi}{2}$ (*c*) $\frac{2\pi}{3}$ (*d*) $-\frac{2\pi}{3}$

7. $\cos\left[\cos^{-1}\left(-\frac{1}{7}\right) + \sin^{-1}\left(-\frac{1}{7}\right)\right] =$

(*a*) –1/3 (*b*) 0 (*c*) 1/3 (*d*) 4/9

8. The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right) + \cos(\tan^{-1} 2\sqrt{2}) =$

(*a*) $\frac{16}{15}$ (*b*) $\frac{14}{15}$ (*c*) $\frac{12}{15}$ (*d*) $\frac{11}{15}$

9. If $\sin^{-1}(1 - x) - 2\sin^{-1} x = \pi/2$, then x equals

(*a*) {0, – 1/2} (*b*) {1/2, 0} (*c*) {0} (*d*) {–1, 0}

[Hint. $\sin^{-1}(1 - x) = \frac{\pi}{2} - 2\sin^{-1} x \Rightarrow 1 - x = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right)$

$\Rightarrow 1 - x = \sin\frac{\pi}{2}\cos(2\sin^{-1} x) - \cos\frac{\pi}{2}\sin(2\sin^{-1} x) = \cos(2\sin^{-1} x)$

$\Rightarrow 1 - x = \cos\cos^{-1}(1 - 2x^2) \Rightarrow 1 - x = 1 - 2x^2 \Rightarrow x = 0, 1/2$. But $x = 1/2$ does not satisfy the equation, so $x = 0$]

10. If $\cos^{-1} x + \cos^{-1} y = \pi$, what is the value of $\sin^{-1} x + \sin^{-1} y$?

(*a*) 0 (*b*) $\frac{\pi}{2}$ (*c*) π (*d*) 2π

11. The value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1} x)$ is :

(*a*) 1/2 (*b*) 1 (*c*) 0 (*d*) – 1/2

12. $\sin^{-1} x + \sin^{-1}(1 - x) = \cos^{-1} x \Rightarrow x \in$

(*a*) {1, 0} (*b*) {– 1, 1} (*c*) {0, 1/2} (*d*) {2, 0}

13. $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} =$

(*a*) $\pi/3$ (*b*) $\pi/4$ (*c*) $\pi/2$ (*d*) 0

14. What is the value of $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) - \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$?

15. What is the value of x that satisfies the equation $\cos^{-1}x = 2\sin^{-1}x$?

16. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is

(*a*) 4 (*b*) 5 (*c*) 1 (*d*) 3

17. What is $\sin\left[\cot^{-1}\left\{\cos\left(\tan^{-1}x\right)\right\}\right]$, where $x > 0$, equal to ?

(*a*) $\sqrt{\frac{(x^2+1)}{(x^2+2)}}$ (*b*) $\sqrt{\frac{(x^2+2)}{(x^2+1)}}$ (*c*) $\frac{(x^2+1)}{x^2+2}$ (*d*) $\frac{(x^2+2)}{(x^2+1)}$

[Hint. See solved Ex. 15].

ANSWERS

1. (*b*)	**2.** (*b*)	**3.** (*d*)	**4.** (*d*)	**5.** (*a*)	**6.** (*a*)	**7.** (*b*)
8. (*b*)	**9.** (*c*)	**10.** (*a*)	**11.** (*d*)	**12.** (*c*)	**13.** (*c*)	**14.** 0
15. $\frac{1}{2}$	**16.** (*d*)	**17.** (*a*)				

HINTS

1. $\sin^{-1}x = \frac{\pi}{2} - \cot^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} \Rightarrow \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}\frac{1}{2}$

$$\Rightarrow \tan^{-1}\frac{\frac{x}{\sqrt{1-x^2}}-\frac{1}{2}}{1+\frac{x}{2\sqrt{1-x^2}}} = 0 \Rightarrow \frac{\frac{x}{\sqrt{1-x^2}}-\frac{1}{2}}{1+\frac{x}{2\sqrt{1-x^2}}} = \tan 0^\circ \Rightarrow \frac{x}{\sqrt{1-x^2}} - \frac{1}{2} = 0$$

3. Let $\cot^{-1}\frac{1}{2} = \theta \Rightarrow \cot\theta = \frac{1}{2} \Rightarrow \sin\theta = \frac{2}{\sqrt{5}}$; Let $\cos^{-1}x = \phi \Rightarrow x = \cos\phi \Rightarrow \tan^{-1}\phi = \frac{2}{\sqrt{5}}$

5. $\tan^{-1}\frac{1}{\sqrt{\cos\alpha}} - \tan^{-1}\sqrt{\cos\alpha} = \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}}-\sqrt{\cos\alpha}}{1+\frac{\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}}} = x \Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

6. $\theta = \sin^{-1}[\sin(-600^\circ)] = \sin^{-1}(-\sin 600^\circ) = \sin^{-1}[-\sin(360^\circ + 240^\circ)]$

$= \sin^{-1}(-\sin 240^\circ) = \sin^{-1}[-\sin(180^\circ + 60^\circ)] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

7. $\sin\left(\tan^{-1}\frac{2/3}{1-1/9}\right) + \cos(\tan^{-1}2\sqrt{2}) = \sin\left(\tan^{-1}\frac{3}{4}\right) + \cos(\tan^{-1}2\sqrt{2})$

$= \sin\left(\sin^{-1}\frac{3}{5}\right) + \cos\left(\cos^{-1}\frac{1}{3}\right)$

10. $\Rightarrow \left(\frac{\pi}{2} - \sin^{-1}x\right) + \left(\frac{\pi}{2} - \sin^{-1}y\right) = \pi \Rightarrow \sin^{-1}x + \sin^{-1}y = 0$

11. $\Rightarrow \sin\left[\cot^{-1}(1+x)\right] = \sin\left(\frac{\pi}{2} \pm \tan^{-1} x\right) \Rightarrow \cot^{-1}(1+x) = \frac{\pi}{2} \mp \tan^{-1} x$

$\Rightarrow \cot^{-1}(1+x) + \tan^{-1}(\pm x) = \frac{\pi}{2} \Rightarrow 1 + x = \pm x$

12. $\sin^{-1} x + \sin^{-1}(1-x) = \frac{\pi}{2} - \sin^{-1} x \Rightarrow \sin^{-1}(1-x) = \frac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow 1 - x = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right) \Rightarrow 1 - x = \cos(2\sin^{-1} x)$

$\Rightarrow 1 - x = 1 - 2\sin^2(\sin 2(\sin^{-1} x) \Rightarrow 1 - x = 1 - 2x^2$

13. $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{3}\right) + \tan^{-1}\frac{1}{3}$