Understanding Mathematics

Textbook for Class IX

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Department of Curriculum and Professional Development
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Royal Government of Bhutan

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MINISTER

FOREWORD

Provision of quality education for our children is a cornerstone policy of the Royal Government of Bhutan. Quality education in mathematics includes attention to many aspects of educating our children. One is providing opportunities and believing in our children's ability to understand and contribute to the advancement of science and technology within our culture, history and tradition. To accomplish this, we need to cater to children's mental, emotional and psychological phases of development, enabling, encouraging and supporting them in exploring, discovering and realizing their own potential. We also must promote and further our values of compassion, hard work, honesty, helpfulness, perseverance, responsibility, *thadamtsi* (for instance being grateful to what I would like to call '*Pham Kha Nga',* consisting of parents, teachers, His Majesty the King, the country and the Bhutanese people, for all the goodness received from them and the wish to reciprocate these in equal measure) and *ley-ju-drey* — the understanding and appreciation of the natural law of cause and effect. At the same time, we wish to develop positive attitudes, skills, competencies, and values to support our children as they mature and engage in the professions they will ultimately pursue in life, either by choice or necessity.

While education recognizes that certain values for our children as individuals and as citizens of the country and of the world at large, do not change, requirements in the work place advance as a result of scientific, technological, and even political advancement in the world. These include expectations for more advanced interpersonal skills and skills in communications, reasoning, problem solving, and decision-making. Therefore, the type of education we provide to our children must reflect the current trends and requirements, and be relevant and appropriate. Its quality and standard should stem out of collective wisdom, experience, research, and thoughtful deliberations.

Mathematics, without dispute, is a beautiful and profound subject, but it also has immense utility to offer in our lives. The school mathematics curriculum is being changed to reflect research from around the world that shows how to help students better understand the beauty of mathematics as well as its utility.

The development of this textbook series for our schools, *Understanding Mathematics,* is based on and organized as per the new School Mathematics Curriculum Framework that the Ministry of Education has developed recently, taking into consideration the changing needs of our country and international trends. We are also incorporating within the textbooks appropriate teaching methodologies including assessment practices which are reflective of international best practices. The *Teacher's Guides* provided with the textbooks are a resource for teachers to support them, and will definitely go a long way in assisting our teachers in improving their efficacy, especially during the initial years of teaching the new curriculum, which demands a shift in the approach to teaching and learning of Mathematics. However, the teachers are strongly encouraged to go beyond the initial ideas presented in the Guides to access other relevant resources and, more importantly to try out their own innovations, creativity and resourcefulness based on their experiences, reflections, insights and professional discussions.

The Ministry of Education is committed to providing quality education to our children, which is relevant and adaptive to the changing times and needs as per the policy of the Royal Government of Bhutan and the wish of our beloved King.

I would like to commend and congratulate all those involved in the School Mathematics Reform Project and in the development of these textbooks.

I would like to wish our teachers and students a very enjoyable and worthwhile experience in teaching, learning and understanding mathematics with the support of these books. As the ones actually using these books over a sustained period of time in a systematic manner, we would like to strongly encourage you to scrutinize the contents of these books and send feedback and comments to the Curriculum and Professional Support Division (CAPSD) for improvement with the future editions. On the part of the students, you can and should be enthusiastic, critical, venturesome, and communicative of your views on the contents discussed in the books with your teachers and friends rather than being passive recipients of knowledge.

Trashi Delek!

Lyonpo Thinley Gyamtsho
MINISTER
Ministry of Education

January of 2007

INTRODUCTION

HOW MATHEMATICS HAS CHANGED

Mathematics is a subject with a long history. Although newer mathematical ideas are always being discovered, much of what you will be learning is mathematics that has been known for hundreds of years, if not longer.

Mathematics is a study of quantity, space, structure, patterns and change. This study at the school level is divided into 5 strands of content, namely, numbers and operations, algebra, geometry, measurement, and data and probability.

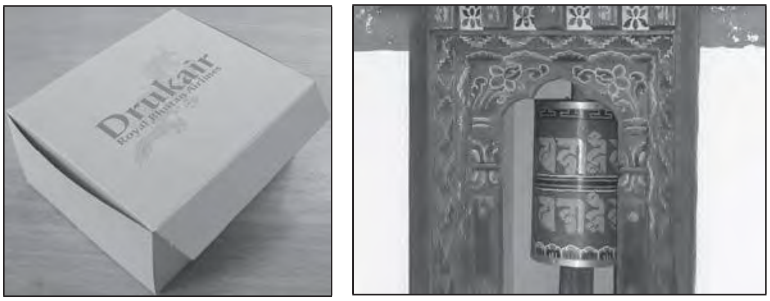
Nowadays, greater emphasis is given to conceptual understanding rather than on memorizing and applying rote procedures. There are many reasons for this.

- In the long run, it is very unlikely that you will remember the mathematics you learn unless it is meaningful. It is much easier to understand and memorise something that relates to what you already than to memorise something that does not make sense.
- Some approaches to mathematics have not been successful; there are many adults who are not comfortable with mathematics even though they were successful in school. This indicates that a change in approach is necessary.

In your textbook, the mathematics is made meaningful in many ways:

- Mathematics should be taught using contexts that are meaningful to you. They can be mathematical contexts or real world contexts. The textbook uses both Bhutanese and international real world contexts. For example, in Unit 6 (Measurement), tasks with international contexts involve estimating the volume of the Great Pyramid in Giza and the Swayambhunath Stupa in Kathmandu. Tasks with Bhutanese contexts involve calculating the surface area of a snack box used by Druk Air or estimating the volume of a cylindrical prayer wheel. Meaningful contexts will help you see and appreciate the value of mathematics.

Calculating the volume of the Great Pyramid and the Swayambhunath Stupa



Calculating the surface area of a snack box used by Druk Air and a prayer wheel

- You will be asked to explain why something is true, not simply to state that it is true. For example, you might be asked to demonstrate that the volume of a cube with a particular surface area is greater than the volume of any other rectangular prism with that same surface area.
- When you discuss mathematical ideas, you will be expected to use the processes of problem solving, communication, reasoning, making connections (connecting mathematics to the everyday world and connecting mathematical topics to each other), and representation (representing mathematical ideas in different ways, such as graphs and tables). For example, when you work with polynomials, you will connect operations with polynomials to operations with numbers, use reasoning to see why different representations of polynomials are equivalent, and communicate your thinking while solving problems.
- The reason you learn mathematics is to help you solve problems. In the real world, you are not told when to factor or when to multiply. You will be given opportunities to figure out when and how to apply the concepts and skills you are learning in order to solve problems.

USING YOUR TEXTBOOK

Each unit has

- a *Getting Started* section
- two or three chapters, which divide the content of the unit into sections
- regular lessons and at least one *Explore* lesson
- a *Game* (usually)
- at least one *Connections* feature
- a *Unit Revision*

Getting Started

There are two parts to each *Getting Started* section: *Use What You Know* and *Skills You Will Need*. Both will help you know whether you have the critical knowledge you need in order to proceed. They will remind you of knowledge and terminology you have already learned that will be useful in the unit.

- *Use What You Know* is an activity that you complete with a partner or in a small group.
- *Skills You Will Need* is a review of the skills you will use in the unit.

Regular Lessons

- Lessons are numbered #.#.#—the first number tells the unit, the second number the chapter, and the third number the lesson within the chapter. For example, Lesson 4.2.1 is Unit 4, Chapter 2, Lesson 1 (first lesson in Chapter 2 of Unit 4).
- Each regular lesson is divided into five parts:

- A *Try This* task
- The exposition (the main ideas of the lesson)
- A question that revisits the *Try This* task
- *Examples*
- *Practising and Applying*

Try This

- The *Try This* task is in a shaded box, like the example below from lesson 1.1.1 on page 2.

$$2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

> **A.** Suppose you write $2^9 = 2^a \times 2^b$.
> **i)** Find two pairs of values for *a* and *b* that would make this true. **ii)** Find another pair of values.
> **iii)** What do you notice about how the values of *a* and *b* are related in each pair?

- The *Try This* is a brief task that you do with a partner or in a small group. It is related to the new learning, but you can complete it without the concepts and skills that are the focus of the lesson. The new mathematics you are able to learn in the exposition will make more sense to you if you do some related mathematics before the teacher presents the lesson.

The Exposition

- The exposition appears in a box immediately following the *Try This*.
- The exposition presents the main concepts and skills of the lesson.
- Key mathematical terms are introduced and described. When a key term first appears in a unit, it is highlighted in **bold type** to indicate that it is found in the glossary (at the back of the book).
- You are not expected to copy the exposition into your notebook either directly from the book or from your teacher's lecture.

Revisiting the Try This

- The revisiting the *Try This* question(s) follows the exposition and appears in a shaded lozenge, like this example from lesson 1.1.1 on page 3.

> **B.** Was **part A** an example of the product law or the quotient law? Explain.

- The question shows how your new learning relates to what you already learned from the *Try This* task.

Examples

- The *Examples* provide additional instruction by modelling how to approach the questions you will meet in *Practising and Applying*. Each example is a bit different from the others so that you have many models from which to work.
- Sometimes you work through the examples independently, sometimes in pairs or in small groups, and sometimes with your teacher.
- What is special about the examples is that they show not only the formal mathematical work in the left hand *Solution* column, but also what a student might be thinking in the right hand *Thinking* column. This is intended to help you learn to think mathematically. Many of the examples present two or even three different solutions. The example below from lesson 1.1.1 on page 3 shows two possible ways to approach the task, *Solution 1* and *Solution 2*.

Example 1 Expressing Powers Using the Product Law	
List at least two possible ways of writing 3^8 as a product of powers of 3.	
Solution 1 $3^8 = (3 \times 3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^5 \times 3^3$ $3^8 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)$ $= 3^4 \times 3^4$ $3^8 = (3 \times 3) \times (3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^2 \times 3^3 \times 3^3$	**Thinking**  • I knew I needed 8 threes multiplied together because that's what 3^8 means. • I grouped them in different ways to get different possibilities.
Solution 2 $3^8 = 3^{2+6} = 3^2 \times 3^6$ $3^8 = 3^{1+7} = 3^1 \times 3^7$ $3^8 = 3^{3+5} = 3^3 \times 3^5$	**Thinking**  • I used the product law: $a^r \times a^s = a^{r+s}$ and looked for two exponents that added to 8. • I could have written $3^1 \times 3^7$ as 3×3^7 because any number to the power of 1 is the number itself.

Practising and Applying

- Depending on your teacher’s wish, you might work on the *Practising and Applying* questions independently, with a partner, or in a group. You can use the exposition and examples as references.
- The questions start out like those in the *Examples* and progress to questions requiring more problem solving and more explanations. The last question brings you back to one of the main points of the lesson.

Explore Lessons

- An *Explore* lesson provides an opportunity for you to work with a partner or in a small group to investigate some mathematics.
- Your teacher does not lecture in an *Explore* lesson. Instead, you work through a problem by following the questions that direct your investigation.

Connections Feature

- The *Connections* feature takes many forms. Sometimes it is a relevant and interesting historical note. Sometimes it relates the mathematical content of the unit to the content of a different unit. Other times it relates the mathematical content to a real world application. For example, in Unit 1, the *Connections* on page 18 is about the Richter scale, which is used to measure the intensity of an earthquake. The mathematics behind the scale will make sense to you after learning about scientific notation in the previous lesson.
- There is always one or more *Connections* feature in a unit.
- You usually work in pairs or small groups to complete the task or answer the question(s).

Game

- There is usually at least one *Game* in each unit.
- The *Game* is a way to practise skills and concepts introduced in the unit with a partner or in small group.
- The required materials and rules are listed in the book. Usually there is a sample shown to help you understand the rules.

Unit Revision

- The *Unit Revision* is an opportunity to review the lessons in the unit.
- There is always a mixture of skill, concept, and problem solving questions.
- The order of the questions in the *Unit Revision* usually follows the order of the lessons in the unit.
- You can work with a partner or on your own, as your teacher suggests.

Glossary

At the end of the book, there is a glossary of new mathematical terminology and definitions. The glossary also contains other important mathematical terms from previous classes. There is also a set of instructional terms commonly used in the units (for example, justify, explain, predict, …). These are intended to help you understand what is expected of you.

Answers

- Answers to most of the numbered questions are provided in the back of the textbook. Answers that are lengthy explanations are not included; your teacher has these answers.
- Questions with letters, such as A or B, do not have answers in the back of the book. Your teacher has the answers to these questions.
- There is often more than one possible answer to a question. This is indicated in the answers by the phrase *Sample Response*. When you see an answer prefaced with *Sample Response*, your answer may still be correct even if it does not match the answer given.

ASSESSING YOUR MATHEMATICAL PERFORMANCE

Forms of Assessment

Your teacher will observe and report on your mathematical performance. Sometimes your teacher will collect information about what you understand in order to change the way you are taught. Other times your teacher will use information about your performance to give you a mark.

Assessment Criteria

- Your teacher should inform you about what mathematical content will be assessed and how it will be assessed. For example, you should know if the intent of the assessment is to focus on skills and application or on problem solving.
- Your mark and all assessments should reflect the curriculum for Class IX. The proportions of the mark assigned for each unit should reflect both the time spent on the unit and the importance of the unit.
- All assessment should have a balance of skills, applications, concepts, and problem solving. The balance will vary depending on the unit and purpose of the assessment.
- Your teacher should inform you whether a test is being marked numerically, using a letter grade, or whether a rubric is being used. A rubric is a chart that describes criteria for your work, usually in four levels of performance. If a rubric is used, your teacher should let you see it before you start on the task.

Determining a Mark or Grade

In determining your mark, your teacher might use a combination of tests, assignments, projects, performance tasks, and homework.

THE CLASSROOM ENVIRONMENT

In almost every lesson, you will be engaged in some work either in in pairs or in small groups (either in the *Try This* or during an *Explore* lesson). Being engaged in your learning helps you learn better.

While you are working on your own, in pairs or in groups, communication plays a significant role in every lesson. Through communication you can clarify your thinking and show your teacher and classmates what you understand.

You should always share your responses, even if they are different from those offered by other students. It is only in this way that you will really be engaged in the mathematical thinking instead of being a spectator.

MATHEMATICAL TOOLS

Manipulatives

- All students, including those who are already good at mathematics, can benefit from using manipulative materials. For example, Unit 2 makes frequent use of algebra tiles to represent polynomials concretely. Although some students can be successful without these materials, everyone can benefit from their use. You will start to see not only how to perform algebraic manipulations, but why they are done the way they are.
- Manipulative materials are important in Class IX in the units on polynomials, probability, geometry, and measurement.

Algebra tiles for polynomials

Appropriate Calculator Use

- In Class IX, the calculator should be used as a regular tool. At this point in your mathematical education, you are no longer being asked simply to perform routine calculations. Calculations are now part of more sophisticated mathematical tasks that are the real focus of your learning.
- You may not have the same type of calculator as your classmates, so specific instructions for how to use your calculator are not provided in the textbook. Your teacher can help you learn to use your calculator correctly.

YOUR NOTEBOOK

- It is valuable for you to have a well-organized, neat notebook to look back at to review the main mathematical ideas you have learned. However, it is also important for you to feel comfortable doing rough work in that notebook rather than doing it elsewhere and then wasting valuable time copying your rough work neatly into your notebook. If you do rough work on other paper, which will certainly happen from time to time, it may not be necessary to copy it into your notebook.
- Your teacher will sometimes point out important points to record in your notebook. You should also make your own decisions about which ideas to include in your notebook.

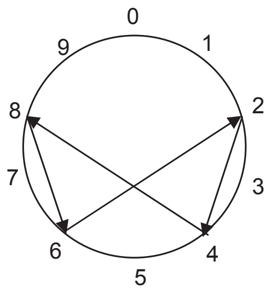
UNIT 1 NUMBER AND OPERATIONS

Getting Started

Use What You Know

The powers of 2 are, in order, $2^1, 2^2, 2^3, 2^4, 2^5, \ldots$. Their values are 2, 4, 8, 16, 32, … . You can show the pattern of the ones digits (2, 4, 8, 16, 32, …) on a circle. The pattern is started on the right.

Ones Digits for Powers of $2^{\blacksquare}$



A. i) What shape does the first five powers of 2 form?

ii) If you continued the pattern, what would happen? Explain.

B. i) Draw another circle and label it clockwise with the numbers 0 to 9.

ii) Show the pattern of the ones digits for the powers of 3.

iii) Repeat **parts i) and ii)** for the powers of 4, 5, 6, 7, 8, 9, and 10.

iv) What observations can you make from looking at all the circle patterns for the powers of 2 to 10?

v) How might you have predicted some of these patterns?

C. How could you use patterns to predict the ones digits for $\blacksquare^{20}$ for different values of ■ between 2 and 10?

Skills You Will Need

1. Write each as a decimal.

a) 10^{-3} **b)** 5×10^{-2} **c)** 3×10^{2} **d)** 4×10^{-1}

2. Calculate each.

a) $(-2)^3$ **b)** -4^2 **c)** $(-3)^2$

3. List all positive perfect squares that are less than 300.

4. Draw a picture to show one meaning for $\sqrt{121}$.

5. Between what two whole numbers is $\sqrt{150}$?

6. Deki says that $\sqrt{2} = 1.414$. How do you know that Deki is wrong?

Chapter 1 Exponents

1.1.1 Introducing the Exponent Laws

Try This

$$2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

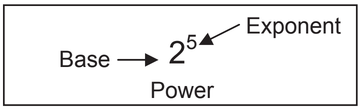
A. Suppose you write $2^9 = 2^a \times 2^b$.

i) Find two pairs of values for *a* and *b* that would make this true.

ii) Find another pair of values.

iii) What do you notice about how the values of *a* and *b* are related in each pair?

$2 \times 2 \times 2 \times 2 \times 2$ can be written as the **power** 2^5 and read as “2 raised to the 5” or *“2 to the fifth.”* 2 is called the **base** and 5 is called the **exponent**.



Note that the term "power" is also used to describe the exponent.

- The base of the power is the number that is repeatedly multiplied. The exponent, if it is a whole number, tells the number of times that the base is multiplied. Just as multiplication is a shortcut for addition, exponentiation is a shortcut for multiplication. It takes a lot less space to write 2^5 than $2 \times 2 \times 2 \times 2 \times 2$.
- Any number that can be written in the form a^b, where *b* is a whole number, is called a power of *a*. For example, 32 can be written as 2^5, so 32 is “a power of 2.”
- Any number, *n*, can be written as a power, n^1, since an exponent of 1 indicates the base occurs only once in the product, $n^1 = n$. For example, $3^1 = 3$ or $45 = 45^1$.
- If you multiply two powers of the same base, you can simplify the calculation by adding the exponents. This is called the **product law.**

For example, $2^3 \times 2^5 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2) = 2^8$. Notice that the exponent 8 in the product is the sum of the exponents 3 and 5 in the factors. This works since there are 3 twos + 5 twos, which is 8 twos multiplied together.

$$\boxed{\text{Product law: } a^r \times a^s = a^{r+s}}$$

Not only can you use the product law to multiply two powers of the same base, but you can also use it to write a power as the product of other powers, for example, $5^{10} = 5^3 \times 5^7$.

- There is also a **quotient law** for dividing powers of the same base, where you subtract the exponents.

$$\boxed{\text{Quotient law: } a^r \div a^s = a^{r-s}}$$

For example, $5^{10} \div 5^3 = \dfrac{5\times5\times5\times5\times5\times5\times5\times5\times5\times5}{5\times5\times5} = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$.

Notice that the exponent 7 in the quotient is the difference between the exponents 10 and 3 in the dividend and divisor. This works because, when divide the numerator into the denominator, $\dfrac{5\times5\times5\times5\times5\times5\times5\times(5\times5\times5)}{(5\times5\times5)}$, you get 5^7.

B. Was **part A** an example of the product law or the quotient law? Explain.

Examples

Example 1 Expressing Powers Using the Product Law

List at least two possible ways of writing 3^8 as a product of powers of 3.

Solution 1 $3^8 = (3 \times 3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^5 \times 3^3$ $3^8 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)$ $= 3^4 \times 3^4$ $3^8 = (3 \times 3) \times (3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^2 \times 3^3 \times 3^3$	**Thinking**  • I knew I needed 8 threes multiplied together because that's what 3^8 means. • I grouped them in different ways to get different possibilities.
Solution 2 $3^8 = 3^{2+6} = 3^2 \times 3^6$ $3^8 = 3^{1+7} = 3^1 \times 3^7$ $3^8 = 3^{3+5} = 3^3 \times 3^5$	**Thinking**  • I used the product law: $a^r \times a^s = a^{r+s}$ and looked for two exponents that added to 8. • I could have written $3^1 \times 3^7$ as 3×3^7 because any number to the power of 1 is the number itself.

Example 2 Expressing Powers Using the Quotient Law

Consider the quotient:

$$\frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}$$

a) Express the numerator and denominator as powers of 2. State the value of the quotient as a power of 2.

b) Express the result as the quotient of two other pairs of powers of 2. Provide more than one example.

Solution **a)** $\frac{2^{14}}{2^{11}} = 2^{14} \div 2^{11} = 2^3$	**Thinking**  **a)** I counted the number of 2s in the top and bottom of the fraction—there were three more twos in the top. That means if you divide the numerator by the denominator, the quotient is 2 × 2 × 2, which is 2^3.
b) $2^3 = 2^{18} \div 2^{15}$ $2^3 = 2^{100} \div 2^{97}$ $2^3 = 2^4 \div 2^1$	**b)** I knew I needed two numbers for exponents that were 3 apart because of the quotient law, $a^r \div a^s = a^{r-s}$.

Practising and Applying

1. Complete the statements by filling in the missing values.

a) $2^3 \times 2^{■} = 2^9$

b) $11^{18} \div 11^3 = 11^{■}$

c) $7^{10} \div 7^2 = ■^8$

2. Which of the following expressions are equivalent to one another?

A. $3^7 \times 3^2 \div 3^3$

B. $3 \times 3 \times 3$

C. $3^{27} \div 3^{21}$

D. $3^2 \times 3^2 \times 3^2$

3. Write the number 5^8 each way.

a) as a product of other powers of 5

b) as a quotient of other powers of 5

c) as a power of 25

d) as a product of other powers of 25

e) as a quotient of other powers of 25

4. Express 64 each way.

a) as a power of 2

b) as the product of two powers of 2

c) as the product of three powers of 2

5. A recipe uses one tablespoon of oil. The recipe is doubled.

a) How many tablespoons of oil are needed? Express this as a power.

b) You double the recipe again. How many tablespoons are needed now? Express this as a power.

c) You double the recipe five more times. How many tablespoons are needed? Express this as a power.

6. Explain how you know $3^4 = 9^2$ without calculating the values.

7. Write $\sqrt{7^4}$ as a power of 7. Explain your thinking.

8. $5^a \times 5^b = 5^{11}$

a) Can both *a* and *b* be even? Explain.

b) How many solutions are there for *a* and *b* if *a* and *b* are positive integers? How do you know?

9. Explain how $5^4 \times 5^7 \times 5^{19}$ can be simplified to a single power.

10. Explain the rule for dividing powers with the same base. Include an example to support your explanation.

GAME: Rolling Powers

Play in groups of two or three. Players take turns rolling one die to get a base, and then rolling two more dice to get two exponents. They create a multiplication from the numbers rolled. For example, with a base of 4 and exponents 3 and 5, the player would write $4^3 \times 4^5$. Players multiply the powers and express the product as a single power. The number of points is based on the product:

- 1 point if the exponent is greater than 8
- 1 point if the power has a whole number square root
- 1 point if the power can be expressed as a power of 2 that is greater than 1

The player who first accumulates 15 points wins.

For example, a multiplication such as $4^3 \times 4^5 = 4^8$ would get 2 points:

- 1 point because 4^8 has a whole number square root, namely 4^4
- 1 point because 4^8 can be written as 2^{16}

1.1.2 The Power Law of Exponents

Try This

A. Write each expression as a single power.

i) $2^5 \times 2^5 \times 2^5 \times 2^5$ **ii)** $3^4 \times 3^4 \times 3^4$ **iii)** $4^2 \times 4^2 \times 4^2 \times 4^2 \times 4^2$

B. Describe an efficient way to find the exponent of the single power.

There are several other laws that allow you to simplify expressions involving powers.

- A power raised to a power is a special case of the exponent product law you learned in the previous lesson, since it involves the product of powers of the same base. This special case has its own rule called the **power law**.

Power law: $(a^r)^s = a^{rs}$

For example, $(3^6)^4 = 3^6 \times 3^6 \times 3^6 \times 3^6 = 3^{6+6+6+6} = 3^{6 \times 4} = 3^{24}$.

- When products of numbers are raised to the same power, you can use the **power of a product law** to simplify the calculations.

Power of a product law: $(ab)^r = a^r b^r$

For example, $2^5 \times 5^5 = (2 \times 5)^5 = 10^5$. This makes the calculation much simpler—10^5 is much easier to calculate than $2^5 \times 5^5$ since 10^5 is 100,000 but $2^5 \times 5^5$ is 32×3125.

- The **power of a quotient law** lets you simplify quotients. It can be used with quotients in fractional form that have numerators and denominators with different bases that are raised to the same power. It can also be used in division situations, as shown below.

Power of a quotient law: $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$ or $(a \div b)^r = a^r \div b^r$

For example:

$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2^4}{3^4}$ and $(2 \div 3)^4 = 2^4 \div 3^4$

The power of a quotient law can also be used "in reverse."

For example, $\frac{2^4}{3^4}$ can be written as $\left(\frac{2}{3}\right)^4$ and $2^4 \div 3^4$ can be written as $(2 \div 3)^4$.

C. Which of the exponent laws could you have used to help you answer **part A**? Explain why it works.

Examples

Example 1 Comparing Quantities

Which expression represents the greater quantity in each pair? Show your work.

a) 6^4 or $3^5 \times 2^5$ **b)** $(-5)^{10}$ or $(5^3)^3$ **c)** 3^{12} or 9^7 **d)** 5^4 or $30^3 \div 6^3$

Solution	**Thinking**
a) $3^5 \times 2^5 = (3 \times 2)^5 = 6^5$ $6^5 > 6^4$ $3^5 \times 2^5$ is greater than 6^4	**a)** I noticed that 3 and 2 were raised to the same power so I used the power of a product law to simplify $3^5 \times 2^5$ to a single power. 
b) $(5^3)^3 = 5^{3 \times 3} = 5^9$ $(-5)^{10} = 5^{10}$ $5^{10} > 5^9$ $(-5)^{10}$ is greater than $(5^3)^3$	**b)** I made them both powers of 5 and then compared them. To make $(-5)^{10}$ a power of 5, I thought about what it meant — (–5) multiplied together 10 times. Since there are 10 negative signs multiplied together, the result will be positive so I knew $(-5)^{10}$ was the same as $(5)^{10}$.
c) Since $9 = 3^2$, 9^7 can be written as $(3^2)^7 = 3^{2 \times 7} = 3^{14}$ $3^{14} > 3^{12}$ 9^7 is greater than 3^{12}	**c)** I made them both powers of 3 and compared them. To make 9^7 a power of 3, all I did was substitute 3^2 for 9 and then used the exponent power law.
d) $30^3 \div 6^3 = (30 \div 6)^3 = 5^3$ $5^4 > 30^3 \div 6^3$	**d)** I used the quotient law to change $30^3 \div 6^3$ into a power of 5 so I could compare them.

Example 2 Solving Equations Involving Exponents

Solve for *n* in each equation.

a) $7^{10} = (7^2)^n$ **b)** $1{,}000{,}000 = (10^n)^2$ **c)** $-4^{15} = -8^n$ **d)** $3^5(n^5) = 30^5$

Solution	**Thinking**
a) $7^{10} = (7^2)^n$ $7^{10} = 7^{2 \times n}$ $7^{10} = 7^{2n}$ $10 = 2n$ $n = 5$	**a)** Since they both had the same base, all I had to do was match the exponents. I used the power law to get rid of the brackets in $(7^2)^n$. 
b) $1{,}000{,}000 = (10^n)^2$ $1{,}000{,}000 = 10^{n \times 2}$ $10^6 = 10^{2n}$ $6 = 2n$ $n = 3$	**b)** I gave them both the same base so all I had to do was match the exponents — I changed 1,000,000 to a base of ten because I knew 1,000,000 was 10^6 ($10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$). • I used the power law to simplify $(10^n)^2$ and get rid of the brackets.

Solution	**Thinking**
c) $-4^{15} = -8^n$ $-(2^2)^{15} = -8^n$ $-2^{30} = -8^n$ $-2^{30} = -(2^3)^n$ $-2^{30} = -2^{3n}$ $30 = 3n$ $10 = n$	 **c)** I knew that if I gave them both the same base, I could just work with the exponents. • I replaced 4 with 2^2 and then used the power law to get a single exponent. • I then replaced 8 with 2^3 and used the power law to simplify it and get rid of the brackets.
d) $3^5(n^5) = 30^5$ $(3n)^5 = 30^5$ $3n = 30$ $n = 10$	**d)** I knew if I gave them both the same exponent, I could just work with the bases. • I used the power of a product law to simplify $3^5(n^5)$ to $(3n)^5$. • Since $(3n)^5$ and 30^5 had the same exponent, I was able to solve for n using only the bases.

Practising and Applying

1. Write each as a single power:

a) $(5^3)^5$ **b)** $(13^4)^9$

c) $(9^5)^5$ **d)** $(2^7)^3$

2. Find the value of b.

a) $5^{18} = (5^b)^3$

b) $5^b = (5^3)(5^3)^3$

c) $5^{18} = 25^b$

3. Order these from least to greatest.

$(2^4)^4$, 8^5, $(4^3)^3$, $(2^7)^2$, $(-2)^8$, -2^{30}

4. Find values of m and n to make each true.

a) $(10^m)^n = 1000$ **b)** $(10^m)^n = 1{,}000{,}000$

5. Solve for n: $\left(\frac{3}{5}\right)^2 \times 25 = 3^n$

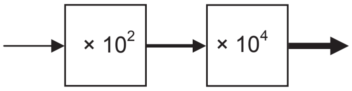
6. Write 8^{10} each of the following ways.

a) using the power law, $(a^r)^s = a^{rs}$

b) as a power of a product, $(ab)^r = a^r b^r$

c) as a power of a quotient, $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

7. To make a sound louder, a scientist puts it through two amplifiers in sequence. The first one amplifies the sound by a factor of 10^2 and the next amplifies by a factor of 10^4. By what factor is the sound increased?



8. Write each number as a perfect square and as a product of perfect squares.

a) 100 **b)** 144 **c)** 1600

9. $a^n \times b^n \times c^n$ is a perfect square, for example, $2^2 \times 3^2 \times 5^2 = 30^2 = 900$. What must be true about n? Explain why.

10. Give an example where each exponent law in this lesson would make a calculation simpler.

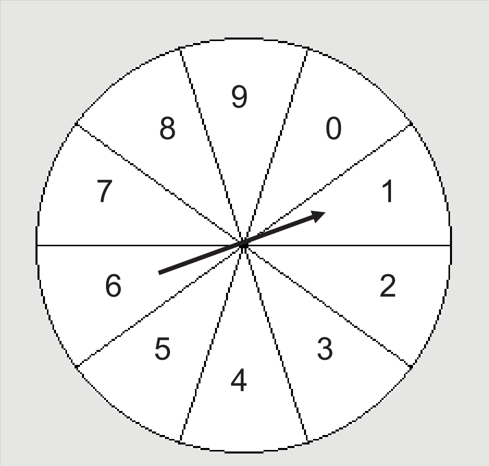
- Power law
- Power of a product law
- Power of a quotient law

1.1.3 Negative and Zero Exponents

Try This

Karma, Dodo, and Tshewang are playing a game with the spinner below.

- Each player spins the spinner twice.
- The player uses the two digits they spin to create a power of the form a^b. The player then chooses which number is the base and which is the exponent.
- The player(s) with the power that has a value closest to 1 wins the round and gets 1 point. If there is a tie, both players get 1 point.



Karma spins a 2 and a 9, Dodo spins a 3 and a 0, and Tshewang spins a 7 and a 1.

A. Which two powers could each player create?

B. Who do you think should NOT get a point for this round?

A power with an exponent that is 0 or negative needs special attention to understand what the exponent means and determine the value of the power.

- To determine the value of a power with 0 as the exponent, such as 2^0, you can use a pattern as shown in the table below:

Power	Value
2^4	16
2^3	8
2^2	4
2^1	2
2^0	?

Each value in the right hand column is half of the value above it. If you extend the pattern to 2^0, 2^0 is half of 2, or 1.

$2^0 = 1$

- You can also use the quotient law to determine the value of a power of 0.

For example:

- If you were calculating $2^8 \div 2^8$, the answer would be 1 because any number divided by itself is 1.
- Using the quotient exponent law, $2^8 \div 2^8 = 2^{8-8} = 2^0$.
- Because $2^8 \div 2^8 = 1$ and the exponent law tells us $2^{8-8} = 2^0$, then 2^0 must be 1.

The example described above would work the same for any base (other than 0) and any exponent.

$$\boxed{a^0 = 1,\ a \neq 0}$$

Note that 0^0 is not 1 because if you multiply zeroes you get zero. Since this is in conflict with the rule $a^0 = 1$, 0^0 is considered to be undefined.

- To determine the value of a power with a negative exponent, such as 2^{-1} or 2^{-2}, you can also use a pattern as shown in the table below.

Power	Value
2^4	16
2^3	8
2^2	4
2^1	2
2^0	1
2^{-1}	?
2^{-2}	?

Each value in the right-hand column is half of the value above it. If you extend the pattern to 2^{-1} and 2^{-2}, 2^{-1} is $\frac{1}{2}$ or $\frac{1}{2^1}$ and 2^{-2} is $\frac{1}{4}$ or $\frac{1}{2^2}$.

It looks like 2^{-n} is $\frac{1}{2^n}$.

$$\boxed{2^{-1} = \frac{1}{2} \text{ and } 2^{-2} = \frac{1}{4}}$$

You can also use the quotient law to understand negative exponents and to determine the value of a power with a negative exponent.

For example:

$3^{-2} = 3^{0-2} = 3^0 \div 3^2 = 1 \div 3^2 = \frac{1}{3^2}$ $\qquad$ $a^{-n} = a^{0-n} = a^0 \div a^n = 1 \div a^n = \frac{1}{a^n}$

$$\boxed{a^{-n} = \frac{1}{a^n}}$$

You can use fractions to understand negative exponents. In any situation where the bases in a numerator and denominator are the same but the exponent in the denominator is greater, for example, $\frac{3^5}{3^7}$, the result will be a fraction with a numerator of 1 and a denominator that is a power of the base.

For example, $3^{-2} = 3^5 \div 3^7 = \frac{3^5}{3^7} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3} = \frac{1}{3 \times 3} = \frac{1}{3^2}$.

C. Would you change your answer to **part B** knowing what you know now about exponents of 0 and 1? Explain.

Examples

Example Calculations with Negative and Zero Exponents

Calculate. **a)** $(3^5)^0$ **b)** 4^{-2} **c)** $10^4 \times 10^{-4}$ **d)** $10^4 \times 10^{-7}$

Solution	Thinking
a) $(3^5)^0 = 3^{5\times 0} = 3^0 = 1$	 **a)** Anything (but 0) to the exponent 0 is 1.
b) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	**b)** Since the exponent is –2, I used the rule $a^{-n} = \frac{1}{a^n}$.
c) $10^4 \times 10^{-4} = 10^{4+(-4)} = 10^0 = 1$	**c)** I added the exponents and got 0 and I knew that $a^0 = 1$, if $a \neq 0$.
d) $10^4 \times 10^{-7} = 10^{4+(-7)}$ $= 10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$	**d)** I added the exponents and got –3. Then I used the rule $a^{-n} = \frac{1}{a^n}$.

Practising and Applying

1. Calculate.

a) $(13^0)^2$

b) 5^{-2}

c) $5^3 \times 5^{-5}$

d) $0^1 + 1^0$

2. a) Calculate 3^{-2} and 9^{-1}.

b) Calculate 3^2 and 9^1.

c) How could you have predicted what you observed in **parts a) and b)**?

3. Solve for n: $4^n \times 2^{-18} = 1$

4. State possible values for a and b in each case:

a) $3^3 \times a^b = 1$ **b)** $3^3 \times a^b = 3$

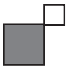
c) $7^a \times 7^0 = b$ **d)** $4^a \times 2^b = 1$

5. a) Express $\frac{1}{2}$ and $\frac{1}{3}$ as powers with negative exponents.

b) Calculate 6^{-1}.

c) Compare $\frac{1}{2} \times \frac{1}{3}$ with the result in **part b)**. What do you notice?

6. Use the numbers 0, 1, or, 2 in the grey box and the numbers –2, –1, 0, 1, 2 in the white box. If the numbers in the boxes must be different, how many different values can you create? List these values.



7. Which is greater in each pair? Justify your choices without calculating.

a) 3^{-5} or $(-5)^3$ **b)** 4^{-5} or $(-5)^4$

c) -9^2 or $(-9)^2$ **d)** -9^3 or $(-9)^3$

8. a) What do you know about the value of $(-6)^n$ when n is an odd positive integer? when n is an even positive integer?

b) Explain why a greater value of n may not result in a greater result for $(-6)^n$.

9. Explain why $\left(\frac{3}{10}\right)^{-2}$ is the reciprocal of $\left(\frac{3}{10}\right)^2$ and equal to $\left(\frac{10}{3}\right)^2$.

10. Explain why $a^0 = 1$, $a \neq 0$.

1.1.4 Fractional Exponents

Try This

A. i) Use the power law to explain why each of these is true.

$(9^4)^2 = 9^8$ $(4^2)^2 = 4^4$ $(7^3)^2 = 7^6$

ii) Use the results from **part i)** to find the square root of each number below. Express the square root as a power. Explain your method.

9^8 5^{10} 17^8 6^1

There are special names for particular types of exponential expressions.

- To **square** a number means to multiply a number by itself. A number is squared when the exponent is 2. "5 squared" means 5^2, which is 5 × 5 or 25.
- To understand how **square roots** and exponents are related, think back to earlier work with the exponent power law.

For example, you saw that $(2^3)^2 = 2^6$, so the square root of 2^6 or $\sqrt{2^6}$ is $2^{6 \div 2}$ or 2^3. Similarly, if $(3^5)^2 = 3^{10}$, then $3^5 = 3^{10 \div 2}$. So it seems reasonable that the square root of any number *n,* or n^1, has an exponent of $\frac{1}{2}$. That is, $\sqrt{n^1} = n^{\frac{1}{2}}$.

- You can also use the product law to understand the connection.

For example, $25^{\frac{1}{2}} \times 25^{\frac{1}{2}} = 25^1$. Since $25^{\frac{1}{2}}$ is multiplied by itself to get 25, $\sqrt{25^1}$ must be $25^{\frac{1}{2}}$ and so, $25^{\frac{1}{2}} = 5$. Since this would be true for any number,

$$\boxed{a^{\frac{1}{2}} = \sqrt{a}}$$

Note that the square root sign or the exponent $\frac{1}{2}$ is assumed to mean the positive, or **principal square root**.

- A number is **cubed** when the exponent is 3. "2 cubed" means 2^3, which is $2 \times 2 \times 2 = 8$. The **cube root** of 8, or $\sqrt[3]{8}$, is 2 because $2^3 = 8$.
- The cube root of any number can be represented using the exponent $\frac{1}{3}$ since $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1$. $\boxed{a^{\frac{1}{3}} = \sqrt[3]{a}}$
- The power law can be used to simplify complicated fractional exponents, using $a^{\frac{b}{c}} = (a^{\frac{1}{c}})^b$. For example, $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (2)^2 = 4$.

B. Use what you have learned about square roots and exponents to explain how to find the square roots of the powers in **part A ii)**.

Examples

Example Calculating Fractional Exponents

Evaluate.

a) $64^{\frac{1}{3}}$ **b)** $(8^{\frac{1}{3}})^2$ **c)** $8^{\frac{2}{3}}$ **d)** $8^{-\frac{2}{3}}$

Solution	**Thinking**
a) $64^{\frac{1}{3}} = \sqrt[3]{64}$ $= 4$	 **a)** 64 was raised to the power $\frac{1}{3}$, so I knew I needed its cube root, which was 4 (because $4 \times 4 \times 4 = 64$).
b) $(8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2$ $= 2^2 = 4$	**b)** I did what is inside the brackets first. I found the cube root of 8, which was 2, and then I squared 2 to get 4.
c) $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = 4$	**c)** Using the power law, $a^{\frac{b}{c}} = (a^{\frac{1}{c}})^b$, I knew $8^{\frac{2}{3}}$ was $(8^{\frac{1}{3}})^2$, which was **part b)**, so I knew it had the same answer, 4.
d) $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4}$	**d)** I knew a negative exponent meant $8^{-\frac{2}{3}}$ was $\frac{1}{8^{\frac{2}{3}}}$. I saw that the denominator was the power in **part c)**, so I knew it was 4 and the answer was $\frac{1}{4}$.

Practising and Applying

1. Calculate.

a) $144^{\frac{1}{2}}$ **b)** $125^{\frac{1}{3}}$ **c)** $64^{-\frac{1}{2}}$ **d)** $16^{\frac{3}{2}}$

2. $6 \times 6 \times 6 \times 6 = 1296$

a) What is $1296^{\frac{1}{4}}$?

b) What is $1296^{\frac{2}{4}}$?

c) Explain why $1296^{\frac{2}{4}}$ is the square root of 1296.

3. Simplify the exponents.
(Do not calculate the values.)

a) $(63^{48})^{\frac{1}{2}}$ **b)** $(30^{62})^{\frac{1}{2}}$ **c)** $(118^{26})^{\frac{1}{2}}$

4. Without calculating, show why all these have the same value.

$49^{\frac{3}{2}}$ $(49^{\frac{1}{2}})^3$ $(49^3)^{\frac{1}{2}}$ 7^3

5. a) List five values of k for which $(5^k)^{\frac{1}{3}}$ would result in an integer.

b) Verify two of your answers in **part a)** by substituting each value into $(5^k)^{\frac{1}{3}}$ and evaluating.

c) Would the result be an integer if $k = 100$? Explain.

6 Solve for n.

a) $(7^n)^{\frac{1}{2}} = 49$

b) $(125)^{\frac{n}{3}} = 1$

c) $2^{-n} = \left(\frac{1}{2}\right)^3$

7. Explain how to determine the square root of an expression with an even exponent.

Chapter 2 Scientific Notation

1.2.1 Scientific Notation with Large Numbers

Try This

A. i) Which expression below appears to have the greatest value? the least?

$(2 \times 10^4) \times (3 \times 10^5)$ $(15 \times 10^4) \times (4 \times 10^4)$ 6000 $\times (10^3)^2$

ii) Write each expression above in the form $k \times 10^n$. What do you notice?

Large numbers can be difficult to compare. One of the reasons we use symbols, or numerals to write numbers is that it makes them easier to compare. With numerals it is easy to see that, for example, 5,000,000 is greater than 3,200,000.

If the numbers are very large, for example, with twenty or thirty zeroes at the end, they can be very difficult to compare even when they are in numeral form. This is why scientific notation was developed as a way to write numbers.

- **Scientific notation** is a special way of representing numbers. A number in scientific notation is written in the form:

$$\boxed{m \times 10^a \text{ with the multiplier, } m\text{, being at least 1 but less than 10}}$$

For example, in scientific notation: $5{,}000{,}000 = 5 \times 10^6$

$3{,}200{,}000 = 3.2 \times 10^6$

Notice that the digits 5 and 3 move 6 places to the right as they change from representing millions to representing the number of 10^6.

$$\mathbf{5}{,}000{,}000 = \mathbf{5}00{,}000 \times 10^{\mathbf{1}} = \mathbf{5}0{,}000 \times 10^{\mathbf{2}} = \mathbf{5}000 \times 10^{\mathbf{3}} = \mathbf{5}00 \times 10^{\mathbf{4}} = \mathbf{5}0 \times 10^{\mathbf{5}} = \mathbf{5} \times 10^{\mathbf{6}}$$

$$\mathbf{3}{,}200{,}000 = \mathbf{3}20{,}000 \times 10^{\mathbf{1}} = \mathbf{3}2{,}000 \times 10^{\mathbf{2}} = \mathbf{3}200 \times 10^{\mathbf{3}} = \mathbf{3}20 \times 10^{\mathbf{4}} = \mathbf{3}2 \times 10^{\mathbf{5}} = \mathbf{3}.2 \times 10^{\mathbf{6}}$$

- It is important to know why the multiplier must be at least 1 and less than 10. When the multiplier of a number in scientific notation is within this set range, we can focus on the power of ten to understand the magnitude of the number.

- When the multiplier is always at least 1 and less than 10, numbers in scientific notation are also very easy to compare.

\- When you compare some large numbers, like 2 billion or 5 million, it is not the 5 or the 2 that is important; it is the million or billion, which is represented by the power of ten.

For example, $2 \times 10^9 > 5 \times 10^6$ since the first number is in the billions (10^9) and the second number is only in the millions (10^6). You do not even have to look at the multiplier.

\- When you compare large numbers like 3.2 billion or 9 billion, then the 3.2 and the 9 are important because the power of ten is the same.

For example, $3.2 \times 10^9 < 9.0 \times 10^9$ since they are both in the billions (10^9) and $3.2 < 9.0$.

- Numbers in scientific notation are easy to multiply and divide using the exponent product and quotient laws because powers of ten all have the same base.

For example, to calculate $(3.2 \times 10^6) \times (1.4 \times 10^4)$, you can change the order of the factors and multiply the number parts and the power parts separately.

$$(3.2 \times 10^6) \times (1.4 \times 10^4) = (3.2 \times 1.4) \times (10^6 \times 10^4) \quad [10^6 \times 10^4 = 10^{6+4}]$$
$$= 4.48 \times 10^{10}$$

$$(3.2 \times 10^6) \div (1.4 \times 10^4) = (3.2 \div 1.4) \times (10^6 \div 10^4) \quad [10^6 \div 10^4 = 10^{6-4}]$$
$$\approx 2.29 \times 10^2$$

If the result is a multiplier that is greater than or equal to 10, the multiplier must be decreased and the power of ten increased accordingly, or the answer will not be in scientific notation.

For example:

$$(3.2 \times 10^6) \times (4.4 \times 10^4) = 14.08 \times 10^{10} \quad [14.08 \geq 10, \text{ so } 14.08 \rightarrow 1.408 \times 10^1]$$
$$= 1.408 \times 10^1 \times 10^{10}$$
$$= 1.408 \times 10^{11}$$

- To add or subtract numbers in scientific notation, you first express the numbers using the same power of ten and then add or subtract the multipliers. If the powers are different, the standard practice is to begin by changing the powers to the same lower power.

For example:

$$3.45 \times 10^3 + 2.67 \times 10^2 = 34.5 \times 10^2 + 2.67 \times 10^2$$
$$= (34.5 + 2.67) \times 10^2$$
$$= 37.17 \times 10^2 \quad [37.17 \geq 10, \text{ so } 37.17 \rightarrow 3.717 \times 10^1]$$
$$= 3.717 \times 10^1 \times 10^2$$
$$= 3.717 \times 10^3$$

- On your calculator, if you enter a number in scientific notation, such as 3.45×10^{12}, it may appear as 3.45 ^{12}. If the exponent is less than 10, it is usually displayed with a leading 0 in the exponent. For example, 3.45×10^2 may appear as 3.45 ^{02}. For very large numbers beyond the capacity of the calculator display, the calculator will display a result in scientific notation form even if you did not enter the numbers in scientific notation. (Note that the number of digits that are displayed on your calculator will depend on how your calculator is programmed. For a display such as 3.45000 ^{12}, the calculator would have been programmed to display five decimal places.)

B. Write the numbers in **part A i)** in scientific notation to compare them. What do you notice?

Examples

Example 1 Expressing Large Numbers in Scientific Notation

Write each number in scientific notation.

a) 35,689

b) 0.25 million

c) 178,000,000,000,000

d) 198,434,892.73

Solution	**Thinking**
a) 35,689 is about 35 thousand so $35{,}689 = 35.689 \times 10^3$ $= 3.5689 \times 10^4$	 **a)** I needed a number between 1 and 10 as the multiplier to use with a power of ten. To decrease 35.689 to 3.5689 so it's less than ten, I had to increase 10^3 to 10^4.
b) 1 million = 10^6 so, 0.25 million $= 0.25 \times 10^6$ $= 2.5 \times 10^{-1} \times 10^5$ $= 2.5 \times 10^5$	**b)** Since 0.25 was less than 1, I knew I had to change it to 2.5. Since $0.25 = 2.5 \times 10^{-1}$, I substituted that into 0.25×10^6 and used the product law: $2.5 \times 10^{-1} \times 10^6 = 2.5 \times 10^5$.
c) 178,000,000,000,000 $= 178 \times 10^{12}$ $= 1.78 \times 10^{14}$	**c)** I knew that the number was 178×10^{12} since there are 12 zeroes after the digits 178. Since $178 = 1.78 \times 10^2$, I substituted that into 178×10^{12} and used the product law: $178 \times 10^{12} = 1.78 \times 10^2 \times 10^{12} = 1.78 \times 10^{14}$
d) 198,434,892.73 is about 198,000,000 or 198×10^6 So, 198,434,892.73 $= 198.43489273 \times 10^6$ $= 1.9843489273 \times 10^8$	**d)** I knew that the number was about 198×10^6 since there are 6 digits after 198 but before the decimal. To change the multiplier from 198.43489273 to 1.9843489273, I divided by a factor of 100, which meant I had to multiply the power of ten by a factor of 100, or 10^2 and $10^6 \times 10^2$ is $10^{8.}$

Example 2 Comparing Large Numbers using Scientific Notation

Which number in each pair is greater? Explain.

a) 3.4×10^7 or 9.5×10^6

b) 35 billion or 4098 million

c) $(7.2 \times 10^3) \times (2.5 \times 10^6)$ or 1.9×10^{10}

[Continued]

Example 2 Comparing Large Numbers using Scientific Notation [Continued]

Solution	**Thinking**
a) $3.4 \times 10^7 > 9.5 \times 10^6$ because $10^7 > 10^6$	 **a)** I knew that since both numbers were in scientific notation with different powers, I only had to compare the powers of ten.
b) 35 billion > 4098 million because 35 billion = 3.5×10^{10} 4098 million = 4.098×10^9 $10^{10} > 10^9$	**b)** I changed both numbers to scientific notation and then compared them: 4098 million $= 4098 \times 10^6$ $= 4.098 \times 10^3 \times 10^6$ $= 4.098 \times 10^9$ 35 billion $= 35 \times 10^9 = 3.5 \times 10^1 \times 10^9$ $= 3.5 \times 10^{10}$
c) $1.9 \times 10^{10} > (7.2 \times 10^3) \times (2.5 \times 10^6)$ because $(7.2 \times 10^3) \times (2.5 \times 10^6)$ $= 1.8 \times 10^{10}$ $1.9 \times 10^{10} > 1.8 \times 10^{10}$	**c)** I changed the product to a number in scientific notation so I could compare: $(7.2 \times 10^3) \times (2.5 \times 10^6)$ $= (7.2 \times 2.5) \times (10^3 \times 10^6)$ $= 18 \times 10^9 = 1.8 \times 10^{10}$

Example 3 Calculating using Scientific Notation

The earth travels at a speed of 107,210 km per hour around the sun.

a) Express the distance it travels in one day using scientific notation.

b) Using the result from **part a)**, estimate the distance it travels in one year.

Solution	**Thinking**
a) $107{,}210 = 1.07210 \times 10^5$ $24 \times 1.07210 \times 10^5$ $= 25.7304 \times 10^5$ $= 2.57304 \times 10^6$ km	 **a)** I wrote the speed in scientific notation. Then I multiplied by 24 since the speed was km per hour and I wanted km per day.
b) $365 \times 2.57304 \times 10^6$ $\approx 400 \times 2.5 \times 10^6$ $= 4 \text{ x } 10^2 \times 2.5 \times 10^6$ $= (4 \times 2.5) \times (10^2 \times 10^6)$ $= 10^1 \text{ x } 10^8$ $= 10^9$ Earth travels about 1 billion km a year.	**b)** From **part a)**, I knew how far it travelled in a day so all I had to do was multiply by 365 to estimate how far it travelled in one year. I rounded 365 to 400 to make it easy to calculate. • My estimate is probably high since I rounded 365 up quite a lot to 400.

Practising and Applying

1. Express each in scientific notation:

a) 2,000,000 **b)** 4,357,893,389

c) $(1.2 \times 10^7) \times (3 \times 10^6)$

d) 20,478,389.5

2. Express $(64 \times 10^{15}) \times (25 \times 10^{12})$ in scientific notation.

3. About 1.4 billion bytes of computer storage are needed to store all the information in an encyclopaedia. Write that number in scientific notation.

4. a) Write the number 42,357,200 as the product of two numbers, one being a power of ten. Find three answers.

b) Are any of the numbers you wrote in **part a)** in scientific notation? Explain.

5. Which is greater in each pair? Explain.

a) $(2 \times 10^6) \times (7 \times 10^{13})$ or 10^{20}

b) $(2 \times 10^4)^3$ or 250 billion

c) 3.5893×10^8 or 375×10^6

6. a) Calculate $(15 \times 10^{23}) \times (250 \times 10^{13})$.

b) Write the result in scientific notation.

7. It is reported that a computer pioneer in the United States has a net worth of 46.6 billion United States (U.S.) dollars. Use scientific notation to help you estimate the computer pioneer's worth in ngultrums, if 1 U.S. dollar = Nu 63.5.

8. Assume that a person blinks his or her eyes every 5 s. Estimate how many times you have blinked your eyes in your life. Record your answer in scientific notation.

9. A 45-year-old parrot has a heart beat rate of about 550 beats per minute. A 75-year-old man has a heart beat rate of about 70 beats per minute. Whose heart has beat more times in their life so far? Explain.

10. **a)** About 1.5×10^9 people live in China and about 7.5×10^5 people live in Bhutan. The number of people living in China is *N* times the number of people living in Bhutan. Find *N*.

b) About 3.4×10^7 people live in Canada. It is claimed that there are more than 500 times as many people in Canada as in Bhutan. Is this claim valid? Explain.

11. Why is □.□ × 10^9 always less than □.□ × 10^{10}?

12. The population of the world on August 22, 2006 was 6,536,211,569. Describe how you would convert this number to scientific notation.

CONNECTIONS: The Richter Scale

The magnitude of most earthquakes is measured on the Richter scale, invented by Charles F. Richter in 1934. The Richter magnitude is calculated from the amplitude of the largest seismic wave recorded for the earthquake.

Tsunami damage in Indonesia: caused by an earthquake

The Richter scale is designed to measure how severe an earthquake feels.

Magnitude	**Earthquake effects**	**Number each year**
2.5 or less	Usually not felt, but can be measured	900,000
2.5 to 5.4	Often felt, but only minor damage	30,000
5.5 to 6.0	Slight damage to buildings and other structures	500
6.1 to 6.9	May cause a lot of damage in populated areas	100
7.0 to 7.9	Major earthquake with serious damage	20
8.0 or greater	Great earthquake, which can totally destroy communities near the epicenter	1 every 5 to 10 years

The scale is based on the same principle as scientific notation. For example, a magnitude of 1.0 describes an earthquake that may go unnoticed. Each increase of 1 on the scale corresponds to an earthquake that is 10 times as strong.

- Magnitude 2 compared to magnitude 1 is 10^{2-1} or 10^1 times as strong.
- Magnitude 3 compared to magnitude 1 is 10^{3-1} or 10^2 times as strong.
- Magnitude 4.5 compared to magnitude 3 is $10^{4.5-3}$ or $10^{1.5}$ times as strong.

1. An earthquake measuring 5.7 on the Richter scale was felt in Bhutan on February 15, 2006. An earthquake estimated at 8.1 was experienced in 1897. About how many times stronger was the 1897 earthquake than the 2006 one?

1.2.2 Scientific Notation with Small Numbers

Try This

A. How do you know each statement is true?

i) $0.0001 = 1 \times 10^{-4}$ **ii)** $0.0003 = 3 \times 10^{-4}$

Scientific notation can also be used to help with comparing and calculating very small numbers less than 1.

- When a number is expressed in scientific notation, it is written in the form $m \times 10^a$ with the multiplier, m, being at least 1 but less than 10.

For example, $2{,}300{,}000{,}000 = 2.3 \times 10^9$

- Suppose you had a very small number less than 1, such as the speed at which the Japanese ma-drake bamboo grows, which is about 0.00005 km/h. If you wanted to express this in scientific notation, it would make sense that the multiplier would be 5, but it does not make sense to multiply it by a positive power of ten, which indicates that the multiplier is increased.

- To understand scientific notation for numbers less than 1, it is helpful to think about place value in terms of powers of ten.

For example, consider the number 0.00005.

millions decimal millionths

10^6	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
						0 ●	0	0	0	0	5	

The final digit of 0.00005 is in the 10^{-5}, or the hundred thousandths column, so you would read it as 5 hundred thousandths.

0.00005 or 5 hundred thousandths is 5×10^{-5} in scientific notation.

Another example, 0.000027, is shown below:

millions decimal millionths

10^6	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
						0 ●	0	0	0	0	2	7

The final digit of 0.000027 is in the 10^{-6}, or the millionths column, so you would read it as 27 millionths.

0.000027 or 27 millionths is $27 \times 10^{-6} = 2.7 \times 10^1 \times 10^{-6} = 2.7 \times 10^{-5}$ in scientific notation.

- You use numbers in scientific notation with negative exponents to compare and calculate the same way as you use numbers with positive exponents.

- It is helpful to become familiar with some of the common values:

$10^{-1} = 0.1$ $10^{-2} = 0.01$ $10^{-3} = 0.001$ $10^{-4} = 0.0001$ $10^{-5} = 0.00001$

B. i) Write the numbers 0.0001 and 0.0003 in scientific notation.

ii) How is writing the two numbers in scientific notation like what you did in **part A**?

Examples

Example 1 Expressing Numbers Less than 1 in Scientific Notation

Write each amount in scientific notation.

a) 0.00000234 **b)** 45 thousandths **c)** 0.000000000000003

Solution	**Thinking**
a) 0.00000234 = $2.34 \times 10^{?}$ = 2.34×10^{-6}	 **a)** I needed a multiplier that was between 1 and 10, so the decimal had to go between 2 and 3. That meant the digit that was in the millionths place, which was 2, had to move 6 places to the left to get to the ones place. So the power of ten had to be 10^{-6}.
b) 45 thousandths = 0.045 = $4.5 \times 10^{?}$ = 4.5×10^{-2}	**b)** I first changed the number to a decimal. I knew the multiplier was going to be 4.5 and that meant the digit 4 had to move from the hundredths place to the ones place, which was a move of 2 places. So the power of ten had to be 10^{-2}.
c) 0.000000000000003 = 3×10^{-15}	**c)** The digit 3 moved 15 places to the left to get to the ones place, so the power of ten had to be 3×10^{-15}.

Example 2 Calculator Displays and Scientific Notation

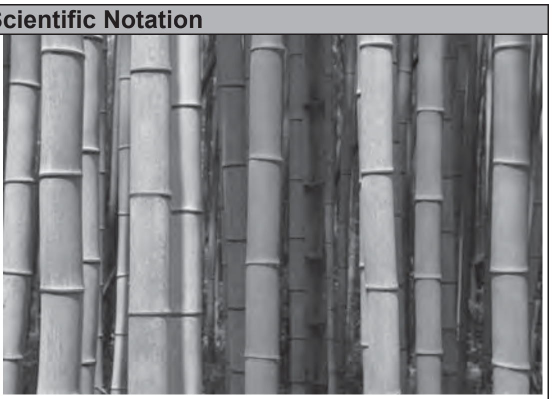
Write the value that corresponds to the following calculator displays:

a) $3.15620^{\ -07}$ **b)** $9.32388^{\ 07}$

Solution	**Thinking**
a) $3.15620^{\ -07}$ = 0.00000031562	 **a)** The 3 that was in the ones place in the multiplier had to move 7 places to the right because the exponent of the power of ten was –7. I had to add zeros in front of the 3 in order to do that.
b) $9.32388^{\ 07}$ = 93,238,800	**b)** The 9 in the ones place had to move 7 places to the left because the exponent of the power of ten was 7. That meant 9.32388×10^{7} changed to $932388\underline{00}$. (I added two zeros at the end in order to do that.)

Example 3 Calculating Using Scientific Notation

Some bamboo plants grow at a rate of 0.0000125 km per hour. Use scientific notation to find out how many metres they would grow in 6 weeks.



Solution	**Thinking**
Growth rate in scientific notation: 0.0000125 km/h $= 1.25 \times 10^{-5}$ km/h	• I expressed the growth rate in scientific notation. 
Growth rate in metres per hour: 1 km = 1000 m = 10^3 m $1.25 \times 10^{-5} \times 10^3$ m/h $= 1.25 \times 10^{-2}$ m/h	• The growth rate was in kilometres per hour, but I changed it to metres per hour by multiplying by 1000, or 10^3.
Hours in 6 weeks: $6 \times 7 \times 24 = 1008$ $1008 = 1.008 \times 10^3$	• I calculated the number of hours in 6 weeks by multiplying by the number of days in a week, 7, and the number of hours in a day, 24. • Then I wrote the number of hours in scientific notation.
Growth, in metres, in 6 weeks: $(1.25 \times 10^{-2}) \times (1.008 \times 10^3)$ $= 1.25 \times 1.008 \times 10^{-2} \times 10^3$ $= 1.26 \times 10^1$ = 12.6 m	• The actual growth is the growth rate (in m/h) multiplied by the number of hours.
The bamboo plants would grow 12.6 m in 6 weeks.	• Finally, I changed the number in scientific notation back to standard notation because it's easier to understand that way.

Practising and Applying

1. Write each in scientific notation:

a) 0.00007

b) 0.00134893

c) $(2.2 \times 10^{-6}) \times (4 \times 10^{-8})$

d) 356,158.7

2. On some calculators, the key stroke sequence 567 [SCI] [=] results in the display 5.67000 ^{02} on the screen. What would be displayed after each of the following calculator sequences?

a) 0.45 [SCI] [=]

b) 0.07394 [SCI] [=]

3. A calculator was used to evaluate 4864 × 2176. The result was displayed as 1.05841 ^{07}.

a) What value does the displayed result represent?

b) The answer is not exact in this case. How do you know without doing the calculation in full?

4. a) Which fractions are represented by the following calculator displays?

i) 3.33333 $^{-01}$

ii) 5.00000 $^{-01}$

b) For each expression in **part a)**, explain whether you think it is an exact or approximate value of the fraction.

5. Express the product of the following multiplication in scientific notation:

$(3.5 \times 10^{4}) \times (1.5 \times 10^{-17}) \times (4.0 \times 10^{8})$

6. Pema begins to enter the following expression into his calculator:

$(7.3 \times 10^{-5}) \times (3.1 \times 10^{32}) \times (4.09 \times 10^{-18})$

He enters the multipliers in each number first. He finds that 7.3 [×] 3.1 [×] 4.09 produces a result of 9.25567 ^{01}. What will the final display look like?

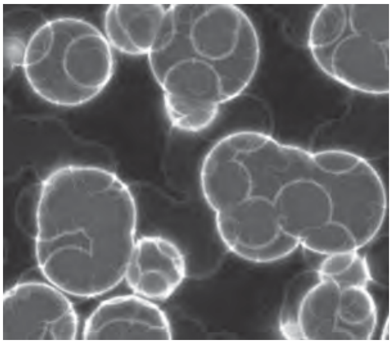
7. The average growth rate of a child between birth and age 18 is 1.1×10^{-8} km per hour.

a) How fast is that in kilometres per day?

b) How fast is that in centimetres per year?

8. The speed that a bacterium can move on a kitchen table is 0.00016 km per hour. Which is a better estimate for how long it would take to move 1 m?

35 min or 350 min



9. Consider the product of any two quantities written in scientific notation, $(m \times 10^{p}) \times (n \times 10^{q})$. When the value of the product is written in scientific notation, is it possible for the exponent of the power of ten to be

a) more than $(p + q)$?

b) equal to $(p + q)$?

c) less than $(p + q)$?

Justify your answers with explanations or suitable examples.

Chapter 3 Rational and Real Numbers

1.3.1 Estimation with Rational Numbers

Try This

A. You are dividing one fraction by another. The quotient is about 1.5.
What might the fractions be? List three possible pairs of fractions.

A **rational number** is a number that can be expressed as a quotient, or ratio of the form $\frac{a}{b}$, where a and b are **integers**. Integers are rational numbers because they can be thought as fractions with a denominator of 1, for example, $-4 = -\frac{4}{1}$ and $7 = \frac{7}{1}$. Decimals like 2.3 and 0.333… are also rational numbers because 2.3 as a fraction is $2\frac{3}{10}$ or $\frac{23}{10}$ and 0.333… is $\frac{1}{3}$.

- When you calculate using rational numbers, it is appropriate to estimate when an exact answer is not needed or not suitable. Estimation is also appropriate when an exact calculation is otherwise difficult.

For example, suppose you find out that during re-entry a space shuttle is travelling at about 28,000 m/s and you want to compare that with the speed of a car travelling at about 55 km/h. Since the speeds of a car and a shuttle vary and the speeds you are working with are estimates to begin with, it is appropriate to estimate.

Change 55 km/h to m/s:

55 km/h = 55,000 m/h [1 h = 3600 s]
= 55,000 m/3600 s [Round 55,000 and 3600 to easy-to-divide numbers.]
≈ 60,000 m/4000 s
≈ 15 m/s

Compare 28,000 m/s and 15 m/s:

28,000 ÷ 15 [Round 28,000 to 30,000 to get an easy-to-divide number.]
≈ 30,000 ÷ 15
= 2000

28,000 m/s is about 2000 times as fast as 55 km/h.

- Notice the use of the ≈ sign above. This sign is used instead of the equal sign when estimated or approximated values are being used.

For example:

$37 \times 42 \approx 40 \times 40 = 1600$ $0.666... \approx 0.67$

- Sometimes an exact calculation does not make sense.

For example, suppose Dorji walks about 3 km to school each morning and returns the same way in the afternoon. About how many kilometres does Dorji walk to and from school in one month?

Since the number of days of school in a month varies, you estimate there are about 25 school days in a month.

25 days in a month × 6 km a day = 150 km in a month

- Estimation is also used when you want to know if an amount is reasonable.

For example, Kuenzang has Nu 1000 available to purchase a uniform, ten notebooks, writing instruments, and a geometry set to begin Class IX.

- A uniform costs between Nu 500 and Nu 600.
- Each notebook will cost between Nu 10 and Nu 20.
- The writing instruments will total about Nu 100.
- The geometry set costs between Nu 40 and Nu 60.

Does Kuenzang have enough money?

$600 + 10 \times 20 + 100 + 60$ [Use the greatest value for each item to be sure.]
$= 960$

Since he has Nu 1000 and his materials will cost no more than Nu 960, he will have enough money.

B. Explain how you selected your fractions in **part A**.

Examples

Example 1 Estimating Products and Quotients

Estimate the value of each.

a) 13.57×0.54 **b)** $\left(\frac{1}{3}\right)^{50}$ **c)** $(35.82 + 28.1) \div 12.56 \times 19.89$

Solution	**Thinking**
a) 13.57×0.54 $\approx 14 \times \frac{1}{2} = 7$	 **a)** 0.54 is about one-half and 13.57 is about 14 so I knew the answer was about 7.
b) $\left(\frac{1}{3}\right)^{50} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \ldots$ ≈ 0	**b)** Each time I multiplied by another $\frac{1}{3}$, the value got lower and lower. After 50 times, the answer must be close to zero.
c) $(35.82 + 28.1) \div 12.56 \times 19.89$ $\approx (36 + 28) \div 12 \times 20$ $= 64 \div 12 \times 20$ $\approx 60 \div 12 \times 20$ $= 5 \times 20 = 100$	**c)** I rounded each number to a whole number so it was easier to calculate. Then I rounded 64 to 60 so I could divide it easily by 12.

Example 2 Solving a Problem by Estimating

The bus company offers two bus rides daily from Phuntsholing to Thimphu and two more bus rides from Thimphu to Phuntsholing. Each bus has a capacity of 50 passengers. Each passenger pays a one-way fare of Nu 190. If the bus company expects $\frac{5}{6}$ of the seats to be occupied on each ride, estimate the total revenue expected from this route each day.

Solution

Number of passengers per ride:

$$\frac{5}{6} \times 50 = \frac{250}{6}$$
$$= 250 \div 6$$
$$\approx 240 \div 6 = 40$$

Revenue per ride:

$40 \times 190 \approx 40 \times 200 = 8000$

Total revenue for four rides:

$4 \times 8000 = 32,000$

Total revenue is about Nu 32,000.

Thinking

- First I estimated how many passengers were $\frac{5}{6}$ of 50. I rounded 250 to 240 so it was easy to divide by 6.
- Then I estimated how much money 40 passengers would pay per trip. I rounded 190 to 200 so it was easy to multiply by 40.
- Finally, I multiplied by 4 because there were 4 rides per day—2 from Phuntsholing to Thimphu and 2 from Thimphu to Phuntsholing.

Practising and Applying

1. Estimate.

a) 25.5×7.8

b) $14.5324 \div 1.98$

c) $\frac{3}{5} \times 297$

d) $(2^7)^3$

2. The oranges in the basket on Dema's back weigh about 18 kg. On average there are 11 oranges per kilogram. Estimate the number of oranges Dema is carrying.

3. Usually it takes Penjor's family three days to complete the rice harvest with seven workers. This year there are only four workers.

a) Estimate how many days it will take to complete the rice harvest this year. Is your estimate lower or higher than the actual amount? How do you know?

b) Why might your estimate be different from someone else's estimate?

4. a) Estimate the number of seconds in one week.

b) Calculate the actual number of seconds. How does it compare with your estimate?

5. There are seven lower secondary schools in a dzongkhag with an average of 485 students per school. If about 55% of the students are male, about how many females attend these schools?

6. Describe two different ways to estimate each.

a) $-7 \times \left(\frac{2}{3}\right)^4$

b) $7.06 \div 0.3$

c) 0.25×465

d) 1078×512

7. Describe a situation involving large numbers in scientific notation where estimation would be appropriate.

1.3.2 Order of Operations

Try This

You can arrange the numbers 1, –2, 3, –4, and 5 in any order with any operation sign (+, –, ×, ÷) between each of pair of numbers to make an expression equal to 3. Brackets may also be included.

$$1 \square (-2) \square 3 \square (-4) \square 5 = 3$$

For example, one possible expression is 3 + 1 + 5 + (–2) + (–4) = 3.

A. **i)** Find another solution.

ii) Insert or move brackets, without changing any other operations in your expression, to create three expressions that result in an answer other than 3.

The order of operations you learned for working with integers and fractions also applies to rational numbers.

- Anything inside a bracket is done first.

For example: $\left(3\frac{3}{4}+\left(-\frac{3}{5}\right)\right) \times 3$ [Calculate $\left(3\frac{3}{4}+\left(-\frac{3}{5}\right)\right)$ first.]

$= 3\frac{3}{20} \times 3$

$= 9\frac{9}{20}$

Square brackets can be used when there are brackets inside brackets. For example: $\left(3\frac{3}{4}+\left(-\frac{3}{5}\right)\right)$ could be written as $\left[3\frac{3}{4}+\left(-\frac{3}{5}\right)\right]$.

- Exponents are next to brackets in order. Exponents are simplified before doing division, multiplication, addition, or subtraction.

For example: $-6.3 + \left[7.2^{5-3} + \left(-\frac{5}{6}\right)\right] \div 11$ [Simplify 7.2^{5-3} first.]

$= -6.3 + \left[51.84 - \frac{5}{6}\right] \div 11$ [Calculate $\left[51.84 - \frac{5}{6}\right]$ next.]

$\approx -6.3 + 51.007 \div 11$ [Calculate 51.007 ÷ 11 next.]

$= -6.3 + 4.637 = -1.663$

- Division and multiplication are done ahead of subtraction or addition.

For example: $-3.2 + 5 \times 6$ [Calculate 5 × 6 first.]

$= -3.2 + 30 = 26.8$

The order of operations is listed below:

Brackets

Exponents

Division or Multiplication in the order they appear from left to right

Addition or Subtraction in the order they appear from left to right

B. How did inserting, moving, or removing brackets in your expression from **part A i)** change the result for **part A ii)**? Explain using the order of operations.

Examples

Example 1 Describing the Order of Operations

For each, complete the calculation using the order of operations.

a) $(3.8 - 4.23 \times 4.6)^2$ **b)** $[4.5 + (-7.5^2 \div 3)] + 1.2^{-3}$ **c)** $20 \div \frac{1}{2}$ of 8

Solution

a) $(3.8 - 4.23 \times 4.6)^2$

$= (3.8 - 19.458)^2$

$= (-15.658)^2$

≈ 245.173

b) $[4.5 + (-7.5^2 \div 3)] + 1.2^{-3}$

$= [4.5 + (-56.25 \div 3)] + 1.2^{-3}$

$= [4.5 + (-18.75)] + 1.2^{-3}$

$= [4.5 - 18.75] + \frac{1}{1.2^3}$

$\approx -14.25 + 0.579 = -13.671$

c) $20 \div \frac{1}{2}$ of $8 = 20 \div (\frac{1}{2} \times 8)$

$= 20 \div 4 = 5$

Thinking

a) I did what was inside the brackets first. Inside the brackets, I multiplied before adding. Once I finished inside the brackets, I did the exponent.

b) First I calculated what was inside the round brackets because they were inside the square brackets. Inside the round brackets, I calculated the exponents and then divided. Then I added to complete the calculation inside the square brackets. I then calculated with the exponent. The last thing I did was add.

c) I know that when I see a calculation like $\frac{1}{2}$ of 8 it really means $(\frac{1}{2} \times 8)$.

Example 2 Calculations using Order of Operations

Calculate.

a) $40.5 - 3 \times (-2)^5 \div [10 + 3 \times (-2.4)]$ **b)** $\left(3\frac{1}{3} \times \left(-2\frac{1}{10}\right) + 3\frac{3}{5}\right)^{-1}$

Solution

a) $40.5 - 3 \times (-2)^5 \div [10 + 3 \times (-2.4)]$

$= 40.5 - 3 \times (-32) \div (10 - 7.2)$

$= 40.5 - 3 \times (-32) \div 2.8$

$= 40.5 + 96 \div 2.8$

$\approx 40.5 + 34.29 = 74.79$

b) $\left(3\frac{1}{3} \times \left(-2\frac{1}{10}\right) + 3\frac{3}{5}\right)^{-1}$

$= \left(-\frac{10}{3} \times \frac{21}{10} + 3\frac{3}{5}\right)^{-1}$

$= \left(-7 + 3\frac{3}{5}\right)^{-1}$

$= \left(-3\frac{2}{5}\right)^{-1} = \left(-\frac{17}{5}\right)^{-1} = -\frac{5}{17}$

Thinking

a) First I did what was in the square brackets. Inside the brackets, I multiplied before adding. I did the exponent work at the same time, since it didn't affect anything else. I then multiplied –3 by –32 and then divided the product, 96, by 2.8. Finally I added.

b) Inside the brackets, I multiplied first after changing each of the two mixed numbers to a fraction, and then I added. After I had finished all the calculations inside the brackets, I used the exponent.

Practising and Applying

1. Which calculation should you do first in each situation?

a) $(4 – 4.3 × 5.7)^2$

b) $(5.8 ÷ 3.6 × 2.7)^5$

c) $(–6.7) ÷ 3.2 × 5.7$

d) $0.8 × (2.3)^3 – 0.5$

2. Calculate:

a) $15.3 + 3.1 × 4 – 8 × (–0.6)$

b) $\frac{1}{9} + \left(\frac{5}{8} - \frac{1}{2}\right)^{-1}$

c) $(2.8 + (–1)^4) × (3.4^0 – 4 × 0.5)$

d) $2.8 ÷ (–3.1 + 5^2)$

3. Which, if any, of the following expressions equals 2.4?

A. $3.4 + 2 × (0.9 – 2.8 ÷ 2)$

B. $0.3^3 – 1.5 × 0.2$

C. $2.4 × (4.5 + 1.3 × 0.18)^0$

D. $–[1.5 – 1.3 × (2 + 0.5 × 2)]$

4. Place brackets, as needed, to make each equation correct.

a) $10 × \left(\frac{5}{2}\right)^{-1} – 20 × 0.3^2 = 2.2$

b) $\left(\frac{2}{3}\right)^{-1} – 1 – \frac{2}{3} × 3 = \frac{1}{2}$

c) $\frac{1}{4} × \frac{1}{4} – \frac{1}{4} + \frac{1}{4} ÷ \frac{1}{4} = 1$

d) $\frac{2^3}{5} – 2 × 3 ÷ 4 + 1 = \frac{2}{5}$

5. a) Insert brackets, if necessary, to make the expression below as great as possible.

$$1 × \frac{2}{3} + 3 × 4 + \frac{2}{5}$$

b) Insert brackets, if necessary, to make the expression as small as possible.

6. Five rational numbers are added together. Some are positive and some are negative. Would the placement of brackets affect the answer? Use an example to explain.

7. A combination of addition and subtraction signs is used to describe a calculation involving four rational numbers. Would the placement of the brackets usually affect the answer? Provide an example to support your explanation.

8. Explain why this calculation is easy to do mentally even with so many calculations in it.

$$7 + (4 + 11 × 3 + 5^3 × 0.43)^0 × 3$$

9. Yeshey needed to estimate the total area of two square walls in order to decide how much paint to buy. The first square wall had a side length of about 2.5 m and the second had a side length of about 3.7 m. Yeshey added the lengths together and then squared the number to get a total area of 38.44 m^2. When Kinley heard the result, she thought it was too great. Is Kinley right? Explain.

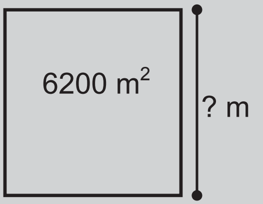
10. Why is the order of operations important in calculating? Provide an example to support your explanation.

1.3.3 Square Roots

Try This

A park is in the shape of a square. The area is 6200 m^2.

A. About how long is the side of the park?



The square root of a number can be multiplied by itself to get the number. You can write the square root using the symbol $\sqrt{\ }$.

For example, since 10 × 10 = 100, $\sqrt{100} = 10$.

- A square root may be calculated by multiplying mentally, by hand, or by using a calculator. The result will depend on the method you choose.

For example: $\sqrt{200} \approx 14$, since $14 \times 14 = 196$ and $196 \approx 200$

$\sqrt{200} \approx 14.1$, since [$\sqrt{\ }$] 200 [=] 14.14213562

- All real numbers other than 0 are actually squares of two pairs of numbers, positive and negative opposites. For example, 196 = 14 × 14 and (–14) × (–14). The principal square root of a number is the positive value. This value is appropriate in most practical situations, such as working with lengths.

For example, you want to know the side length of a square with an area of 25 cm^2. The answer would be $\sqrt{25}$, which means the positive square root of 25 since a negative square root would not make sense.

- Some, but not all, square roots are rational numbers.

For example, $\sqrt{25}$ is rational, as are $\sqrt{\frac{25}{81}}$ and $\sqrt{0.49}$ because they can each be written as the quotient of two integers ($\sqrt{25} = 5 = \frac{5}{1}$, $\sqrt{\frac{25}{81}} = \frac{5}{9}$, and $\sqrt{0.49} = 0.7 = \frac{7}{10}$).

- Many other square roots, such as $\sqrt{2}$, cannot be written as the quotient of integers and are therefore not rational. These are called **irrational** numbers.

- The square root of a number greater than 1 is always less than the number itself and the square root of a number between 0 and 1 is always greater than the number itself.

For example: $\sqrt{2}$ is about 1.414, which is less than 2.

$\sqrt{0.64} = 0.8$, which is greater than 0.64.

- It is easy to describe the square root of a number that is written as a power of even numbers.

For example: $\sqrt{7^{10} \times 5^4} = (7^{10} \times 5^4)^{\frac{1}{2}} = 7^{10 \div 2} \times 5^{4 \div 2} = 7^5 \times 5^2$

- It is also useful to write a number as even powers of numbers.

For example:

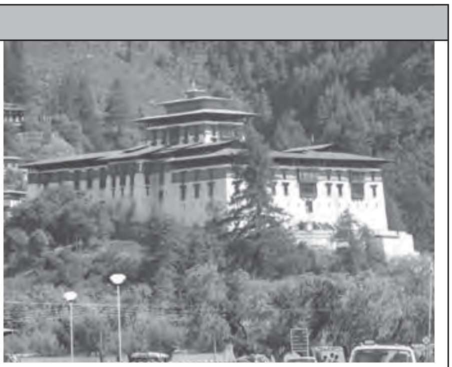
Since $3600 = 36 \times 10^2$, $\sqrt{3600} = \sqrt{36 \times 10^2} = \sqrt{36} \times \sqrt{10^2} = 6 \times 10$.

B. How do you know the side length of the park in **part A** is $\sqrt{6200}$ m?

Examples

Example 1 Multiplying by Hand

The area of the land inside the square wall around a dzong is about 3500 m^2. About how long is each side of the wall?



Solution	**Thinking**
$\sqrt{3500}$ = ? → ? × ? = 3500	 • The square root can be thought of as the side length of a square with a certain area, so I knew I needed to find $\sqrt{3500}$.
50 × 50 = 2500 too low 60 × 60 = 3600 too high	• I tried squaring 50 and then 60 (since 5 × 5 = 25 and 6 × 6 = 36 and 35 is between 25 and 36). • 50 was too small and 60 too big, but I knew it was closer to 60 so I tried 59 next.
59 × 59 = 3481 a bit low	• 59 was still a bit low, so I tried 59.2 next.
59.2 × 59.2 = 3504.64 close enough If $59.2^2 = 3504.64$, then $\sqrt{3500} \approx 59.2$. Each side of the wall is about 59.2 m.	• When I squared 59.2, it was about 3500.

Example 2 Using a Calculator

The area of Nepal is about 140,800 km^2. If Nepal were shaped like a square, what would its side length be?

Solution	**Thinking**
$\sqrt{140{,}800} = ?$ $300 \times 300 = 90{,}000$ $400 \times 400 = 160{,}000$ $\sqrt{140{,}800} \approx 375.2$ The side length would be about 375.2 km.	• I knew I wanted to find $\sqrt{140{,}800}$. • I estimated first so I could check my answer. 300 was way too small. 400 was a bit big so I knew it was going to be a little less than 400. • I used a calculator: [$\sqrt{\ }$] 140800 [=] 3.75233^{02}, which is about 375.2. • 375.2 is just under 400, as I estimated.

Practising and Applying

1. To which whole number is each closest?

a) $\sqrt{39}$ **b)** $\sqrt{97}$ **c)** $\sqrt{290}$ **d)** $\sqrt{6438}$

2. Estimate to decide which one of these answers is wrong. Explain your thinking.

A. $\sqrt{5612} \approx 74.9$ **B.** $\sqrt{91{,}230} \approx 30.2$

C. $\sqrt{517{,}432} \approx 719.3$

3. Use $\sqrt{68} \approx 8.2$ to estimate each.

a) $\sqrt{6800}$ **b)** $\sqrt{680{,}000}$

4. How do you know $\sqrt{0.444...}$ is a rational number?

5. A square has an area of 4823 m^2.

a) How do you know the side length is about 70 m?

b) Is it less than or more than 70 m?

c) Calculate the side length to the nearest tenth of a metre.

6. a) i) Compare $\sqrt{28}$ with $\sqrt{4} \times \sqrt{7}$.

ii) Compare $\sqrt{300}$ with $\sqrt{30} \times \sqrt{10}$.

iii) What do you notice?

b) Show that $\sqrt{20 \times 4} = \sqrt{20} \times \sqrt{4}$.

7. Why might you suspect that $\sqrt{8}$ is not a rational number?

8. What number multiplied by itself equals 864? Why are there two possible answers to this question?

9. The formula to estimate the number of seconds it takes for an object to fall h metres is $0.45 \times \sqrt{h}$.

a) About how many seconds will it take for an object to fall from each height?

i) 100 m **ii)** 1000 m **iii)** 50,000 m

b) Which square root did you use for your answers in **part a)?** Why?

10. Why is the square root of a whole number greater than 1 always less than the number?

11. a) Pema factored 142,884 to find its square root: $142{,}884 = 9^2 \times 7^2 \times 6^2$

Why does $\sqrt{142{,}884} = 9 \times 7 \times 6$?

b) Factor 9216 to calculate $\sqrt{9216}$.

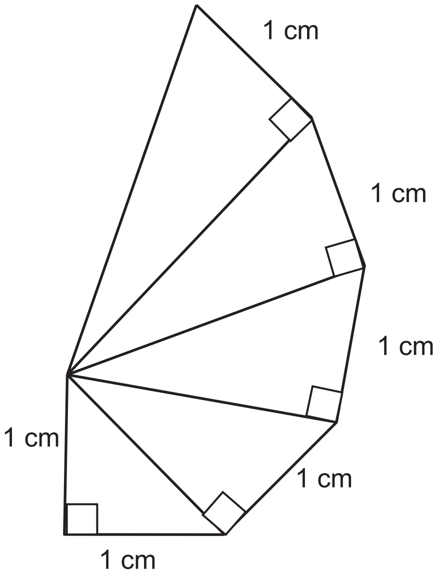
12. Write each as the product of a multiplier between 1 and 100 and an even power of ten and then estimate the square root.

a) 46,216 **b)** 626,147

13. Describe how to estimate $\sqrt{39{,}417}$ without a calculator.

1.3.4 EXPLORE: Representing Square Roots

The spiral below is called an Archimedes spiral, after a famous ancient Greek mathematician. It is made of right triangles, each with a base of 1 cm. Each additional right triangle is built on the hypotenuse of the last triangle.



Note that this diagram is not drawn to scale.

A. Draw accurately the spiral above.

B. Extend the spiral by drawing six more right triangles. Build each triangle on the hypotenuse of the last triangle and give it a base of 1 cm.

C. Measure the hypotenuse of each of the 11 triangles in your spiral. Record the lengths to the nearest tenth of a centimetre.

D. Use the Pythagorean theorem to calculate each hypotenuse length. Express the length as a square root.

E. i) What patterns do you notice?
ii) Do you think those patterns will continue? Explain.

F. Which triangle will have a hypotenuse that measures $\sqrt{40}$ cm?

G. Which triangles have hypotenuse lengths that you are confident are rational numbers? Explain.

1.3.5 Representing Real Numbers

Try This

A. List five rational numbers that fit each statement below. Include no more than two integers in each list.

i) the number is less than –2

ii) the number is greater than 2π

iii) the number is between $\sqrt{2}$ and $\sqrt{5}$

Rational numbers can be represented as a ratio, or quotient of integers. Irrational numbers cannot. You already know some irrational numbers, such as $\sqrt{2}$ and π.

• Rational numbers can always be represented with terminating or non-terminating repeating decimals.

For example, $3\frac{5}{8}$ as 3.625 or $-3\frac{4}{9}$ as –3.4444….

• For irrational numbers, the decimal part of the representation is both non-terminating and non-repeating.

For example, $\sqrt{2}$ is 1.414213562… and π is 3.14159…. Another example of an irrational number is 0.1234567891011121314….

• When you add, subtract, multiply, or divide (other than by 0) rational numbers, the result is always rational. But when you add, subtract, multiply, or divide irrational numbers, the result is sometimes rational and sometimes irrational.

For example:

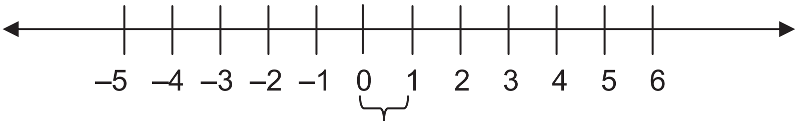
- *Addition*: $\sqrt{2} + (-\sqrt{2})$ is 0, which is rational, but $\sqrt{2} + \sqrt{3}$ is irrational.
- *Subtraction*: $\sqrt{2} - \sqrt{2}$ is 0, which is rational, but $\sqrt{2} - \sqrt{3}$ is irrational.
- *Multiplication*: $\sqrt{2} \times \sqrt{50} = 2^{\frac{1}{2}} \times 50^{\frac{1}{2}} = (2 \times 50)^{\frac{1}{2}} = \sqrt{100} = 10$, which is rational,

but $\sqrt{2} \times \sqrt{3} = 2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (2 \times 3)^{\frac{1}{2}} = \sqrt{6}$ is irrational.

- *Division*: $\sqrt{60} \div \sqrt{15} = \sqrt{4}$ is 2, which is rational, but $\sqrt{30} \div \sqrt{15} = \sqrt{2}$, which is irrational.

• **Real numbers** are made up of rational and irrational numbers. Each point on a number line can be represented by a real number — between any two integers there are an infinite number of rational and irrational numbers.

For example:

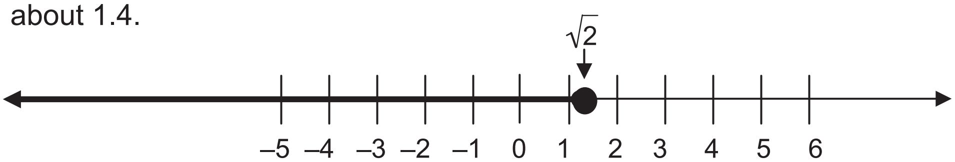


[Continued]

Between 0 and 1 are all rational numbers of the form $\frac{a}{b}$ where $a < b$ and a and b are positive whole numbers, as well as many irrational numbers, for example, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{2}}{5}$ and $\frac{\pi}{4}$.

- A number line can be used to graph **inequalities** related to real numbers. Inequalities are statements that describe how one quantity is greater than or less than another.

For example, the inequality statement $n \leq \sqrt{2}$ indicates that n can be any number less than or equal to $\sqrt{2}$. On the number line below, those values are shown with the thick line and arrow pointing left and the solid circle placed at $\sqrt{2}$, which is about 1.4.

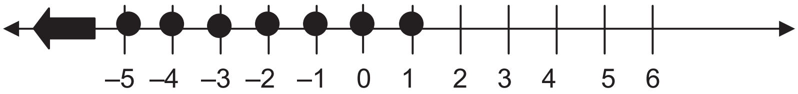


$n \leq \sqrt{2}$

If the inequality statement were $n < \sqrt{2}$, you would draw an open circle at the approximate location of $\sqrt{2}$ (≈ 1.4) to show that it was not included.

$n < \sqrt{2}$

If only integer solutions are allowed, you would draw a solid circle at each integer, rather than a thick line and arrow. You would write the possible values that would solve this inequality in this format: {…, –3, –2, –1, 0, 1}.



$n < \sqrt{2}$, n is an integer

B. i) Graph one of the inequalities you solved in **part A** on a number line.
ii) Find one irrational and one rational number that would solve each inequality**.**

Examples

Example 1 Deciding Whether a Number is Rational

Which of these are rational?

a) $\sqrt{2} \times \sqrt{2}$ **b)** $\sqrt{2} \times \sqrt{\frac{2}{9}}$ **c)** 3.23232323… **d)** $\frac{3}{2} \times \pi$

Solution	**Thinking**
a) $\sqrt{2} \times \sqrt{2} = (\sqrt{2})^2 = 2$ 2 is rational.	 **a)** I knew that when you square a square root, you get the original number, $(\sqrt{2})^2 = 2$.
b) $\sqrt{2} \times \sqrt{\frac{2}{9}} = \sqrt{\frac{2 \times 2}{9}}$ $= \sqrt{\frac{4}{9}} = \frac{2}{3}$ $\frac{2}{3}$ is rational.	**b)** Since $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, I knew that $\sqrt{2} \times \sqrt{\frac{2}{9}} = \sqrt{2 \times \frac{2}{9}} = \sqrt{\frac{2 \times 2}{9}}$.
c) 3.23232323… is rational.	**c)** The decimal repeats, so I knew it was a rational number.
d) $\frac{3}{2} \times \pi$ is irrational.	**d)** I knew that π is irrational so if it is multiplied by a rational number other than 0, it will stay irrational.

Example 2 Solving and Graphing Inequalities

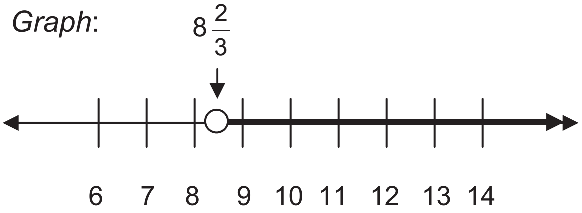
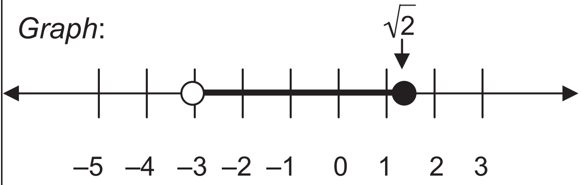
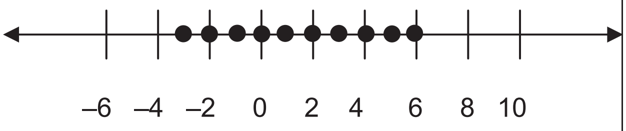
a) Find three rational and three irrational numbers to solve each inequality. Then graph each inequality.

i) $n > 8\frac{2}{3}$, n is a real number **ii)** $-3 < n \leq \sqrt{2}$, n is a real number

b) Graph $-\sqrt{10} \leq n < 2\pi$, n is an integer.

Solution	**Thinking**
a) i) $n > 8\frac{2}{3}$, n is a real number *Solutions*: Rational solutions: $9\frac{2}{3}$, $10\frac{2}{3}$, and $11\frac{2}{3}$ Irrational solutions: $9\sqrt{2}$, $10\sqrt{2}$, and $11\sqrt{2}$ [Continued]	 **a) i)** I knew that I needed numbers that were greater than but not including $8\frac{2}{3}$. • For the rational numbers, I used a combination of $\frac{2}{3}$ with integers greater than 8. • For the irrational numbers, I used multiples of $\sqrt{2}$ that were greater than $8\frac{2}{3}$. Since $\sqrt{2}$ is about 1.4, multiplying $\sqrt{2}$ by 9, 10, and 11, would result in irrational numbers greater than $8\frac{2}{3}$.

Example 2 Solving and Graphing Inequalities [Continued]

Solution	**Thinking**
a) i) $n > 8\frac{2}{3}$, n is a real number *Graph*: 	**a) i)** I knew the graph had to have an open circle at $8\frac{2}{3}$ because $n > 8\frac{2}{3}$ does not include $8\frac{2}{3}$. I also knew that it had to have a solid line and arrow pointing to the right because n is a real number > $8\frac{2}{3}$.
ii) $-3 < n \leq \sqrt{2}$, n is a real number *Solutions*: Rational solutions: –2, –1, and 0 Irrational solutions: $\sqrt{2}$, $\sqrt{2} - 1$, and $\sqrt{2} - 2$ *Graph*: 	**ii)** I knew that all the solutions for n were between –3 and $\sqrt{2}$, not including –3 but including $\sqrt{2}$, or about 1.4. So I knew –2, –1, and 0 would work. • For the irrational numbers, I knew I could use $\sqrt{2}$. I also knew I could subtract 1 or 2 from $\sqrt{2}$ and the result would still be irrational and in the right range. • I knew the graph had to have an open circle at –3, because $-3 < n$ does not include –3, and a solid circle at $\sqrt{2}$ (about 1.4), because $n \leq \sqrt{2}$ includes $\sqrt{2}$. I also knew that it had to have a solid line between the open and solid dots because n is a real number.
b) $-\sqrt{10} \leq n < 2\pi$, n is an integer 	**b)** Since 2π is a bit more than 6 and n is an integer, I knew n had to be 6 or less. Since $-\sqrt{10}$ is a bit less than –3 and n is an integer, I knew n had to be –3 or greater. I drew solid dots at each integer value from –3 to 6 because n is an integer.

Practising and Applying

1. Which values below are rational? Explain how you know.

A. 0.135135135… **B.** 3.457

C. 0.23141414… **D.** 1.35363738…

E. $\sqrt{2} + 8$ **F.** $3\pi^0$

2. Four students described the area of a circle with a radius of 2 units. Some estimated and some did not. Which of the four values for the area below were meant to be estimates?

$\frac{88}{7}$ 3.14×4

4π $3.14 \times \sqrt{16}$

3. Which of these calculations below will result in a rational number? Explain how you know.

A. $\frac{4}{5} + 0.4$ **B.** $\frac{4}{5} - \frac{2}{3}$

C. $\frac{4}{5} \times \sqrt{2}$ **D.** $\sqrt{2} \div \sqrt{2}$

E. $\sqrt{2} + 8.3$ **F.** 0.234 + 0.6121212…

4. Give an example of each number.

a) rational between 4 and 5

b) irrational between 4 and 5

c) rational between $\sqrt{6}$ and $\sqrt{7}$

d) irrational between $-\sqrt{6}$ and $-\sqrt{7}$

5. a) If $3 \leq \sqrt{n}$ and $4 \geq \sqrt{n}$, what are the least and greatest possible values of n, if n is a real number?

b) Graph all of the values of n that would be possible solutions for the inequality in **part a)**.

c) Suppose n has to be an integer. How would this affect your answer to **part b)**?

6. Tell whether each statement is true or false. Justify each choice.

a) $\sqrt{0^5}$ is an irrational number.

b) $\pi + 5$ is an irrational number.

c) $\sqrt{2}^{\,0}$ is a rational number.

7. Graph the following sets of numbers:

a) {4, 5, 6, 7}

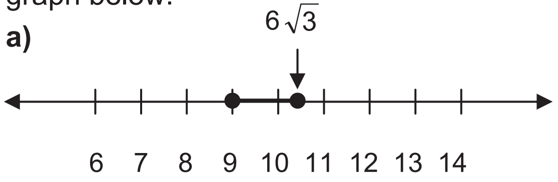
b) {4, 5, 6, ...}

c) $n > 4 + \pi$, n is a real number

d) positive integers greater than $3\sqrt{2}$ but less than $8\sqrt{2}$

8. Write an inequality statement for each graph below.

a)



b)

9. The set of real numbers is called dense. This means that between any two real numbers there is another real number. Demonstrate what this means using the values $-\frac{7}{9}$ and $-\frac{8}{9}$.

UNIT 1 Revision

1. Simplify each. Express as a power.

a) $8^4 \times 8^7 \times (8^9)^2$ **b)** $(5^4)^7 \div 5^9$

c) $(2^3 \times 3^4) \times (3^2)^2$

2. Evaluate each.

a) $(2^4 \times 5^4)$ **b)** $(30^5 \div 3^5)$

3. Solve for *n*.

a) $8^{4n} = 2^{48}$ **b)** $6.4 \times 10^n = (4 \times 10^5)^3$

c) $7^n = 3^n$

4. If $2^a = \frac{1}{2}$ and $5^b = 125$, what is the value of $a^b + b^a$?

5. Calculate.

a) $64^{-\frac{2}{3}}$ **b)** $(14^{12})^{\frac{1}{6}}$ **c)** $\sqrt{3.6 \times 10^{11}}$

6. $M = a \times 10^3$ and $N = b \times 10^4$.
If both numbers are in scientific notation, which is greater, M or N? Explain.

7. A calculator displays 2.34000 ^{06}.

a) What number does this represent?

b) How would you explain to someone unfamiliar with this display how you determined the number in **part a)**?

8. a) Express 0.0000003518 in scientific notation.

b) Is 0.0000003518 greater than 4×10^{-7}? How do you know?

9. Describe two ways to estimate $0.8 \times (-7.6)$.

10. A claim is made that someone who is 15 years old is about one million minutes old. Is this claim valid? Show your work to justify your answer.

11. Evaluate:

a) $4 + (7 - 10 \times 2)^2 + 30 \div (5 - 7)$

b) $(4 + 7 - 1 \div 5) \div 1.2 + 2.5 \times 3 - 0.2$

12. How many solutions are there to each? Explain.

a) a number that, when squared, is 12

b) a number that, when squared, is –12

13. The formula for calculating the required thickness of ice to support a certain mass is:

Required thickness = $0.38\sqrt{m}$, where *m* is the mass to be supported

Why is the formula set up to allow for only a positive square root of the mass?

14. Estimate.

a) $\sqrt{856}$ **b)** $\sqrt{0.8}$

c) $\sqrt{6.42 \times 10^6}$ **d)** $\sqrt{2.23 \times 10^9}$

15. Show how factoring 1764 can help you calculate its square root.

16. A number is between 0.37 and 0.5. What do you know about the square root of that number?

17. Which are rational? Explain.

A. $2 + \sqrt{2}$ **B.** $\sqrt{8}$

C. 2.343434… **D.** $6\pi + 3$

E. $\sqrt{\frac{9}{49}}$ **F.** $\sqrt{\frac{2}{49}}$

18. Tell whether the statement is true or false. Use an example to explain.

a) Some irrational numbers can be squared to become rational numbers.

b) Decimals always repeat or terminate.

19. Graph each inequality. List three rational and three irrational values for *n* that satisfy each inequality.

a) $\sqrt{2} + 2 \leq n \leq \sqrt{2} + 3$, *n* is a real number

b) $n > 3\pi$, *n* is a real number

UNIT 2 POLYNOMIALS

Getting Started

Use What You Know

A. Recall how to represent x, $-x$, $+1$, and -1 using algebra tiles.

x	$-x$	$+1$	-1
▭	▬	□	■

i) Show how to represent $(3x + 2) + (4x - 1)$. Combine like terms to represent the sum.

ii) Explain why you use 5 tiles to represent the first expression and 5 tiles to represent the second expression, but only 8 tiles to represent the sum.

B. i) To represent $(7x + 2) - (4x + 1)$, represent $7x + 2$ and then subtract $4x + 1$ to find the difference.

ii) Explain why you used 9 tiles to represent the first expression, subtracted 5 tiles to represent the second expression, and have 4 tiles left to represent the difference.

C. To calculate $(3x - 1) + (-3x + 1)$, you use 4 tiles to represent the first expression and 4 tiles to represent the second expression, but 0 tiles to represent the sum. Explain why this happens.

D. To calculate $(2x + 1) - (3x + 2)$ using tiles, you can subtract the second expression from the first expression by first adding 4 tiles to the first expression:
a $+x$-tile, a $-x$-tile, a $+1$-tile, and a -1-tile.

i) Explain why this works.

ii) How many tiles are there in the difference?

E. Create an algebraic expression to meet each condition.

	First expression	Second expression	Sum or difference
i)	9 tiles	5 tiles	Sum is represented by 10 tiles.
ii)	10 tiles	8 tiles	Sum is represented by 2 tiles.
iii)	10 tiles	7 tiles	Difference is represented by 7 tiles.
iv)	6 tiles	6 tiles	Difference is represented by 6 tiles.

F. Make up your own problem like the ones in **part E** for a classmate to solve. Trade with your classmate so that each of you can solve the other's problem.

Skills You Will Need

1. Add, subtract, multiply, or divide.

a) $-3 + -4$

b) $- 3 - 4$

c) $-3 \times (-2)$

d) $-8 \div (-2)$

2. Multiply to simplify.

a) $3(2x + 4)$

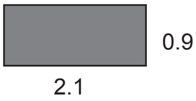
b) $4(5x - 2)$

c) $-3(2 - x)$

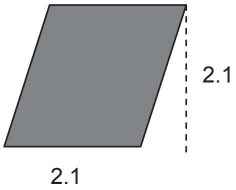
d) $-4(2x - 5)$

3. Calculate the area of each shape.

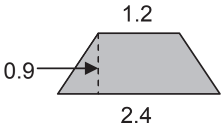
a)



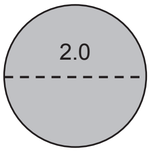
b)



c)

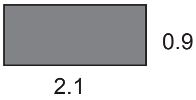


d)

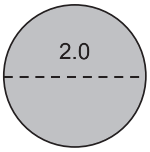


4. Calculate the perimeter or circumference of each shape**.**

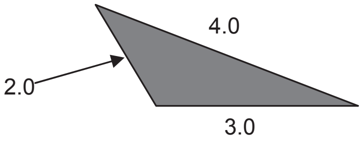
a)



b)



c)



5. Evaluate.

a) $2x + 5$ when $x = -3$

b) $3 - 4x$ when $x = -4$

c) $-3x - 3$ when $x = 0.5$

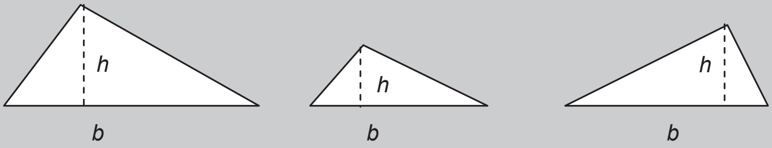
d) $4(2x - 3)$ when $x = 1.2$

Chapter 1 Introducing Polynomials

2.1.1 Interpreting Polynomials

Try This

These triangles all have heights that are half their base lengths.



$$h = \frac{1}{2}b$$

A. i) What is the formula for the area of a triangle?

ii) What formula could you use to calculate the area of any of these triangles if you only know the length of the base? (Recall that $h = \frac{1}{2}b$.)

B. Why are there two variables in the formula for **part A i)** but only one variable in the formula for **part A ii)**?

A **polynomial** is an algebraic expression that includes at least one variable. It usually involves numbers and operation signs as well. The variable in the expression can be raised to one or more whole number powers, for example, $t + t^2$. The variables cannot be raised to any fractional or negative power. For example, $5x^{-2}$ and $5x^{\frac{1}{2}}$ are not polynomials.

- Each part of the polynomial separated by addition or subtraction signs is called a **term**. Polynomials with 1, 2, or 3 terms have special names.

Type of polynomial	Number of terms	Examples
monomial	1	x or $2y$ or xy
binomial	2	$2x + 2y$ or $x + 4$
trinomial	3	$2x + 2y – xy$ or $x – y + x^2$

- The **degree** of the polynomial is determined by the highest power. For example:
  - The binomial $5x^2 – 3x$ is of degree 2 since the highest power of x is 2.
  - The trinomial $2 – 3m – m^3$ is of degree 3 since the highest power of m is 3.
  - The monomial xy appears to be of degree 1 since there is no exponent. However, it is actually of degree 2. You add the degrees of the two variables, x (which is x^1) and y (which is y^1), because the variables are multiplied.
  - The binomial $2x + y$ is of degree 1 since each variable is of degree 1 and no variables are multiplied together.

- When a variable is multiplied by a number, whether an integer, a fraction, a decimal, or an irrational number like π, that number is called a **numerical coefficient**. For example, 4 is a coefficient of x^2 in the monomial $4x^2$. If the coefficient is 1, it is usually not written; for example, instead of $1y$, it is more common to write y.

- Polynomials can have **variable terms** and **constant terms**. For example, in the binomial $2y + 8$, $2y$ is the variable term and 8 is the constant term.

- Sometimes polynomials include **like terms**. These are terms involving exactly the same variables raised to exactly the same powers. Any other pair of terms would be called **unlike terms**. For example, in the polynomial below, the terms $3y$ and y are like terms. Notice that $2x$ and $2xy$ are unlike terms because, even though both contain a 2 and an x, the variable parts of the terms are not identical.

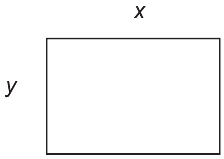
$$2x + \underbrace{3y + 2xy - y}_{\text{Like terms}}$$

A polynomial can only be classified after all the like terms have been combined. For example, $2x + 3x$ may appear to be a binomial, but since you can combine the two like terms to create $5x$, it is actually a monomial. Similarly, $2x – 5x + 13$ can be simplified to $–3x + 13$, making it a binomial. Combining like terms is a way to **simplify** the polynomial expression.

- The terms of a polynomial are sometimes described by considering the operation signs in front of them. For example, in the trinomial $3x^2 + 2xy – 3$, the third term is generally considered to be $–3$ and not 3.

- You can **evaluate** a polynomial by substituting a value for the variables. For example, to evaluate $3x – 4$ when $x = 2$, calculate $3(2) – 4 = 2$.

- Polynomials can be used to represent certain situations. For example, consider the rectangle below with side lengths x and y.



Some polynomials related to this rectangle are
- x to describe one dimension
- y to describe the other dimension
- $2x + 2y$ to describe the perimeter
- xy to describe the area

C. i) What type of polynomial did you use in **part A** for the area formula of the triangles?

ii) What formula could you use to calculate the area of any of the triangles, if you only know the height? (Recall that $h = \frac{1}{2}b$.)

Examples

Example 1 Describing Situations with Polynomials

The distance from Thimphu to Paro is 65 km. It takes about 1.5 h and about 5.4 L of petrol to drive that distance. List some polynomials that describe this information.

Explain what each polynomial describes and how it could be used.

Solution	**Thinking**
$65 \times n$ or $65n$ represents the number of kilometres travelled in n trips from Thimphu to Paro. If I know the number of trips, I could use this monomial to calculate the total distance travelled.	 • I used n as the variable to represent the number of trips from Thimphu to Paro. • Since there are 65 km in each trip, there are n times as many kilometres in n trips.
$3 \times r$ or $3r$ represents the number of hours needed to drive r round trips. If I know the number of round trips, I could use this monomial to calculate the total travelling time.	• I used the variable r to represent the number of round trips. This time I used it to create a monomial with a decimal coefficient to represent travelling time in hours. Since it takes 1.5 h to drive one way, it takes 3 h to drive a round trip. If you take r round trips, the time would be r times the time for one round trip.
$10.8 \times r$ or $10.8r$ represents the number of litres of petrol used to drive r round trips. If I know the number of round trips, I could use this monomial to calculate the total amount of petrol used.	• This time I used r to create a monomial that represents the amount of petrol (litres) used in r round trips. You would use about 5.4 L of petrol to drive one way, so you'd use 10.8 L for a round trip. If you take r round trips, the number of litres of petrol would be $10.8r$.
	• I thought of using the variable t for time in the expression $\frac{65}{t}$ to represent the average driving speed for a different driving time (faster or slower). But that's not a polynomial because the variable t has a negative exponent, $\frac{65}{t} = 65t^{-1}$.

Example 2 Using Variables and Numbers to Create Polynomials

Create ten polynomials using at least one of the variables x and y and at least one of the numbers 2 and 3. For each polynomial, state the type and degree.

Solution	**Thinking**
	 I wanted to include monomials, binomials, trinomials, and one other polynomial. I knew I could use 2 or 3 for the exponent of x or y but the exponent couldn't be negative or a fraction because the expression wouldn't be a polynomial.
Monomials: $3x$, $2y$, $2x^2$, $2xy$ The first two are of degree 1, and the last two are of degree 2.	• To create monomials, I knew that the 2 and 3 could be exponents as long as they are positive. Or, they could be coefficients.
Binomials: $2x + 3$, $2x^3 + 3y$, $y – 2$ The first and last are of degree 1 and the middle one is of degree 3.	• To create binomials, I knew that I could use a + sign or a – sign between the two terms. The 2 and 3 could be exponents, as long as they are positive. They could also be coefficients or they could be the constant term.
Trinomials: $2x + 3y – xy$, $x^2 – 3y – 3$ Both are of degree 2.	• To create trinomials, I knew I had to use two + signs, two – signs, or a + sign and a – sign between the three terms. I also knew that the 2 and 3 could be exponents, as long as they are positive. Or, they could be coefficients or the constant term.
Another polynomial: $2x + x^2 – y – 3y^2$ It is of degree 2.	• I wanted one other kind of polynomial, so I made one with four terms.

Practising and Applying

1. What type of polynomial is each?

a) $2x – 3y$
b) $3x^3 – 2x + 4$
c) $5mn$
d) $–6 + 3y$
e) $2x^3y – 3y$
f) $6x^4$

2. What is the numerical coefficient of each term?

a) m
b) $–2x^2$
c) $2.3xy$
d) $\frac{3}{4}t^3$

3. Which are not polynomials?

A $3 - 2m^3 + p^5$ **B** $6k^{-2} - 6k^{-2}$

C $\frac{3}{x^3}$ **D** $4x^{\frac{1}{3}}$

4. What is the degree of each polynomial?

a) $3t - 3t^2 + 7t^3$ **b)** $3m^4 + 8m$

c) $3p^2 + 4qp$ **d)** $2x - 2y$

5. Create two polynomials, each with a constant term of –3.

6. Evaluate each polynomial.

a) $3m^3 - 2m$ when $m = 4$

b) $6 - x - x^2$ when $x = 1$

c) $4xy - 2x + 3y^2$ when $x = 1$ and $y = 5$

d) $8r - 2rs$ when $r = 3$ and $s = 5$

7. For each polynomial, identify the like terms. Then, combine them to form a simplified polynomial.

a) $3t - 3t^2 + 7t$

b) $3 + 8m - 2n - 7n$

c) $3p + 4q - 2q - 17p$

d) $16 + 16m + m^2 - 16m^3$

8. Create a polynomial to fit each description.

a) monomial of degree 3

b) coefficients that are all even

c) trinomial with all odd coefficients

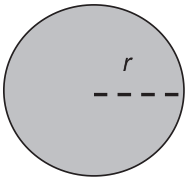
d) one-variable polynomial that has a value of 7 when the variable has a value of 3

e) two-variable binomial that has a value of –4 when the variables have values of –2 and +2

9. Create 12 binomials, each using at least one of m and m^2 and at least one of the numbers 2 and –3.

10. There are 6 large bags and 4 small bags of stones. Each bag holds the same number of stones as the other bags of the same size. How could you represent the total number of stones with a polynomial?

11. Create two polynomials, using r (radius) as the variable, that describe measurements of a circle.



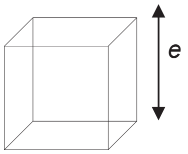
12. For each situation below, create a polynomial. Explain what the polynomial describes.

a) 1 kg of beef costs Nu 80 and 1 kg of chicken costs Nu 85

b) a trip from Thimphu to Punakha is about 77 km and takes about 2 h 15 min

c) a final mark is calculated by counting an exam for 70% and a project for 30%

d) a cube has an edge length of e



13. How are the polynomials $3s + 5$ and $3s^2 + 5$ the same? How are they different?

2.1.2 Adding and Subtracting Polynomials

Try This

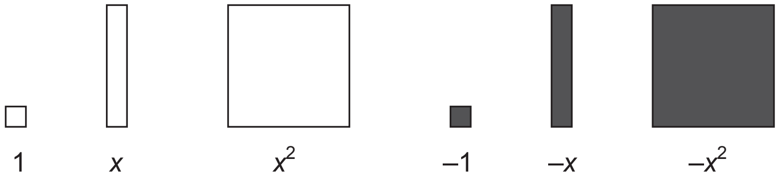
Dorji is calculating his average test score after ten tests.
His ten scores are: 60, 60, 45, 45, 60, 60, 45, 60, 45, and 60.

A. How could he efficiently calculate his average?

- One of the ways to model or represent polynomials is using algebra tiles. These are rectangular and square tiles with particular lengths and widths.

- It is customary to use one colour to represent positive terms and a different colour to represent negative terms. Often the positive terms are white and the negative terms are darker. For example:

You could represent 1, x, and x^2 and -1, $-x$, and $-x^2$ as shown here.



1 or -1 is represented by a square tile that has a side length of 1 unit.
x or $-x$ is represented by a rectangular tile that is 1 unit wide and x units long.
x^2 or $-x^2$ is represented by a square tile that is x units wide and x units long.

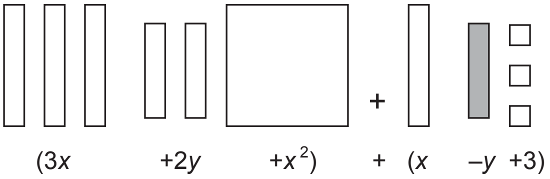
- To represent y or y^2, you use rectangular and square tiles with different lengths and widths than you used to represent x and x^2.

- To represent xy or $-xy$, you use rectangular tiles that are x units long and y units wide.

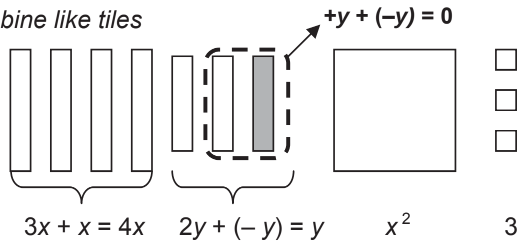
- Adding polynomials means putting them together by combining like terms.

For example, to calculate $(3x + 2y + x^2) + (x - y + 3)$ using tiles, you model both polynomials and then combine like tiles.

Model with tiles



Combine like tiles



Notice that $-y$ is treated as $+ (-y)$, just as you would do to add negative numbers, for example, $3 - 5 = 3 + (-5)$.

$$(3x + 2y + x^2) + (x - y + 3) = 4x + y + x^2 + 3$$

You can also add symbolically, without using the tiles. You still combine like terms. Recall that x is $1x$, $-y$ is $-1y$, and $-y$ is $+ (-y)$.

$$\begin{aligned}(3x + 2y + x^2) + (x - y + 3) &= [3x + x] \mathbf{+} [2y + (-y)] + x^2 + 3\\ &= 4x + y + x^2 + 3\end{aligned}$$

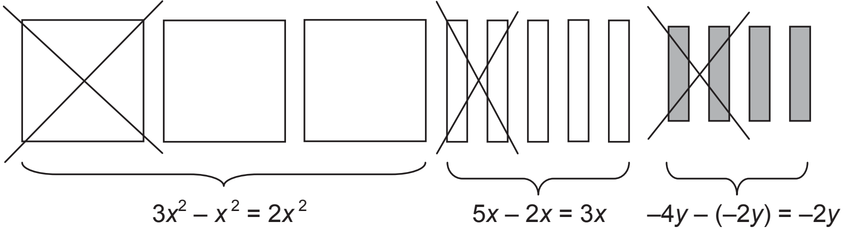
When you combine like terms, you are simplifying the polynomial.

- Subtracting polynomials can be done in several ways.

\- One way is to take the subtrahend away from the minuend. What remains is the difference.

For example, to subtract $(3x^2 + 5x - 4y) - (x^2 + 2x - 2y)$ using tiles, model $3x^2 + 5x - 4y$ and then take away tiles worth $x^2 + 2x - 2y$.

Model with tiles and then take away



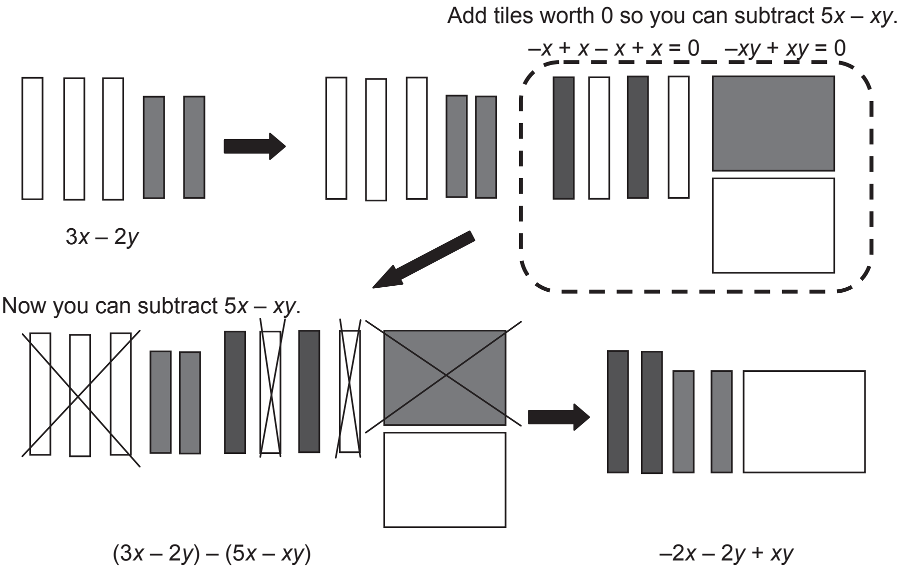
$$\underset{\text{Minuend}}{(3x^2 + 5x - 4y)} - \underset{\text{Subtrahend}}{(x^2 + 2x - 2y)} = \underset{\text{Difference}}{2x^2 + 3x - 2y}$$

You can also subtract symbolically, taking like terms in the subtrahend away from like terms in the minuend.

$$(3x^2 + 5x - 4y) - (x^2 + 2x - 2y) = [3x^2 - x^2] + [5x - 2x] + [-4y - (-2y)]$$
$$= 2x^2 + 3x - 2y$$

- Sometimes you need to use the **zero principle** to allow you to takeaway the required tiles.

For example, to subtract $(3x - 2y) - (5x - xy)$ using tiles, you model $3x - 2y$ and then think about what you have to takeaway, $5x - xy$. In this case, you need to takeaway $5x$ and you only have 3x and you need to takeaway $-xy$ and you have no $-xy$. To do this you add extra tiles with a value of zero so you can take some of them away.

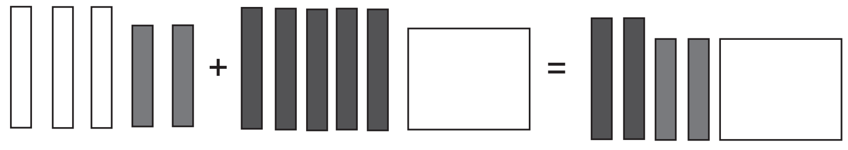

Add tiles worth 0 so you can subtract $5x - xy$.

Now you can subtract $5x - xy$.

- Another way to subtract is to add the opposite of the subtrahend. It is like subtracting negative integers, like $5 - (-3) = 5 + (+3) = 8$.

For example:

$(3x - 2y) - (5x - xy) = (3x - 2y) + (-5x + xy)$ $-[(5x - xy)$ becomes $+(-5x + xy)]$



You can also add the opposite symbolically.

$$(3x - 2y) - (5x - xy) = (3x - 2y) + (-5x + xy)$$
$$= [3x + (-5x)] - 2y + xy$$
$$= -2x - 2y + xy$$

- Yet another way to subtract is to think about what to add. It's like subtracting 5 – 2 by thinking 2 + ? = 5.

$(3x - 2y) - (5x - xy) \rightarrow (5x - xy) + ? = 3x - 2y$

$5x + (-2x) + (-2y) - xy + xy = 3x - 2y$

$5x - xy + [-2x - 2y + xy] = 3x - 2y$

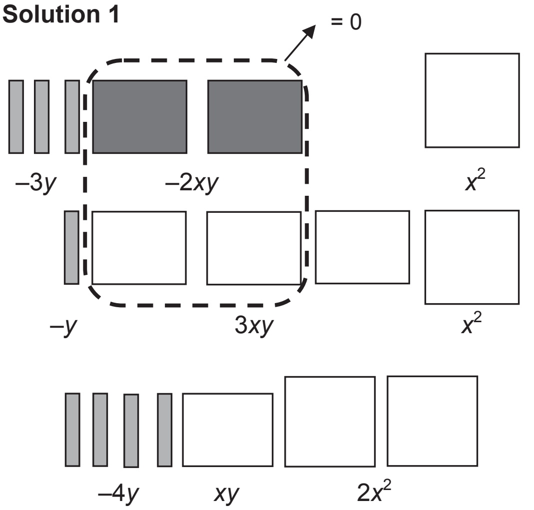
$(3x - 2y) - (5x - xy) = -2x - 2y + xy$

B. How is what you did with calculating Dorji's test scores in **part A** similar to adding like terms?

Examples

Example 1 Adding Polynomials

Add $(-3y - 2xy + x^2) + (-y + 3xy + x^2)$.

Solution 1  $(-3y - 2xy + x^2) + (-y + 3xy + x^2) = -4y + xy + 2x^2$	**Thinking**  • I modelled each polynomial with tiles and then put the like tiles together. • There were two positive *xy*-tiles and two negative *xy*-tiles that I could eliminate because together they're worth 0. • The tiles that were left modelled the sum.
Solution 2 $(-3y - 2xy + x^2) + (-y + 3xy + x^2)$ $= [-3y + (-y)] + [-2xy + 3xy] + [x^2 + x^2]$ $= -4y + xy + 2x^2$	 • I recorded the two polynomials to add. • Next, I combined like terms. • What I ended up with was the sum.

Example 2 Subtracting Polynomials

Subtract $(-3y - 2xy + x^2) - (-y + 3xy + x^2)$.

Solution 1

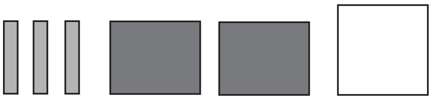
$$(-3y - 2xy + x^2) - (-y + 3xy + x^2)$$
$$= (-3y - 2xy + x^2) \mathbf{+} (\mathbf{+}y \mathbf{-} 3xy \mathbf{-} x^2)$$
$$= [-3y + y] + [-2xy + (-3xy)] + [x^2 - x^2]$$
$$= -2y - 5xy + 0$$
$$= -2y - 5xy$$

Thinking

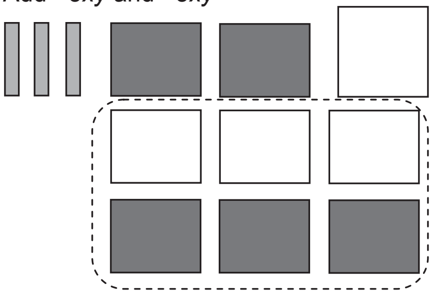
- I subtracted by adding the opposite.
- I works just like when you subtract negative numbers, for example, 6 – (-5) = 6 + (+5) = 11.

Solution 2

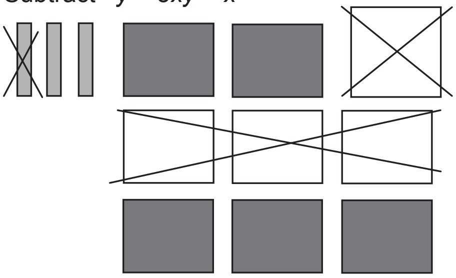
$(-3y - 2xy + x^2) - (-y + 3xy + x^2)$



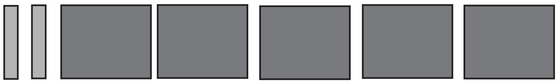
Add $+3xy$ and $-3xy$



Subtract $-y + 3xy + x^2$



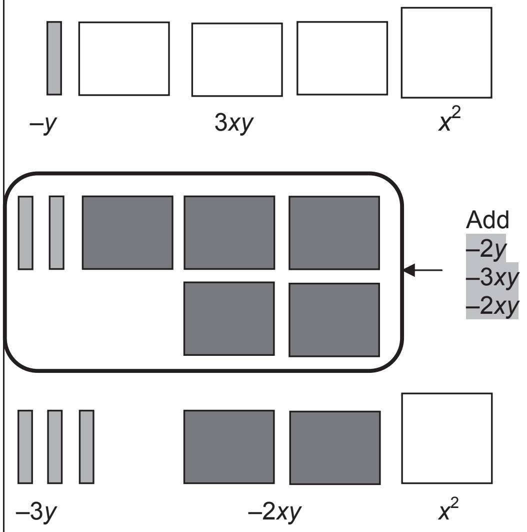
$(-3y - 2xy + x^2) - (-y + 3xy + x^2) = -2y - 5xy$



Thinking

- There were two $-xy$-tiles but I needed to take away three $+xy$-tiles. I used the zero principle to add tiles so that I would have three $+xy$-tiles to take away.
- I added three $+xy$-tiles and three $-xy$-tiles, which I knew had a value of 0.
- I subtracted one $-y$-tile, three $+xy$-tiles, and one x^2-tile.
- I was left with two $-y$-tiles and five $-xy$-tiles.

Solution 3



If $(-y + 3xy + x^2) + (-2y - 5xy) = -3y - 2xy + x^2$,
then
$(-3y - 2xy + x^2) - (-y + 3xy + x^2) = -2y - 5xy$

Thinking

- I needed to take away $+3xy$ and I only had $-2xy$. So I decided to subtract by figuring out what I had to add to $-y + 3xy + x^2$ to get $-3y - 2xy + x^2$.
- I modelled $-y + 3xy + x^2$.
- To end up with $-3y - 2xy + x^2$, I knew I had to
  - add two more $-y$-tiles
  - get rid of the three xy-tiles by adding three $-xy$-tiles
  - add two more $-xy$-tiles
- What I added was the difference.

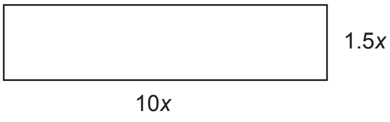
Example 3 Solving a Problem using Polynomials

A rachu is about 10 hand spans long and 1.5 hand spans wide. Its size is based on the hand span of the person wearing the rachu. (A handspan is measured from the tip of the thumb to the tip of the middle finger while the hand is stretched.) Manju and Sonam each have a rachu. Manju has a wider hand span than Sonam so her rachu is larger. Suppose Sonam's hand span is x cm and Manju's hand span is y cm.

- How much longer is the perimeter of Manju's rachu than Sonam's?
- How much greater is the area of Manju's rachu than Sonam's?

Express your answers as polynomials.

Solution



Sonam's rachu:

$P = 2 \times 11.5x = 23x$

$A = 10x \times 1.5x = 15x^2$

[Continued]

Thinking

- I used x to represent Sonam's hand span and drew a picture to model her rachu. Her rachu is $10x$ long and $1.5x$ wide.
- I know perimeter is $2 \times (l + w)$ and area is $l \times w$. I created polynomials by substituting $10x$ for the length and $1.5x$ for the width in each formula.

Example 3 Solving a Problem using Polynomials [Continued]	
Solution	**Thinking**
Manju's rachu: $P = 2 \times 11.5y = 23y$ $A = 10y \times 1.5y = 15y^2$ The perimeter of Manju's rachu is $23y - 23x$ longer than Sonam's. The area of Manju's rachu is $15y^2 - 15x^2$ greater than Sonam's.	• I called Manju's hand span y. Manju's rachu is $10y$ long and $1.5y$ wide so I just replaced the x with a y in the perimeter and area formulas for Sonam's rachu. • I subtracted the polynomials to calculate the differences.

Practising and Applying

1. Model with tiles and add.

a) $(3x + 2x^2 - 4) + (5x^2 - 8x)$

b) $(2y - y^2 + 5x) + (3y^2 - 2y - 4x)$

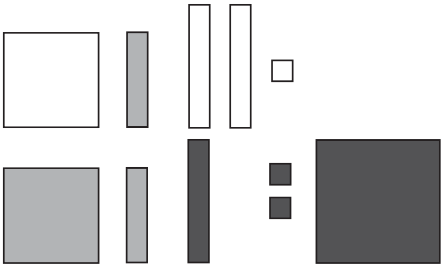
2. Add.

a) $(-2x - 6x^2 - 4) + (5x^2 - 2x)$

b) $(4m^2 - 1.5m + 2) + (9m - 2m^3)$

c) $(y + 2y^2 - 3x) + (7y^3 - 3y^2 - 8x)$

3. What two polynomials are being added below? What is the sum?



4. Model each with tiles by taking away.

a) $(-3y + 2y^2 - 6x) - (-2y + y^2 - 3x)$

b) $(4y^2 - 3x^2 - 2y) - (-2x^2 - y)$

5. Subtract using the zero principle.

a) $(-3y + 2y^2 - 6x) - (3y + 4y^2 + x)$

b) $(2y^2 - x^2 - 3y) - (-2x^2 + y)$

6. Subtract.

a) $(-4m + 3m^3 - t) - (7t - 2m + 2m^3)$

b) $(m^2 - 2m + 40) - (2m^2 + 64)$

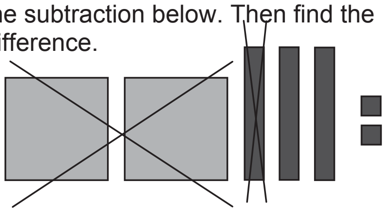
7. Simplify each polynomial by combining like terms.

a) $3x - 2y + 8x - 2y^2 + 6y$

b) $5k + 3h - 6y^2 - 8h - 10k$

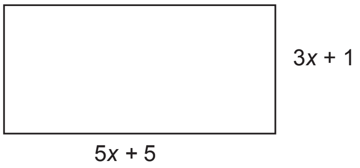
c) $4m^2 + 3m - 2r - 8 - 5m + 9r$

8. Write the expression that represents the subtraction below. Then find the difference.

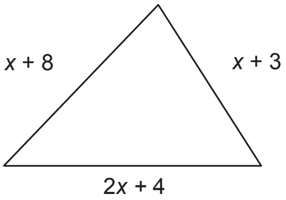


9. Express the perimeter of each as a polynomial. Remember to simplify.

a)



b)



10. Why is subtracting polynomials more complicated than adding them?

Chapter 2 Multiplying Polynomials

2.2.1 Multiplying a Polynomial by a Monomial

Try This

Exchange rates for currencies tell how much of one currency you can buy using another currency.

A Thai tourist who is visiting Bhutan has Nu 2300 and wants to exchange 1200 baht for more ngultrums.

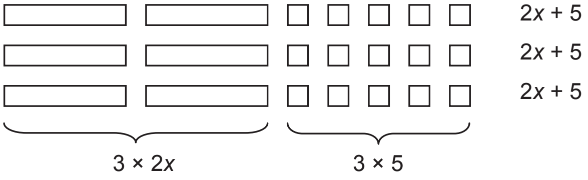
A. The polynomial 2300 + 1200x represents how much money the traveller will have in ngultrums. What does x represent?

B. What if the tourist had twice as many ngultrums and baht? Write a polynomial that would represent how much money the tourist would have in ngultrums.

When a polynomial is multiplied by a monomial, each term of the polynomial is multiplied by the monomial and the products are added.

- When a polynomial is multiplied by a monomial that is a constant, each term of the polynomial is multiplied by the constant and then the products are added.

For example, to multiply 3 × (2x + 5), or 3(2x + 5), using tiles, you model it as 3 groups of 2x + 5, which is 3 groups of 2x, or 3 × 2x, and 3 groups of 5, or 3 × 5:



3(2x + 5) is
3 groups, or rows of
2x + 5.

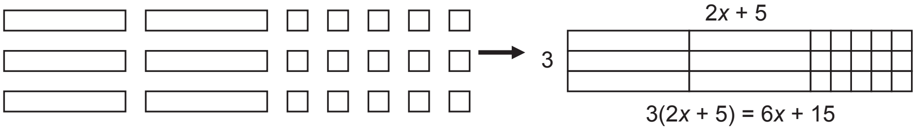
$$\begin{aligned}3(2x + 5) &= 3 \times 2x + 3 \times 5\\ &= 6x + 15\end{aligned}$$

You can also multiply symbolically, still multiplying each term of the polynomial by the constant and adding the products.

$$\begin{aligned}3(2x + 5) &= 3 \times 2x + 3 \times 5\\ &= 6x + 15\end{aligned}$$

Multiplying like this is often called **expanding**. The property that allows us to do this is called the **distributive property**.

You can push the tiles together to make a rectangle. The dimensions of the rectangle are the factors and the area of the rectangle is the product. This is called an area model for multiplication.



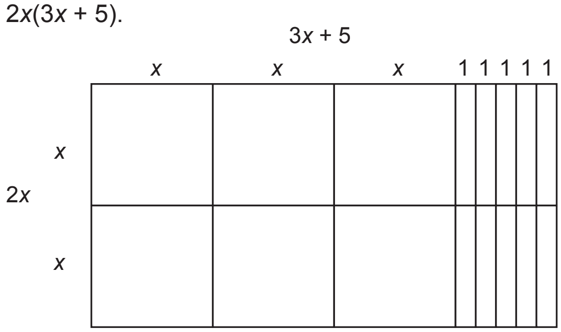
3(2x + 5) = 6x + 15

- When a polynomial is multiplied by a monomial that has a variable, you still multiply each part of the polynomial by the monomial and then add the parts.

For example, to multiply $2x \times (3x + 5)$, which can be written as $2x(3x + 5)$, symbolically you multiply each term of the polynomial by the monomial and then add the products:

$$2x(3x + 5) = 2x \times 3x + 2x \times 5 = 6x^2 + 10x$$

To multiply $2x \times (3x + 5)$ using tiles, you can use the area model for multiplication. Create a rectangle with dimensions 2x by 3x + 5. Since the area of any rectangle is the product of its side lengths, the area of the rectangle, $6x^2 + 10x$, will be the product of $2x(3x + 5)$.



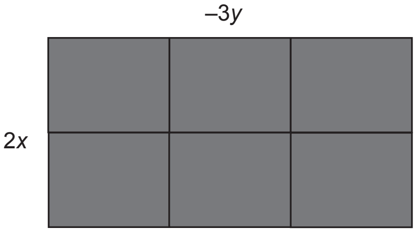
The area of the rectangle is made up of 6 x^2-tiles and 10 x-tiles.

$2x(3x + 5) = 6x^2 + 10x$

- When you multiply a monomial with a positive sign by a monomial with a negative sign, the product monomial will have a negative sign, just as when you multiply negative and positive integers.

For example, to multiply $2x(–3y)$ symbolically: $2x(–3y) = 2(–3)xy = –6xy$

To model a multiplication such as $2x(–3y)$ using tiles and the area model, you create a rectangle with dimensions $2x$ units by $–3y$ units long using negative tiles because the area, or product, will have a negative sign.



The area of the rectangle is made up of six $–xy$ -tiles, or $–6xy$.

$2x(–3y) = –6xy$

- It is not convenient to use algebra tiles to multiply two monomials with negative signs. In this case, multiply symbolically. For example:

$$–2x(–3y) = (–2)(–3)xy = 6xy$$

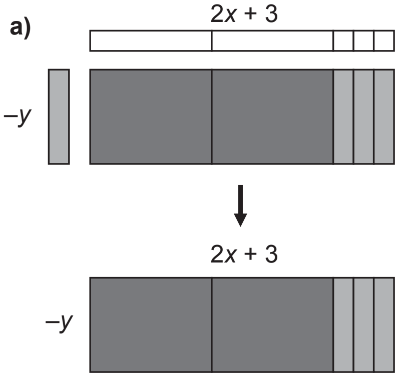
C. Write a polynomial multiplication to represent how you used the polynomial in **part A** to create the polynomial in **part B**.

Examples

Example 1 Multiplying by a Monomial With a Different Variable

a) What is the product of $-y(2x + 3)$? Show your work.

b) How would the answer change if you multiplied $-y(2x - 3)$?

Solution	**Thinking**
a)  Area: two $-xy$-tiles and three $-y$ tiles $-y(2x + 3) = -2xy - 3y$	 **a)** I modelled $-y(2x + 3)$ with tiles by making a rectangle that was $2x + 3$ long and $-y$ wide. • To make sure my dimensions were right, I used guide tiles—I laid out tiles to show the length, $2x + 3$, and a $-y$ tile to show the width. Then I filled the area in with tiles and removed the guide tiles. • I used only negative tiles for the area because multiplying negative by positive terms results in a negative product. Both $-y \times 2x$ and $-y \times 3$ are negative. • I knew that the tiles that made up the area of the rectangle represented the product.
b) $-y(2x - 3) = -y(2x) + (-y)(-3)$ $= -2xy + 3y$ The product changed from $-2xy - 3y$ to $-2xy + 3y$.	**b)** I solved $-y(2x - 3)$ symbolically because I knew I had to multiply two negatives. Even though $-y \times 2x$ was a negative by a positive, $-y \times -3$ was a negative by a negative. • I multiplied $-y$ by each term in $2x - 3$ and then added.

Example 2 Solving a Problem by Multiplying by a Monomial

Every month, Kuenzang pays Nu 5000 for rent and buys several Nu 100 prepaid cell phone cards. If he buys p phone cards each month, how much will he have to pay for rent and phone cards in m months? Express your answer as a polynomial.

[Continued]

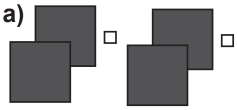
Example 2 Solving a Problem by Multiplying by a Monomial [Continued]

Solution	**Thinking**
Each month: $5000 + 100p$	 • Each month, he spends $5000 + 100p$ since rent is Nu 5000 and he buys p phone cards for Nu 100 each.
In m months: $m(5000 + 100p)$ $m(5000 + 100p) = 5000m + 100mp$	• I multiplied the number of months, represented by m, by how much he spends each month, $5000 + 100p$.
In m months, Kuenzang spends $5000m + 100mp$ ngultrums on rent and p cell phone cards.	• I expanded to multiply and then rearranged each term so the coefficient came first.

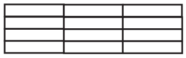
Practising and Applying

1. What multiplication does each tile model represent? All expressions have x or x^2 terms.

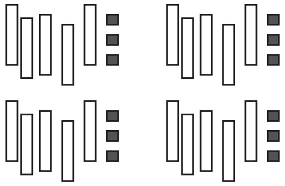
a)



b)



c)



2. Model each with tiles to multiply.

a) $2(-2x - 3)$ **b)** $3y(y + 2)$

c) $4(2y - 2x)$ **d)** $2x(3 - 4y)$

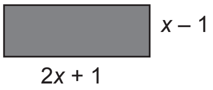
3. Multiply each.

a) $-x(3x + 2y)$ **b)** $6(4 - t + 3t^2)$

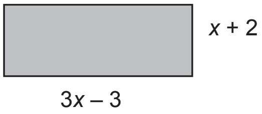
c) $2jk(k - 3)$ **d)** $5(3m + m^3 - 2r)$

4. What is the perimeter of each?

a)



b)



5. An entrepreneur rents some houses at Nu 6000 per month and some at Nu 6050 per month. Write and simplify a polynomial to describe each.

a) renting x houses at the lower rate and renting y houses at the higher rate

b) renting x houses at the lower rate and twice as many at the higher rate

c) renting x houses at the lower rate and half as many at the higher rate

6. Each polynomial below is the product of a monomial and a polynomial. For each, list one possibility for what might have been multiplied.

a) $6t^2 -3t + 15$ **b)** $8xy - 10y^2 + 6y$

c) $14x - 16x^2$ **d)** $8 - 16x + 12y$

7. Expand and simplify.

a) $3(6 - 2c) + 4(8 + c)$

b) $5x(2x - y) - 3y(2 + 3x)$

c) $3y - 4(5x - 3y + y^2)$

d) $xy(2 - 3x + 4y)$

8. A multiplication like $3x(2x)$ will result in fewer terms than a multiplication like $3x(2 + x)$. How do you know?

9. Use tiles to show that each is true.

a) $2(3x - 4) = 6x - 8$

b) $2x(x + 3) = 2x^2 + 6x$

2.2.2 Multiplying a Binomial by a Binomial

Try This

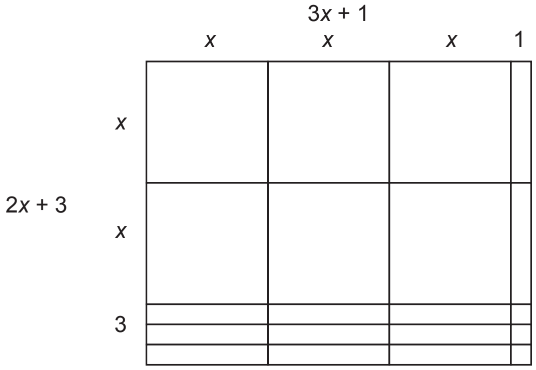
Drakpa has a number trick that he is trying out on his friend Meto.

- He tells Meto to think of a secret number and add 1 to it, then subtract 1 from it, and multiply the two numbers. For example, for the secret number 11, Meto would multiply 10 × 12 to get the product 120.
- Meto tells Drakpa the product.
- Drakpa can tell what Meto's secret number just from hearing the product. For example, if Meto says her product is 48, Drakpa knows that the secret number is 7. If Meto says her product is 99, Drakpa knows that the secret number is 10.

A. How does Drakpa's number trick work?

- If you want to use tiles to multiply two polynomials of degree 1, you create a rectangle whose dimensions are the two polynomials. You then determine the area of the rectangle in tiles.

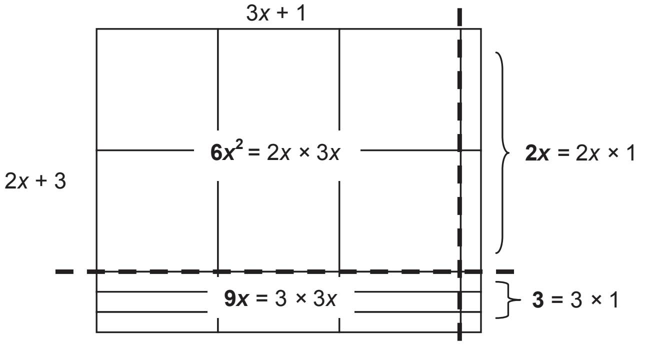
For example, $(2x + 3)(3x +1)$ is the area of a rectangle where one dimension is $2x + 3$ and the other is $3x + 1$.



$(2x + 3)(3x +1) = 6x^2 + 11x + 3$

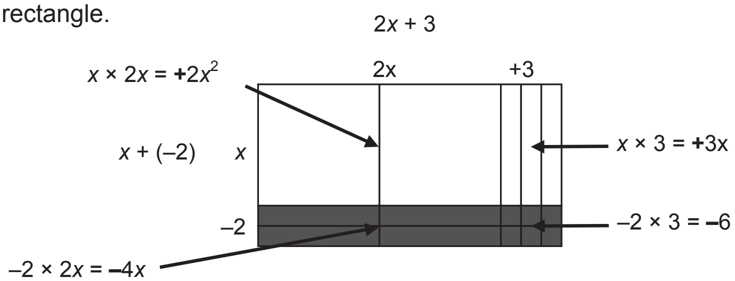
Notice that the area consists entirely of positive tiles. This is because both terms in each binomial are positive. As a result, none of the multiplications of the terms are a negative by a positive.

If you examine the area of the rectangle below, you will see that it is made up of four smaller rectangles. Each smaller rectangle is the product of one term from the first binomial and one term from the second binomial.



$$\begin{aligned}(2x + 3)(3x + 1) &= 6x^2 + 2x + 9x + 3 \quad \text{Simplify: } 2x + 9x = 11x\\ &= 6x^2 + 11x + 3\end{aligned}$$

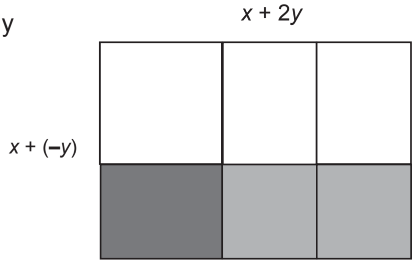
- The rectangle below models $(x – 2)(2x + 3)$. Note that if you want to represent a width such as $x – 2$, you have to think of it as $x + (– 2)$. Notice that the area consists of some positive and some negative tiles. This is because you use negative or positive tiles depending on the terms being multiplied in each smaller rectangle.



$$\begin{aligned}(x – 2)(2x + 3) &= 2x^2 + 3x – 4x – 6\\ &= 2x^2 – x – 6\end{aligned}$$

- You can also use algebra tiles to multiply two binomials that involve different variables. For example, the model to the right shows $(x – y)(x + 2y)$.

$$\begin{aligned}(x – y)(x + 2y) &= x^2 + 2xy – xy – 2y^2\\ &= x^2 + xy – 2y^2\end{aligned}$$



B. i) Suppose Meto's secret number from **part A** is represented by x. What two binomials represent the two numbers Meto is multiplying?

ii) How does expressing the multiplication this way help explain the trick?

C. Try the trick with a classmate.

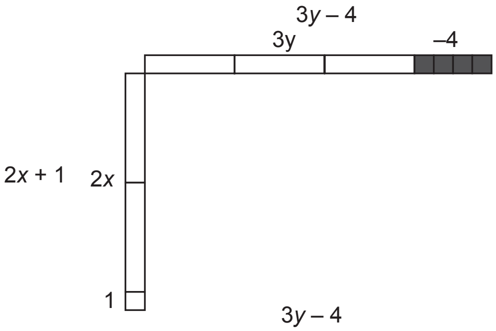
Examples

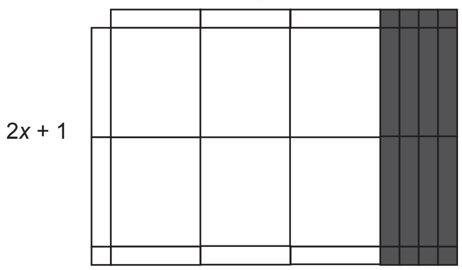
Example Solving a Problem by Multiplying Polynomials

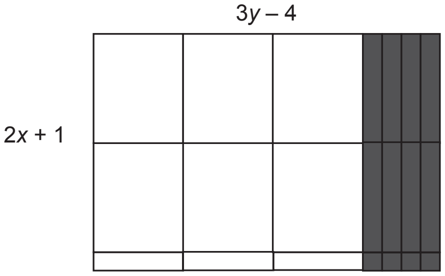
Deki has two rectangular pieces of fabric. The large piece has one edge that is 1 cm longer than twice the width of the small piece. The other edge of the large piece is 4 cm shorter than triple the length of the smaller piece. How much greater is the area of the large piece than the area of the small piece? Use algebra tiles. Express your answers as a polynomial.

Solution

If the small piece is x wide by y long, the large piece is $2x + 1$ wide by $3y – 4$ long.

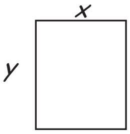






Thinking

- I called the dimensions of the small piece x by y.



- 1 cm longer than 2 times the width is $2x + 1$ and 4 cm shorter than 3 times the length is $3y - 4$ so I knew that the dimensions of the large piece were $2x + 1$ and $3y - 4$.
- I used guide tiles to form the length and width of a rectangle with those dimensions.
- I filled in the rectangle with tiles to find the area. I knew that I needed negative tiles for the part that was multiplied by –4 (that is, $2x \times -4$ and 1×-4).
- I removed the guide tiles.
- I counted the tiles in the area of the rectangle.

[Continued]

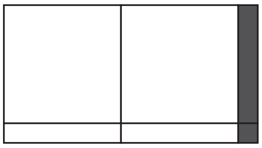
Example Solving a Problem by Multiplying Polynomials [Continued]

Solution	**Thinking**
If the area of the large piece is $6xy - 8x + 3y - 4$ and the area of the small piece is xy, then the difference is $(6xy - 8x + 3y - 4) - xy = 5xy - 8x + 3y - 4$ The area of the large piece is $5xy - 8x + 3y - 4$ bigger than the area of the small piece.	• I knew the small piece was x wide by y long. • I subtracted the area of the small piece from the area of the large piece to find the difference.

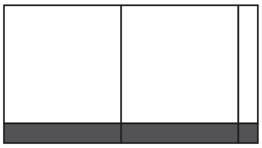
Practising and Applying

1. Which model below represents $(2x - 1)(x + 1)$? How do you know?

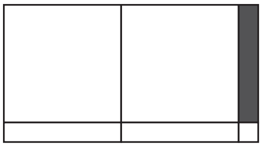
A



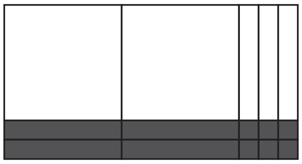
B



C



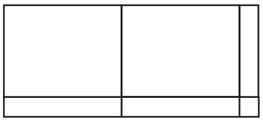
2. Explain how this model represents the product of $(x - 2)(2x + 3)$.



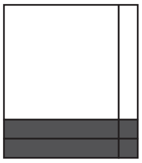
3. Which model requires more tiles—the area model for $(2x - 4)(3x + 1)$ or the area model for $(2x + 1)(3x - 4)$? Explain.

4. What two polynomials are being multiplied in each?

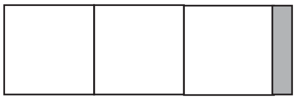
a)



b)



c)



5. Model each multiplication. Write the product.

a) $(2x + 1)(3x + 2)$ **b)** $(3x + 4)(x - 1)$

c) $(y - 3)(5y + 2)$ **d)** $(x + 3)(x - y)$

e) $(2 - x)(3 + y)$

6. For each description below, sketch the rectangle and find its area. Show your work.

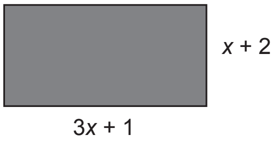
a) The height is $x + 1$ and the width is 3 units more than the height.

b) The height is $x - 1$ and the width is 4 units more than 3 times the height.

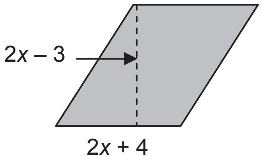
c) The width is $4x + 2$ and the height is 2 units more than half the width.

7. Calculate the area of each shape.

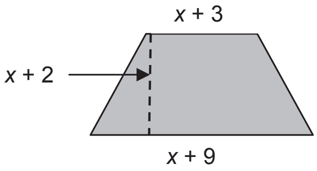
a)



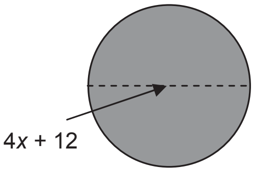
b)



c)



d)



8. a) Multiply $(2 – 3x)(4 + x)$ using tiles.

b) Evaluate the product from **part a)** for $x = –3$.

c) Evaluate $2 – 3x$ for $x = –3$.

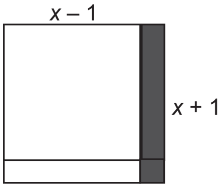
d) Evaluate $4 + x$ for $x = –3$.

e) How are your answers in **part c)** and **part d)** related to the answer in **part b)**?

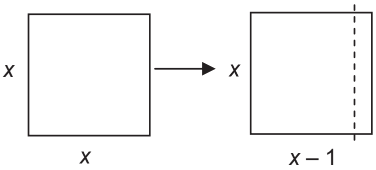
9. Use algebra tiles to find all the possible dimensions that a rectangle with an area of $6x^2 + 12x$ could have.

10. You are multiplying two binomials using tiles and the area model for multiplication. The model uses 15 algebra tiles. List five different pairs of binomials you might be multiplying.

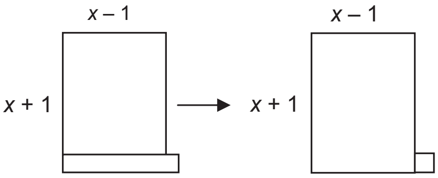
11. Kinley models $(x –1)(x + 1) = x^2 – 1$ as shown below.



Pema says that there is another way to do it and shows what you see below. He starts with an x^2 tile and then cuts off a strip 1 unit wide all the way down one side.



He then moves the cut-off strip to the bottom of what is left of the square.



He notices that there is now a rectangle with dimensions $x + 1$ and $x – 1$ and an extra 1-unit square. How does Pema's approach explain why $(x + 1)(x – 1)$ is equal to $x^2 – 1$?

12. Suppose you use tiles to multiply two binomials of degree 1. How is this the same as multiplying a monomial of degree 1 by a binomial of degree 1? How is it different?

2.2.3 Multiplying Polynomials Symbolically

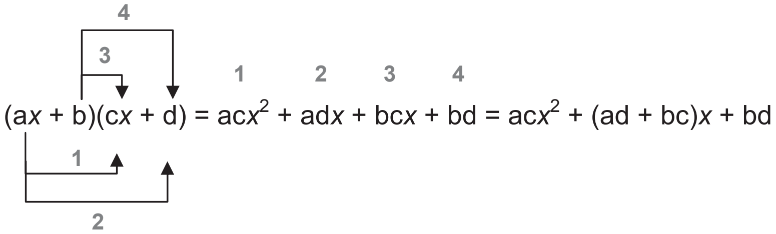
Try This

A. i) Model the following with tiles to determine each product:

- $(2x + 1)(x + 1)$ • $(2x + 1)(x + 2)$ • $(2x + 1)(x + 3)$

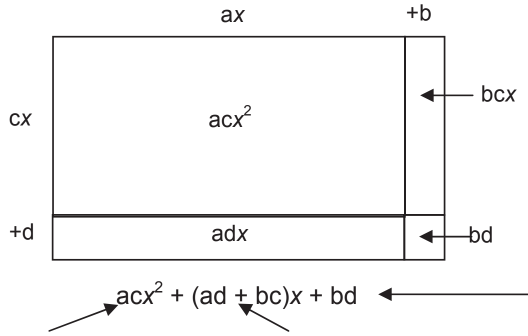
ii) Predict the product of $(2x + 1)(x + 30)$. Explain your thinking.

- When you multiply a binomial by a binomial symbolically, each term in the first binomial is multiplied by each term in the second binomial and then like terms are combined. This is an application of the distributive property for multiplication over addition. That is, when you multiply by a sum of terms, you multiply each part and then add the parts, as shown below:



$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd$$

Below is a sketch of the algebra tile model for the multiplication above. Notice how the four terms above relate to the four smaller rectangles in the sketch.



This sketch of the area model shows only the outline of each of the four smaller rectangles in the model.

$$acx^2 + (ad + bc)x + bd$$

The coefficient of x^2 is the product of the two coefficients of x in the binomials, a × c.

The coefficient of the middle term is the sum of two products: the coefficient of x in the 1st binomial multiplied by the constant term in the 2nd binomial, and the constant term in the 1st binomial multiplied by the coefficient of x in the 2nd binomial.

The constant term is the product of the two constants, b × d.

For example, $(2x – 3)(4x + 3) = 8x^2 + 6x – 12x – 9$

$= 8x^2 – 6x – 9$

- The same procedure can be used when two different variables are involved. For example, to multiply $(2x + 3y)(x^2 + 4)$, each term in one binomial is multiplied by each term in the other:

$$(2x + 3y)(x^2 + 4) = 2x^3 + 8x + 3yx^2 + 12y$$

B. Use what you have learned about multiplying binomials to check your prediction for the product of $(2x + 1)(x + 30)$ in **part A**. Was your prediction correct?

Examples

Example Counting Terms in the Product of Binomials

Pema noticed that when he multiplied two binomial factors, the number of terms in the product varied. What are the least and greatest number of terms possible? How do you know?

Solution	**Thinking**
There are three terms when ad + bc ≠ 0: $(ax + b)(cx + d)$ $= acx^2 + (ad + bc)x + bd$ For example: $(2x – 1)(x – 3)$ $= 2x^2 – 6x – 1x + 3$ $= 2x^2 – 7x + 3$	 • I tried to get three terms by making sure none of the coefficients or constants in the product is zero: $ad + bc = 2 \times -3 + (-1 \times 1)$ $= -6 + (-1) = -7$
There are two terms if ad + bc = 0: $acx^2 + 0x + bd = acx^2 + bd$ For example: $(6x – 3)(4x + 2)$ $= 24x^2 + 12x – 12x – 6$ $= 24x^2 – 6$	• I made two terms by making $(ad + bc)x = 0$. This happened when ad = -bc: $ad + bc = 6 \times 2 + (-3 \times 4)$ $= +12 + (-12) = 0$
There are four terms when there are two different variables or different powers involved: $(ax + by)(cx + d)$ $= acx^2 + adx + bcyx + bdy$ $(ax^2 + b)(cx + d)$ $= acx^3 + adx^2 + bcx + bd$	• I made four terms by using two variables so that the two middle terms would not be like terms. I couldn't combine them to make one middle term, so there had to be two terms in the middle.
For example: $(2x + y)(x – 2) = 2x^2 + xy – 4x –2y$ $(x + 5)(y – 2) = xy – 2x + 5y – 10$ Two, three, and four terms are possible.	• There couldn't be more than four terms since I was only multiplying two terms by two other terms.

Practising and Applying

1. Multiply.

a) $(5x + 2)(3x + 4)$

b) $(6y – 2)(4y + 5)$

c) $(2x + 8)(–3y – 4)$

d) $(–2y – 6)(–3y – 8)$

2. How many terms are there in each product after like terms are collected?

a) $(2y – 4)(3y + 4)$

b) $(2y – 4)(2y + 4)$

c) $(2y – 4)(2y + 4x)$

d) $(2y – 4)(3y + 6)$

3. By how much is the first polynomial greater than the product of the binomial multiplication?

a) $3y^2 + 4y – 3$ and $(2y + 4)(–y – 8)$

b) $2x – 4xy + 3$ and $(4x –3)(y + 5)$

4. a) Multiply $(7 – 3s)(5 + 4s)$.

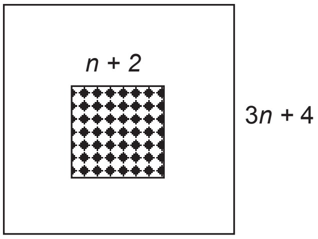
b) Evaluate the product from **part a)** for $s = –3$.

c) Evaluate $7 – 3s$ for $s = –3$.

d) Evaluate $5 + 4s$ for $s = –3$.

e) How are your answers in **part c)** and **part d)** related to the answer in **part b)**?

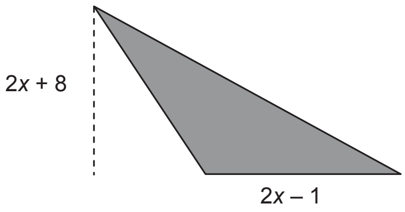
5. A square picture is inserted into a square frame. Write an expression that can be used to find the area of the white space around the picture. Show your work.



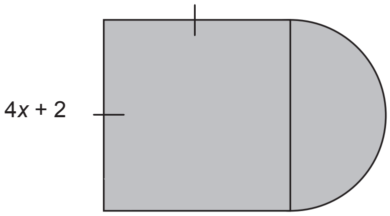
6. When you multiply $(3x –1)(2x + ▲)$, the coefficient of x in the product is 4 more than the coefficient of x^2. What is the value of ▲? Show your work.

7. Write a polynomial to describe the area of each shape. Show your work.

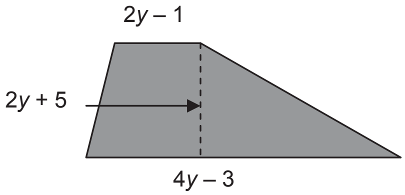
a)



b)



c)



8. A binomial multiplication such as $(x + a)(x + a)$ can be expressed as $(x + a)^2$. How much greater is $(x + a)^2$ than $(x – a)^2$?

9. Show how you can use what you learned about multiplying polynomials to help you calculate each mentally.

a) $42^2 = (40 + 2)^2$

b) $79^2 = (80 - 1)^2$

c) $53^2 - 47^2$

10. You are trying to teach a friend how to multiply $(3x + 2)(4x + 3)$. How could you use a numerical example such as $(30 + 2)(40 + 3)$ to help explain?

GAME: Polyprod

Polyprod is a game that will allow you to practise polynomial multiplication.

- Players take turns. Two to four players can play.
- On each player's turn, he or she rolls a die four times and records the results in the blanks below to create two binomials to multiply.

$$\left(\underset{\text{1st roll}}{\underline{\qquad}}\,x + \underset{\text{2nd roll}}{\underline{\qquad}}\right)\left(\underset{\text{3rd roll}}{\underline{\qquad}}\,x + \underset{\text{4th roll}}{\underline{\qquad}}\right)$$

- The player then multiplies the polynomials and determines the coefficient of x in the product. The coefficient of x^2 and the constant do not matter in this game.

Scoring

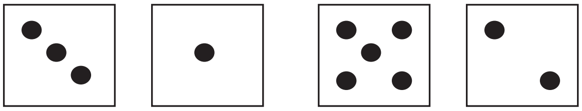
If the coefficient of x in the product is less than 10, you get 1 point.

If the coefficient of x is 10 to 20, you get 2 points.

If the coefficient of x is more than 20, you get 3 points.

The first player to get 20 points wins the game.

For example, if a player rolls 3, 1, 5, and 2:



The polynomials would be:

$$(\mathbf{3}x + \mathbf{1})(\mathbf{5}x + \mathbf{2}) = 15x^2 + 11x + 2$$

The player wins 2 points for a coefficient of 11 because 11 is in the 10 to 20 range.

Chapter 3 Dividing Polynomials

2.3.1 Dividing a Polynomial by a Monomial

Try This

Buthri and Devika bought x items that each cost Nu 250 and y items that each cost Nu 60. They decided to share the cost equally.

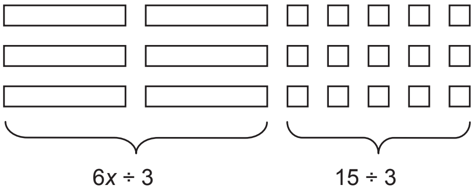
A. i) How much will the items cost altogether? Express your answer as binomial.

ii) How much will each girl need to spend?

- When a polynomial is divided by a monomial, each term of the polynomial is divided by the monomial and the quotients are added.

When a polynomial is divided by a monomial that is a constant, each term of the polynomial is divided by the constant and the quotients are added.

For example, to divide $(6x + 15) \div 3$ using tiles, it is modelled as $6x + 15$ divided into 3 groups. $6x$ is divided into 3 equal groups, or $6x \div 3$, and 15 is divided into 3 equal groups, $15 \div 3$.



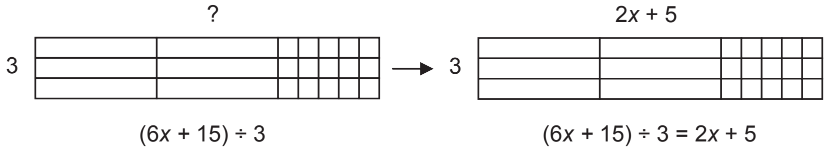
$(6x + 15) \div 3$ is $6x + 15$ divided into 3 rows, or groups. Each row represents the quotient $2x + 5$.

$$\begin{aligned}(6x + 15) \div 3 &= 6x \div 3 + 15 \div 3\\ &= 2x + 5\end{aligned}$$

You can also divide symbolically, dividing each term of the polynomial by the constant and then adding the quotients:

$$\begin{aligned}(6x + 15) \div 3 &= (6x \div 3) + (15 \div 3)\\ &= 2x + 5\end{aligned}$$

You can push the tiles together to make a rectangle, or area model. An area model can be used for division because the area represents the dividend and one of the side lengths represents the divisor. The unknown side length is the quotient.

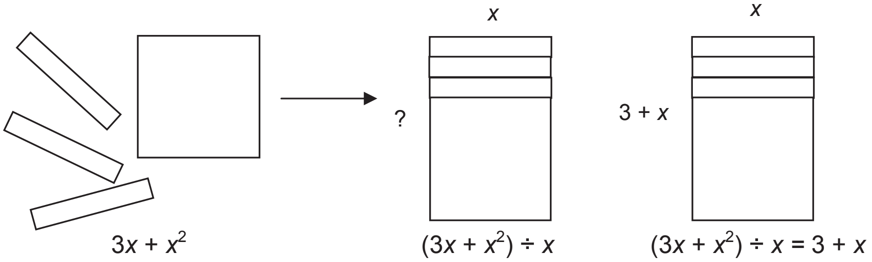


- When you divide a by a monomial that has a variable, you still divide each term in the polynomial by the monomial and then add the quotients.

For example, to divide $(3x + x^2) \div x$ symbolically, $3x$ is divided by x and x^2 is divided by x and the quotients are added:

$$(3x + x^2) \div x = (3x \div x) + (x^2 \div x) = 3 + x$$

To divide $(3x + x^2) \div x$ using tiles, you can use the area model. You arrange the tiles representing the dividend, $3x + x^2$, into a rectangle that has a side length equivalent to the divisor, x. The other side length will be the quotient.



- If the monomial you are dividing by is not a factor of one or more the terms in the polynomial, the quotient will not be a polynomial.

For example:

$(x^2 + 3x + 5) \div x = x + 3 + \frac{5}{x}$ $\quad$ $x + 3 + \frac{5}{x}$ is not a polynomial

because x is not a factor of 5.

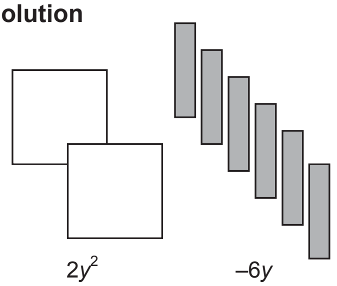
B. In **part A**, you created a binomial to represent the total cost of x items and y items. Then, you divided the binomial to find how much it would cost each girl. Write the division as a binomial divided by a monomial.

Examples

Example 1 Dividing a Polynomial by a Monomial Using Tiles

Show how to use tiles to divide $(2y^2 - 6y) \div 2y$.

Solution



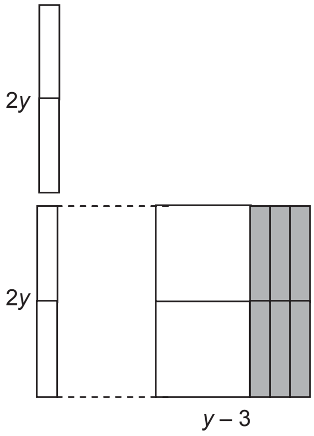
[Continued]

Thinking

- When I divide a polynomial by a monomial, I need to think of the polynomial as the area of a rectangle and of the monomial as one of its sides.
- I collected two y^2-tiles ($2y^2$) and six $-y$-tiles ($-6y$) to represent the area.

Example 1 Dividing a Polynomial by a Monomial Using Tiles [Continued]

Solution



$(2y^2 - 6y) \div 2y = y - 3$

Thinking

- Then I tried to shape the tiles into a rectangle with a side length of 2*y*.
- To help me do this I set up guide tiles to show me one side length.
- I arranged the area tiles to stay within the guide side length.
- I knew that the quotient was the remaining side length.

Example 2 Dividing Expressions With More than One Variable

Calculate $(3xy + 6y^2) \div 3y$.

Solution

$$(3xy + 6y^2) \div 3y = \frac{3xy + 6y^2}{3y}$$

$$= \frac{3xy}{3y} + \frac{6y^2}{3y}$$

$$= x + 2y$$

$$(3xy + 6y^2) \div 3y = x + 2y$$

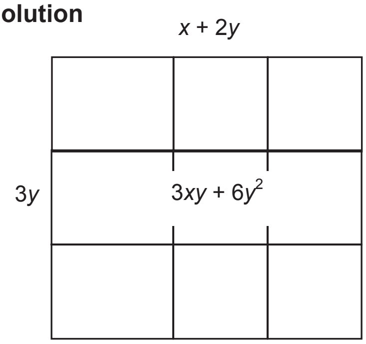
Thinking

- Sometimes I think of division as a fraction. If the numerator is a sum, it works just like it does with numbers, for example,

$$(10 + 15) \div 5 = \frac{10+15}{5} = \frac{10}{5} + \frac{15}{5}$$

- I simplified each fraction by looking for a common factor:

- The common factor in $\frac{3xy}{3y}$ is 3*y*, so there was an *x* left in the numerator and a 1 in the denominator when I simplified.
- The common factor in $\frac{6y^2}{3y}$ is 3*y*, so there was 2*y* left in the numerator and a 1 in the denominator when I simplified.

Solution	**Thinking**
 $(3xy + 6y^2) \div 3y = x + 2y$	• I checked to see if a rectangle with an area of $3xy + 6y^2$ had dimensions $3y$ and $x + 2y$, and it does.

Practising and Applying

1. Model with algebra tiles to divide.

a) $(6y + 2y^2) \div 2y$

b) $(5xy - 3x - x^2) \div x$

c) $(3x^2 + 24x) \div 3$

d) $(2y - 3xy) \div y$

2. Divide.

a) $(6m^3 - 2m^2n) \div 2m$

b) $(4s^3 + 2s - 10st) \div 2s$

c) $(16m^3 - 8m^2) \div 4m$

3. For each fraction below:

- Divide to find the quotient.
- Evaluate the quotient for $k = -2$ and for $m = 3$.
- Evaluate the numerator for $k = -2$ and for $m = 3$.
- Evaluate the denominator for $k = -2$.

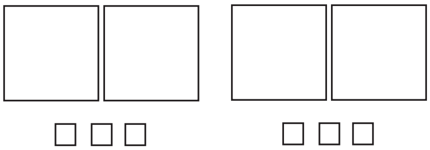
How does the value of the quotient relate the values of the numerator and denominator?

a) $\dfrac{4k^2 + 6km}{2k}$

b) $\dfrac{6k^2 - 6k}{3k}$

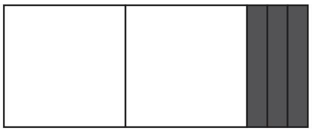
4. What division is being modelled?

a)



b)

c)



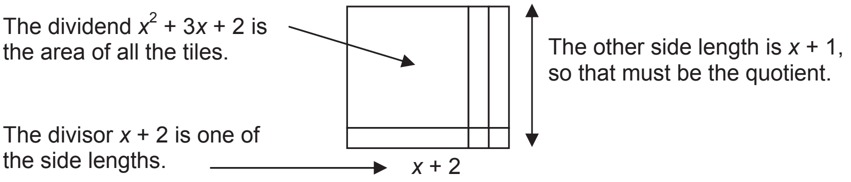
5. The quotient of a polynomial divided by a monomial is $3y - 2$. What could the polynomial and monomial be? List four possibilities.

6. How can you use what you know about multiplying polynomials to help you divide a polynomial by a monomial?

2.3.2 EXPLORE: Dividing a Polynomial by a Binomial

To divide polynomials, you can model the dividend as the area of a rectangle with one side length equal to the divisor.

For example, $(x^2 + 3x + 2) \div (x + 2)$ can be modelled as shown:



$$(x^2 + 3x + 2) \div (x + 2) = (x + 1)$$

A. Use algebra tiles to determine each quotient.

i) $(3x^2 + 7x + 2) \div (x + 2)$

ii) $(2x^2 + 7x + 6) \div (x + 2)$

iii) $(x^2 + 7x + 10) \div (x + 2)$

B. i) In what ways were all of your models in **part A** the same?

ii) Within each model, why were some of the *x*-tiles vertical and some horizontal?

C. Use algebra tiles to determine each quotient.

i) $(2x^2 + 7x - 4) \div (2x - 1)$

ii) $(4x^2 + 4x - 3) \div (2x - 1)$

iii) $(6x^2 - x - 1) \div (2x - 1)$

D. i) In what ways were all of your models in **part C** the same?

ii) Within each model, why were some of the *x*-tiles positive and some negative?

E. Look back at the divisions you did in **parts A and C**. If you divide the first term of the dividend by the first term of the divisor, how does this help you begin to figure out the quotient?

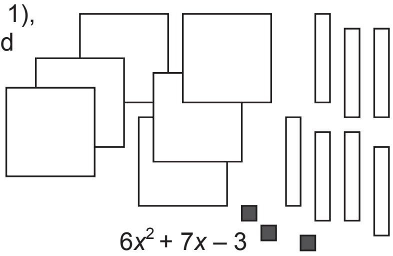
2.3.3 Dividing a Polynomial by a Binomial

Try This

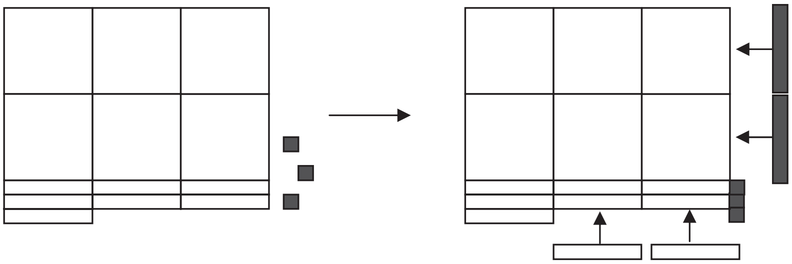
A. Nima is dividing $12x^2 + x – 6$ by another polynomial with integer coefficients. The quotient is also a polynomial with integer coefficients. How do you know he was not dividing by $5x$ + ▲?

- When you divide polynomials, you can use tiles and the area model. The area of a rectangle represents the dividend and one of the side lengths is the divisor. The other side length is the quotient.

For example, to divide $(6x^2 + 7x – 3) \div (3x – 1)$, gather together six x^2-tiles, seven x-tiles, and three –1-tiles.



Arrange the tiles in a rectangle where one side has a length of $3x – 1$. The other side length will be the quotient.

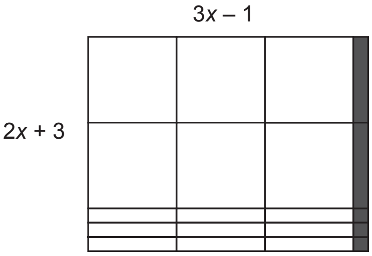


It's not a rectangle yet, nor is the width $3x – 1$. You need two $–x$-tiles on the right side to make the width $3x – 1$.

If you add two $–x$-tiles to the right and two x-tiles to the bottom, you can make rectangle whose width is $3x – 1$ without changing the value of the area because $–2x + 2x = 0$.

The quotient is the other side length, $2x + 3$.

$(6x^2 + 7x – 3) \div (3x – 1) = 2x + 3$



- You can also divide polynomials symbolically by thinking about how whole numbers are divided. When you calculate with whole numbers, you focus on the leftmost digits to give you an estimate, because those digits represent the most significant part of the number.

For example, to divide two numbers such as 1920 ÷ 60, you might estimate the quotient by dividing 1800 ÷ 60. Then you would multiply to get a product equal to the dividend, 1920.

$1920 \div 60 = ?$

Estimate — Multiply

$1920 \div 60 \approx 1800 \div 60 = 30 \longrightarrow 30 \times 60 = 1800$

31 — $\times\ 60 = 1860$

$32 \times 60 = 1920$ 32 works

$1920 \div 60 = 32$

You can use a similar approach to divide polynomials.

For example, to divide $3x^3 + 6x^2 + x + 2$ by $x + 2$,

- "Estimate" first by dividing $3x^3$ (in the dividend) by x (from the divisor). Just as with numbers, these leftmost terms are the greatest powers of each.
- Use your "estimate" to create a binomial with an unknown term.
- Multiply the binomial by the divisor.
- Use the coefficient of x or the value of the constant in the dividend to help you figure out the unknown term.

$(3x^3 + 6x^2 + x + 2) \div (x + 2) = ?$

The coefficient of x in the dividend is 1, so $\blacktriangle$ must be 1.

Estimate — Multiply

$(3x^3 + 6x^2 + x + 2) \div (x + 2)$

$\approx 3x^3 \div x = 3x^2 \longrightarrow (3x^2 + \blacktriangle)(x + 2) = 3x^3 + 6x^2 + \blacktriangle x + 2\blacktriangle$

$= (3x^2 + \mathbf{1})(x + 2) = 3x^3 + 6x^2 + x + 2$

$(3x^3 + 6x^2 + x + 2) \div (x + 2) = 3x^2 + 1$

- Sometimes the greatest powers are not at the left. In this case, you begin by rearranging the dividend and divisor. For example:

$(-32 - 8x + 4x^2) \div (4 + 2x) \rightarrow (4x^2 - 8x - 32) \div (2x + 4) = ?$

The constant in the dividend is –32, so

$4\blacktriangle = -32$

$\blacktriangle = -8$

Estimate — Multiply

$(4x^2 - 8x - 32) \div (2x + 4)$

$\approx 4x^2 \div 2x = 2x \longrightarrow (2x + \blacktriangle)(2x + 4) = 4x^2 + 8x + \blacktriangle 2x + 4\blacktriangle$

$= (2x + \mathbf{-8})(2x + 4) = 4x^2 - 8x - 32$

$(4x^2 - 8x - 32) \div (2x + 4) = 2x - 8$

- When working with polynomials of the same variable, the degree of the quotient is always the difference between the degrees of the dividend and the divisor.

For example, when you divide a degree 4 polynomial in one variable by a degree 3 polynomial of the same variable, the quotient has degree 1, as shown below:

$(3x^4 - x^3) \div x^3 = 3x - 1$ degree 4 ÷ degree 3 = degree 1

This makes sense if you think of the exponent laws: $x^4 \div x^3 = x^1$.

- Sometimes when you divide polynomials the quotient is not a binomial.

For example: $(x^3 - 1) \div (x - 1) = x^2 + x + 1$

B. In **part A**, you had to explain how you knew that Nima did not divide $(12x^2 + x - 6)$ by $(5x + ▲)$. How could estimating a quotient by dividing the first term of the dividend by the first term of the divisor have helped you explain?

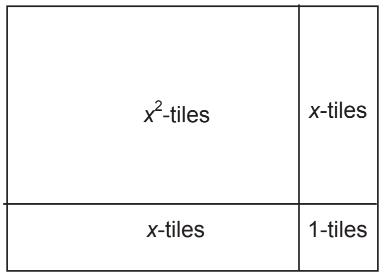
Examples

Example 1 Representing Division with Algebra Tile Diagrams

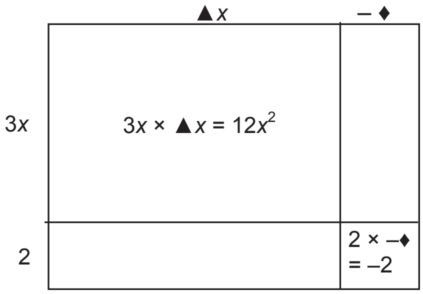
You can sketch diagrams based on algebra tile area models to help you divide polynomials. Divide $(12x^2 + 5x - 2) \div (3x + 2)$ using algebra tile diagrams. Show your work.

Solution

$(12x^2 + 5x - 2) \div (3x + 2)$



$(12x^2 + 5x - 2) \div (3x + 2)$



[Continued]

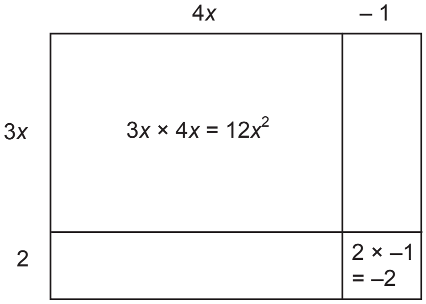
Thinking

- I sketched a diagram of an algebra tile model. I knew it had to have four parts — one part for x^2-tiles, two parts for x-tiles (vertical and horizontal), and one part for 1-tiles.

- I labelled one side $3x + 2$, which was the divisor.
- I labelled the unknown side $▲x - ♦$ because $3x \times ▲x = 12x^2$ (so I knew the x-term had to be positive) and $2 \times -♦ = -2$ (so I knew the constant term had to be negative).

<table>
<tr><th colspan="2">Example 1 Representing Division with Algebra Tile Diagrams [Continued]</th></tr>
<tr><th>Solution</th><th>Thinking</th></tr>
<tr><td>

$(12x^2 + 5x - 2) \div (3x + 2)$



$(12x^2 + 5x - 2) \div (3x + 2) = 4x - 1$

</td><td>

- Using $3x \times \blacktriangle x = 12x^2$, I knew the value of ▲ had to be 4 because $3x \times 4x = 12x^2$.
- Using $2 \times -\blacklozenge = -2$, I knew the value of ♦ had to be 1 because $2 \times -1 = -2$.
- The unknown side length, or quotient, was $4x - 1$.

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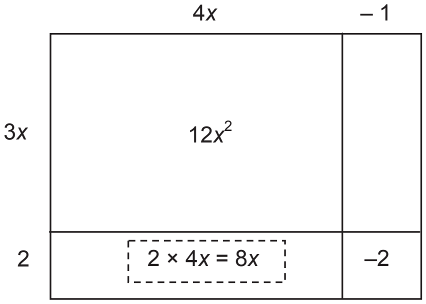

$(12x^2 + 5x - 2) \div (3x + 2) = 4x - 1$

</td><td>

- I decided to check by figuring out the value of the two x-tile parts of the model to see if they matched the x-term in the dividend.
- For the horizontal x-tiles: $2 \times 4x = 8x$, so I knew there were 8 x-tiles there.

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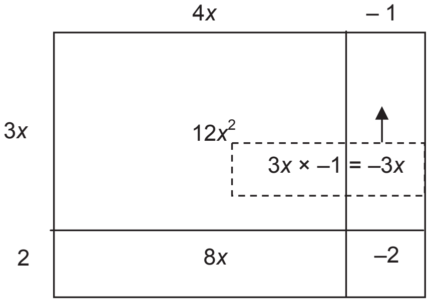


$(12x^2 + 5x - 2) \div (3x + 2) = 4x - 1$

</td><td>

- For the vertical x-tiles: $3x \times -1 = -3x$, so I knew there were three $-x$-tiles there.
- That made sense because the x-term in the dividend is $5x$ and $8x + (-3x) = 5x$.

</td></tr>
<tr><td>

$(3x + 2) \times (4x - 1) = 12x^2 + 5x - 2$

</td><td>

- I multiplied to double check.

</td></tr>
</table>

Example 2 Dividing Symbolically

Calculate $(x^4 + 2x^2 - 15) \div (x^2 + 5)$. Show your work.

Solution

$$(x^4 + 2x^2 - 15) \div (x^2 + 5)$$

$$\approx x^4 \div x^2 = x^2$$

$$(x^2 + \blacktriangle)(x^2 + 5) = x^4 + 5x^2 + \blacktriangle x^2 - \blacktriangle 5$$

$$\blacktriangle 5 = -15 \text{ so } \blacktriangle = -3$$

$$(x^2 - 3)(x^2 + 5) = x^4 + 2x^2 - 15$$

so

$$(x^4 + 2x^2 - 15) \div (x^2 + 5) = (x^2 - 3)$$

$$\begin{aligned}(1^4 + 2(1)^2 - 15) \div (1^2 + 5) &= 1^2 - 3\\ (1 + 2 - 15) \div (1 + 5) &= 1 - 3\\ -12 \div 6 &= -2\\ -2 &= -2\end{aligned}$$

Thinking

- The powers in the polynomials were already in descending order so I didn't have to rearrange them.
- I divided x^4 by x^2 to get an estimate of x^2.
- I created a binomial using x^2 and an unknown term and then I multiplied.
- I knew the last term in the product, $\blacktriangle 5$, was the constant in the dividend, –15, so the missing term had to be –3.
- I evaluated using $x = 1$ to check and found that my answer was correct.

Practising and Applying

1. Divide.

a) $(10x^2 + 19x + 6) \div (5x + 2)$

b) $(6x^2 + x - 12) \div (2x + 3)$

c) $(2x^2 - x - 15) \div (x - 3)$

d) $(8x^2 - 6x - 54) \div (2x - 6)$

2. Divide.

a) $\dfrac{6x^3 + 4x^2 - 3x - 2}{3x + 2}$ **b)** $\dfrac{22x - 14 + 12x^2}{3x + 7}$

c) $\dfrac{x^4 - x^3 + x - 1}{x - 1}$ **d)** $\dfrac{20 - 11x^2 - 3x^4}{x^2 + 5}$

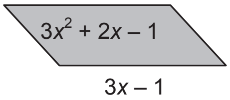
3. For each part in **question 2**, evaluate the numerator and denominator for $x = -1$. Then evaluate the quotient for $x = -1$. What do you notice?

4. List three possibilities for what Polynomial A and Polynomial B below might be.

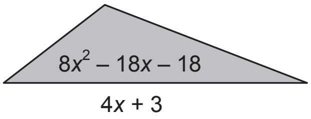
Polynomial A $\div$ $(2x + 1)$ = Polynomial B

5. Describe the height of each shape as a polynomial. The area and base length are given.

a)



b)



6. Describe a situation where the quotient of two polynomials is not a polynomial. Explain how you know this is the case.

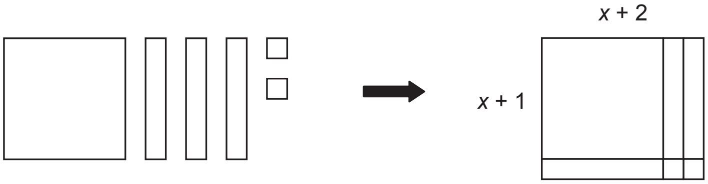
7. You divide a polynomial of degree 4 by a polynomial of degree 2 of the same variable. Predict the degree of the quotient. Explain your prediction.

2.3.4 EXPLORE: Creating Rectangles to Factor

When you divide a polynomial by another polynomial, you know one factor of the dividend, which is the divisor, and you are looking for the other factor, the quotient. If you think of a rectangle, the dividend is the area and the factors, which are the divisor and the quotient, are the dimensions of the rectangle.

When you divide a polynomial and you do not know either of the factors, it is like looking for both dimensions of a rectangle with a particular area. This is sometimes called **factoring**, or **factorising**.

For example, to factor $x^2 + 3x + 2$ using algebra tiles, take one x^2-tile, three x-tiles, and two 1-tiles and arrange them to form a rectangle. The dimensions of the rectangle will be the factors of the polynomial.

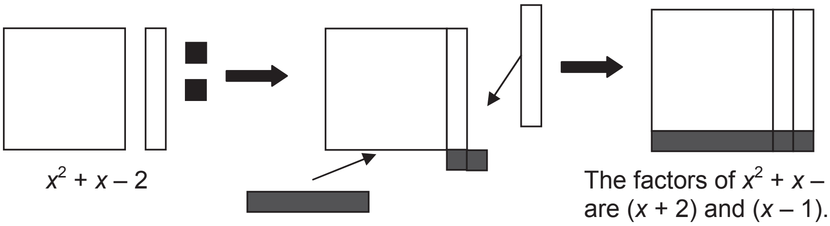


$x^2 + 3x + 2$

The factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$.

Sometimes, if there are negatives involved, you might have to add tiles before you can make a rectangle. To do this without changing the value of the polynomial you are factoring, you add positive and negative tiles of equal value.
For example, $x^2 + x – 2$ can be factored as shown below:

If you add a vertical x tile and a horizontal $–x$ tile, you can make a rectangle without adding any value to the polynomial.



$x^2 + x – 2$

$1x + –1x = 0$

The factors of $x^2 + x – 2$ are $(x + 2)$ and $(x – 1)$.

A. Factor these polynomials using algebra tiles. Sketch your models and list the two factors.

i) $x^2 + 5x + 6$	**ii)** $x^2 – 2x – 3$	**iii)** $2x^2 + 10x + 8$
iv) $2x^2 + 11x + 12$	**v)** $3x^2 + 8x + 4$	**vi)** $4x^2 – 2x – 6$

B. For each polynomial in **part A**,

i) compare the coefficient of the x^2-term in the polynomial with the coefficients of the x-terms in the factors. What do you notice?

ii) compare the constant term in the polynomial with the constant terms in the factors. What do you notice?

iii) compare the x-coefficient in the polynomial with the various coefficients and constants in your factors. What do you notice?

CONNECTIONS: Using Number Patterns to Factor

Sometimes you can use number patterns to help you factor a polynomial.

For example, suppose you want to factor $x^2 + 2x + 1$. Substitute various values for x to evaluate the polynomial and then factor each value.

Evaluate

x	$x^2 + 2x + 1$
1	4
2	9
3	16
4	25

Factor

x	$x^2 + 2x + 1$
1	4 = 2 × 2
2	9 = 3 × 3
3	16 = 4 × 4
4	25 = 5 × 5

Observe the pattern:
- When x is 1, both the factors are 1 more than x.
- When x is 2, both the factors are 1 more than x.
- It is the same when x is 3 or 4. Both the factors are 1 more than x.

If each factor is 1 more than the value of x, the factors are $(x + 1)$ and $(x + 1)$.

Multiply to check: $(x + 1)(x + 1) = x^2 + 1x + 1x + 1 = x^2 + 2x + 1$

Here is another example. Suppose you want to factor $x^2 + 3x + 2$. Substitute various values for x to evaluate the polynomial and then factor each value.

Evaluate

x	$x^2 + 3x + 2$
1	6
2	12
3	20
4	30

Factor

x	$x^2 + 3x + 2$
1	6 = 2 × 3
2	12 = 3 × 4
3	20 = 4 × 5
4	30 = 5 × 6

Observe the pattern:
- When x is 1, one factor (2) is 1 more than x and the other factor (3) is 2 more than x.
- When x is 2, one factor (3) is 1 more than x and the other factor (4) is 2 more than x.
- It is the same when x is 3 or 4. One factor is 1 more than x and the other factor is 2 more than x.

If one factor is 1 more than x and the other is 2 more than x, the factors are $(x + 1)$ and $(x + 2)$.

Multiply to check: $(x + 1)(x + 2) = x^2 + 2x + 1x + 2 = x^2 + 3x + 2$

Factor the polynomials below using patterns, as shown above.

1. $x^2 + 5x + 6$
2. $4x^2 + 4x + 1$
3. $x^2 - 1$
4. $x^2 - x - 2$
5. $9x^2 - 6x + 1$

UNIT 2 Revision

1. For each polynomial, tell each.

- its degree
- the type of polynomial it is
- whether there are like terms and what they are
- a situation the polynomial could describe

a) $3x - 2y + 4x + 6y$ **b)** $16 \times x^2$

2. Use algebra tiles to determine each sum or difference. Sketch your models.

a) $(-2x - 4) + (6x + 1)$

b) $(2x - 4) - (x - 2)$

c) $(-2x - 4) - (6x + 1)$

3. Add or subtract.

a) $(3x^2 -2x + 8) + (-5x^2 + 3x - \text{y})$

b) $(2x + 7) + (x^2 - 6x - 2)$

c) $(3x - 6x^2 + 8x^3) - (-x + x^2 - y^2)$

4. Why should you combine like terms when you add or subtract polynomials?

5. Multiply.

a) $2y(3 - 4y)$

b) $4(2 - 3x)$

c) $-6x(y + 4)$

d) $(5 - 2x)(-3x)$

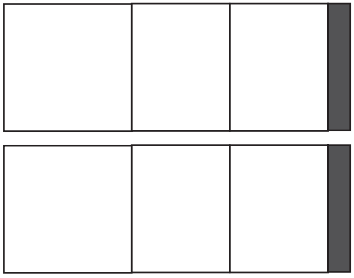
6. For each part of **question 5,** evaluate each factor for $x = -2$ and $y = 3$ and then evaluate the product for $x = -2$ and $y = 3$. What do you notice?

7. What two polynomials are being multiplied in each model below?

a)

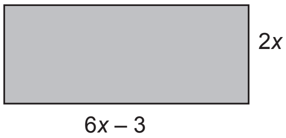
b)

c)

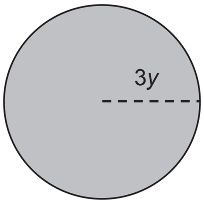


8. Calculate the area of each shape.

a)

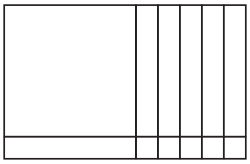


b)

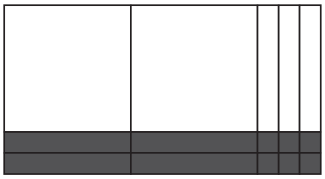


9. What two polynomials are being multiplied in each model below? Write a multiplication statement that includes the product.

a)



b)



10. Model each multiplication using algebra tiles and find the product.

a) $(2x + 3)(x + 2)$

b) $(2x + 3)(–x + 2)$

c) $(2x – 3)(x + 2)$

d) $(2y + 3)(–y – 2)$

11. Two binomials are multiplied. The product is modelled with 20 algebra tiles. List one possible pair of factors. Now find another possibility.

12. You multiply two binomials and the product is a polynomial of degree 3. What might have been the degrees of the polynomials you multiplied? How do you know?

13. Show how you can use what you know about multiplying polynomials to calculate these mentally.

a) 71^2

b) $71^2 – 69^2$

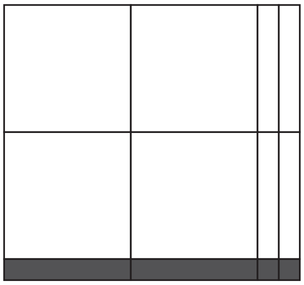
14. Model each division using algebra tiles and find the quotient.

a) $(4xy – 2y^2) \div 2y$

b) $(6y^2 – 4xy) \div 2$

c) $(4x^2 + 14x + 6) \div (x + 3)$

15. What division does this algebra model show?



16. Divide.

a) $(16x^2 + 6x^3 – 6x) \div (6x – 2)$

b) $(4 – 10x – 6x^2) \div (4 + 2x)$

17. Factor each polynomial.

a) $9x^2 + 3x – 2$

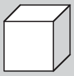
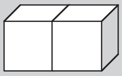
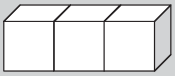
b) $x^2 – 9$

UNIT 3 LINEAR RELATIONS AND EQUATIONS

Getting Started

Use What You Know

Imagine that you join whole cubes to form “trains” and then paint the exterior faces of each cube.

1-cube train	2-cube train	3-cube train
		

A. List the number of painted faces for each train shown.

B. Complete this table of values to show the number of painted faces in each train up to ten cubes long.

C. Plot the points in your table to show the relationship between the number of painted faces and the train length.

D. For each increase of 1 cube in the length, what is the increase in the number of painted faces?

E. How can you predict the number of painted faces if you know the length of the train?

F. Use the relationship described in **part E** to predict the number of painted faces for a train that is 25 cubes long.

Train length	Painted faces
1	6
2	10
3	

Skills You Will Need

1. Evaluate.

a) 3^2 **b)** $(–2)^3$

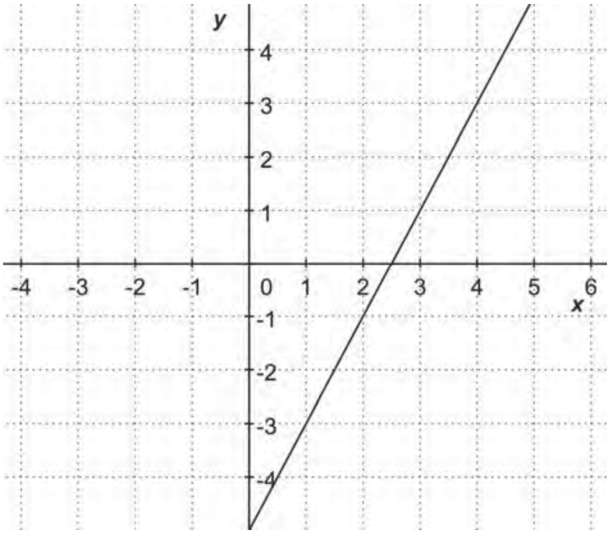
c) 5^2 **d)** 4^0

2. a) Use the graph to complete this table of values.

x	y
1	
	0
3	
4.5	

b) What is the slope of the line?

c) If x is 1.5, what is y?



3. Solve each equation.

a) $2a + 5 = 3$ **b)** $3x – 2 = –4$

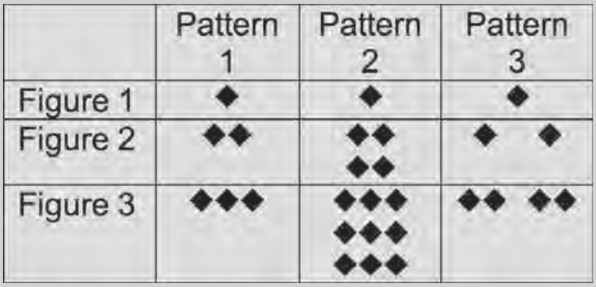
4. Calculate the area and circumference of a circle with radius 10 cm. Show your work.

Chapter 1 Linear and Non-Linear Relation Graphs

3.1.1 Patterns and Relations in Tables

Try This

- *Pattern 1* grows by adding one diamond to the previous figure.
- *Pattern 2* grows by forming squares of diamonds, each square one diamond wider than the one above it.
- *Pattern 3* grows by doubling the number of diamonds in the previous figure.



	Pattern 1	Pattern 2	Pattern 3
Figure 1	◆	◆	◆
Figure 2	◆◆	◆◆ ◆◆	◆ ◆
Figure 3	◆◆◆	◆◆◆ ◆◆◆ ◆◆◆	◆◆ ◆◆

A. Create a table of values for each pattern that tells the number of diamonds for each figure number. Extend each pattern to Figure 5.

B. Without extending the pattern, predict the number of diamonds there will be in Figure 10 for each pattern. Explain your prediction.

When two **variables** or two sets of values are connected in some way, you can often find a relationship between them. The property that connects one set of values to another is called a **relation.**

- A relation may be described in different ways:

- by listing connected values as **ordered pairs**;
- by writing connected values in a **table of values**;
- by plotting ordered pairs to create a graph of the relation;
- by stating a pattern rule expressed in words; or
- by using algebraic expressions to form an equation.

- Relations are often named according to the type of algebraic expression used to describe them.

Form of expression	**Type of relation**	**Example**
$ax + b$	**linear**	$y = 2x + 1$
$ax^2 + bx + c$	**quadratic**	$y = x^2 + 1$
$ab^x + c$	**exponential**	$y = -3^x + 4$

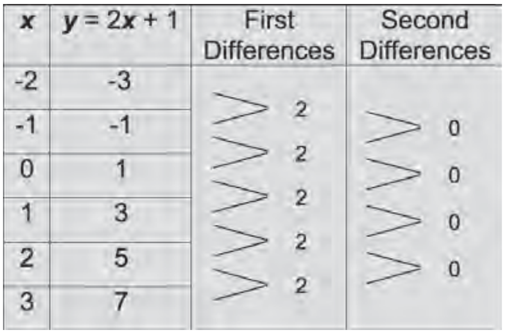
- One way you can determine whether a relation is linear, quadratic, or exponential is by calculating **first differences** and **second differences**. To do this, you

- create a table of values where the values in the first column are equally spaced;

- calculate the differences between the values of consecutive numbers in the second column (these are called first differences); and then

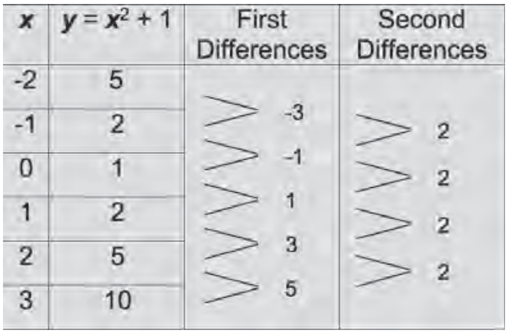
- calculate the differences between first differences (these are called second differences).

- For a linear relation, first differences are equal. Second differences are 0.



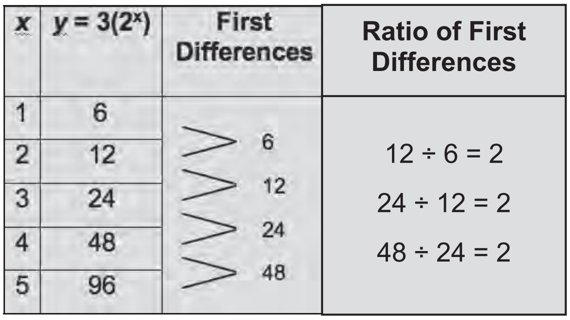
x	$y = 2x + 1$	First Differences	Second Differences
-2	-3	2	0
-1	-1	2	0
0	1	2	0
1	3	2	0
2	5	2	
3	7		

- For a quadratic relation, second differences are equal but not 0.



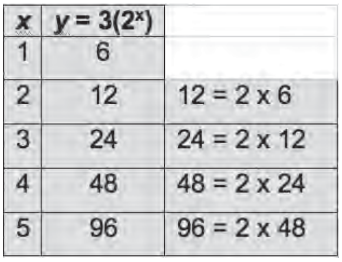
x	$y = x^2 + 1$	First Differences	Second Differences
-2	5	-3	2
-1	2	-1	2
0	1	1	2
1	2	3	2
2	5	5	
3	10		

- For an exponential relation, the ratios of consecutive first differences are equal.



x	$y = 3(2^x)$	First Differences	Ratio of First Differences
1	6	6	12 ÷ 6 = 2 24 ÷ 12 = 2 48 ÷ 24 = 2
2	12	12	
3	24	24	
4	48	48	
5	96		

Note that, in an exponential relation you multiply each term by the same factor (the ratio) to get the next term.



x	$y = 3(2^x)$	
1	6	
2	12	12 = 2 x 6
3	24	24 = 2 x 12
4	48	48 = 2 x 24
5	96	96 = 2 x 48

- Another way to tell whether a relation is exponential is when the variable appears in the exponent of the algebraic expression, for example y = $3(2^x)$.

C. For each diamond pattern from **part A**, there is a relation between the figure number and the number of diamonds.

i) What type of relation is Pattern 1? How do you know?

ii) What type of relation is Pattern 2? How do you know?

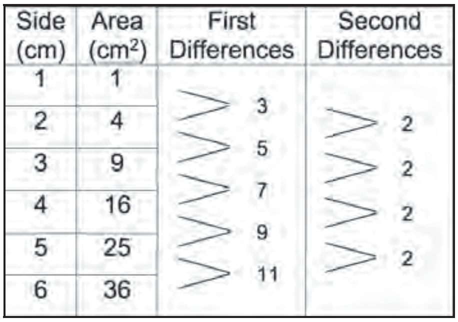
iii) What type of relation is Pattern 3? How do you know?

Examples

Example 1 Identifying a Quadratic Relation

Identify the type of relation that predicts the area of a square from its side length.

Solution



Side (cm)	Area (cm²)	First Differences	Second Differences
1	1	3	2
2	4	5	2
3	9	7	2
4	16	9	2
5	25	11	
6	36		

It is a quadratic relation.

Thinking

- I made a table of values that showed the areas for several squares. I used the formula $A = s^2$ to calculate each area.
- I used side lengths that were spaced apart by the same amount.
- The second differences are all equal but not zero, so this must be a quadratic relation.

Example 2 Determining Linear, Quadratic, or Exponential Relations

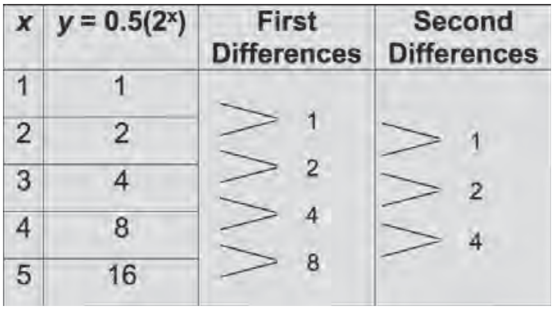
The algebraic expression $y = 0.5(2^x)$ is exponential since the variable x appears in an exponent in the expression.

a) How can you tell from the table of values that it is neither linear nor quadratic?

b) How can you tell from the table of values that it is exponential?

x	$y = 0.5(2^x)$
1	1
2	2
3	4
4	8
5	16

a) Solution



x	$y = 0.5(2^x)$	First Differences	Second Differences
1	1	1	1
2	2	2	2
3	4	4	4
4	8	8	
5	16		

The first and second differences show that the relation is neither linear nor quadratic.

Thinking

- The first differences are not equal, so the relation can't be linear.
- The second differences are not equal, so the relation can't be quadratic.

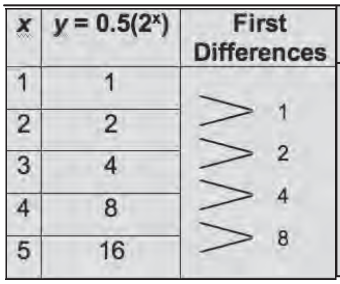
b) Solution 1

Each term is 2 times the term above it in the table of values. That means the relation is exponential.

Thinking

- I looked at the table of values to see if there was a multiplication pattern that connected the values in each row of the table.

b) Solution 2



x	y = 0.5(2^x)	First Differences	Ratio of 1st Differences
1	1		
		1	
2	2		2 ÷ 1 = 2
		2	
3	4		4 ÷ 2 = 2
		4	
4	8		8 ÷ 4 = 2
		8	
5	16		

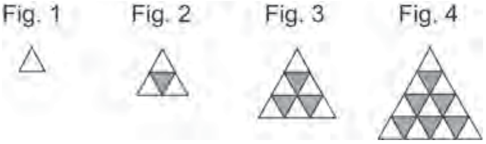
The relation is exponential.

Thinking

- The ratio of the first differences is the same for each row in the table, so the relation must be exponential.

Practising and Applying

1. Look at this pattern made from equilateral triangles with sides of 1 cm.



a) Create a table to show the relationship between the perimeter of the outside triangle and figure number.

b) Create a table to show the relationship between the number of white triangles and figure number.

c) Create a table to show the relationship between the total number of small triangles and figure number.

d) For each relation in **part a), part b), and part c)**, state whether it is linear, quadratic or exponential. Explain.

2. For each figure after Figure 1, there is a square of pentagons with one pentagon above it. Is the relation between the number of pentagons and the figure number linear, quadratic, or exponential? Explain how you know.

Figure 1

Figure 2

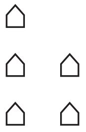
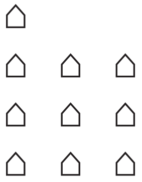


Figure 3



3. For each table of values, state whether the relation it represents is

Linear
Quadratic
Exponential
None of these

a)

x	*y*
0	1
1	3
2	9
3	27
4	81

b)

x	*y*
-2	5
-1	7
0	9
1	11
2	13

c)

x	*y*
1	1
2	8
3	27
4	64
5	125

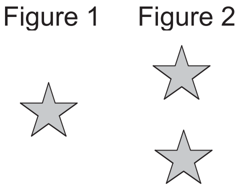
d)

x	*y*
-2	0
0	-4
2	0
4	12
6	32

e)

x	*y*
1	3
2	6
5	9
6	12
8	15

4. These are the first two figures in a pattern.



Draw the next three figures for each:

a) if the pattern were a linear relation

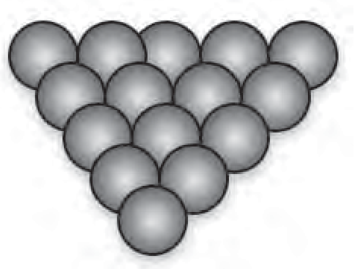
b) if the pattern were an exponential relation

5. a) Create a table of values to show the relationship between the radius and the circumference of a circle.

b) Create a table of values to show the relationship between the radius and the area of a circle.

c) Determine whether each relation in **parts a) and b)** is linear, quadratic, or exponential. Justify your decisions.

6. Fifteen balls are arranged in a triangle to form the base of a pyramid.



a) How many more balls are needed to build the layer above this?

b) How many more balls are needed in total to complete the pyramid?

c) How many balls would you need if you wanted to build a layer underneath the base layer?

d) Determine whether the relationship between the total number of balls used and the number of layers in the pyramid is linear, quadratic, exponential, or none of these.

7. The table shows the length of an elastic band from which a mass was suspended. Determine whether the relationship between the length of the elastic and the mass is linear, quadratic, or neither.

Mass (g)	5	10	15	20	25	30
Length (cm)	1	2	3	4	5	6

8. A skydiver jumps from a plane at an altitude of 6000 m. Her altitude is recorded every 4 s before she opens her parachute. Is the relation described by the table linear, quadratic, or neither?

Time (s)	Altitude (m)
0	6000
4	5920
8	5680
12	5280
16	4720
20	4000
24	3120

9. a) Create tables of values that represent linear, quadratic, and exponential relations.

b) Use first and second differences or patterns in the tables to show which table represents which type of relationship.

3.1.2 Scatter Plots of Discrete and Continuous Data

Try This

Dawa is loading a 30 kg cart with 1 kg bottles of water.
Therchu is filling a 30 kg tank with water (1 L of water is 1 kg).

A. Use a table of values to show

i) how the total mass of Dawa's cart of water bottles increases as it is loaded one bottle at a time.

ii) how the total mass of Therchu's tank of water increases as it is filled.

B. Sketch a graph for each table of values in **part A**.

You can use scatter plots to show a relationship between variables.

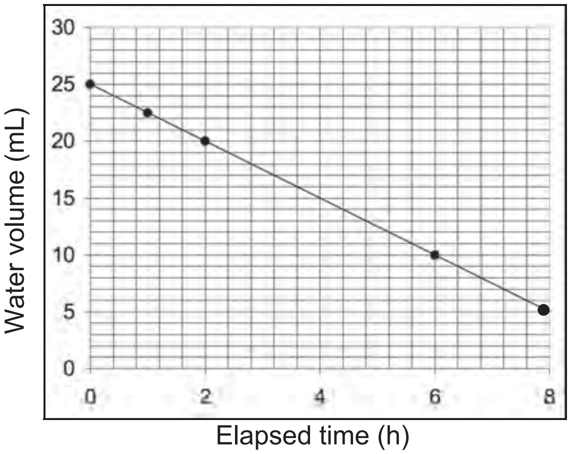
- The **independent variable** is the variable you control. Its values usually appear in the left column of a table of values and are the *x*-coordinates of the plotted points.
- The value of the **dependent variable** depends on the value of the independent variable. Its values usually appear in the right column of a table of values and are the *y*-coordinates of the plotted points.
- Sometimes the plotted points are joined by a line or a curve. This makes it easier to see whether there is a relationship between the variables.
- A solid line is usually drawn to show that any value between the plotted ones could have been used as a value of the independent variable. In this case, the data is said to be **continuous**.
- If no values or a limited number of values exist between the ones plotted, a dashed graph is usually used and the data is said to be **discrete**.
- When you predict or estimate between known coordinates you are **interpolating.** Predicting beyond known coordinates is called **extrapolating**.

For example, this table shows how the volume of water in a glass changes as the water evaporates.

Elapsed time (h)	Water volume (mL)
0	25
1	22.5
2	20
6	10
8	5

- Elapsed time is the independent variable.
- Water volume is the dependent variable.

Evaporation of Water



Since any amount of elapsed time between the plotted points is possible, the data is continuous and a solid line is used to join the points. That means it is possible to interpolate values between plotted points. For example, at 4 h the volume was 15 mL.

C. i) Which graph from **part B** represents continuous data? discrete data?

ii) For which graph does it make sense to interpolate values? For which graph does it make sense to extrapolate?

Examples

Example 1 Using a Scatter Plot with Discrete Data

The first three figures in a pattern are shown. Use a graph to predict the number of squares in Figure 20.

Figure 1

3 squares

Figure 2

5 squares

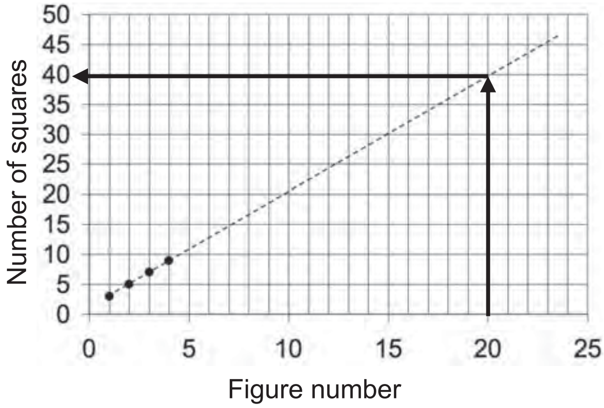
Figure 3

7 squares

Solution

Figure number	Number of squares
1	3
2	5
3	7
4	9

Squares in the Pattern



From the graph, it looks like the 20th figure will have about 40 squares but it is more likely 41.

Thinking

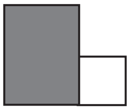
- I made a table of values that related the number of squares in the pattern to the figure number.
- I plotted the data points from my table of values in a scatter plot.
- The data is discrete since the figure number is a whole number, so I used a dashed line to show the relationship.
- I used the graph to extrapolate, or predict the number of squares in the 20th figure to be about 40. Then I noticed that each figure has an odd number of squares so I adjusted my prediction from 40 to 41 squares.

Example 2 Using a Scatter Plot with Continuous Data

Imagine that you are building shapes like those below. Each white square has a side length of 1 cm. Each grey rectangle has a height of 2 cm but the width can be any value. Use the graph in **example 1** to estimate the area of each shape

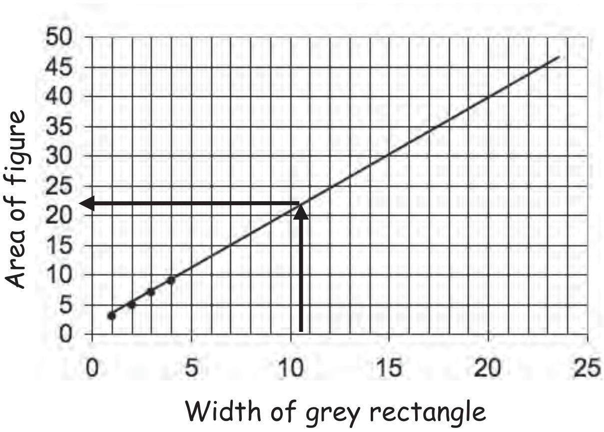
a) if the width of the grey rectangle is 10.5 cm

b) if the width of the grey rectangle is 3.5 cm



Solution

a)



When the grey rectangle is 10.5 cm wide, the area of the figure is about 22 cm^2.

[Continued]

Thinking

- The graph from example 1 has the number of squares in each figure along the vertical axis. Because each square has an area of 1 cm^2, I knew I could think of the vertical axis as the area of each figure instead.
- The graph from example 1 has the figure number along the horizontal axis. This is the same as the width of the grey rectangle for whole number values. So I knew I could think of the horizontal axis as the width of the grey rectangle instead.
- Because the widths of the grey rectangles can have any value, the data set is continuous. So I visualized a solid line instead of a dashed line.
- To find the area of the grey rectangle that is 10.5 cm wide, I located 10.5 cm on the horizontal axis and then found the point on the graph for that x-coordinate. Then I located the value on the vertical axis for that point.

Example 2 Using a Scatter Plot with Continuous Data [Continued]

Solution

b) When the grey rectangle is 3.5 cm wide, the area of the figure is about 8 cm^2.

Thinking

- To find the area of the grey rectangle that is 3.5 cm wide, I interpolated. I located 3.5 cm on the horizontal axis and then found the point on the graph for that *x*-coordinate. Then I located the value on the vertical axis for that point.

Example 3 Joining Points with a Smooth Curve

The first three figures in a pattern are shown. Draw a graph to show how the number of pentagons is related to the figure number in this pattern.

Figure 1

Figure 2

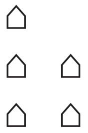
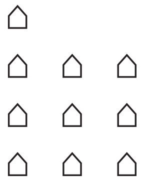
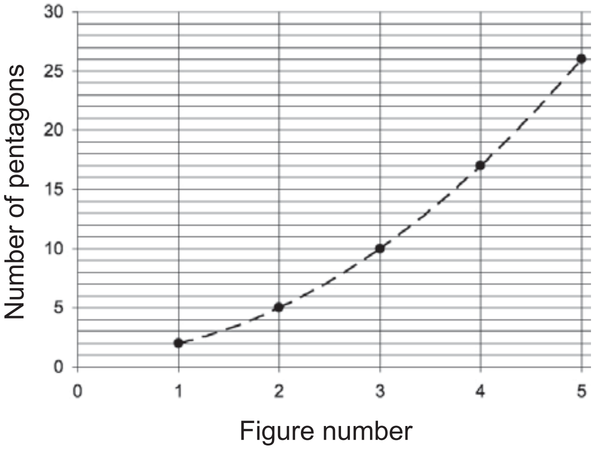


Figure 3



Solution

Figure number	Number of pentagons
1	2
2	5
3	10
4	17
5	26



Pentagon Pattern

Thinking

- I made a table of values for the first several figures in the pattern.
- Then I plotted the points.
- The points do not fall along a line, so I used a smooth curve to join them.
- There can't be a decimal figure number, so I used a dashed curve to show that the data was discrete.
- You can't interpolate values between the plotted points because the figure numbers and number of pentagons can only be whole numbers.

Practising and Applying

1. a) Create the next three figures.

b) Complete the table of values.

Figure number	Number of circles
1	3
2	7
3	11
4	
5	
6	

c) Is this data discrete or continuous? How do you know?

d) Should you use a solid or dashed line to represent the data?

e) Draw a scatter plot to show the relationship between the figure number and the number of circles.

f) Use your scatter plot to extrapolate the number of circles in Figure 10.

2. a) Calculate the area of five rectangles with a width of 5 cm. You might use heights of 1 cm, 1.5 cm, 2 cm, 3 cm, 4.75 cm, and 5 cm.

b) Use your data to complete the following table of values.

Height	Area

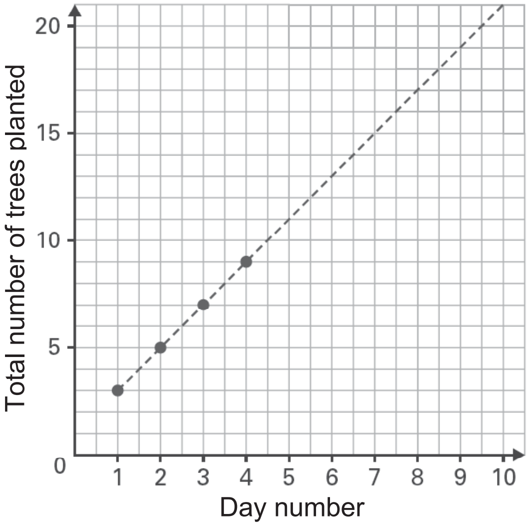
c) Is this data set discrete or continuous? How do you know?

d) Should you use a solid or a dashed line to represent the data?

e) Draw a scatter plot and line of best fit to show the relationship between height and area.

3. This graph shows the total number of trees planted in a park over several days.

Trees Planted in a Park

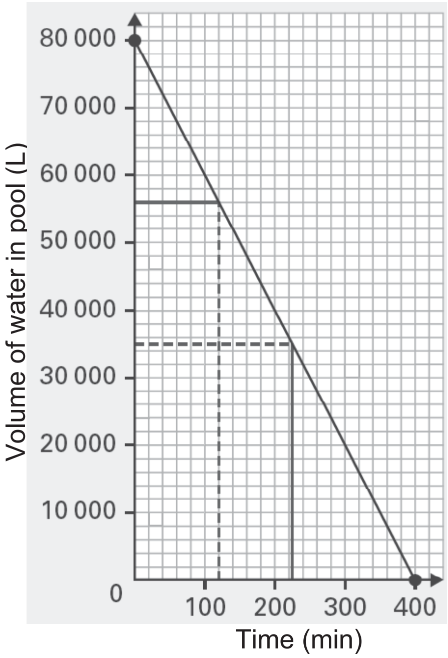


a) Why is a dashed line used instead of a solid line in this graph?

b) Use the graph to extrapolate the total number of trees planted on the 7th and 10th days.

4. The graph shows the amount of water remaining in a swimming pool as it is being drained.

Draining the Pool

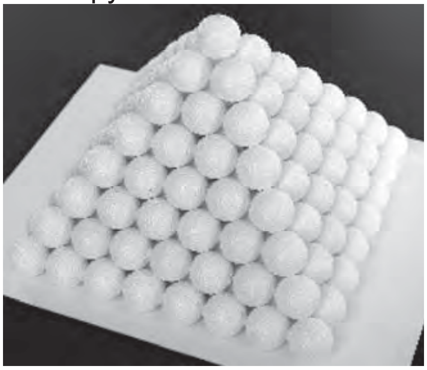


4. [Cont'd] **a)** Why is it appropriate to use a solid line instead of a dashed line for the graph?

b) Two pairs of line segments are used to interpolate values from the graph. What question might each pair answer?

c) Why would it not be reasonable to extrapolate values beyond 400 min?

5. Balls are piled to form a square-based pyramid.



a) Complete the table of values.

Layer number	Number of balls
1	1
2	4
3	
4	
5	
6	
7	
8	

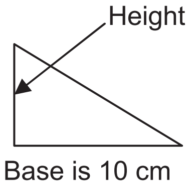
b) Is this data set discrete or continuous? How do you know?

c) Should you use a solid or dashed line to represent the data? Why?

d) Construct a scatter plot for the data. Why would a smooth curve fit the data points better than a line?

6. Sonam drew several different right triangles, each with a base of 10 cm, but with different angles at the base. She measured the height of each triangle and recorded her data in a table.

Angle (°)	Height (cm)
5	0.9
10	1.8
15	2.7
20	3.6
25	4.8
30	5.8
35	7.0
40	8.4
45	10.0
50	11.9
55	14.3
60	17.3

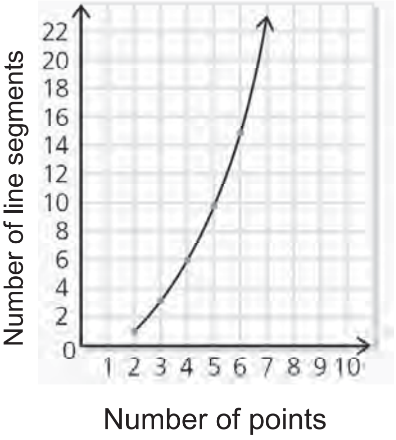


a) Construct a scatter plot using the data.

b) Would you use a line or a smooth curve to connect the points? Explain.

c) Should the graph be dashed or solid? Why?

7. Manju marked points on a page and connected them. She did this for 2, 3, 4, and 5 points. Then she drew a graph. Was she right to use a solid curve? Explain.



3.1.3 EXPLORE: Graphs of Linear and Non-Linear Relations

Linear relations, quadratic relations, and exponential relations not only use different expressions to represent them, but they also have different graphs.

A. Each equation represents a continuous linear relation. Create a table of values for each and then graph it.

i) $y = x$ **ii)** $y = x + 1$ **iii)** $y = 3x - 1$

B. Each equation represents a continuous quadratic relation. Create a table of values for each and then graph it.

i) $y = x^2$ **ii)** $y = x^2 + 1$ **iii)** $y = 3x^2 - 1$

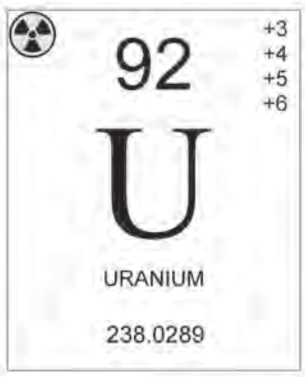
C. Each equation represents a continuous exponential relation. Create a table of values for each and then graph it.

i) $y = 2^x$ **ii)** $y = 2^x + 1$ **iii)** $y = 3(2^x) - 1$

D. How can you tell by just looking at a graph whether it represents a relation that is linear, quadratic, or exponential?

CONNECTIONS: Half-Life

Uranium is a radioactive material. It is used in some countries to generate electricity. As time passes, a piece of uranium will release its energy and lose mass. This process is called radioactive decay. The time that it takes a piece of radioactive material to lose half its mass is called its half-life.



1. Suppose you start with 256 g of radioactive material with a half-life of 1 day. What mass will remain after 1 day? 2 days? 3 days?

2. **a)** What fraction of the original mass will remain after 1 day? 2 days? 3 days?

b) Express each fraction from **part a)** in the form $\frac{1}{2^n}$, where n is the number of days.

c) Predict the fraction that will remain after 10 days. Explain your prediction.

3. There is a relationship between the number of days and the fraction of material remaining. What type of relationship is it? How do you know?

3.1.4 Graphs of Linear and Non-Linear Relations

Try This

There is a myth about a hero who had to choose a reward from a powerful king. The king presented the hero with these three choices for his reward:

Choice 1: 3 coins a day

Choice 2: 1 coin at the end of this week, 2 coins at the end of the next week, 3 coins at the end of the week after that and so on, earning 1 more coin at the end of each week than the week before.

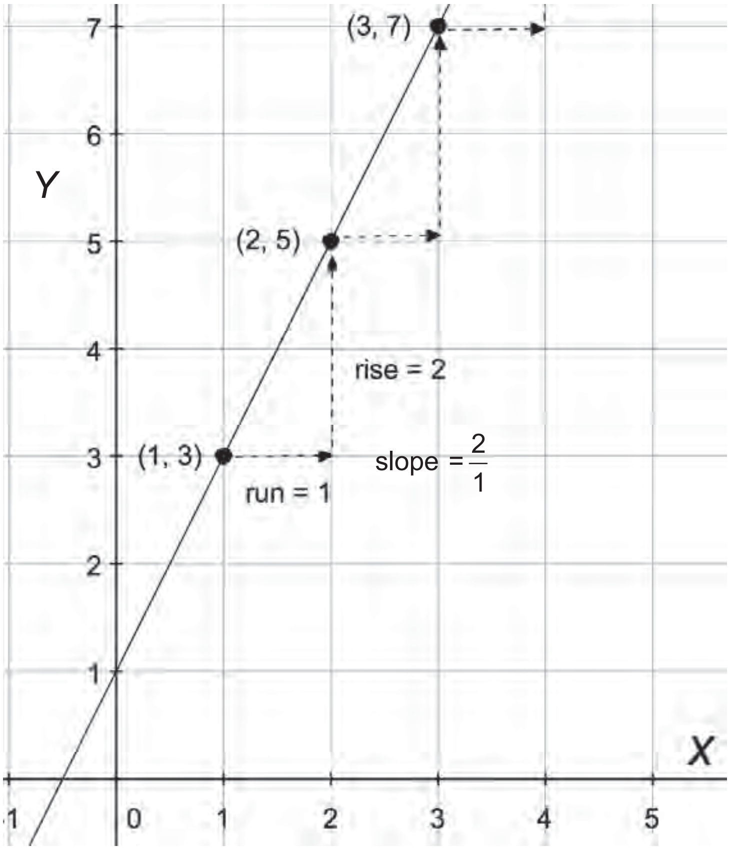
Choice 3: 2 coins at the end of the first four weeks, 4 coins at the end of the next four-week period, 8 at the end of the next four week period, and so on. The number of coins would double each four-week period.

A. For each choice, tell how many coins the hero would have received in total by the end of the 4th week.

B. Repeat **part A** for the end of the 12th week.

The graph of a linear relation is a straight line.

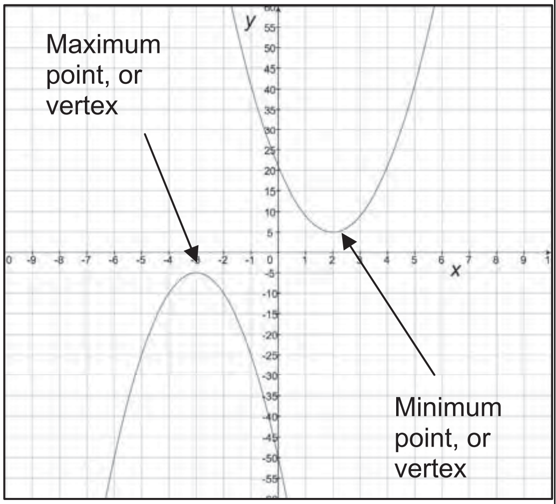
- In a linear relation, the y-coordinates increase by a constant value, the **rise**, as the values of the x-coordinates increase at a constant value, the **run**. The ratio of these values is the **slope** of the line: $\frac{\text{rise}}{\text{run}}$.
- You can choose any two points to determine the rise and run.
- If the line slopes up and to the right, the slope is positive, as shown in the graph to the right.



- If the line slopes down and to the right, the slope is negative because the value of the rise is negative.

The graph of a quadratic relation is non-linear.

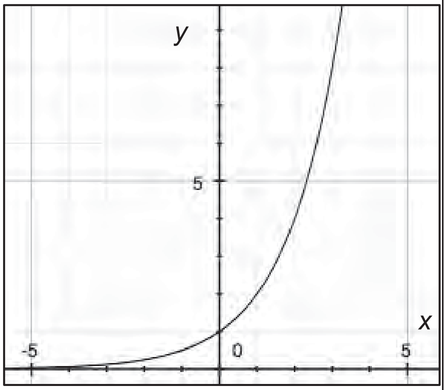
- The graph of a quadratic relation is a U-shaped curve called a **parabola**.
- If the parabola opens upward, like the top curve in the graph to the right, it has a **minimum** point.
- If it opens downward, like the other curve, it has a **maximum** point.
- The maximum or minimum point is called the **vertex** of the parabola.
- Both of the parabolas shown here have a vertical line of symmetry.



- Sometimes it does not make sense to use certain values for the *x*-coordinate such as negative values, so you may only see half of the curve.

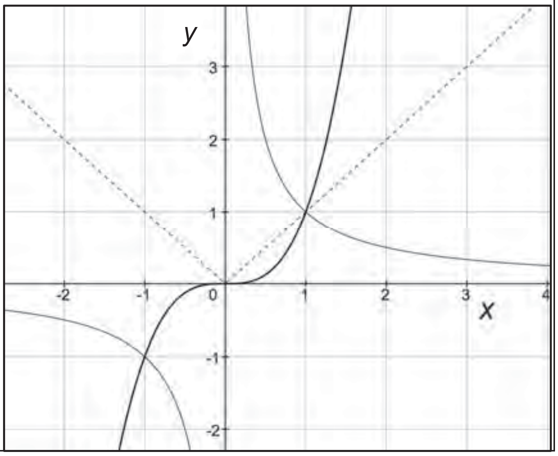
The graph of an exponential relation is also non-linear.

- The graph of an exponential relation becomes nearly parallel to the *x*-axis on one side and then curves upward and becomes nearly parallel to the *y*-axis on the other side.
- Sometimes it does not make sense to use certain values for the *x*-coordinate, so you see only one of the sides becoming nearly parallel to the *y*-axis.



There are other types of relations that have graphs that are neither straight lines, U-shaped, nor curved upwards.

- The term **non-linear** can be used to describe these graphs and relations.



C. i) What choice in **part A** would you recommend to the hero if the time period was 1 year? Explain.

ii) What kind of relation does each choice describe?

Examples

Example 1 Comparing the Graphs of Relations

a) Draw the graphs of the relations represented by $y = 2x$, $y = x^2$, and $y = 2^x$.

b) Compare the graphs.

Solution

a)

$y = 2x$

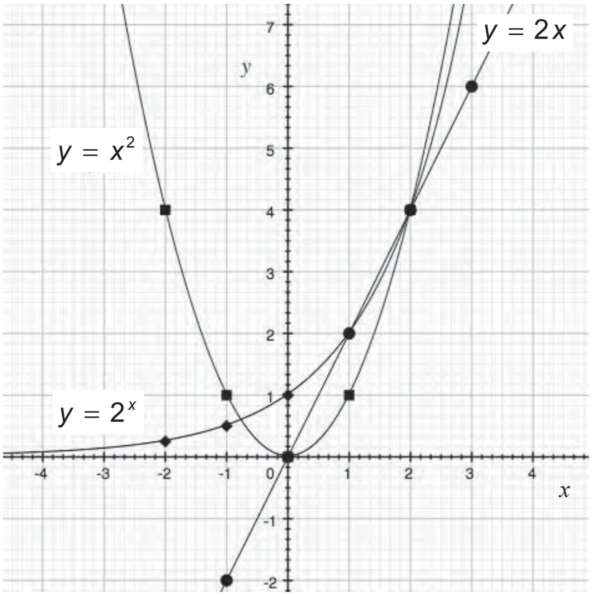
x	y
–2	–4
–1	–2
0	0
1	2
2	4
3	6

$y = x^2$

x	y
–2	4
–1	1
0	0
1	1
2	4
3	9

$y = 2^x$

x	y
–2	0.25
–1	0.5
0	1
1	2
2	4
3	8



b) The graphs for $y = 2x$ and $y = x^2$ go through the origin, but the graph of $y = 2^x$ does not.

- The graphs of $y = x^2$ and $y = 2^x$ are curves, but the graph of $y = 2x$ is a straight line.
- The graph of $y = 2x$ has points where the *y*-coordinates are negative. The others do not.
- The graph of $y = x^2$ is symmetrical about the *y*-axis, but the others are not symmetrical about any vertical line.

Thinking

a) I created a table of values for each relation using some equally spaced values for *x*.

- I used the values to give me ordered pairs, or points, to plot.

- The question did not say I had to use discrete values of *x*, so I assumed it was continuous data and I used a solid curve to show the shape of each relation.

b) I looked at the graphs to compare them. I looked for things that were the same and for things that were different.

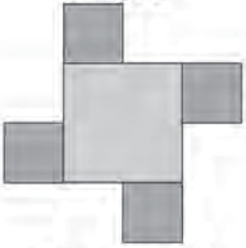
Example 2 Graphing a Non-Linear Relationship

The following diagrams show the stages in a design based on the 8 cm square shown in Stage 1. What kind of relation describes the connection between the stage number and the area at each stage?

Stage 1

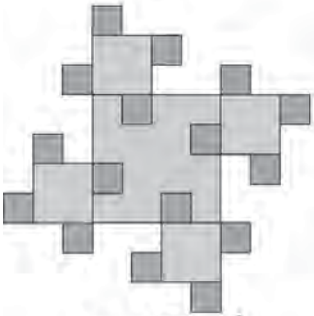
Area = 64 cm^2

Stage 2



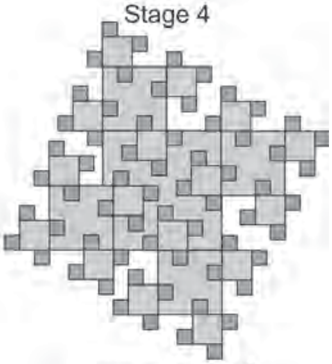
Area = 128 cm^2

Stage 3



Area = 176 cm^2

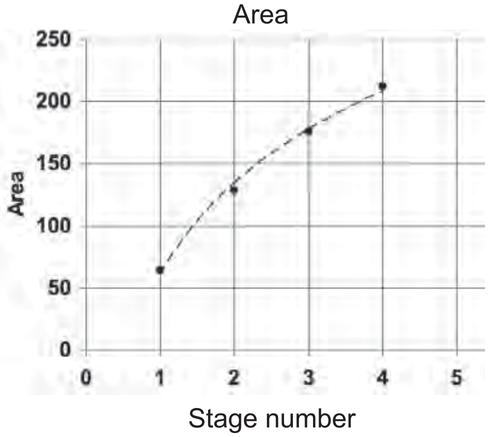
Stage 4



Area = 212 cm^2

Solution

Stage	Area (cm^2)
1	64
2	128
3	176
4	212



This relation is non-linear. It does not seem to be quadratic or exponential.

Thinking

- I decided to graph the relation because the shape of the graph would tell me what type of relation it was.
- I created a table of values that related the stage number to the area of the shaded figure.
- I used the values to create a scatter plot.
- I used a dashed curve to join the points because the data is discrete (you can't have decimal stage numbers).
- The graph isn't a straight line, nor is it U-shaped or curving upwards. I'll just call it a non-linear relation.

ractising and Applying

1. a) Graph these equations on the same grid.

$y = 2x + 1$ $\quad$ $y = 2x - 1$

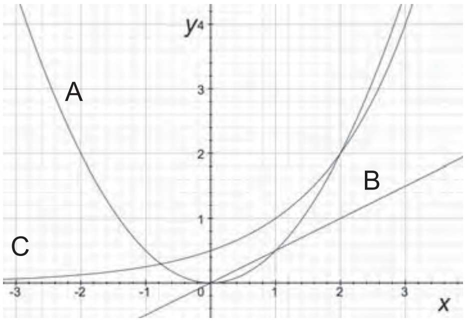
$y = -2x - 1$ $\quad$ $y = -2x + 1$

b) How do you know each represents a linear relation?

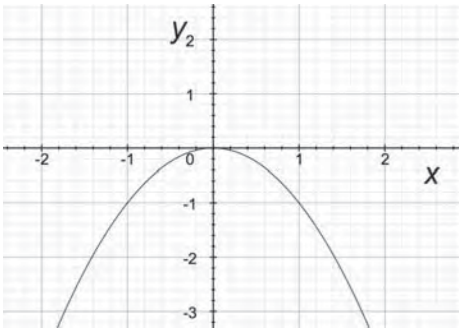
c) Which have the same slope? What is that slope?

2. a) Which graph below represents a quadratic relation? How do you know?

b) Which represents an exponential relation? How do you know?



3. The graph of $y = -x^2$ is below.



a) Sketch the graph of $y = -x^2 + 3$.

b) How are the graphs the same?

c) How are the graphs different?

4. Would a graph that shows the relationship between the radius and the circumference of a circle be a line or a parabola? How do you know?

5. a) Draw a graph that shows the relationship between the length of the edge of a cube and the surface area of the cube.

b) Draw another graph that shows the relationship between the length of the edge of a cube and the volume of the cube.

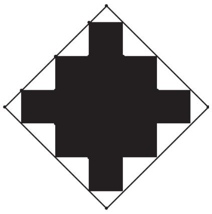
c) How can you tell that neither graph is linear?

6. In this pattern, a large diagonal square is drawn around figures made of black squares. The area of each black square is 1 cm^2.

Figure 1

Figure 2

Figure 3



a) Complete the following table.

Figure number	White area	Black area	Area of big square	Part of big square that is black
1	1	1	2	0.5000
2	3	5	8	0.6250
3	5	13	18	0.7222
4				
5				
6				

b) Use graphs to show the relationship between the figure number and the values in each column of the table.

c) Which of the relationships are linear, quadratic, exponential, or none of these? How do you know?

Chapter 2 Equation of a Line

3.2.1 The Meaning of Slope and *Y*-Intercept

Try This

A climber has climbed to an altitude of 1700 m. He now stands at the bottom of a cliff that he plans to climb. He usually climbs at a rate of 5 m per minute.

A. i) Suppose you were to draw a graph showing what his altitude will be every 5 min for the first 30 min of his climb. How do you know the graph will be a straight line?

ii) Draw the graph.

B. What altitude has he reached

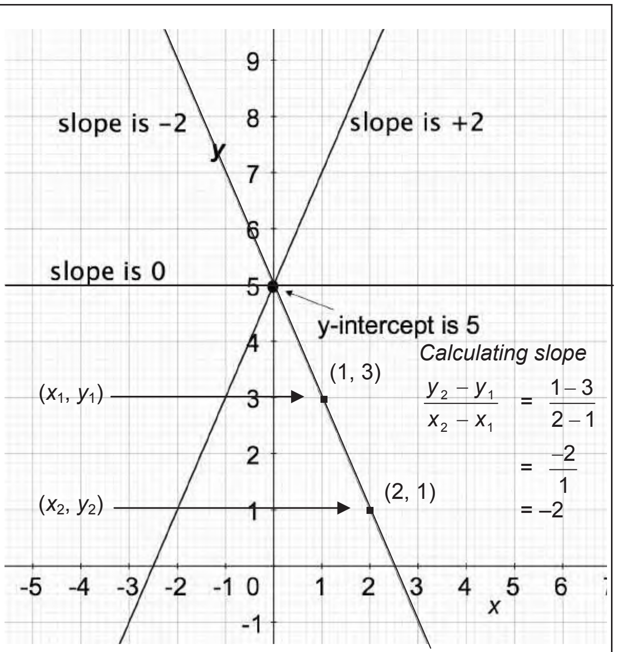
i) at 0 min into the climb? **ii)** at 20 min into the climb?

- The **slope** of a line tells the rate at which the *y*-variable changes in terms of the *x*-variable:

  - If the slope is positive, the *y*-variable increases as the *x*-variable increases.
  - If the slope is negative, the *y*-variable decreases as the *x*-variable increases.
  - If the slope is 0, the *y*-variable does not change as the *x*-variable changes.

- The slope (m) can be calculated using any two points on the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



- The ***y*-intercept** is the value of the *y*-coordinate where the line meets or crosses the *y*-axis.

C. i) How is the climbing rate of 5 m per min reflected in your graph from **part A**?

ii) What is the *y*-intercept of the graph? What does it represent?

Examples

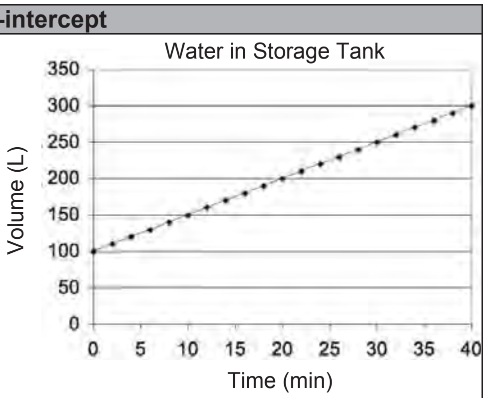
Example 1 Interpreting Slope and *Y*-intercept

This graph shows the volume of water in a storage tank as it was being filled.

a) How much water was in the tank initially?

b) At what rate was the tank being filled?

c) How are the values in **part a)** and **part b)** related to the graph?



Solution	Thinking
a) The tank started with 100 L of water.	 **a)** The graph shows that at 0 min there were 100 L of water in the tank.
b) The tank was filled at the rate of 5 L/min.	**b)** I knew I could use any two points on the graph to calculate the rate at which the tank was filled because the rate is the same as the slope. • I used the points at 0 min (0, 100) and at 10 min (10, 150) to find the slope: $\frac{150 - 100}{10 - 0} = \frac{50}{10} = 5$.
c) 100 L is the *y*-intercept of the graph. 5 L/min is the slope.	

Example 2 Predicting Slope and *Y*-intercept

An empty water barrel weighs 15 kg. It is filled with water at the rate of 0.5 L/s. The mass of 1 L of water is 1 kg. Suppose you drew the graph of the combined mass of the barrel and water as it is filled.

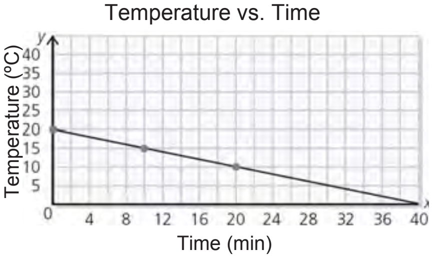
a) How do you know the graph will be a straight line?

b) What will be the slope and *y*-intercept of the graph?

Solution	Thinking
a) The mass of the barrel and water changes at a constant rate so the graph must be a straight line.	 **a)** I knew 0.5 L of water went into the barrel every second. Since 1 L of water is 1 kg that meant the mass increased by 0.5 kg each second.
b) The slope is 0.5. The *y*-intercept is 15.	**b)** A rate of 0.5 L/s means that, for a run of 1 s, the rise would be 0.5 kg. That's a slope of $\frac{0.5}{1} = 0.5$. • I knew that *y*-intercept was the value of the *y*-coordinate when $x = 0$. At 0 s, there was no water in the barrel, so the mass was the mass of the empty barrel, which was 15 kg.

Practising and Applying

1. This graph shows the temperature of water in a glass.



a) What does the slope of the graph represent?

b) The slope is negative. What does that tell you about what happened to the water temperature?

c) What does the *y*-intercept represent?

d) If the water cooled at the same rate but the *y*-intercept had been 25, how would that have changed the graph?

2. In many cities, when you hire a taxi you pay a base amount plus a rate per minute of travel time.

a) Suppose you were to graph the relationship between the time it takes to complete a trip and its cost. Why would the graph be linear?

b) What would the slope represent?

c) What would the *y*-intercept represent?

3. You can calculate your ideal heart beat rate for exercise by first subtracting your age from 220.

- Your maximum ideal heart beat rate is 85% of this value.
- Your minimum ideal heart beat rate is 60% of this value.

a) Suppose you were to graph the relationship between age and the maximum heart beat rate. How do you know the graph will be a straight line?

b) Predict the *y*-intercept.

c) Draw the graph to check your prediction.

d) Repeat **parts a) to c)** for the minimum ideal heart beat rate.

e) Compare the graphs.

4. A pilot wants to reduce her plane's airspeed at a constant rate from 550 km/h to 100 km/h in 5 min.

a) Complete the table to show the airspeed at the end of each one-minute interval.

Elapsed time (min)	Airspeed (km/h)
0	550
1	
2	
3	
4	
5	100

b) Suppose you were to graph the aircraft's airspeed from 0 min to 5 min. How do you know the graph will be a straight line?

c) Draw the graph.

d) Determine the slope and the *y*-intercept of the graph.

e) How could you determine the slope and *y*-intercept from the table?

5. a) Suppose the pilot in **question 4** reduced speed at a constant rate in the same amount of time but to 200 km/h. What would the new graph look like?

b) Suppose the pilot reduced speed at a constant rate from 550 km/h to 100 km/h in 4 min. How would this affect the graph? Explain.

6. **a)** How do you know that a linear relationship exists between the figure numbers and the number of sticks in the following pattern?

Figure 1 Figure 2 Figure 3

b) Suppose you were to graph the relationship. What would be the value of the slope? Justify your answer.

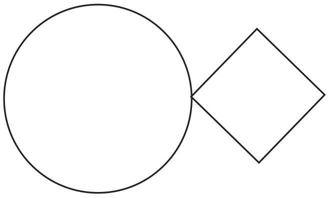
c) What would be the value of the *y*-intercept? Justify your answer.

7. The formula that relates a circle's circumference to its radius is $C = 2\pi r$.

a) What would be the slope of the graph?

b) What radius would you have to consider in order to determine the *y*-intercept? Explain.

8. Suppose you attached a 1 cm square to a circle to create the figure below. Suppose you then graphed the relationship between the radius of the circle and the perimeter of the entire figure.



a) How do you know the graph will be linear?

b) What will be the slope of the graph?

c) What would be the value of the *y*-intercept?

9. When using the coordinates of two points to calculate slope, why is it important to subtract the *x*-coordinates in the same order as the *y*-coordinates? Use an example to help you explain.

10. Temperature is usually measured using one of two common scales—Fahrenheit and Celsius.

There is a linear relationship between temperatures in one scale and temperatures in the other scale.

	Celsius	Fahrenheit
Freezing point of water	0°C	32°F
Boiling point of water	100°C	212°F

a) Draw the graph of the relation between the two scales. Place Celsius temperatures on the horizontal axis.

b) How could you use the graph to determine the temperature in °F for 30°C?

c) Your graph uses the vertical axis to show Fahrenheit temperatures.
What is the meaning of the *y*-intercept of the graph?

d) What is the slope of your graph?

e) What is the meaning of the slope?

3.2.2 EXPLORE: The Equation of a Line

A linear relation can be described with an equation of the form y = mx + b. When the equation is in this form, you can visualize the graph of the relation using its slope and y-intercept.

A. i) Choose any value for m and substitute it into the form y = mx + b.

ii) Create three different equations, each with a different value for b.
Create one equation with a positive value for b, one with a negative value for b, and one where b is 0.

iii) Graph the equations on the same set of axes.

iv) What do you notice about the slope of the graphs?

v) What do you notice about the y-intercepts of the graphs?

B. i) Choose any value for b and substitute it into the form y = mx + b.

ii) Create three different equations, each with a different value for m.
Create one equation with a positive value for m, one with a negative value for m, and one where m is 0.

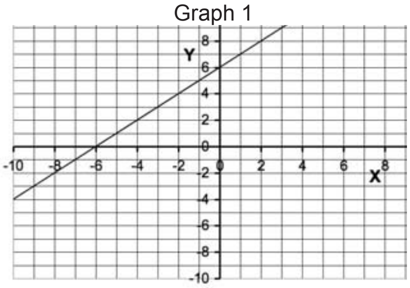
iii) Graph the equations on the same set of axes.

iv) What do you notice about the y-intercepts of the graphs?

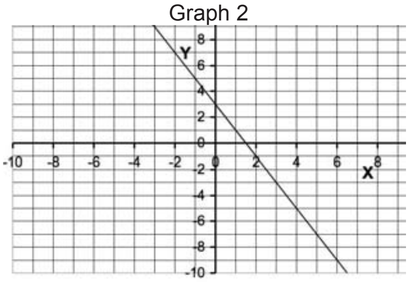
v) What do you notice about the slopes of the graphs?

C. Write the equation for each graph below.

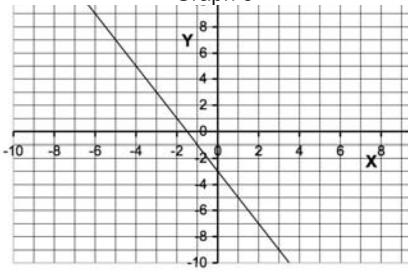
Graph 1



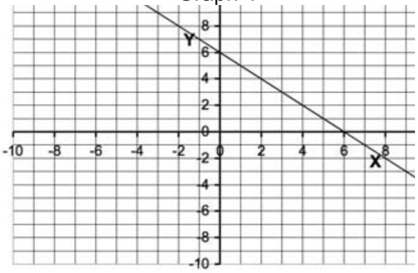
Graph 2



Graph 3



Graph 4



3.2.3 Slope and *Y*-Intercept Form

Try This

Oil is shipped in large drums. An empty drum has a mass of 25 kg and 1 L of oil has a mass of 0.92 kg.

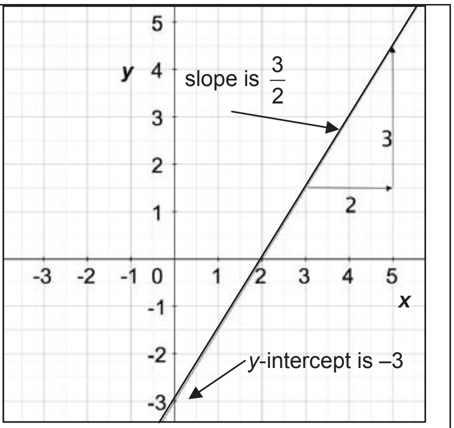
A. Write a formula that you could use to calculate the total mass of the drum and the oil if you know the volume of oil in it. Use the variable *k* for the total mass in kilograms of the drum and the oil and *l* for the volume of oil in litres.

When an equation of a line is in the form ***y* = m*x* + b**, it is called the **slope and *y*-intercept form** of the equation.

- The value of m is the slope of the line.
- The value of b is the *y*-intercept.

If you know the slope of a line and its *y*-intercept, you can use this information to write an equation for the graph.

For example, this graph has a slope of $\frac{3}{2}$ and a *y*-intercept of –3, so the equation of the line is $y = \frac{3}{2}x - 3$.



The equation of this graph is $y = \frac{3}{2}x - 3$.

B. i) How did you use the information about the mass of the drum and the mass of 1 L of oil to create the formula in **part A**?

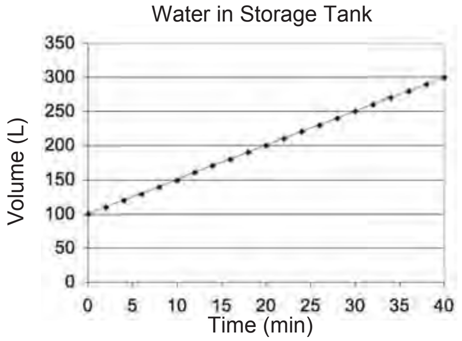
ii) Suppose you were to create a graph of the relationship between the total mass and the volume of oil. What are the slope and *y*-intercept of the graph? How do you know?

Examples

Example 1 Determining the Equation of a Line Given its Graph

The graph shows the volume of water in a storage tank as the tank was being filled over a 40 min period.

Write an equation for this graph.



Solution

The equation of the line is $y = 5x + 100$.

Thinking

- I could see from the graph that the y-intercept was 100.
- I calculated the slope by dividing the rise by the run using the values at 40 m (40, 300) and at 0 m (0, 100):

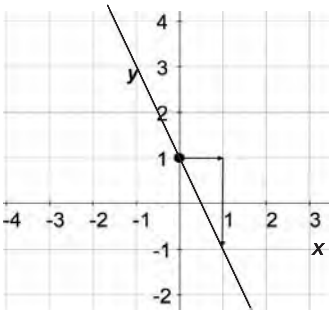
$$\frac{300 - 100}{40 - 0} = \frac{200}{40} = 5$$

- So I knew that m = 5 and b = 100 and I substituted these values into $y = mx + b$.

Example 2 Sketching the Graph of a Line Given its Slope and *Y*-intercept

Sketch the graph of a line that has a slope of –2 and a y-intercept of 1.
Then write the equation for the graph.

Solution



$y = -2x + 1$

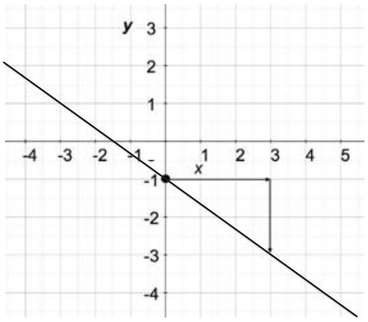
Thinking

- I marked the y-intercept at the point (0, 1) on the y-axis. It's easier to plot the y-intercept before the slope since there is only one spot it can be.
- Since the slope is –2, I knew that, for a run of 1, there had to be a rise of –2. A rise of –2 means the graph goes down.
- I marked a point that was 1 unit to the right (the run) and 2 units down (the rise) from the y-intercept.
- Then I drew a line through the two points.

Example 3 Sketching a Line Given its Equation

Sketch the graph of a line represented by the equation $y = \frac{-2}{3}x - 1$.

Solution



$y = \frac{-2}{3}x - 1$

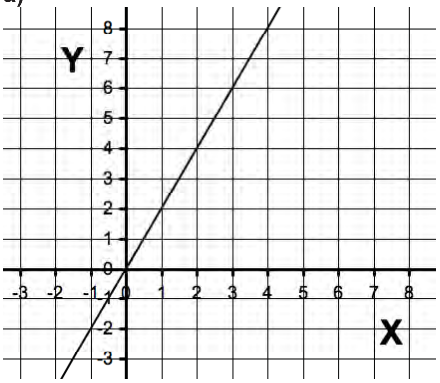
Thinking

- The equation was in the form $y = mx + b$, so I knew the slope (m) was $\frac{-2}{3}$ and the y-intercept (b) was –1.
- I first plotted the y-intercept on the y-axis at (0, –1).
- The slope was negative, so I knew the line had to go down from left to right.
- I plotted a point that was 3 units to the right (a run of 3) and 2 down (a rise of -2) from the y-intercept.
- Then I joined the two points with a line.

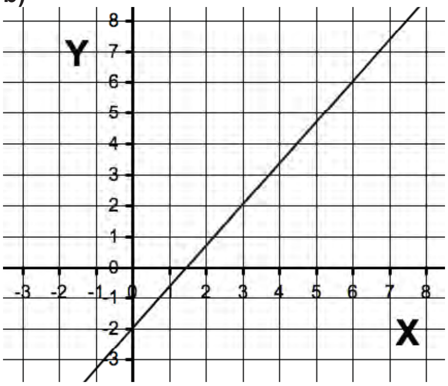
Practising and Applying

1. Determine the slope and y-intercept for each graph.
Then write the equation for each graph.

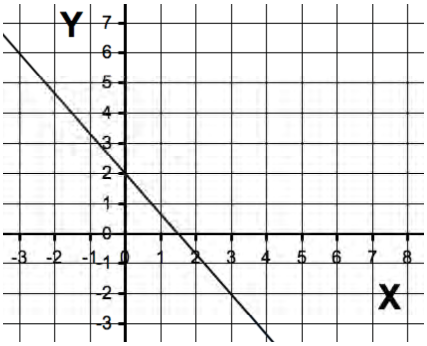
a)



b)



1. c)



2. Sketch the graph of each line below. Then write its equation.

a) slope is 1 and y-intercept is –1

b) slope is –3 and y-intercept is 1

3. Imagine the graph for each equation.

A: $y = 3x + 2$ **B**: $y = -2x - 2$

C: $y = \frac{2}{3}x + 5$ **D:** $y = \frac{1}{3}x + 2$

a) Which graph crosses the *y*-axis at the highest point? at the lowest point?

b) Which graph has the slope closest to vertical? Which has the slope closest to horizontal?

4. Sketch each graph.

a) $y = \frac{1}{2}x - 2$ **b)** $y = \frac{-2}{3}x + 3$

c) $y = \frac{-3}{4}x$ **d)** $y = 1 + \frac{3}{4}x$

e) $y = 0.25x + 1.5$ **f)** $y = -0.5x - 2.5$

5. Forensic scientists sometimes use the following equations to estimate the height (*H*) of a person from the length (*f*) of the person's femur bone. (The femur is the large bone in the upper part of the leg.) All measurements are in centimetres.

Male: $H = 1.9f + 81.3$
Female: $H = 2.0f + 73.0$

5. a) The *y*-intercept in this situation could be called the *H*-intercept. What will be the *y*- or *H*-intercept of the graph of each equation?

b) What will be the slope of each line?

c) Use the information from **parts a) and b)** to sketch both lines on the same axes.

d) Why do the values of the *y*-intercepts not represent the heights of real people?

6. Some doctors in North America use the following formulas to predict adult height (*H*) using a child's height at three years of age (*c*).

Male: $H = 1.3c + 55$
Female: $H = 1.3c + 42$

a) How could you use the equations to predict that the lines would have the same slope?

b) How could you use the equations to predict how far apart the *y*-intercepts of the lines will be?

c) Sketch both lines on the same axes.

7. Suppose you were sketching the graph of $y = \frac{1}{3}x + 2$. Why does it make sense to begin with the *y*-intercept instead of the slope?

8. Imagine that you are given the equation of a line in the form $y = mx + b$. Describe the steps you could use to sketch the graph of the line using only the information in the equation.

3.2.4 The Line of Best Fit

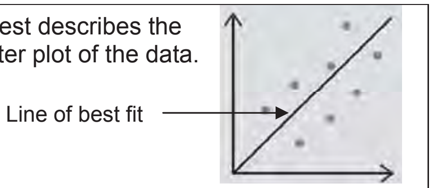
Try This

Mr. Yeshey recorded how long each student studied for a mathematics test and the mark each student received. He wants to show his students that there is a relation between study time and test marks.

Student	A	B	C	D	E	F	G	H	I	J	K	L
Hours of study	0	0	1	1	1	2	2	3	3	3	4	4
Test mark (%)	68	58	71	80	74	76	84	70	90	93	88	87

A. Use the marks in the table to predict a mark for a student who studied for 2.5 h. Explain your prediction.

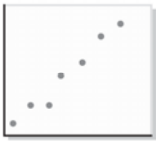
The **line of best fit** is the straight line that best describes the relationship between two variables in a scatter plot of the data.



The **correlation** between variables is a description of the strength of the relationship between two variables. It tells whether you can use the line of best fit to predict the value of one variable if you already know the value of the other variable.

- A **positive correlation** means that as one variable increases, the other variable also increases.

A strong positive correlation



A weak positive correlation

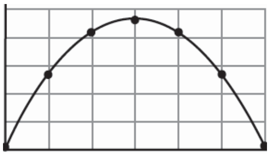
- A **negative correlation** means that as one variable increases, the other variable decreases.

A strong negative correlation

A weak negative correlation

- Sometimes there is **no correlation** between the variables.

- Sometimes a smooth curve provides a better fit to the data than a line.



You can draw a line of best fit for a correlation. The stronger the correlation, the more confident you can be in making predictions based on the line. You can also use the equation of the line of best fit to make predictions about the data.

B. i) Create a scatter plot for the data from **part A**. Draw a line of best fit. Use it to predict a student's test score for a study time of 2.5 h.

ii) How does your prediction compare with the prediction you made in **part A**? Which prediction was easier to make? Explain.

Examples

Example 1 Predicting a Value Using the Line of Best Fit

Length of humerus	Height
38.5	181.6
33.9	169.3
24.3	141.5
25.4	144.8
34.4	169.7
28.5	152.7
30.9	160.7
24.8	142.8
36.8	176.6
41.2	189.4
27.2	149.8
30.1	157.2
31.6	162.7
32.3	164.6
30.8	159.2
31.6	162.7
27.3	149.2
34.5	170.2
25.4	143.8
36.9	176.9

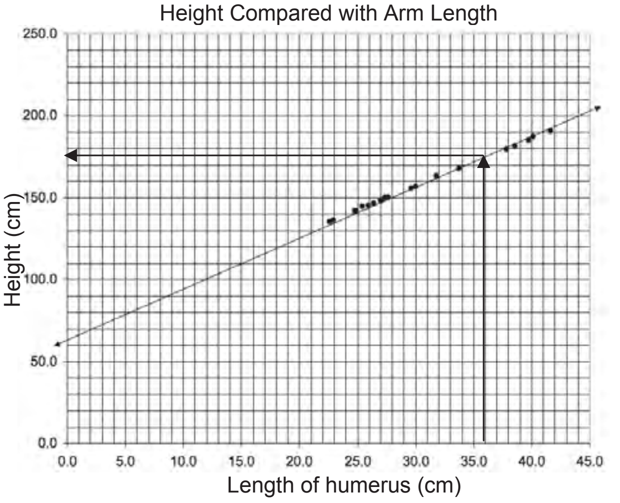
This table, shown here in two parts, shows the relationship between length of the humerus bone (the large bone in upper arm) and height for a group of 20 men.

a) Use a scatter plot and a line of best fit to predict the height of a man whose humerus bone is 36 cm long.

b) Describe the correlation.

c) Why might other students make different predictions?

Solution


Height Compared with Arm Length

A man with a 36 cm humerus would be about 173 cm tall.

b) There is a strong positive correlation between height and humerus length. [Continued]

Thinking

a) I made a scatter plot using the data and then drew a line of best fit.

- I located the point on the graph for a humerus length of 36 cm and then located the point on the vertical axis directly across from it, which was about 173 cm.

b) I noticed that the data points lie fairly closely along a straight line and the line goes up and to the right.

Example 1 Predicting a Value Using the Line of Best Fit [Cont'd]

Solution

c) The way you position the ruler depends on the position of your eyes when you look at the ruler and the plotted points. If other students position the ruler differently than I did, their lines of best fit maybe a bit different.

The scale of the graph also makes a difference. If a student draws a large graph, he or she can use a different scale on each axis than someone who makes a smaller graph. He or she might be able to plot the points more accurately and position the ruler a bit differently.

Thinking

- If someone plots the points correctly, the predictions might be different from mine, but they shouldn't be very different.

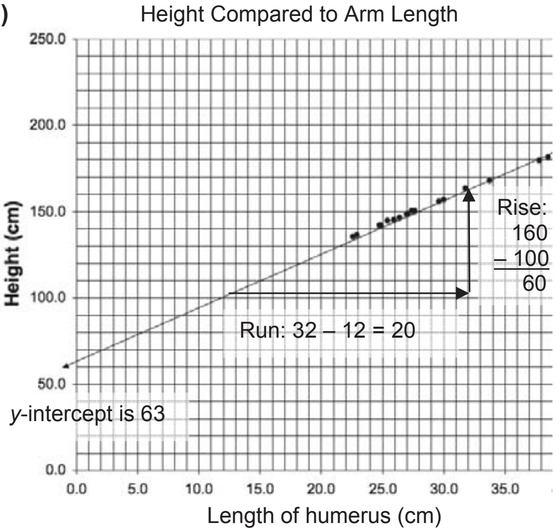
Example 2 Predicting a Value Using the Equation of the Line of Best Fit

a) Determine the equation of the line of best fit for the data in **example 1**.

b) Use the equation in **part a)** to predict the height of a man whose humerus bone is 36 cm long. Show your work.

Solution

a)



$m = \frac{60}{20} = 3$ $b = 63$

The equation of the line of best fit is $y = 3x + 63$.

b)

$$\begin{aligned} y &= 3x + 63 \\ &= 3(36) + 63 \\ &= 108 + 63 \\ &= 173 \end{aligned}$$

A man with a 36 cm humerus would be about 173 cm tall.

Thinking

a) I knew that if I could figure out the y-intercept, b, and slope, m, I could use $y = mx + b$ to write the equation.

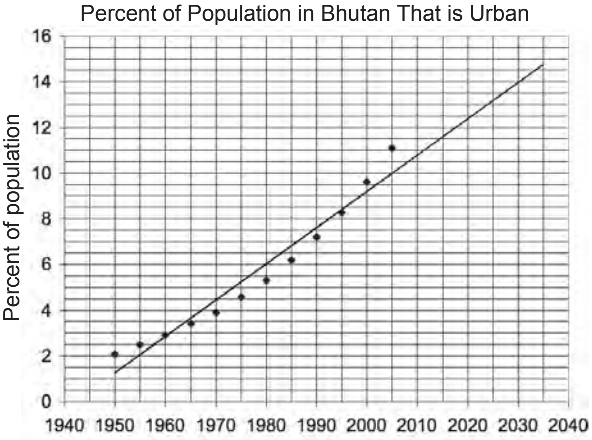
- The y-intercept was easy to figure out; it's about 63.
- To get the slope, I chose two points on the graph and used them to calculate the rise and run and then the slope.
- I substituted for m and b in $y = mx + b$.

b) I substituted 36 for x in my equation.

Practising and Applying

1. This graph shows how the percentage of Bhutan's population living in urban areas has increased since 1950. The data comes from the United Nations.

Many people have moved to urban areas like Paro.



a) Discuss the correlation between the year and the percentage of population that is urban.

b) Use the line of best fit to predict the percentage for 2010. Why is this prediction less than the actual percentage for 2005?

c) Use the line of best fit to predict the percentage for 2030.

d) Use the line of best fit to estimate the percentages for 1992 and 2002.

2. This data, from The Bhutan National Human Development Report 2005, shows the number of licensed businesses in Bhutan. Treat 1998 as year 0 to make the year axis start at 0.

Actual year	Year number	Number of licensed businesses
1998	0	11,896
1999	1	16,663
2000	2	18,134
2001	3	21,067
2002	4	32,035

a) Draw a scatter plot and a line of best fit for this data. Use the year number for the *x*-axis.

b) Determine the equation of the line of best fit. How did you calculate the slope?

c) Use the equation in **part b)** to predict the number of licensed businesses in Bhutan in 2010.

d) Discuss how the number of data points and the correlation between the year and the number of businesses affects your confidence in the prediction.

3. The United Nation reports the average life expectancy for its member nations. This set of data is for Bhutan.

Year	Female	Male
1955	36.1	34.6
1960	37.5	36.0
1965	39.1	37.5
1970	40.7	39.1
1975	42.3	40.8
1980	44.1	42.5
1985	46.7	45.1
1990	52.3	50.5
1995	57.1	54.9
2000	60.9	58.6
2005	63.9	61.5

a) Draw separate scatter plots for female and male life expectancy.

b) Use your scatter plots to extrapolate the average life expectancy for a female and for a male in 2010 and 2030.

c) Interpolate the male and female life expectancies in 1992 and 2002.

4. Match each correlation description with one of the graphs below.

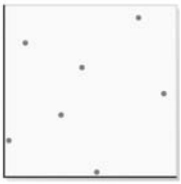
a) no correlation

b) a moderate positive correlation

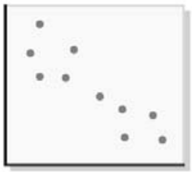
c) a moderate negative correlation

d) a strong positive correlation

Graph I



Graph II

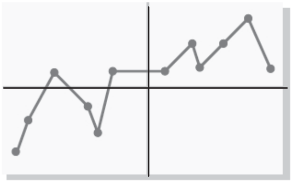


Graph III

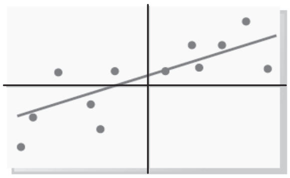
Graph IV

5. Three students graphed the same data in order to make predictions. Which student's graph is the most useful for making predictions? Why?

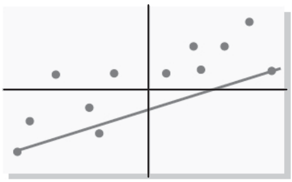
Dorji's



Kinley's



Yamuna's



6. How would you describe each of the following to someone who knows nothing about lines of best fit?

a) a graphing situation in which a line of best fit is appropriate

b) how to determine the line of best fit

c) how to use the line of best fit

3.2.5 Standard Form

Try This

Suppose you withdraw Nu 200 in Nu 20 or Nu 50 notes from your bank account.

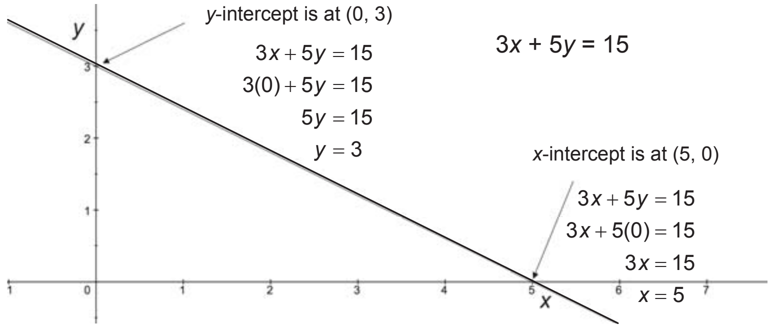
A. Make a table to show the number of ways you can make Nu 200 using only Nu 20 notes and Nu 50 notes. You can use all Nu 20 notes, a combination of Nu 20 and Nu 50 notes, or all Nu 50 notes.

Number of Nu 20 notes	Number of Nu 50 notes

B. Suppose you graphed the combinations of Nu 20 and Nu 50 notes by using the number of Nu 20 notes as the *x*-coordinate and the number of Nu 50 notes as the *y*-coordinate. What would the graph look like?

One form for writing the equation of a line is the slope and *y*-intercept form that you have already used. Another form is called standard form. When an equation for a line is in the form **A*x* + B*y* = C**, it is said to be in **standard form**.

- You can easily determine both the *x*- and *y*-intercepts of the graph from the standard form of the equation:
  - The *y*-intercept is the value of the *y*-coordinate where the line meets or crosses the *y*-axis. The coordinates of the *y*-intercept look like (0, *y*). You can substitute *x* = 0 into the equation and solve it to determine the *y*-intercept.
  - The ***x*-intercept** is the value of the *x*-coordinate where the line meets or crosses the *x*-axis. The coordinates of the *x*-intercept look like (*x*, 0). You can substitute *y* = 0 into the equation and solve it to determine the *x*-intercept.
- You can use the *x*- and *y*-intercepts to draw the graph of an equation.



y-intercept is at (0, 3)

$$\begin{aligned} 3x + 5y &= 15 \\ 3(0) + 5y &= 15 \\ 5y &= 15 \\ y &= 3 \end{aligned}$$

$3x + 5y = 15$

x-intercept is at (5, 0)

$$\begin{aligned} 3x + 5y &= 15 \\ 3x + 5(0) &= 15 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

C. i) What are the *x*- and *y*-intercepts of the graph you visualized in **part A**?
ii) What does each intercept represent?

Examples

Example 1 Graphing Given the Equation in Standard Form

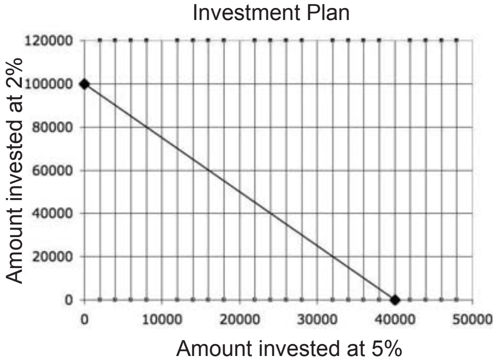
Dawa used the equation $0.05x + 0.02y = 2000$ to represent how many 5% bonds (x) and 2% investment certificates (y) he would invest in to earn Nu 2000.

a) Sketch the graph that shows the possible combinations of bonds and certificates that could be invested to earn Nu 2000.

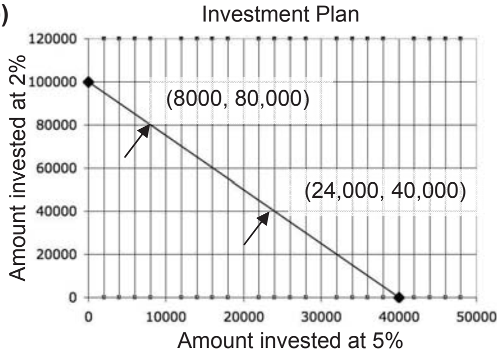
b) Use the graph to determine two possible combinations that will earn Nu 2000.

Solution

a)
$$\begin{aligned} 0.05(0) + 0.02y &= 2000 \\ 0.02y &= 2000 \\ y &= 2000 \div 0.02 = 100{,}000 \\ 0.05x + 0.02(0) &= 2000 \\ 0.05x &= 2000 \\ x &= 2000 \div 0.05 = 40{,}000 \end{aligned}$$



b)



Thinking

a) I knew if I plotted the *x*- and *y*-intercepts, I could just join them to draw the graph. The equation was in the standard form $Ax + By = C$ so it was easy to figure them out:

- The y-intercept occurs when $x = 0$, so I substituted $x = 0$ into the equation and solved for *y*.
- The x-intercept occurs when $y = 0$, so I substituted $y = 0$ into the equation and solved for *x*.

• I marked these points on the axes and then joined them to draw the graph.

b) I chose two points on the graph that had coordinates that were easy to read because they were on the intersection of grid lines.

Solution	**Thinking**
b) Two possible investment combinations that will each earn Nu 2000 are • Nu 8000 in 5% bonds and Nu 80,000 in 2% certificates • Nu 24,000 in 5% bonds and Nu 40,000 in 2% certificates	• I checked to make sure they earned Nu 2000 by substituting their values into the equation: $0.05(8000) + 0.02(80,000)$ $= 400 + 1600 = 2000$ $0.05(24,000) + 0.02(40,000)$ $= 1200 + 800 = 2000$

Example 2 Writing an Equation to Describe a Situation

Yuden wants to invest her money so it will earn interest.

- She will deposit some of her money with a bank that pays 4.2% interest.
- The rest she will use to buy stock that is currently paying dividends of 9.6%.

She wanted to earn Nu 400 from this investment combination.

Write an equation to describe her investment plan.

Solution	**Thinking**
Let *b* represent the amount she invests with the bank and *s* represent the amount Yuden invests in stocks. Her investment plan is described by the following equation: $0.042b + 0.096s = 400$	 • I knew I needed one variable for the amount deposited in the bank and another for the amount invested in stocks. • She hopes to earn 4.2% on her bank deposit, so I knew I had to multiply that investment amount by 0.042. • She hopes to earn 9.6% on stocks, so I had to multiply that investment amount by 0.096.

Example 3 Determining Slope and *y*-intercept Form Given Standard Form

Determine the slope and *y*-intercept form of the line with equation $5x + 2y = 10$.

Solution 1	**Thinking**
$5x + 2y = 10 \rightarrow y = mx + b$ $5x + 2y = 10$ $5x - 5x + 2y = 10 - 5x$ $2y = 10 - 5x$ $2y = -5x + 10$ $\frac{2y}{2} = \frac{-5x + 10}{2}$ $y = \frac{-5}{2}x + \frac{10}{2}$ $y = \frac{-5}{2}x + 5$	 I rearranged the equation so *y* was by itself on the left. • I subtracted $5x$ from both sides. • I knew that $10 - 5x = -5x + 10$. • I divided both sides by 2. • I knew that $\frac{-5x + 10}{2} = \frac{-5}{2}x + \frac{10}{2}$. • I simplified $\frac{10}{2}$ to 5.

Solution 2	**Thinking** 
To determine the y-intercept, set $x = 0$ and solve for y. $5(0) + 2y = 10$ $2y = 10$ $y = 5$ The y-intercept is 5. So, in the $y = mx + b$ form of the equation, b = 5.	• I knew I needed the slope and the y-intercept to write the equation. • I used the standard form of the equation to determine the y-intercept by setting $x = 0$.
To determine the x-intercept, set $y = 0$ and solve for x. $5x + 2(0) = 10$ $5x = 10$ $x = 2$ The x-intercept is 2.	• I determined the x-intercept using the equation by setting $y = 0$.
Slope = m = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-0}{0-2} = \frac{5}{-2} = -\frac{5}{2}$	• I used the coordinates of the intercepts to calculate the slope.
The equation of the line: $y = -\frac{5}{2}x + 5$	• I used the value of the slope and the y-intercept to write the equation.

Practicing and Applying

1. A line has equation $3x + 2y = 6$.

a) Determine the coordinates of the y-intercept.

b) Determine the coordinates of the x-intercept.

c) Calculate the slope of the line.

d) Write the slope and y-intercept form of the equation.

2. Karma wrote a multiple-choice test. The scoring system worked as follows:

- gain 4 points for each correct answer
- lose 1 point for each incorrect answer
- 0 points for unanswered questions

Karma received 60 points on the test.

a) How does the equation $4c - i = 60$ describe all the different combinations of correct and incorrect answers Karma could have had to get a score of 60?

b) Use x- and y-intercepts to graph this equation.

c) What does each intercept mean? Are both intercepts possible? Explain.

d) If Karma got the same number of questions correct as incorrect, how many questions did he answer on the test?

3. Use the equation from **example 2** to represent Yuden's investment plan:

$$0.042b + 0.096s = 400$$

a) Sketch a graph for this equation.

b) Use the graph to determine four different combinations of the two investment choices that will earn her Nu 400.

c) Why do negative values for the variables not make sense in a situation like this?

4. Lobzang works at two different jobs.

- One job pays him Nu 600 an hour.
- The other job pays Nu 500 an hour.

He wants to earn a total of Nu 4500.

a) Write an equation to describe all the different combinations of pay rates that will earn him Nu 4500.

b) Sketch the graph of this equation.

c) Use the graph to determine three combinations of hours Lobzang can work at the two jobs and still earn Nu 4500.

d) Why do negative values for the variables not make sense in a situation like this?

5. Use the intercepts to determine the equation of each line in slope and *y*-intercept form.

a) $2x + 3y = 12$

b) $4x – 5y = 20$

c) $5x + 2y = 5$

d) $10x – 3y = 15$

6. Two students are playing a number guessing game. One says, "If I multiply one of my numbers by 3 and add it to the other, the sum is 27."

a) Write an equation to describe this number relationship.

b) Sketch a graph of this relationship.

c) Use the graph to find three combinations of whole numbers that satisfy the description.

7. Two students are playing a number guessing game.

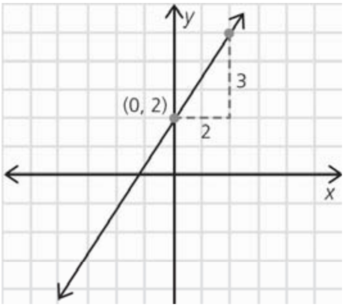
- One student says, "If you multiply my greater number by 3 and the other by 2 and add them, you get 16."
- The other student says, "If you multiply my greater number by 4 and subtract the other, you get 18."

a) Write an equation to describe each number relationship.

b) Sketch the graphs of both relationships on the same set of axes.

c) Use the graph to determine a pair of numbers (x, y) that satisfies both relationships.

8.



a) Write the equation of the graph above in slope and *y*-intercept form and in standard form.

b) Use your answer from **part a)** to determine points on the line that have integer coordinates.

c) Graph $3x – 2y = 4$.

d) How does your graph in **part c)** compare to the graph above? Why do you think this happened?

9. Describe the steps you must follow to sketch the graph of an equation presented in standard form.

10. Explain how to determine an equation in slope and *y*-intercept form from an equation in standard form.

Chapter 3 Linear Equations and Inequalities

3.3.1 Solving Linear Equations Algebraically

Try This

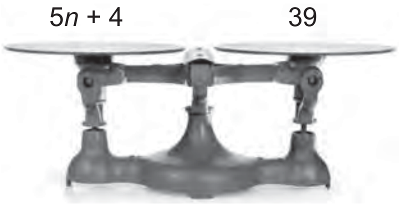
Algebra tiles can be used to represent equations as well as polynomials.
The following algebra tile model represents the equation $2x - 2 = -x + 4$.

$2x - 2 = -x + 4$

A. What is the solution to the equation?

B. How do the algebra tiles represent the equation?

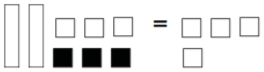
- An **equation** is a mathematical statement in which the value on the left side of the equal sign is the same as the value on the right side of the equal sign. For example, in the equation $5n + 4 = 39$, the right side is 39. That means the left side is also 39, which means that the value of n must be 7 ($5(7) + 4 = 39$).
- The **solution** to a linear equation is the value of a variable that makes the equation true. For example, 7 is the solution to the equation $5n + 4 = 39$ because $5(7) + 4 = 39$. When you **solve** an equation, you find the value of the solution.
- To solve an equation, you can think of the equation as a pan balance. Your goal is to isolate the variable on one side of the pan balance, or equation, because the value on the other side will be the solution. To isolate the variable on one side, you add and subtract values, or multiply and divide by values. Because the equation is like a pan balance, you have to do the same operation on one side of the equation as the other side to make sure the pan balance, or equation, stays balanced.



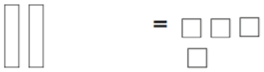
- You can use **inverse operations** to undo the operations in the equation one step at a time.
- When you are able to undo operations so that the variable term is by itself on one side of the equation, you have **isolated the variable**.
- When you have solved the equation, the result is a simpler, **equivalent equation** that has the same solution.

For example, to solve $2n - 3 = 1$, you can represent the equation and the steps of the solution using algebra tiles.

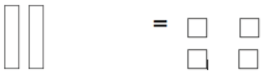
Represent the equation $2n - 3 = 1$ using tiles.



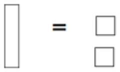
Add 3 to both sides to undo –3 on the left side.



Simplify.



Divide both sides by 2 to undo $2 \times n$ on the left side.



You have isolated variable n and solved the equation.

You can use the same strategy with symbols.

$$5n + 4 = 39$$

$$5n + 4 - 4 = 39 - 4$$ Subtract 4 from both sides to undo +4 on the left side.

$$5n = 35$$

$$\frac{5n}{5} = \frac{35}{5}$$ Divide both sides by 5 to undo $5 \times n$ on the left side.

$$n = 7$$ You have isolated the variable n to solve the equation.

When you isolate the variable, you create a simpler equivalent equation with the same solution: $5n + 4 = 39$ is equivalent to $n = 7$.

C. How could you isolate x to solve the equation in **part A**

i) using inverse operations? **ii)** using algebra tiles?

Examples

Example 1 Using an Equation to Solve a Problem

A photographer charges a sitting fee of Nu 250 and then charges Nu 60 for each photograph ordered. Passang can only afford to spend Nu 1060. How many photographs can Passang order? Show your steps.

Solution	**Thinking**
p represents the number of photographs Passang can order. $250 + 60p = 1060$ $250 - 250 + 60p = 1060 - 250$ $60p = 810$ $60p \div 60 = 810 \div 60$ $p = 13.5$ He can buy 13 photographs.	 • I wrote an equation to represent the situation and then solved it. • I subtracted 250 to isolate the variable term and then divided by 60 to isolate the variable. • The solution was 13.5 but half a photograph was not possible so I knew it had to be 13.

Example 2 Solving an Equation with Fractional Coefficients

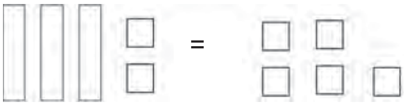
Solve $\frac{3}{4}x + 1= \frac{1}{2}x + 4$. Show your steps.

Solution 1	**Thinking**
$\frac{3}{4}x + 1 = \frac{1}{2}x + 4$ $4 \times (\frac{3}{4}x + 1) = 4 \times (\frac{1}{2}x + 4)$	 • I wanted all numbers to be integers, so I multiplied both sides by the common denominator of 4.
$4 \times \frac{3}{4}x + 4 \times 1 = 4 \times \frac{1}{2}x + 4 \times 4$ $3x + 4 = 2x + 16$	• I used the distributive property to make sure I multiplied every term in the equation.
$3x - 2x + 4 = 2x - 2x + 16$ $x + 4 = 16$	• I subtracted 2*x* from each side to begin isolating the variable *x* on the left.
$x + 4 - 4 = 16 - 4$ $x = 12$	• I subtracted 4 from both sides to isolate the variable *x* and solve the equation.
Solution 2	**Thinking**
$\frac{3}{4}x + 1 = \frac{1}{2}x + 4$ $\frac{3}{4}x - \frac{1}{2}x + 1 = \frac{1}{2}x - \frac{1}{2}x + 4$ $\frac{1}{4}x + 1 = 4$ $\frac{1}{4}x + 1 - 1 = 4 - 1$ $\frac{1}{4}x = 3$	 • I subtracted $\frac{1}{2}x$ and then 1 from both sides to isolate the variable term $\frac{1}{4}x$ on the left.
$4 \times \frac{1}{4}x = 4 \times 3$ $x = 12$	• Then, I multiplied both sides by 4 to isolate the variable *x* on the left and solve the equation.

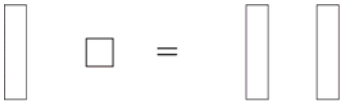
Practising and Applying

1. Write an equation for each algebra tile model.

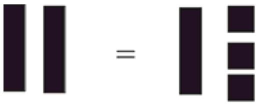
a)



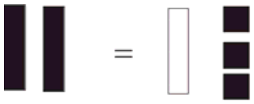
b)



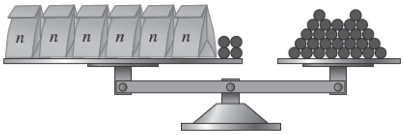
c)



d)



2. Describe the steps needed to solve the equation represented by this pan balance.



$6n + 4 = 28$

3. Solve each equation.

a) $2x + 3 = 11$

b) $5a – 10 = 15$

c) $–2 = 2y + 4$

d) $7 = 4x – 1$

e) $x – 2.3 = 4.6$

f) $1.5x – 6.5 = 53.5$

4. Nima deposited the same amount of money into his bank account each week for five weeks. Then he used some of these savings to buy a gift that cost Nu 60. That left him with Nu 180 in the account.

a) Write an equation to represent the situation. Let *w* represent his weekly deposit.

b) Solve the equation to determine Nima's weekly deposit.

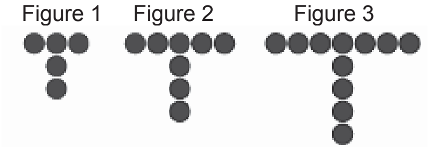
5. A photographer charges a sitting fee of Nu 250 and then charges Nu 50 for each photograph purchased.

a) A customer has Nu 1030 to spend. Write the equation you could solve to determine the number of photographs the customer can buy.

b) Solve the equation.

c) How many photographs can be purchased?

6. a) Write an equation for the relationship between the figure number and number of counters in each figure.



b) Use the equation to determine the figure that can be made with exactly 50 counters.

c) Explain why there is no figure in this pattern that can be made with exactly 100 counters.

7. Solve each equation.

a) $3x – 2 = 2x + 1$

b) $5x + 4 = 8x + 10$

c) $6 – 2x = 3x + 11$

d) $13 + 2x = 4 + x$

e) $x – 2.5 = 3x + 1.5$

f) $2.5x + 0.5 = 2x – 3.5$

8. Maya bought three identical chocolate bars. She gave the storekeeper Nu 100 and received change of Nu 40.

a) Write an equation to represent the situation.

b) Solve the equation to determine the cost of each chocolate bar.

9. Solve each equation.

a) $-3x + 3 = 2x - 1$

b) $6 - 2a = -3a - 4$

c) $\frac{2}{3}x + 1 = \frac{1}{3}x + 3$

d) $\frac{1}{5}x - 1 = \frac{1}{3}x - 3$

10. The equation that converts a Fahrenheit temperature to Celsius is shown below.

$$C = \frac{5}{9}F - \frac{160}{9}$$

Solve the equation to determine the temperature for which the Fahrenheit and Celsius values are the same. (Hint: Let F = C.)

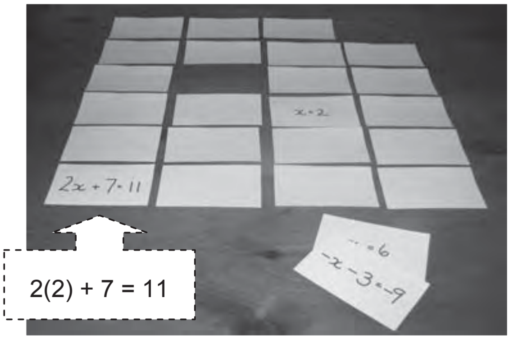
11. Imagine that you have been asked to show another student how to solve the equation $3x + 2 = 5x + 3$.

a) Model how to solve the equation using algebra tiles.

b) Show all the steps needed to solve the equation symbolically.

GAME: Equation Concentration

- Make game cards by cutting a sheet of paper into 12 identical rectangles.
- Make up six different equations and solve them. Write each equation on its own card. Write each solution on its own card. Use pencil so that the contents of the card cannot be read through the back.
- Combine your cards with those of another student and place all the cards face down on the desk in an array.



2(2) + 7 = 11

- Players take turns turning over two cards at a time. If one card shows an equation and the other shows its solution, the player wins those cards and takes another turn. A player can take no more than 2 turns in a row.
- If the cards do not match, turn them both face down again.
- Keep playing until all the cards have been matched.
- The player who has the most cards at the end of the game wins.

3.3.2 Solving Linear Inequalities

Try This

Figure 1

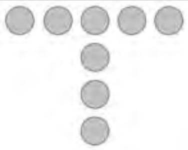


Figure 2

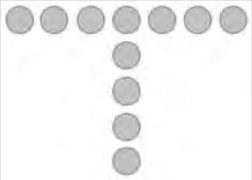


Figure 3

A. i) Think about the largest figure in this pattern that can be made using no more than 50 counters. What is its figure number?

ii) Which figure numbers can be made with 50 or fewer counters?

An **inequality** is a mathematical statement in which the value on the left side is compared with the value on the right side using an inequality symbol.

For example, the inequality $5n + 4 < 39$ is a comparison between the expression $5n + 4$ (4 more than the product of 5 and a number, n) and the value 39. The symbol < indicates that $5n + 4$ is less than 39.

- The **solution** to an inequality is the set of values for the variable that make the inequality true. For example, $n < 7$ is the solution to the inequality $5n + 4 < 39$ because, when any number less than 7 is substituted into the inequality, the left side will be less than 39.

- You solve an inequality using the same steps you would use to solve the related equation. You could use the following steps to solve $4 < 3n – 2$:

$4 < 3n - 2$	
$4 + 2 < 3n - 2 + 2$	Add 2 to each side to isolate the variable term on the right.
$6 < 3n$	
$6 \div 3 < 3n \div 3$	Divide each side by 3 to isolate the variable on the left.
$2 < n$	You have isolated the variable n to solve the equation.

When you isolate the variable you create a simpler equivalent inequality with the same solution: $4 < 3n – 2$ is equivalent to $2 < n$.

- When you solve an inequality, choose operations that keep the coefficient of the variable term positive. If you end up with a negative coefficient and divide or multiply both sides by a negative, the solution will be incorrect.

B. i) The answer to **part A ii)** could be expressed as $n \leq 16$. Explain.

ii) Explain why $n < 17$ could also be used.

Examples

Example 1 Using an Inequality to Solve a Problem

A photographer charges a base fee of Nu 250 and then charges Nu 50 for each photograph ordered. Maya can afford to spend no more than Nu 1060.
What different quantities of photos can Maya afford? Show your steps.

Solution

P represents the number of photographs Maya can order.

$$250 + 50p \le 1060$$

$$250 - 250 + 50p \le 1060 - 250$$

$$50p \le 810$$

$$50p \div 50 \le 810 \div 50$$

$$p \le 16.2$$

She can buy 16 or fewer photographs.

Thinking

- I used ≤ because she can afford to spend *no more* than Nu 1060—that means she could spend exactly Nu 1060 or she could spend less.
- I subtracted 250 from each side to isolate the variable term, 50*p*, on the left.
- I divided by 50 to isolate the variable *p* on the left and solve the inequality.
- The solution was 16.2 but part of a photograph is not possible so it knew it had to be 16 or fewer.

Example 2 Solving an Inequality with the Variable on Both Sides

Solve $3a + 5 > 6a - 7$.

Solution

$$3a - 3a + 5 > 6a - 3a - 7$$

$$5 > 3a - 7$$

$$5 + 7 > 3a - 7 + 7$$

$$12 > 3a$$

$$12 \div 3 > 3a \div 3$$

$$4 > a$$

$$a < 4$$

This means that the value of *a* must be less than 4.

Thinking

- I subtracted 3*a* instead of 6*a* from each side so I would end up with a positive coefficient.
- I added 7 to each side to isolate the variable term 3*a* on the right.
- I divided both sides by 3 to isolate to variable *a* on the right and solve the inequality.
- I switched the variable to the other side because I find it easier to understand that way. To do that, I had to reverse the inequality sign from > to <.

Practising and Applying

1. Solve each inequality.

a) $2x + 1 < 7$ **b)** $4x - 5 \leq 7$

c) $6 > 5x - 4$ **d)** $-3 \geq 2x + 1$

e) $2.5x + 1.25 < 3.75$ **f)** $5.1 \geq 2.4x + 0.3$

2. A water tank starts with 400 L of water in it. It is being filled at the rate of 50 L/min and there are now less than 1000 L of water in the tank.

a) Write an inequality to represent this situation.

b) Solve the inequality to determine the maximum number of minutes the water could have been flowing into the tank.

3. A storage tank contained 100,000 L of oil. It is being emptied at the rate of 150 L/min and there are less than 40,000 L of oil left in the tank.

a) Write an inequality to represent this situation. Use t to represent the number of minutes that the tank could have been emptying.

b) Solve the inequality to determine the minimum number of minutes the oil could have been flowing from the tank.

4. Solve each inequality.

a) $5a + 7 < 3a + 11$ **b)** $6b + 2 > 7b - 5$

c) $5x - 1 \leq 3x + 3$ **d)** $3 - x > 5$

e) $5 - x < 3 + x$ **f)** $-4x + 2 > -2x - 2$

5. a) Which figures in this pattern require more than 50 counters?

Figure 1

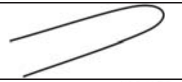
Figure 2

Figure 3

b) Write an inequality to describe the figures in **part a)**.

c) Solve the inequality in **part b)**.

6. A long strand of string is folded in half and then cut into pieces in several steps.

0 cuts	
1 cut	
2 cuts	
3 cuts	

a) Write an equation that relates the number of pieces to the number of cuts.

b) Write an inequality that tells you the maximum number of cuts if you want fewer than 20 pieces.

c) Solve the inequality in **part b)** to determine the maximum number of cuts you can have if you want fewer than 20 pieces.

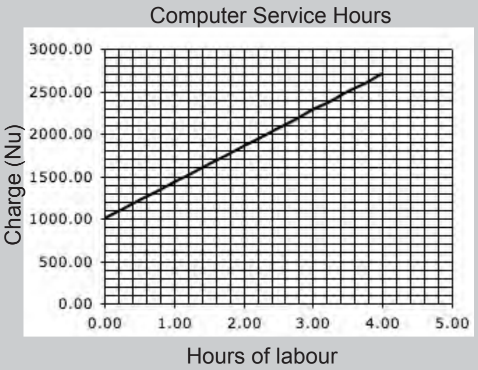
7. A vehicle with an empty fuel tank has a mass of 1050 kg. One litre of petrol has a mass of 737 g. What is the minimum amount of whole litres of petrol that would cause the total mass to exceed 1100 kg?

8. How is solving a linear inequality like solving a linear equation? How is it different?

3.3.3 Solving Linear Equations Graphically

Try This

One of the computers in the school computer lab is not working. A service company has sent the following graph to describe how they charge for service. The school has a monthly budget of Nu 2000 for computer service.



A. How many hours of labour can the school afford this month?

The solution to an equation is the *x*- or *y*-coordinate of a point on its graph.

For example, consider the graph of the linear relation represented by $y = 2x + 3$.

To solve $y = 2x + 3$ for *y* when $x = 11$ using the graph,

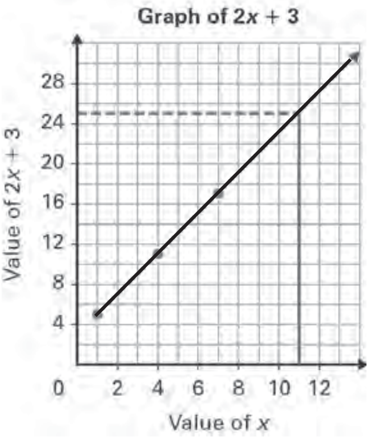
- locate 11 on the *x*-axis,
- locate the point on the graph with that *x*-coordinate (see the solid vertical line)
- determine the *y*-coordinate of that point (see the dashed horizontal line)

The solution is $y = 25$.

To solve $y = 2x + 3$ for *x* when $y = 25$ using the graph,

- locate 25 on the *y*-axis
- locate the point on the graph with that *y*-coordinate (dashed horizontal line)
- determine the *x*-coordinate of that point (solid vertical line)

The solution is $x = 11$.



B. i) What equation did you solve when you answered **part A**?
ii) How did you use the graph to find the solution to the equation?

Examples

Example 1 Using a Graph to Solve a Pattern Equation

What is the number of the figure that uses exactly 39 square tiles?

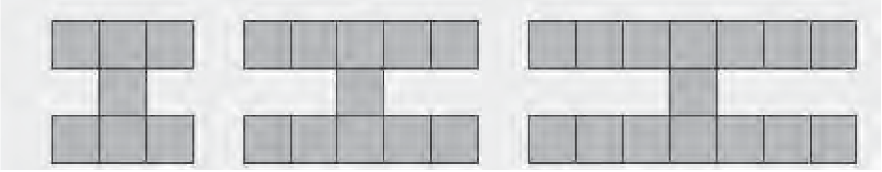


Figure 1 Figure 2 Figure 3

Solution

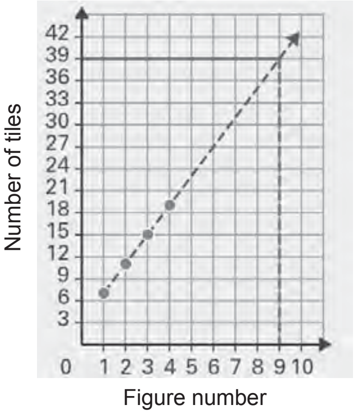


Figure 9 would use 39 square tiles.

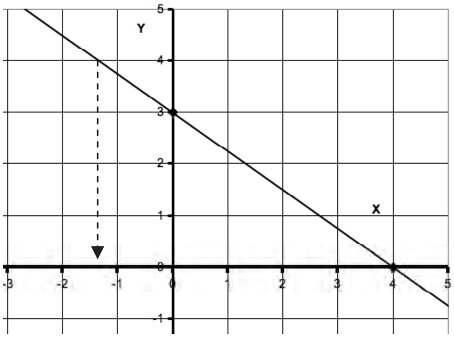
Thinking

- I drew another figure in the pattern and then made a scatter plot that relates the number of tiles to the figure number.
- I located 39 (tiles) on the vertical axis and then located the point on the graph for that coordinate.
- Then I located the *x*-coordinate for that point to determine the figure number.

Example 2 Using a Graph to Solve an Equation in Standard Form

Use a graph to estimate the value of *x* that corresponds to $y = 4$ in $3x + 4y = 12$.

Solution



When *y* is 4, *x* is about –1.3.

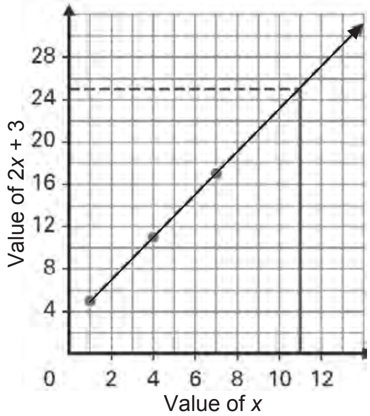
Thinking

- I calculated that the *x*-intercept is 4 and the *y*-intercept is 3. Then I used the *x*- and *y*-intercepts to sketch the graph.
- I found the point on the line that had a *y*-coordinate of 4 and then estimated its *x*-coordinate.

Practising and Applying

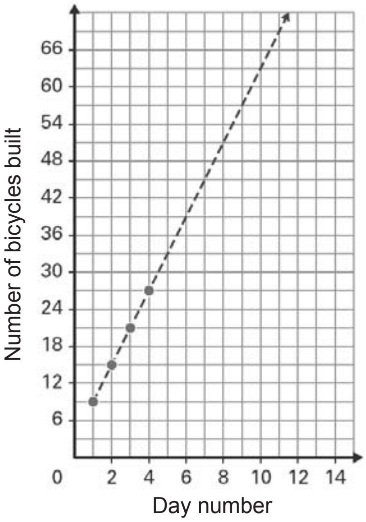
1. Use a graph to determine the solution to the equation $2x - 3 = 21$.

2. a) Write the equation that is being solved in the following graph.

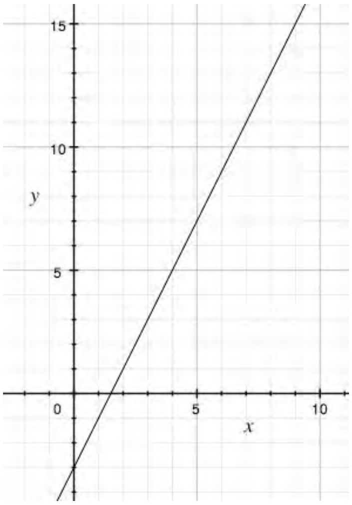


b) Explain why the value determined using the graph may only be a good estimate and not an exact solution.

3. This graph shows the number of bicycles built over several days. Use the graph to determine when 51 bicycles had been built.



4. How could you use the following graph to estimate the solution to $2x - 3 = 14$?



5. Examine the following stick pattern.

a) Write an equation that relates the number of sticks used for a figure to its figure number.

b) Draw a graph that represents the relationship in **part a)**.

c) Describe how you could use the graph to estimate the number of the figure that uses 97 sticks.

d) Solve the equation to determine the figure number discussed in **part c)**.

e) Why is a graphical solution usually only an estimate, while the algebraic solution in **part d)** is exact?

6. A solution to a linear equation is the coordinate of a point on the graph of a linear relation. How does this fact help you when you use a graph to solve a linear equation?

3.3.4 Solving a System of Linear Equations

Try This

Meto wants to work at Mountain Trekkers and Tours as a sales agent selling treks to tourists. He has been offered a choice of two payment plans:

Plan A

5% commission on total sales

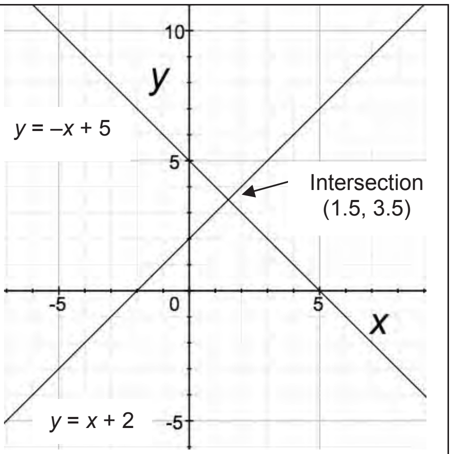
Plan B

3% commission on total sales and an annual base salary of Nu 10,000

A. What should Meto consider before choosing a payment plan?

A **linear system** is a set of linear relations that describe a situation.

- When these relations are described algebraically, the set of equations is called a **system of linear equations.**
- The **solution to a system of linear equations** is the intersection of the graphs of the relations.



For example:

A system of linear equations has been graphed above. The two equations are $y = -x + 5$ and $y = x + 2$. The intersection point, (1.5, 3.5), is a point that both graphs share. It is the solution to the system of equations because it satisfies both equations:

	$y = -x + 5$	$y = x + 2$
	$3.5 = -1.5 + 5$	$3.5 = 1.5 + 2$
3.5	$= 3.5$	$3.5 = 3.5$

B. i) What two equations describe Meto's two payment plans in **part A**?

ii) If you were to graph both equations, what would be the meaning of the intersection point?

Examples

Example Solving a Problem Involving a Linear System

A cell phone company offers two different monthly plans.

Plan	Monthly fee	Price per minute
A	Nu 1200	Nu 12
B	Nu 1400	Nu 10

a) Write an equation to represent the cost of each plan and then graph the system of equations.

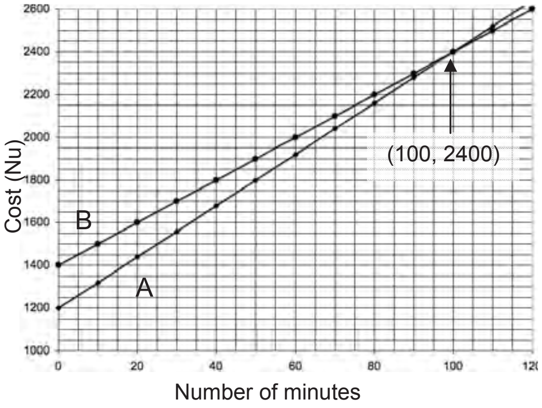
b) For what number of minutes will the cost of the two plans be the same? How do you know?

c) Why is it useful to know this?

Solution

a) Let *c* be the total monthly cost of the plan in ngultrums. Let *m* be the number of minutes used in a month.

Plan	Equation
A	$c = 1200 + 12m$
B	$c = 1400 + 10m$



b) The two plans will charge the same, Nu 2400, if you talk for 100 min. I know because both graphs share that same point, which means for the same number of minutes (100 min, the *x*-coordinate), the cost is the same (Nu 2400, the *y*-coordinate).

c) It is useful to know because it can help you decide which plan is better. If you talk less than 100 min per month, Plan A will cost less. If you talk more than 100 min, Plan B will cost less.

Thinking

a) Each equation represents the relationship between the total monthly cost of the plan and the number of minutes used in a month.

- I drew the graph for each equation on the same coordinate grid.

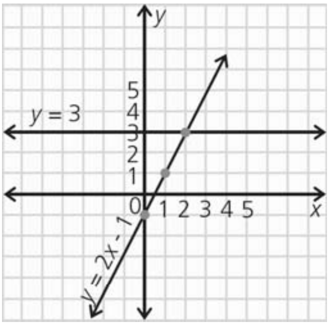
b) The graphs intersected at (100, 2400), which is 100 min and Nu 2400.

c) The graph for Plan A shows that it costs less than Plan B up to the intersection point. Then Plan A starts costing more than Plan B.

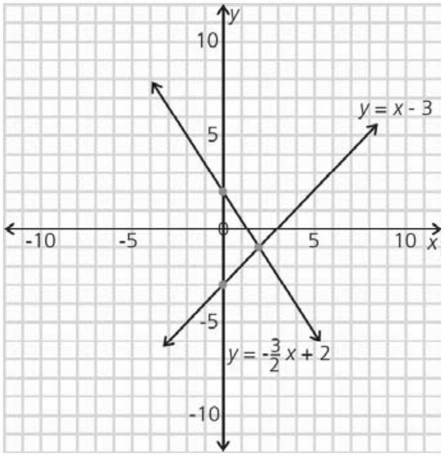
Practising and Applying

1. Use the graphs to solve each system of linear equations.

a) $y = 3$ and $y = 2x - 1$



b) $y = x - 3$ and $y = -\frac{3}{2}x + 2$



2. Create a graph to solve each system of linear equations.

a) $y = 2x - 7$ and $y = 5x - 4$

b) $y = 2x + 3$ and $y = 3x - 2$

c) $y = \frac{3}{4}x + 5$ and $y = -\frac{2}{3}x + 5$

d) $y = \frac{1}{2}x + 1$ and $y = -x + 4$

3. Lobzang works at two different jobs.

- One job pays him Nu 600 an hour.
- The other job pays Nu 500 an hour.
- He wants to earn a total of Nu 4500 and work exactly 8 hours.

a) Write an equation to describe Lobzang's desired income.

b) Write an equation to describe the hours he would like to work.

c) Solve the system of equations to determine the hours he should spend on each job to meet his goal.

4. A vehicle has a mass of 1295 kg and uses petrol. Another vehicle has a mass of 1290 kg and uses diesel fuel.

- 1 L of petrol has a mass of 737 g.
- 1 L of diesel has a mass of 820 g.

What volume of fuel will result in the two vehicles having the same mass?

a) Write an equation to describe the mass of the petrol-powered vehicle with fuel.

b) Write an equation to represent the mass of the diesel-powered vehicle with fuel.

c) You want the vehicles to have the same mass with the same volume of fuel. Solve the system of equations to determine this mass and fuel volume.

5. In the previous lesson, you solved one linear equation by determining the coordinate of a point on the graph of that equation. In this lesson, you solved a system of two linear equations by determining the coordinates of the point of intersection of the graphs of the two equations.

- How are these two situations alike?
- How are they different?

UNIT 3 Revision

1. For each table of values, determine whether the relation is

- linear
- quadratic
- exponential
- none of these

Justify your decision.

a)

x	y
0	–5
1	–3
2	3
3	13
4	27
5	45

b)

x	y
0	–9
1	–8
2	–6
3	–2
4	6
5	22

c)

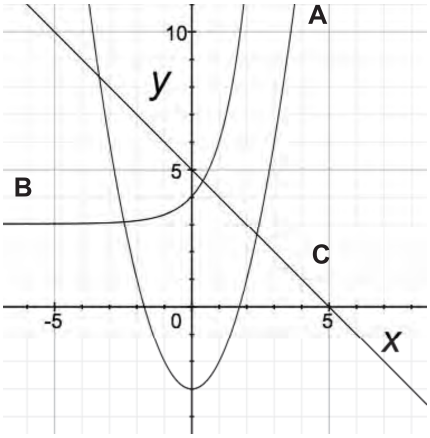
x	y
0	3
1	7
2	11
3	15
4	19
5	23

2. Match each relation with the graph below that most likely represents it, A, B, or C.

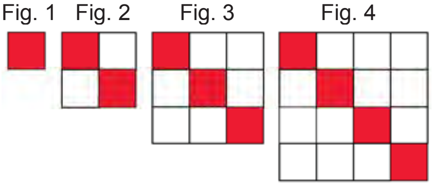
a) $y = x^2 – 3$

b) $y = –x + 5$

c) $y = 3^x + 3$



Use this pattern below to answer questions 3, 4, and 5.



3. a) Complete the table of values.

Side length of square	Number of white squares
1	
2	
3	
4	
5	

3. b) Draw a scatter plot of the data in **part a).** Should you use a line or a smooth curve to join the points? Why?

c) Should your graph be solid or dashed? Why?

4. a) Complete the table of values.

Side length of large square	Total number of small squares
1	
2	
3	
4	
5	

b) Draw a scatter plot of the data in **part a).**

c) Why is it difficult to tell from the graph that the relation is quadratic?

d) What other method could you use to show that the relation is quadratic?

5. Use your graph from question 4 to extrapolate the number of white squares in a large square with a side length of 8.

6. Scientists use a unit called a Pascal (Pa) to measure air pressure. The chart below shows how air pressure decreases with altitude.

Altitude (km)	Air Pressure (Pa)
1	80,000
3	60,000
6	40,000
16	20,000
22	10,000
30	5,000

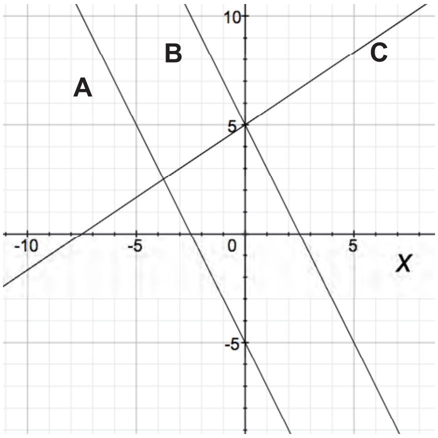
a) Why can you not use first and second differences in this situation to determine whether the relation between altitude and air pressure is linear, quadratic, or exponential? Explain.

b) Is this data discrete or continuous?

c) Draw a scatter plot of the data in the table.

d) Based on the graph, would you say that the data is linear, quadratic, exponential, or none of these? Justify your choice.

7. Write the slope and y-intercept form of the equation for each line, A B, and C.



8. Sketch the graph of each of the following.

a) $y = -3x - 4$

b) $y = -2x + 3$

c) $y = \frac{2}{3}x + 1$

d) $y = -\frac{3}{2}x - 2$

Use the data in the table below to answer questions 9 and 10.

9. The table below, shown here in two parts, shows the relationship between the length of the humerus bone in the forearm and height for a group of 20 men. Anthropologists and police scientists use data like this to estimate height from a single bone.

Humerus length	Height
38.5	181.6
33.9	169.3
24.3	141.5
25.4	144.8
34.4	169.7
28.5	152.7
30.9	160.7
24.8	142.8
36.8	176.6
41.2	189.4
27.2	149.8
30.1	157.2
31.6	162.7
32.3	164.6
30.8	159.2
31.6	162.7
27.3	149.2
34.5	170.2
25.4	143.8
36.9	176.9

a) Draw a scatter plot to show the relationship between length of the humerus bone and height.

b) Describe the strength of the correlation between humerus length and height.

c) Draw a line of best fit for this data.

d) Based on your graph, how tall would you expect a man to be whose humerus bone is 28 cm long? 42 cm long?

10. a) Determine the equation of the line of best fit you drew in **question 9**.

b) Use the equation to check your predictions in **question 9 d).**

11. For each equation in standard form

a) determine the x-intercept

b) determine the y-intercept

c) sketch the graph

A $3x + y = 9$ **B** $2x - 3y = 12$

C $x + 5y = 10$ **D** $-4x + 3y = 12$

12. For each equation in **question 11**

a) determine the slope

b) write the equation in slope and y-intercept form

13. Sangay is an artist. She earns Nu 1,500 for each magazine illustration she draws and Nu 500 for each piece of technical art. Last month she earned Nu 14,000.

a) Write an equation that describes Sangay's earnings if she drew m magazine illustrations and t pieces of technical art.

b) Sketch the graph of this relation.

c) Use the graph to determine four combinations of magazine illustrations and technical drawings Sangay might have sold to earn what she did.

14. Solve the following equations.

a) $3a - 5 = 16$ **b)** $\frac{1}{3}x + 7 = 9$

c) $3 - 2y = 4y - 11$ **d)** $1.5x + 0.5 = 3.5$

e) $-\frac{4}{5}b + 1 = -15$ **f)** $\frac{1}{4}x - 2 = \frac{1}{3}x + 1$

15. Solve the following inequalities.

a) $3y + 1 < 7$ **b)** $4 \geq 2y - 8$

c) $2a + 3 \leq 3a - 2$ **d)** $9 > 3 - 2n$

16. Suppose Sangay, the artist in **question 13**, had sold a total of 12 pieces of art.

a) Write a system of linear equations to represent both her earnings and the total number of pieces of art she sold.

b) Use a graph to solve the system and determine the number of pieces of each type that she sold last month.

UNIT 4 DATA AND PROBABILITY

Getting Started

Use What You Know

A. Make a prediction:

When you roll two dice, are you more likely to roll a difference of 1 or a difference of 2?

5 – 3 = 2

This is a difference of 2.

B. i) Without rolling the dice, copy and complete this chart to show all differences that are possible when you roll two dice. Always subtract the lesser number from the greater number.

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						2
5						
6				2		

For example:
6 – 4 = 2

ii) How many differences did you record in your chart?

iii) How many of those differences are 0? 1? 2? 3? 4? 5?

iv) Use your answers to **parts ii) and iii)** to determine the theoretical probability of rolling each difference: 0, 1, 2, 3, 4, and 5. Express each probability as a fraction with a denominator of 36.

C. Roll the two dice 36 times and keep track of the number of times you roll each difference 0 to 5. Determine the experimental probability for each difference, expressed as a fraction with a denominator of 36.

D. Create a double bar graph to compare the experimental and theoretical probabilities of all possible differences.

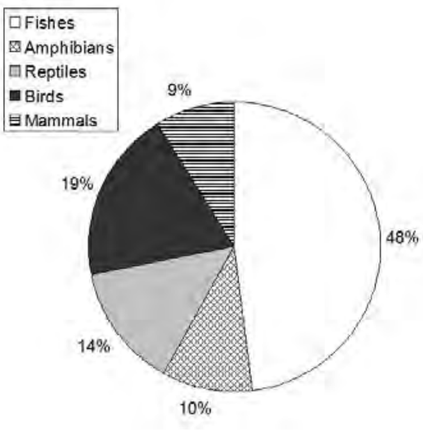
E. Which experimental result more closely matched the theoretical one—rolling a difference of 1 or rolling a difference of 2?

F. Which is more likely—a difference of 1 or a difference of 2?

Skills You Will Need

Use the circle graph on the right to answer questions 1 to 3.

Known Vertebrate Species



1. a) Describe the information in this circle graph.

b) Why is a circle graph a good choice for displaying this data set?

2. Use the graph to answer these questions.

a) Of all the vertebrate species, which represents about half?

b) Are there more amphibian or mammal species?

c) How many more species of birds are there than mammals?

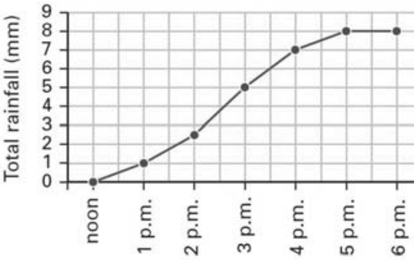
3. The circle graph could have been created without the percent values.

a) Which parts of **question 2** would be difficult to answer without the percent values?

b) Which parts would still be fairly easy to answer?

4. The broken line graph on the right shows the accumulated rainfall over a 6 h period.

Rainfall



a) How much rain fell in total?

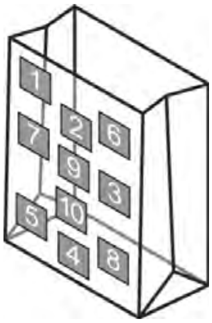
b) When was the rainfall the heaviest?

c) How much rain fell

- **i)** between noon and 3 p.m.?
- **ii)** between 1 p.m. and 2 p.m.?

d) When did it stop raining?

5. Dema reached into this bag of number tiles and picked a tile without looking. What is each theoretical probability?



a) P(a number that is not prime)

b) P(10)

c) P(an odd number)

d) P(an even number)

e) P(a number less than 11)

f) P(not 10)

Chapter 1 Displaying and Analysing Data

4.1.1 Constructing Familiar Data Displays

Try This

This data set shows the number of cell phone minutes 20 people used this month.

75	90	300	250	420	650	350	400	500	600
285	500	150	90	1000	330	200	150	60	750

A. i) What was the maximum amount of time spent on the phone?

ii) What was the minimum amount of time spent on the phone?

iii) What was the range of time spent on the phone?

- One way to display data is to use a **stem and leaf plot**. It organizes the data into place value groupings and shows all the values in order. Because the values are in order, it is easy to determine the **median** and **extreme** values.

For example, this data set shows the distance (in km) 20 people each travelled to attend a meeting.

52	50	35	25	20
120	100 37		5	4
25	36	48	56	72
75	82	36	2	5

The stem and leaf plot for this data set is shown on the right.

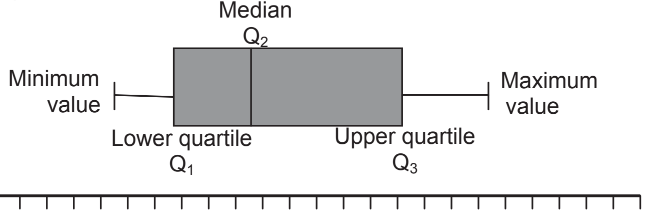
12	0
11	
10	0
9	
8	2
7	2 5
6	
5	0 2 6
4	8
3	5 6 6 7
2	0 5 5
1	
0	2 4 5 5

This extreme is 120.

The median is the average of the two middle values, 36 and 37, which is 36.5.

This extreme is 2.

- A **box and whisker plot**, or **box plot** uses the **median** and **extremes** (the minimum and maximum values) as well as the lower and upper **quartiles** of the data set to organize the data into four groups that each contains an equal number of data values. The plot is constructed using a number line scale. Note that box and whisker plots can be horizontal, like that shown here, or vertical.



- The box in the plot represents the middle half of the data values.
- The median is represented by a vertical line inside the box. Its position in the box corresponds to how the median relates to the middle half of the data.

- The width of the box indicates how spread out the middle half of the data is—the wider the box, the more spread out the middle half of the data is.
- Each whisker represents either the upper or lower fourth of the data and connects the upper and lower quartiles to the extremes. The lengths of the whiskers tell how far away the extremes are from these quartiles—a longer whisker means that the upper or lower fourth of the data is more spread out.
- The **range** of each data set is the difference between the extremes.

• To create a box plot for data on the previous page, follow these steps:

Step 1: Arrange the data in numerical order from minimum to maximum value.

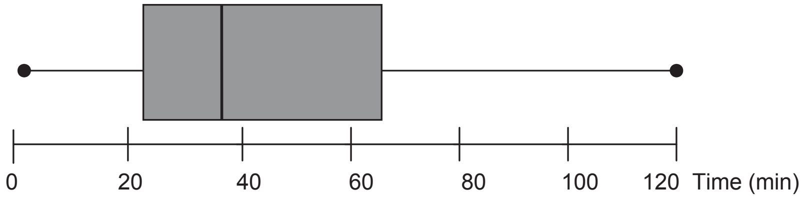
2 4 5 5 20 25 25 35 36 36 Maximum = 120 Minimum = 2
37 48 50 52 56 72 75 82 100 120 Range = 118 (120 – 2)

Step 2: Determine the median (Q2).

The data set has 20 values so there are two middle values (the 10th and 11th), so the median is the mean of those values: (36 + 37) ÷ 2 = 36.5

Step 3: Determine the lower quartile (Q1) and upper quartile (Q3).
- The lower quartile is the median of the first half of the data (the lower 10 data values). Because there are two middle values in the first half of the data (the 5th and 6th), the lower quartile is the mean of those values: (20 + 25) ÷ 2 = 22.5
- The upper quartile is the median of the upper half of the data (the upper 10 data values). Because there are two middle values in the upper half of the data (the 15th and 16th), the upper quartile is the mean of those values: (56 + 72) ÷ 2 = 64

Step 4: Draw a scale that accommodates the range. Draw the box and whiskers.
A scale from 0 to 120 with an interval of 20 is appropriate for a range of 118. Mark the positions of the extremes with dots. Mark the median and the lower and upper quartiles using vertical lines. Draw a box between the upper and lower quartiles. Draw lines from each side of the box to the extremes. Sometimes you will see the extremes marked with vertical lines or just by the ends of the whiskers.

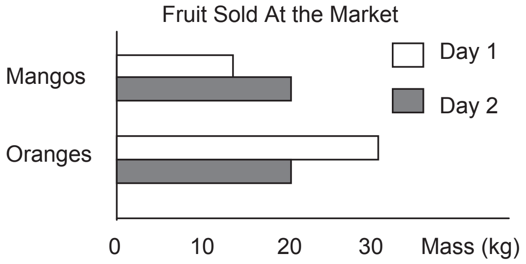


A box and whisker plot for the data in the stem and leaf plot on page 137

• A **circle graph** or pie chart shows how a set of data is broken into parts. Each category in a data set is represented by a section, or fraction, of the circle. The actual fraction or percentage is sometimes included on the graph. To construct the graph by hand, multiply 360° by the appropriate percentage to determine the size of the angle to create. **Example 1** describes how to create a circle graph.

• A **line graph** shows trends, or change over time. Time is plotted on the horizontal axis and the other variable is on the vertical axis. Points are used to represent the data. They are joined with straight lines. Their sometimes-jagged appearance is the reason they are often called broken line graphs. **Multiple line graphs** are used to compare trends in two or more variables over time on the same graph. **Example 2** describes how to create this type of graph.

- A **bar graph** compares the sizes of various categories in a set of data. Bar graphs are constructed using the width or height of bars to represent numbers (as bar graphs can be horizontal, as shown below, or vertical). The bars are separated by spaces. To compare two sets of data on the same graph, a **double bar graph** with pairs of bars is used.



A double bar graph is used to compare two sets of data on the same graph.

B. Which of the plots or graphs described on the last few pages could be used to describe the cell phone data in **part A**? Explain.

Examples

Example 1 Constructing a Circle Graph

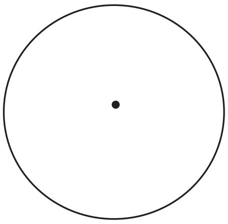
The Population and Housing Census of Bhutan 2005 reported this information about major sources of lighting in households.

Explain the steps you would follow to construct a graph of the data.

Source	Percent
Electricity	57.1
Kerosene	36.5
Others	6.4

Solution

Step 1: *Draw a circle and mark the centre.*



Step 2: *Calculate the angle for each sector.*

Source	Percent	Angle
Electricity	57.1	$0.571 \times 360^\circ$ $= 206^\circ$
Kerosene	36.5	$0.365 \times 360^\circ$ $= 131^\circ$
Others	6.4	$0.064 \times 360^\circ$ $= 23^\circ$

[Continued]

Thinking

- The percentages add up to 100 so I knew I could use a circle graph to show the fraction each electricity source represents of the whole.
- I used a compass to draw the circle.
- I changed each percentage to a decimal by dividing by 100. Then I calculated each central angle by multiplying 360° by the decimal and rounding to the nearest degree. I used a calculator.

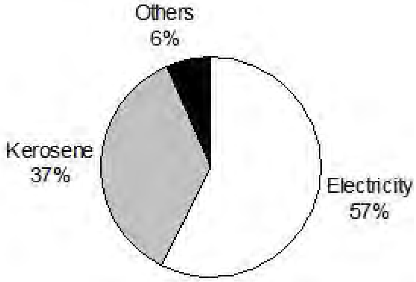
Example 1 Constructing a Circle Graph [Continued]

Solution

Step 3: Draw a central angle with the degree measure for each source.

Step 4: Colour, label, and title the graph.

Sources of Light in Households in Bhutan



Thinking

- I drew a line from the centre to the circumference of the circle to mark the start of the first sector. Then I used a protractor to measure the first angle and drew another line to outline the first sector. I knew it did not matter which angle I drew first.
- For the next sector, I measured the angle from the end of the first sector.
- For the last sector, I just double-checked to make sure the remaining angle was correct.

Example 2 Constructing a Multiple Line Graph

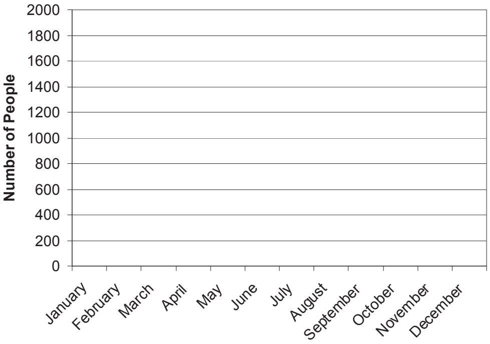
The Department of Tourism in Thimphu reported the number of tourists that visited Bhutan each month during 1997 and 1999.

Explain the steps you would follow to construct a graph of the data.

Month	1997	1999
January	108	148
February	254	322
March	1062	1145
April	662	604
May	275	395
June	90	108
July	123	132
August	231	348
September	276	1069
October	1488	1856
November	640	841
December	154	190

Solution

Step 1: Draw and label the axes.



Thinking

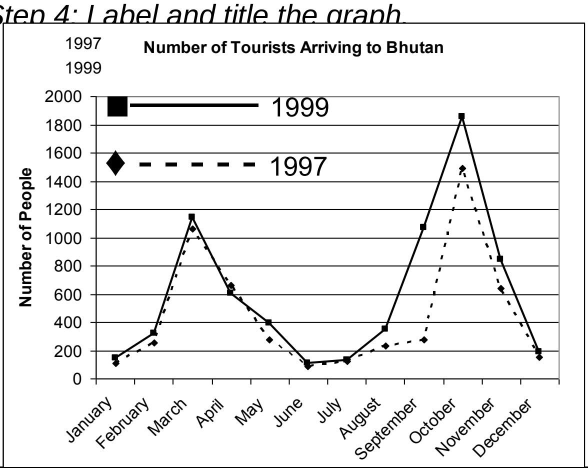
- There were two sets of data, each involving change over time. It made sense to use a multiple line graph to compare the trends.
- I spaced the 12 months equally along the horizontal axis.
- The data went from 90 to 1856 so I used an interval of 200 so I only needed 10 intervals from 0 to 2000.

Solution

Step 2: Plot the data points for each data set.

Step 3: Join the corresponding points.

Step 4: Label and title the graph.



Thinking

- I used a different symbol for the points of each set of data in order to distinguish them—diamonds for 1997 and squares for 1999.
- I used a ruler to join the points. I used a different line for each set of data—dotted lines for 1997 and solid lines for 1999.

Example 3 Constructing a Box Plot for an Odd Set of Data

Construct a box and whisker plot for the following set of data. Describe your steps. Since there is an odd number of data values, you can choose to include the median or not when calculating the lower and upper quartiles.

59	96	108	152	201	99	170	175	95	12
43	101	150	156	70	51	80	200	85	

Solution 1 (including the median in Q1 and Q3)

Step 1a: Arrange the data in order.

12	43	51	59	70	80	85	95	96	99
101	108	150	152	156	170	175	200	201	

Step 1b: Determine the extremes and the range.

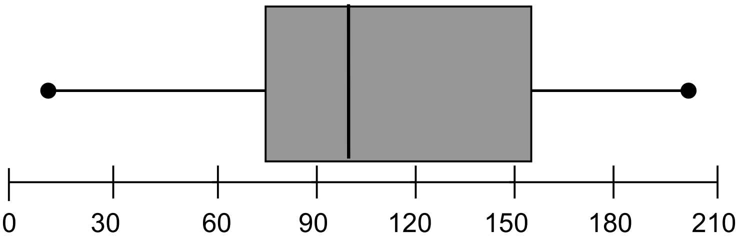
Minimum: **12** Maximum: **201** Range: 201 – 12 = **189**

Step 2: Determine the median (Q2). **99**

Step 3a: Determine the lower quartile (Q1).
(70 + 80) ÷ 2 = **75**

Step 3b: Determine the upper quartile (Q3).
(152 + 156) ÷ 2 = **154**

Step 4: Draw a scale, box, and whiskers.



Thinking

- There are 19 values so Q2 is the 10th value.
- Q1 is the median of the first half of the data (the lower 10 data values, which includes the median), so it's the mean of the 5th and 6th values.
- Q3 is the median of the upper half of the data (the upper 10 data values, which includes the mean), so it's the mean of the 14th and 15th values.

Example 3 Constructing a Box Plot for an Odd Set of Data [Continued]

Solution 2 (not including the median in Q1 and Q3)

Step 1a: Arrange the data in order.

12 43 51 59 70 75 85 95 96 99
101 108 150 152 156 170 175 200 201

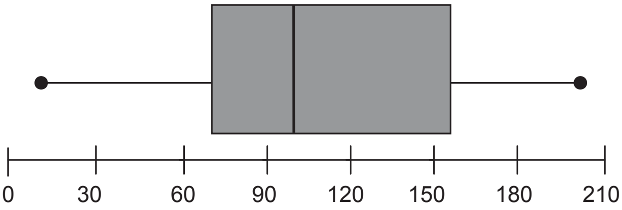
Step 1b: Determine the extremes and the range.

Minimum: **12** Maximum: **201** Range: 201 – 12 = **189**

Step 2: Determine the median (Q2). **99**

Step 3a: Determine the lower quartile without including the median (Q1). **70**
Step 3b: Determine the upper quartile without including the median (Q3). **156**

Step 4: Draw a scale, box, and whiskers.



Thinking

- There are 19 values so Q2 is the 10th.
- Q1 is the median of the 9 values below the median, not including the median, so it's the 5th value.
- Q3 is the median of the 9 values above the median, not including the median, so it's the 15th value.
- A scale from 0 to 210 with an interval of 30 is appropriate for a range of 189 that goes from 12 to 201.

Practising and Applying

1. a) Construct a stem and leaf plot for this set of data about cell phone use (from the Try This task).

75	90	300	250	420
285	500	150	90	1000
650	350	400	500	600
330	200	150	60	750

b) Construct a box and whisker plot for the same data.

2. Construct a circle graph for this data.

Percent of Time Teens in a School in Thimphu Spend on Leisure Activities

Activity	Time spent (%)
Reading	5
TV	70
Games	15
Song and dance	5
Other	5

3. Construct a bar graph for this set of data about population growth.

Population of the Earth between 1750 and 1900

Year	Population (billions)
1750	0.80
1800	0.95
1850	1.20
1900	1.70

4. Construct a double bar graph for the following data.

Number of Students by Gender in Four Classes

Class	Male	Female
Class I	23	19
Class II	16	21
Class III	19	18
Class IV	17	23

5. Listed below are the maximum daily temperatures (in °C) for Toronto, Ontario, Canada for 13 consecutive days in September.

29.3, 29.1, 28.2, 19.1, 18.8, 22.4, 18.4, 17.0, 20.2, 25.0, 25.8, 24.1, 22.1

a) Determine the median temperature.

b) Determine the range.

c) Determine the upper and lower quartiles.

d) Construct a box and whisker plot.

Toronto Skyline

6. Construct a box and whisker plot for each data set.

a) The number of televisions sold each month at an electronics store over the last 12 months:

51, 17, 25, 39, 7, 49, 62, 41, 20, 6, 43, 13

b) The length (in cm) of 15 trout caught in a river:

14.5, 34.6, 45.9, 56.1, 49.4, 22.7, 37.0, 19.6, 59.3, 31.4, 28.3, 36.2, 41.2, 13.6, 44.2

7. Construct a multiple line graph for this set of Olympic data.

Winning times for Men's and Women's Olympic 100 m Sprint Finals

Year	Men's time (s)	Women's time (s)
1928	10.8	12.2
1936	10.3	11.5
1948	10.3	11.9
1956	10.62	11.82
1964	10.06	11.49
1972	10.14	11.07
1980	10.25	11.06
1988	9.92	10.54
1996	9.84	10.94
2004	9.85	10.93

8. Construct an appropriate graph for each data set.

a) Amount of Space Devoted to Various Sections of the *Bhutan Times*

Section	Percent
Bhutan News	60
World News	30
Humour	10

b) A biologist who is studying the Common Kestrel measured and recorded these wingspans (in cm).

65	59	62	73	78	65 79
54	62	73	68	75	69 71
82	73	72	70	63	68

Common Kestrel

4.1.2 Using Graphs to Compare and Organize Data

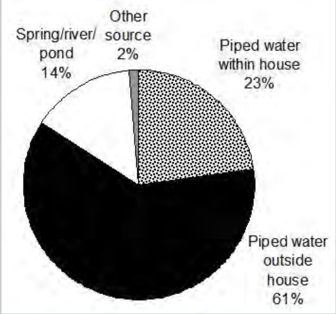
Try This

After reading the fact sheet from the Population and Housing Census of Bhutan 2005, Dorji used the information about drinking water sources to create two different graphs for the same set of data.

A. i) What type of graph is each?
ii) Describe how each graph displays the data.

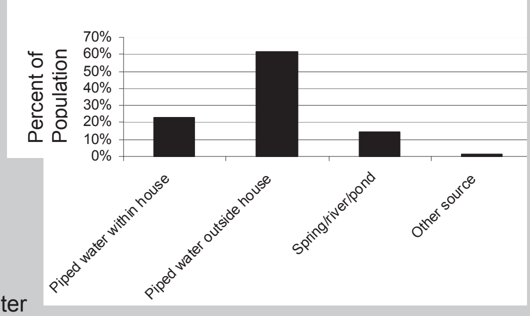
B. Which graph do you prefer? Why?

Main Source of Drinking Water



Paro River is a source of drinking water

Main Source of Drinking Water



Some graphs allow you to see very quickly how different sets of data compare. Bar graphs and circle graphs are often used for this purpose.

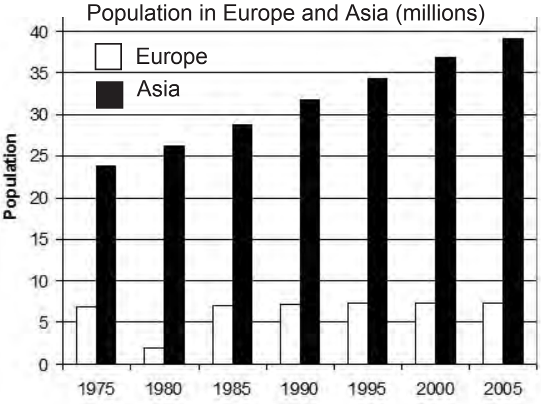
- The double bar graph here makes it easy to see that

- Asia's population is steadily growing

- Europe's population has been growing very slightly, except for a sudden dip in 1980

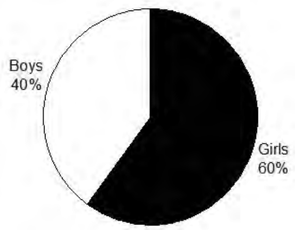
- Asia's population is growing very quickly compared to Europe's

- Asia's population has been considerably greater than Europe's, at least since 1975

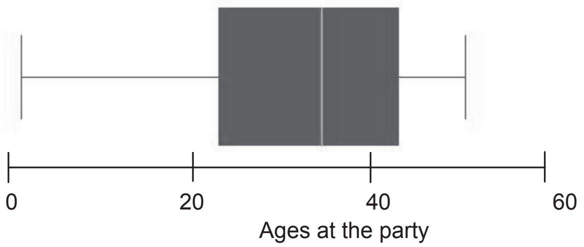


- Circle graphs are particularly good for comparing each part to the whole. For example, the circle graph below makes it easy to see that there are more girls than boys in Pema's class and that girls make up more than half the class.

Students in Pema's Class



- Some graphs show visually how data values in a set of data are grouped or clustered. Box and whisker plots, are commonly used for this purpose.
- A box and whisker plot helps you quickly see where most of the data values are found and where the more extreme pieces of data are located. For example, the box plot below shows that half the people at a party were between about 24 and 43 years old and that the spread of ages below 24 was greater than the spread above 43. It also shows the youngest was 2, the oldest was 51, and half the people were under 36. Note that it is not possible to tell from the graph what each specific age was.



C. Examine Dorji's two graphs from **part A**. Why would a box plot and a double bar graph both be less appropriate ways to show the data?

Examples

Example 1 Justifying the Use of a Box and Whisker Plot

The table shows the distances thrown in the women's shot put final at the 2004 Summer Olympics in Athens, Greece.

Shot put involves "putting" (throwing with a pushing motion) a 4 kg metal ball called the shot as far as possible.

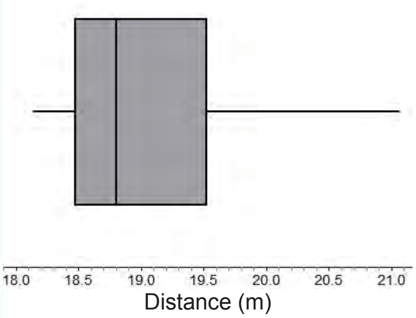
Ranking/name	Country	Distance (m)
1. Korzhanenko	Rus	21.06
2. Cumba	Cub	19.59
3. Kleinert	Ger	19.55
4. Krivelyova	Rus	19.49
5. Ostapchuk	Blr	19.01
6. Khoroneko	Blr	18.96
7. Zabawska	Pol	18.64
8. Gonzalez	Cub	18.59
9. Adams	Nzl	18.56
10. Li	Chn	18.37
11. Borel	Tri	18.35
12. Tunks	Ned	18.14

[Continued]

Example 1 Justifying the Use of a Box and Whisker Plot [Continued]

Chencho used this box and whisker plot to represent the shot put data.

Explain how his box plot describes the data and why he might have chosen that graph.



Solution

The median distance is the vertical line inside the box, located at 18.8 m.

The left side of the box, the lower quartile, is 18.47 m.

The right side of the box, the upper quartile, is 19.52 m.

The left extreme, or minimum distance, is 18.14 m.

The right extreme, or maximum distance, is 21.06 m.

The range of the data is 2.92 m.

The box plot shows that
- half the athletes threw the shot between 18.47 m and 19.52 m
- half the data is closely clustered around the median, because the box is narrow
- the spread of values above 19.52 m was much greater than the spread of values below 18.47 m

Chencho may have chosen to use a box plot because he wanted to see at a glance how the data values are clustered or distributed around the median.

Thinking

- Since there are 12 data values altogether, I knew the median would be the mean of the 6th and 7th values, (18.96 + 18.64) ÷ 2.
- Since there are six data values in the lower half, I made the lower quartile the mean of the 3rd and 4th values, (18.56 + 18.37) ÷ 2.
- Since there are six data values in the upper half, I made the upper quartile the mean of the 9th and 10th values, (19.49 + 19.55) ÷ 2.
- I put the extremes, or the maximum and minimum data values, at the ends of the whiskers.
- The range is the difference between the two extremes, 21.06 – 18.14 = 2.92 m.

Example 2 Justifying the Choice of a Double Bar Graph

The people who reported on the Population and Housing Census of Bhutan for 2005 used the following graph to display data about the state of happiness in the country.

a) Why do you think they chose this type of graph?

b) Make an observation about the data in the graph.

Solution	**Thinking**
a) A bar graph is used when the data categories are discrete and when you want to compare the data in each category. The state of happiness data set was divided into three discrete categories. It made sense to use a bar graph to show how many people chose each of the three categories so they could be compared.	 **a)** There are spaces between the bars on a bar graph to show that the categories of data are separate, or discrete.
A double bar graph was used to make it easy to compare the state of happiness of urban and rural Bhutanese citizens in each category.	• People who live in rural versus urban areas deal with different social issues that could affect their happiness so it makes sense to want to compare the data for each.
b) About half of the citizens surveyed in both the rural and urban areas indicated that they were happy.	**b)** You can get lots of information from the graph. I just chose to talk about the biggest group.

Practising and Applying

1. Choose the letters of all the statements that describe each graph.

a) circle graph

A. shows each number in the data set

B. shows parts of the whole

C. shows the minimum and maximum data values

D. uses the whole shape to represent the set of data and sections to represent the parts

b) bar graph

A. always has bars that touch

B. is constructed from data that can be counted (discrete data)

C. uses the length of each bar to show the number in each category

D. can have bars that are either horizontal or vertical [Cont'd]

1. [Cont'd] **c)** box and whisker plot

A. shows the median and quartiles of a data set

B. shows the range of the data set

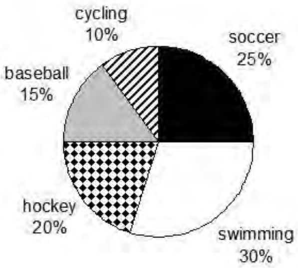
C. shows the mode of the data set

D. shows each number in the data set

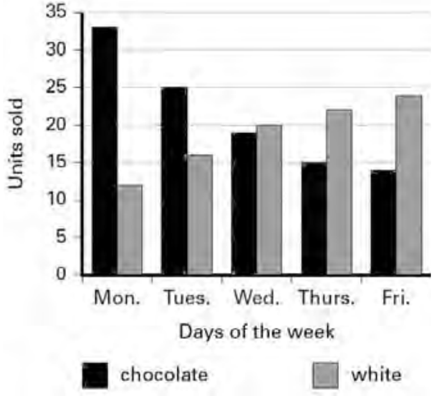
2. For each graph that follows,

- make an observation about the data in the graph
- explain why you think that type of graph was used
- suggest another type of graph (if any) that might be suitable

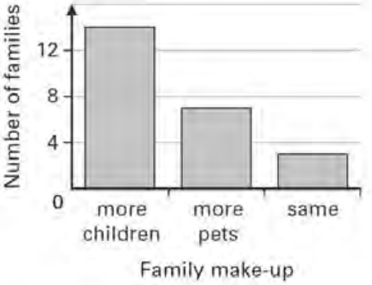
a) Yan's Sports Survey



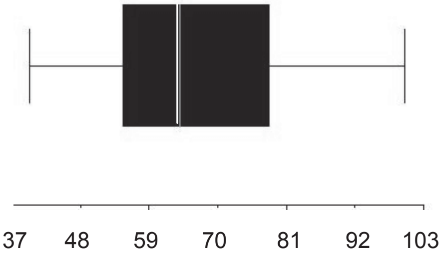
b) Weekly Milk Sales



2. c) Comparing Number of Children and Number of Pets



d) Math Test Scores



3. For each set of data, tell which type of graph you think is most suitable and why.

a) Major Sources of Cooking Fuel in Bhutan

Source	Percent
Electricity	30.6
Firewood	37.2
LPG	25.5
Others	6.7

b) Bhutan Households by Number of Members

Members	Number of households
1 – 2	26,139
3 – 4	39,381
5+	60,595

3. c) The times, in seconds, for 20 runners to run 100 m:

12.8 12.1 13.5 11.8 13.2
12.6 12.3 13.0 11.9 11.5
12.5 12.7 13.9 14.0 13.2
11.8 12.0 13.1 13.8 12.4

4. What type of graph do you think is a good choice for displaying each data set? Justify your choice.

a) the distribution of the mass of 100 students so you can see the middle 50% of the data

b) a comparison of the different types of transportation people use to get to work

c) the number of books each student in a class borrowed from the library in one month

d) the shoe sizes of all the students in your class

e) the heights of the world's 10 tallest buildings

The Petronas Twin Towers in Kuala Lumpur are the tallest twin towers in the world.

5. This chart shows the most common types of things found in the municipal waste site of a big city in North America.

Type of waste	Percent
Paper	43
Yard waste	15
Food waste	12
Glass	8
Plastic	8
Steel	6
Other	8

City waste site in North America

a) Create two different graphs for the data.

b) Which graph do you think best shows the comparison between the different types of waste at the site? Explain.

6. Describe a situation that has not already been presented where you might use each type of graph:

a) a double bar graph

b) a bar graph

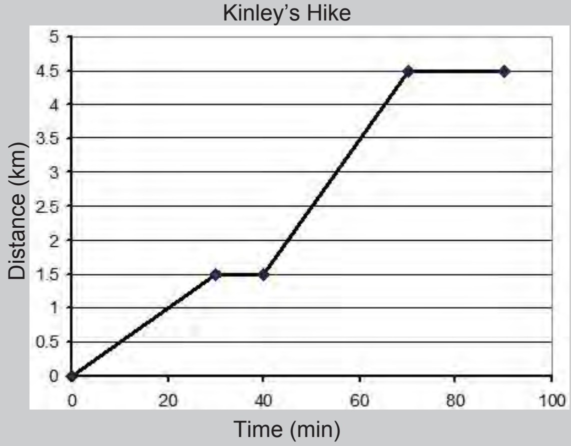
c) a box and whisker plot

d) a circle graph

4.1.3 Using Graphs to Examine Change

Try This

This graph describes how Kinley travelled on an afternoon hike. It shows his distance from home against time.

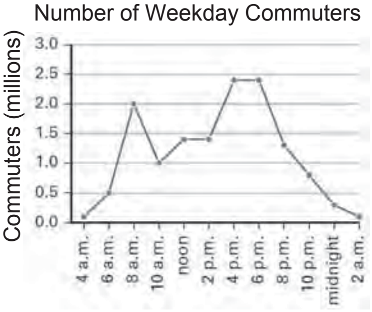


A. Describe Kinley’s hike.

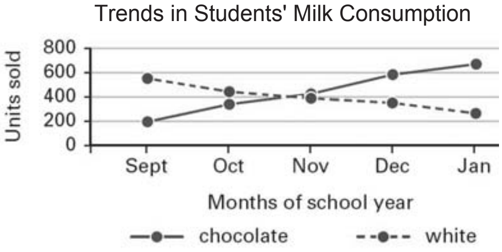
Some graphs are best at showing how things change over time. **Line graphs** are the most common type.

- **Broken line graphs** are used when you want to examine how one quantity changes in relationship to another, often time. Points are plotted and then joined with line segments. By looking at the line you can tell if a quantity is increasing, decreasing, or staying the same over time.
- **Multiple broken line graphs** are used when you want to examine how one quantity changes in relationship to another for several sets of data. For example:

This broken line graph shows the numbers of Canadians who travel to or from work at different times throughout the day. The graph shows peak times at 8 a.m. and between 4 p.m. and 6 p.m.



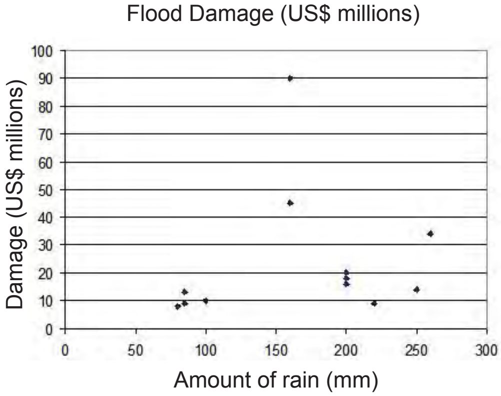
This multiple broken line graph shows the changes in white milk sales and chocolate milk sales at a school in Canada over a 5-month period. Chocolate milk sales increased as white milk sales decreased.



- **Scatter plots** are used to determine if two sets of data are related. And, if there is a relationship, they can be used to determine the nature of the relationship. You plot corresponding numbers from each data set as ordered pairs and then examine the shape of the plotted points. For example:

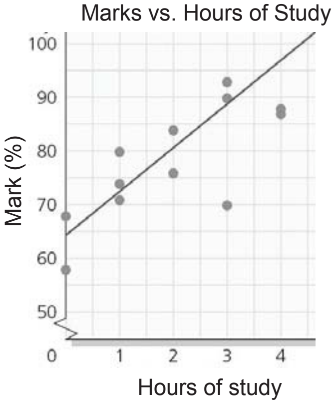
This scatter plot shows the flood damage caused by different amounts of rain in the United States.

The points are all over the graph, so there is no evidence of a relationship between the amount of rain and the damage caused.



This scatter plot shows how students' marks on a test relate to the length of time they spent studying.

The points are in a roughly linear pattern, so there appears to be a linear relationship—the greater the study time, the higher the mark. A solid line of best fit is used because there is a linear relationship and the data values are continuous.

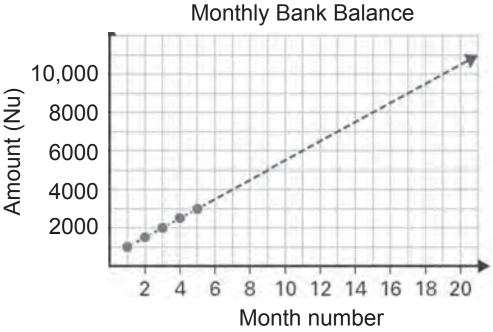


- Sometimes the points on a scatter plot are joined with dashed lines. This indicates there is a trend but that the data values are discrete (not continuous).

For example:

This scatter plot shows the balance in Karma's bank account (in ngultrums) as he makes monthly deposits.

A clear linear trend is evident, but the data values are discrete so a dashed line of best fit is used to show the trend.



B. What kind of graph is the graph of Kinley's hike in **part A**? Why does that type of graph make sense for the situation?

Example Examining Change Using a Broken Line Graph

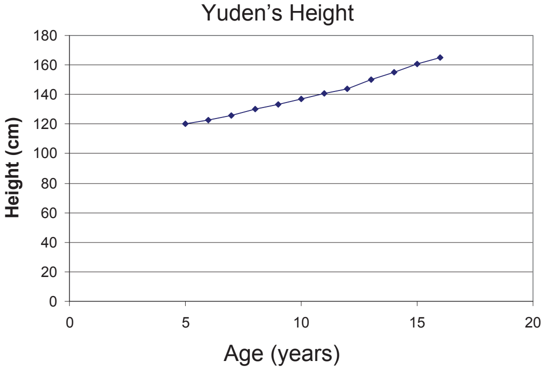
Yuden made a graph of her height at different ages.

a) How many times do you think she measured herself?

b) What do you observe about her growth?

c) Was Yuden correct to join the points with a solid line?

d) Why is this type of graph appropriate?



Solution	Thinking
a) She measured herself 12 times.	 **a)** Each dot represents a time she measured herself. It looks like once a year between the ages of 5 and 16.
b) She grew steadily from 120 cm to 165 cm in 11 years (about 3 cm a year) but she grew a little faster between 12 and 15.	**b)** I looked at the slope to see how much she grew and how quickly. When it was steeper, I knew she grew a bit faster. I also looked at the range, 165 cm – 120 cm = 45 cm.
c) Growth is continuous, so it made sense to use a solid line.	**c)** When something is continuous, the data values between the points make sense—even though she only measured once a year, you can assume that she was still growing in between.
d) Growth is about change over time and line graphs are good for showing when things increase, decrease, or stay the same over time.	**d)** I can see from this graph how quickly she grew over time.

Practising and Applying

1. Choose the letters of all the statements that describe each graph.

a) broken line graph

- **A.** has points that are joined
- **B.** has all points lying on a single line
- **C.** shows how one quantity changes in relationship to another
- **D.** shows all numbers in the data set

b) scatter plot

- **A.** shows each number in the data set
- **B.** shows if two quantities are related
- **C.** shows the mean in the data set without requiring any calculation
- **D.** is a series of points connected with lines

1. c) multiple broken line graphs

A. has points that are joined

B. is used for single sets of data

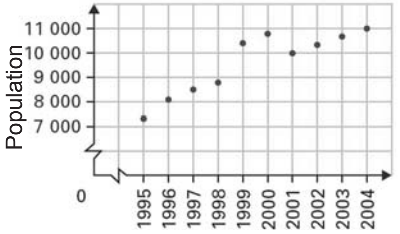
C. shows how several quantities change in relationship to another

D. often uses time as one of the variables

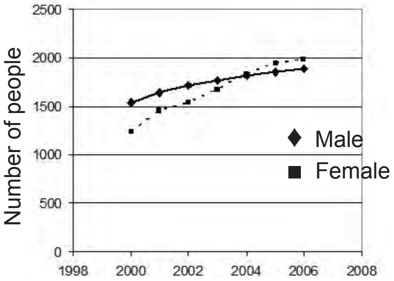
2. For each graph below,

- make an observation about the data
- explain why that graph type was used
- suggest another type of graph (if any)

a) Population Growth (1995–2004)



b) Cell Phone Use



3. For each set of data, what graph do you think is most suitable and why?

a) Height and Arm Length of a Group of Students

Height (cm)	Arm length (cm)
159	43
157	41
160	45
159	43
160	46
157	41

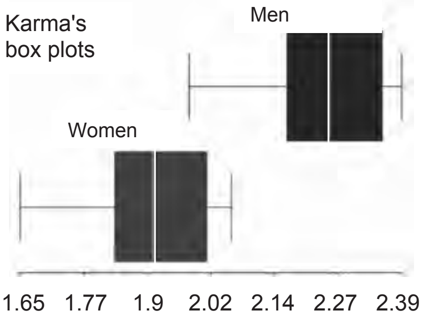
3. b) Bhutan: Imports and Exports of Merchandise Trade (US$ millions)

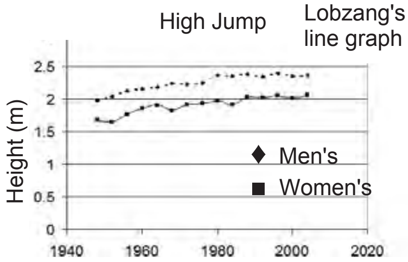
Year	Exports	Imports
1980	17	50
1990	70	81
1995	103	112
2000	103	203
2002	108	165
2003	108	171

c) Bhutan: Tourist Arrivals

Year	Tourists (thousands)
1980	1.0
1990	1.5
1995	4.8
2000	7.6
2002	5.6
2003	6.3

4. Karma and Lobzang created these graphs for data about the men's and women's winning Olympic high jumps.





a) Why do you think each student used the type of graph he did?

b) Are both graphs equally useful for showing change? Explain.

4.1.4 Misleading Graphs

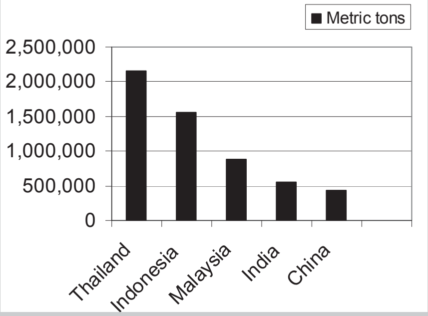
Try This

Dechen, Sangay, and Sonam each created a bar graph to compare the quantities of natural rubber produced by the world's leading producers.

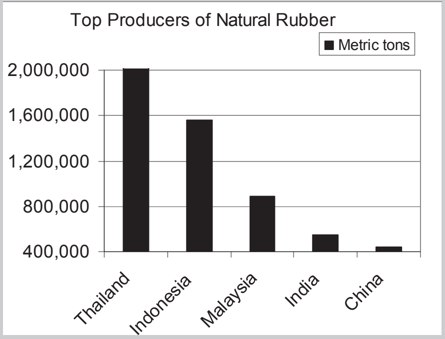
Top Producers of Natural Rubber

Country	Metric tons, or tonnes
Thailand	2,162,411
Indonesia	1,564,324
Malaysia	885,700
India	550,000
China	440,000

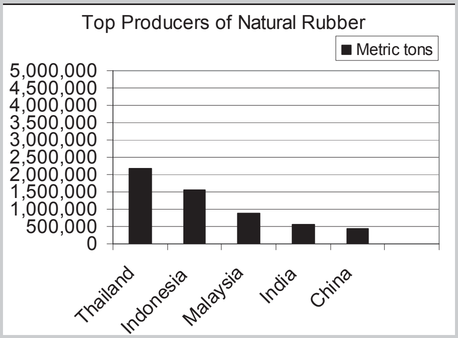
Sangay's Graph



Sonam's Graph



Dechen's Graph



All three students have come to different conclusions about the same data.

- One student says that the data set indicates that the difference among the five countries is significant.
- Another says that there is a difference but that it is not very significant.
- The third student says that the difference is somewhere between these two.

A. i) Compare the graphs. Comment on the similarities and differences.

ii) Which features of these graphs might lead the students to come to three different conclusions about the same set of data?

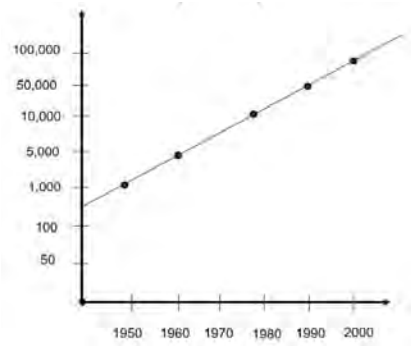
There are several reasons why a graph might be misleading. Sometimes the creator of the graph does this deliberately. Other times it is not the creator's intent to mislead.

- **Poor use of scale**

- The scale should be constant on each axis of the graph.
- The scale should be presented as increasing from left to right on the horizontal axis and from bottom to top on the vertical axis.
- The scale should be appropriate for the range of the data.

For example, in this graph, the vertical scale does not increase by the same increment. This gives the false impression that the population has been growing at a constant rate since 1950.

Population Growth (1950 to 2000)

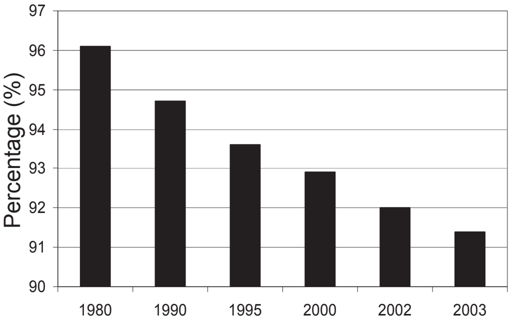


- **Misplaced zero on the axes**

Most people read a graph with the assumption that the zero point is at the bottom of the vertical axis or the left of the horizontal axis. If the vertical scale of a graph does not start at 0, there should be an obvious indication of a break in the axis (which is usually done with a jagged axis line at the point where the break occurs). This is most critical with bar graphs. Otherwise, one can get a very misleading impression of the data. Note that, for bar graphs, it is preferable to not use a break at all and, instead, ensure the scale chosen can accommodate all the data.

For example, in this bar graph, the vertical scale starts at 90 and not at 0. This gives the false impression that a significant number of people are leaving rural areas.

Percentage of Bhutanese in Rural Areas



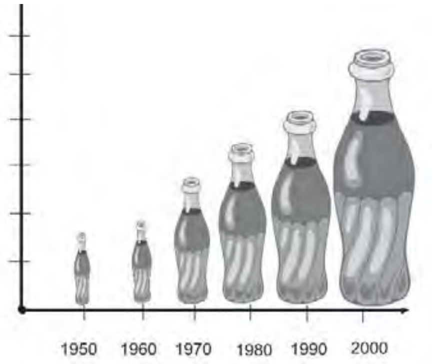
- **Inconsistent shading or visual effects**

These are often added to make a graph more attractive. In some cases, though, they distort the graph and mislead the person reading it.

For example, this graph uses different-sized bottles instead of bars to represent the sales each decade. The graph gives the misleading impression that sales increased rapidly between 1950 and 2000 since the "bars" are getting bigger both horizontally and vertically. In a standard bar graph, only the length or height of the bar changes.

Note that you really cannot conclude anything from this graph because there is no vertical scale.

Drink Sales for Lucky Beverage Company

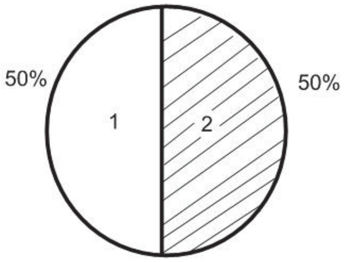


- **Wrong choice of graph type**

For example, someone might use a circle graph to compare data, but if the total data set does not represent a whole, this would be misleading.

For example, in this graph, half the students seem to have 2 siblings and half have 1 sibling. However, many categories of data were left out, such as no siblings, 3 siblings, and so on.

Students With 1 or 2 Siblings



B. i) Which graph or graphs in **part A** do you think represent the data about natural rubber production in a misleading way? Explain.

ii) Which student do you think made each claim below? How do you know?

- The difference among the five countries is significant.
- There is a difference but it is not very significant.
- The difference is somewhere between these two.

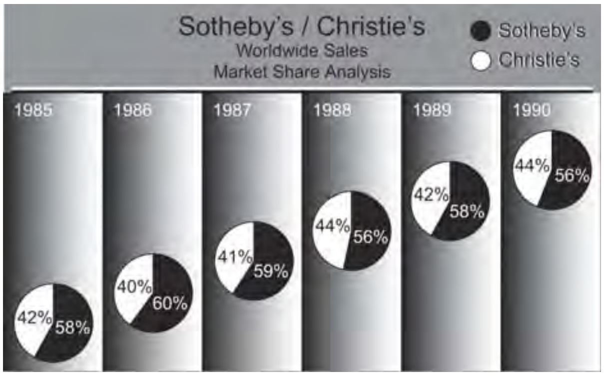
Examples

Example Examining Misleading Graphs

Sotheby's and Christie's are two large auction companies based in England. Auction companies sell items for people by having buyers come to their auctions and bid on the items.

a) Explain why this graph might be misleading.

b) How should the data have been graphed? Explain.

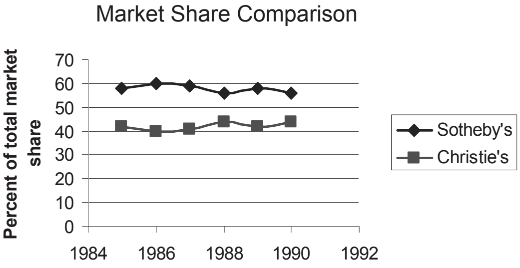


Solution

a) The graph misleads you to think that the market shares of both companies are increasing because the graph shows the circle rising, as if something is increasing. If you look at the data in each circle graph, the values for each company actually go up and down.

The fact that circle graphs have been used is also misleading because there may be more auction companies than just these two. If that is the case, the market share of both Sotheby's and Christie's will be less than what is shown on the circle graphs.

b) The data should have been graphed in a multiple broken line graph because line graphs are good for showing trends over time. A multiple line graph allows you to show and compare multiple trends on the same graph.



Thinking

a) The actual values for Christie's are 42, 40, 41, 44, 42, 44.

For Sotheby's, they are 58, 60, 59, 56, 58, 56.

• Circle graphs should only be used when you are comparing parts to the whole. A set of two auction companies likely does not represent the whole.

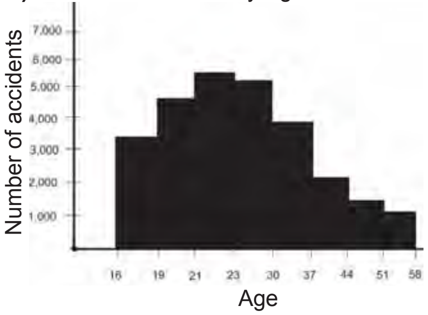
b) The line graph I drew shows that the market share of Sotheby's is generally decreasing while Christie's is increasing, although there are ups and downs in both.

• My graph accurately reflects the given data but I still wonder if there are more auction companies that should be included.

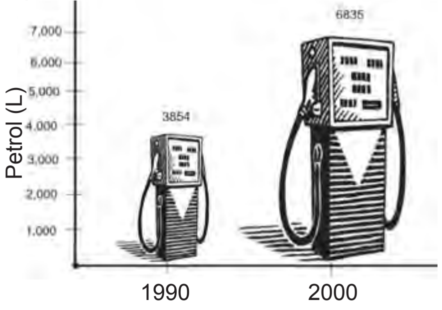
Practising and Applying

1. Identify the features of each graph that might cause the graph to be misleading. Explain why the graph is misleading.

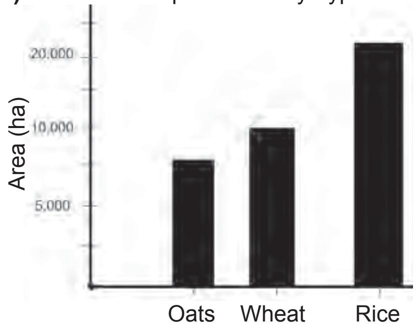
a) Car Accidents by Age of Driver



b) Petrol Sold



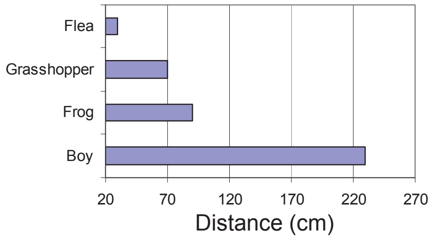
c) Crops Planted by Type



2. You need to decide whether a graph is misleading. Describe three things to look for in the graph.

3. Is each statement true or false?

Length of Standing Jump



a) The boy's jump is more than twice as long as the frog's jump.

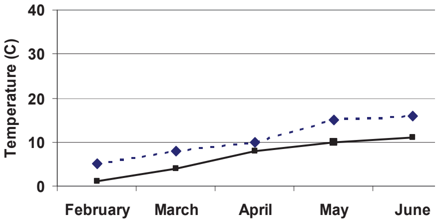
b) The flea's jump is about a quarter of the length of the grasshopper's jump

c) The boy's jump is less than three times as long as the grasshopper's.

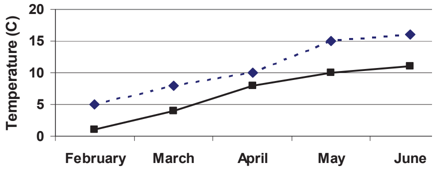
4. Why can you not just compare the bars to answer **question 3**?

5. These graphs show temperatures in Thimphu. Which graph might mislead you to think that the temperature does not change much? Why?

Maximum and Minimum Temperatures in Thimphu, February to June



Maximum and Minimum Temperatures in Thimphu, February to June



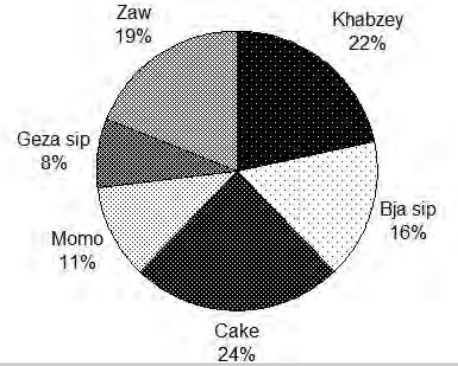
6. Create a graph of your own that is misleading. Explain how it misleads.

4.1.5 Drawing Conclusions From Graphs

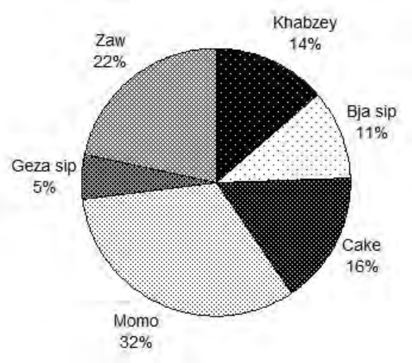
Try This

Anjali's class was surveyed about snack preferences one month before and one month after hearing a guest speaker talk about nutrition. The circle graphs below show the survey results.

Before Hearing the Speaker



After Hearing the Speaker



A. What conclusion can you draw from the data? Is it reasonable? Explain.

- Conclusions can often be drawn from a graphical display. However, the conclusions must follow reasonably from the data. Assuming that the data set has been collected correctly and is free of bias, the following questions can help you judge the reasonableness of a conclusion:

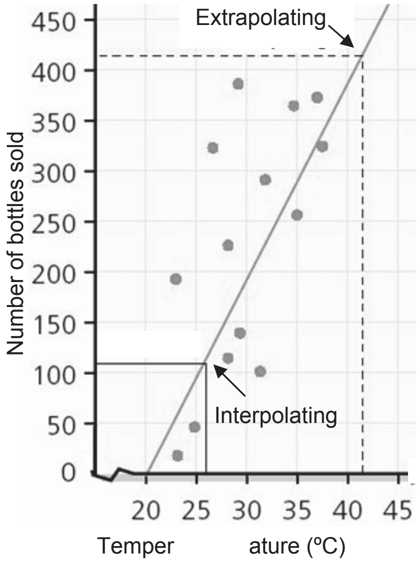
- *What reasoning was used to draw the conclusion?*

- *Are there any features of the graph that might be misleading?*

- Sometimes you can draw conclusions from a graph for data values that were not included in the actual data collection. Graphs that show trends, such as line graphs and scatter plots, are often suitable for interpolating and extrapolating.

For example, in this scatter plot, the number of bottles sold when the temperature outside is 26ºC has been interpolated to be about 110 and the number of bottles sold when the temperature is 41ºC has been extrapolated to be about 410.

Water Sales vs. Temperature



You should be cautious when extrapolating and interpolating data. The relationship between the variables may not remain the same beyond the limits of your data or it may not make sense to extrapolate. As well, if the data is discrete, interpolating might not make sense.

B. Can the data about snack preferences in **part A** be used to predict snack preferences three months from now? Explain.

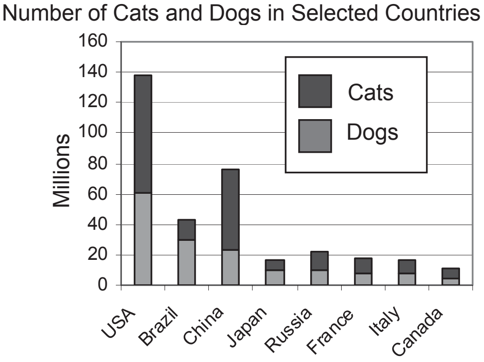
Examples

Example 1 Drawing Conclusions

The graph on the right shows the number of dogs and cats in several countries.

a) What type of graph is this?

b) What conclusions can be drawn from this data?

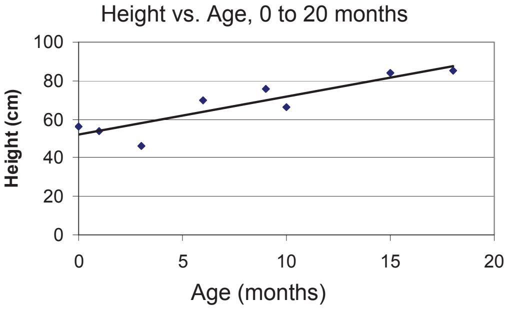

Number of Cats and Dogs in Selected Countries

Solution	**Thinking**
a) It is a stacked bar graph, which is a type of double bar graph.	 **a)** A stacked bar graph is like a double bar graph but it shows the two bars for each country (one for cats and one for dogs), stacked on top of each other.
b) *Comparing total cats and dogs*: Among the eight countries, the USA has the greatest total population of cats and dogs and Canada has the lowest. *Comparing cats versus dogs*: • In the USA, China, Canada, and Russia the number of cats is greater than the number of dogs. • In Brazil and Japan, the number of dogs is greater than cats. • In the other countries, the numbers of cats and dogs are about equal.	**b)** Because the graph shows both the total number of cats and dogs in each country and the number of cats and the number of dogs in each country, I was able to come to conclusions that involved - comparing the total number of cats and dogs in the eight countries - comparing the number of cats versus dogs in each country and among the eight countries

Example 2 Interpolating and Extrapolating

Doctors recorded the ages and heights of a sample of baby boys. A scatter plot was then created and a line of best fit drawn.

Age (months)	Height (cm)
0	56
15	84
18	85
9	76
1	54
3	46
6	70
10	66



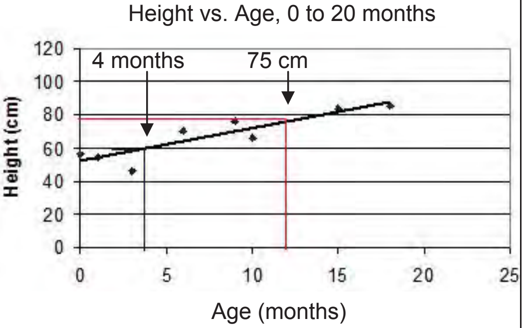
Use the graph to make these estimates or predictions. Explain your thinking.

a) the age of a baby boy who is 60 cm tall and the height of a 12-month-old boy

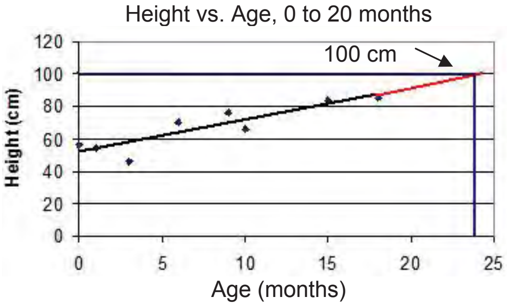
b) the height of a 24-month-old boy

Solution

a) A 60 cm baby boy is about 4 months old.
A 12-month-old baby boy is about 75 cm tall.



b) Assuming the trend in growth continues at the same rate, a 24-month-old boy would be about 100 cm tall.



Thinking

a) I found the height, 60 cm, on the vertical axis and on the line of best fit. Then I looked for the corresponding value on the horizontal axis, 4 months.

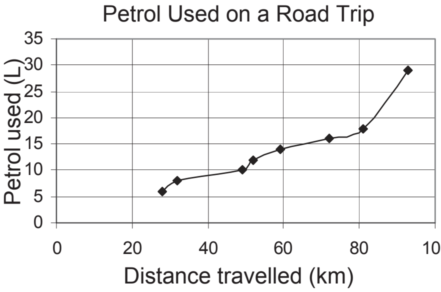
- I found the age, 12 months, on the horizontal axis and on the line of best fit. Then I looked for the corresponding value on the vertical axis, 75 cm.

b) I extended the line of best fit assuming the trend continued but I know you have to be careful when extrapolating.

- I don't think my prediction is reasonable because I don't think the trend would continue. Babies grow really fast at first and then their growth slows down.

Practising and Applying

1. This graph shows the relationship between the amount of petrol used and the distance travelled.

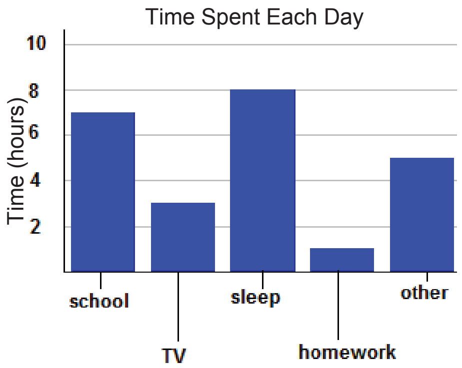


a) About how much petrol would you expect to use to travel 40 km? 55 km?

b) About how many kilometres would you have travelled if you had used 15 L of petrol? 25 L of petrol?

c) What conclusions can you draw from the graph?

2. This graph shows the average number of hours North American teenagers spend doing various activities each day.



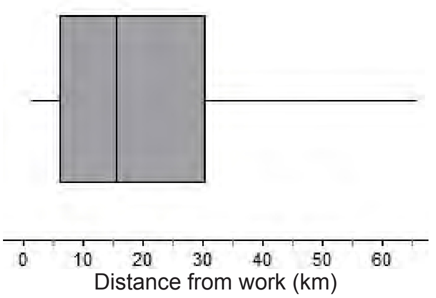
a) What is one conclusion you can draw from the graph?

b) About how many hours is spent watching TV?

c) What activities do teens spend less than 2 h doing each day?

d) About what percentage of the day is spent at school?

3. This graph shows the distances some people live from where they work.

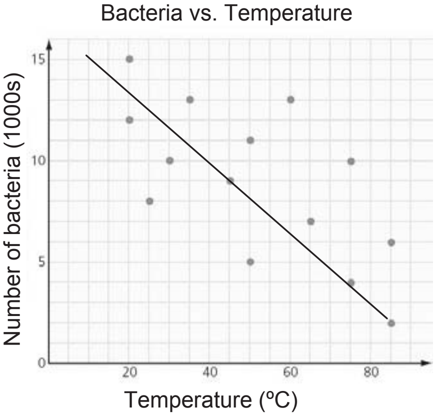


a) Estimate the range of distances between people's homes and workplaces.

b) Can this graph be used to predict how many people live more than 20 km away from their work? Explain.

c) What conclusions can you draw?

4. This graph shows the number of bacteria present in a laboratory culture at different temperatures.



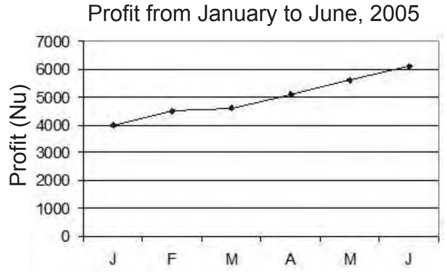
a) About how many bacteria would be present at 70 ºC?

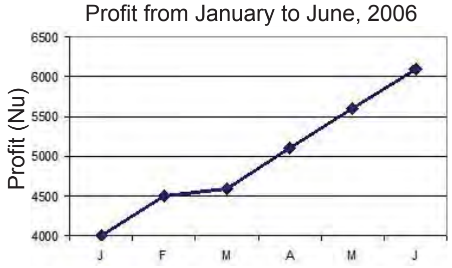
b) At about what temperature would you find about 14,000 bacteria?

c) Can you predict the number of bacteria present at 200 ºC? Explain.

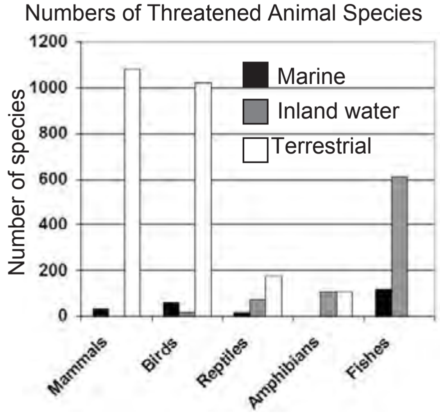
d) What conclusions can you draw?

5. Based on these two graphs, Koyelle suggests that her company's profit over the first six months of 2006 increased dramatically in comparison to 2005. Is her conclusion valid? Explain.


Profit from January to June, 2005

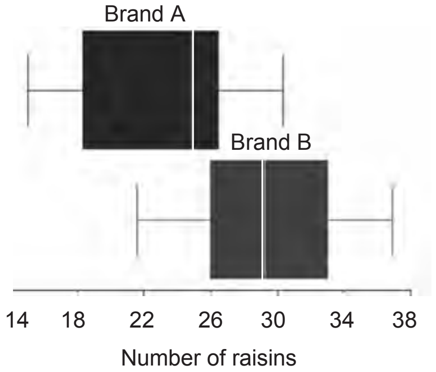

Profit from January to June, 2006

6. Based on the data in this graph, Jamyang suggests that Marine species face little threat of extinction. Is her conclusion valid? Explain.


Numbers of Threatened Animal Species

7. Data about two different brands of small raisin boxes were collected. Boxes were selected at random and then the number of raisins in each box was counted and graphed.

The box plots below display the data about the number of raisins in the boxes for each brand.



Based on the data in these graphs, what conclusions can you draw about the two brands of raisins?

8. Name some types of graphs that cannot be used to interpolate and extrapolate. Explain why they cannot be used in this way.

Chapter 2 Probability

4.2.1 Determining and Comparing Probabilities

Try This

You are about to roll two dice.

A. Make a prediction. Which do you think is more likely, I or II? Explain your prediction.

I. Both numbers will be even.

II. The sum of the numbers will be 8.

B. Roll the dice 36 times to see what happens. Was your prediction correct?

- When you want to determine the probability that an event will or will not occur, you can either conduct an experiment many times so that you have some confidence in the result (which is called **experimental probability**) or you can determine the probability theoretically (which is called **theoretical probability**).

For example, consider the probability of NOT rolling a multiple of 3 with one die.

Experimental probability

Roll the die 30 times. Record the results.

1	2	3	4	5	6
/	𝍸 𝍸	𝍸	///	𝍸 //	////

The non-multiples of 3 are 1, 2, 4, and 5 and these were rolled 21 times out of 30.

The experimental probability is $\frac{21}{30} = \frac{7}{10}$.

Theoretical probability

Do a mathematical analysis by listing all the possible outcomes, the **sample space**, and then looking for those that are **favourable outcomes**, in this case, non-multiples of 3:

Sample space (all possible outcomes): 1, 2, 3, 4, 5, and 6 (6 altogether)

Favourable outcomes (non-multiples of 3): 1, 2, 4, and 5 (4 altogether)

There are 6 possible equally likely outcomes in the sample space and 4 of them are favourable.

The theoretical probability is $\frac{4}{6} = \frac{2}{3}$.

Notice the experimental and theoretical probabilities are close, but not identical.

- Sometimes events involve a combination of two outcomes. Again, you can determine experimental and theoretical probabilities and then compare them.

For example, consider the probability that the sum of two dice is 9.

Experimental probability

Roll a pair of dice 40 times and record the results.

Sum of 9	Sum not 9
5	35

The experimental probability is $\frac{5}{40} = \frac{1}{8}$.

Theoretical probability

Analyse the situation by making a chart to record all possible outcomes.

There are 36 equally likely outcomes in the sample space and the number of 9s in the chart (the favourable outcomes) is 4.

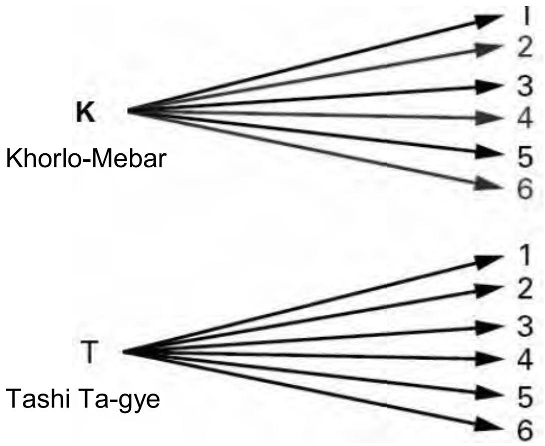
The theoretical probability is $\frac{4}{36} = \frac{1}{9}$.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Although a chart was used above to determine all possible outcomes, sometimes you might choose to use a list or a tree diagram.

For example, to determine the theoretical probability of tossing a coin and then rolling a die and getting the result Khorlo-Mebar (burning wheel) and an even number, you can create a tree diagram.

Tossing a Coin Rolling a Die List of Outcomes



(K, 1)
(K, 2) √
(K, 3)
(K, 4) √
(K, 5)
(K, 6) √

(T, 1)
(T, 2)
(T, 3)
(T, 4)
(T, 5)
(T, 6)

There are 12 equally likely outcomes and 3 of them are favourable (K, even).

P(K, even) = $\frac{3}{12} = \frac{1}{4}$.

C. i) What is the theoretical probability for each event described in **part A**?

ii) How did your experimental results from **part B** compare to the theoretical probabilities you determined in **part C i)**?

Examples

Example Determining Theoretical Probability

You have a deck of 100 number cards: 10 of each number 1 to 10. You shuffle the deck, pick a card, read its value, and then return it to the deck. You repeat this to draw a second card.

a) What is the theoretical probability that the second card you draw has a value that is greater than the value of the first card?

b) What is the theoretical probability that the product of the two values you draw is greater than 30?

[Continued]

Example Determining Theoretical Probability [Continued]

Solution 1

a) Value of Second Card Compared to First Card

value of second card drawn

value of first card drawn

	1	2	3	4	5	6	7	8	9	10
1	=	>	>	>	>	>	>	>	>	>
2	<	=	>	>	>	>	>	>	>	>
3	<	<	=	>	>	>	>	>	>	>
4	<	<	<	=	>	>	>	>	>	>
5	<	<	<	<	=	>	>	>	>	>
6	<	<	<	<	<	=	>	>	>	>
7	<	<	<	<	<	<	=	>	>	>
8	<	<	<	<	<	<	<	=	>	>
9	<	<	<	<	<	<	<	<	=	>
10	<	<	<	<	<	<	<	<	<	=

45 favourable outcomes out of 100 possible equally likely outcomes

$$P\text{ (2nd card > 1st card)} = \frac{45}{100} = \frac{9}{20}$$

b) Product of Both Cards is Greater Than 30

value of second card drawn

value of first card drawn

	1	2	3	4	5	6	7	8	9	10
1	X	X	X	X	X	X	X	X	X	X
2	X	X	X	X	X	X	X	X	X	X
3	X	X	X	X	X	X	X	X	X	X
4	X	X	X	X	X	X	X	√	√	√
5	X	X	X	X	X	X	√	√	√	√
6	X	X	X	X	X	√	√	√	√	√
7	X	X	X	X	√	√	√	√	√	√
8	X	X	X	√	√	√	√	√	√	√
9	X	X	X	√	√	√	√	√	√	√
10	X	X	X	√	√	√	√	√	√	√

39 favourable outcomes out of 100 possible equally likely outcomes

$$P\text{ (1st card} \times \text{2nd card} > 30) = \frac{39}{100}$$

Thinking

a) I made a 10-by-10 chart to show all the possible outcomes of drawing two cards.

- I included a > sign when the second card was greater than the first, a < sign when the second card was less than the first, and an = sign when the cards were equal.
- I analysed the chart to look for favourable outcomes—outcomes where the value of the second card was greater.

b) I made another 10-by-10 chart because I know there were 10 × 10 = 100 outcomes.

- I used a checkmark √ to show when the product was greater than 30 and an X when it wasn't.
- I analysed the chart to look for favourable outcomes — a product greater than 30.
- There were 100 possible outcomes and 39 of them were favourable.

Solution 2

a) Total possible equally likely outcomes: 10 × 10 = 100

Favourable Outcomes for Second Number Drawn to be Greater Than First Number

1st number	Favourable 2nd number(s)	
1	2 to 10	9 outcomes
2	3 to 10	8 outcomes
3	4 to 10	7 outcomes
4	5 to 10	6 outcomes
5	6 to 10	5 outcomes
6	7 to 10	4 outcomes
7	8 to 10	3 outcomes
8	9 or 10	2 outcomes
9	10	1 outcome
10	impossible	0 outcomes
Total favourable outcomes 45 outcomes		

45 favourable outcomes out of 100 possible outcomes

P (2nd card > 1st card) = $\frac{45}{100} = \frac{9}{20}$

b)

Favourable Outcomes for Second Number Drawn for Product to be Greater Than 30

1st number	Favourable 2nd numbers
4	8, 9, 10
5	7, 8, 9, 10
6	6, 7, 8, 9, 10
7	5, 6, 7, 8, 9, 10
8	4, 5, 6, 7, 8, 9, 10
9	4, 5, 6, 7, 8, 9, 10
10	4, 5, 6, 7, 8, 9, 10
Total favourable outcomes 45 outcomes	

P (1st card × 2nd card > 30) = $\frac{39}{100}$.

Thinking

a) I reasoned that there had to be 100 possible outcomes because there are 10 possible numbers in the first draw and for each of these there are 10 possible numbers you can draw in the second draw and 10 × 10 = 100.

- I then reasoned that, if I drew a 1 in the first draw that I'd have to draw a 2, 3, 4, 5, 6, 7, 8, 9, or 10 in the second draw for the second number to be bigger. I knew I could use this same reasoning for each first draw.
- I created an organized list to list all the favourable outcomes in the second draw for each first draw.

b) I used the same reasoning I did for **part a)**. For example, if I drew a 1, 2, or 3 in the first draw, a product greater than 30 would be impossible. If I drew a 4 in the first draw then I'd have to draw an 8, 9, or 10 in the second draw for the product to be greater than 30. I used this same reasoning for each first draw.

- I created an organized list to list all the favourable outcomes in the second draw for each first draw.

Practising and Applying

1. You roll two dice. Determine each theoretical probability.

a) sum is greater than 7

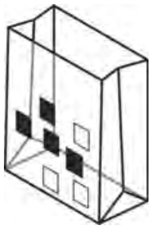
b) difference is less than 2

c) product is less than 10

d) quotient of the greater value divided by the lesser value is less than 2

2. Determine an experimental probability for each part of **question 1**. Roll the dice 30 times for each experiment. Compare your results to your results from **question 1**.

3. A bag contains 3 white tiles and 4 black tiles. You reach in and draw one tile, return it to the bag, and then draw another tile.



Determine each theoretical probability.

a) both tiles drawn are black

b) both tiles drawn are the same colour

c) neither tile drawn is black

d) tiles drawn are different colours

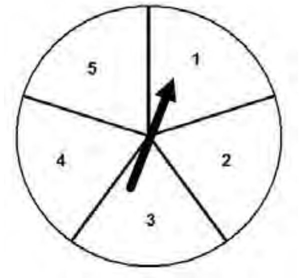
4. You have a deck of 40 number cards: five of each number 2 to 9. You shuffle the deck, pick a card, look at it, return it to the deck, and then repeat for a second card. Determine each theoretical probability.

a) sum is 12

b) difference is less than 5

c) product is odd

5. You spin this spinner twice.



Determine each theoretical probability.

a) both spins are odd

b) first spin is greater than the second

c) sum of the spins is greater than 8

6. Repeat each part of **question 5** but change the spinner so that the sections numbered 1 and 2 are combined into a single section numbered 2.

7. Copy and label the spinner below with the letters A and B so that the theoretical probability of spinning A and B, each once, in two spins is $\frac{12}{25}$.

8. You roll a die twice. The theoretical probability that outcome A happens twice is $\frac{1}{9}$. What could outcome A be? Explain.

9. Explain why charts, tree diagrams, lists are useful for determining theoretical probabilities.

4.2.2 Calculating Probability of Two Independent Events

Try This

Pema was rolling two dice. He knew the events were **independent events** because the outcome of one roll does not affect the outcome of the other.

He found that he rolled two even numbers about the same number of times as he rolled two odd numbers. He wondered about the theoretical probabilities of rolling two even numbers and two odd numbers. He created an outcome chart to show all the possible outcomes of rolling two dice.

	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

A. i) How many possible outcomes are there altogether?
ii) How many of the outcomes are (even, even)? How many are (odd, odd)?

B. What is the probability of rolling two even numbers? two odd numbers?

- You can determine the probability of two independent events using an outcome chart, such as Pema's above, or you can use a tree diagram.

For example, to find the probability of tossing a coin and getting Tashi Ta-gye (the Eight Auspicious Signs) facing up and rolling a die and getting a prime number, a tree diagram could be created as follows:

Step 1: Start with the coin toss. List the possible outcomes: K for Khorlo facing up and T for Tashi Ta-gye facing up.

Tossing a Coin

K

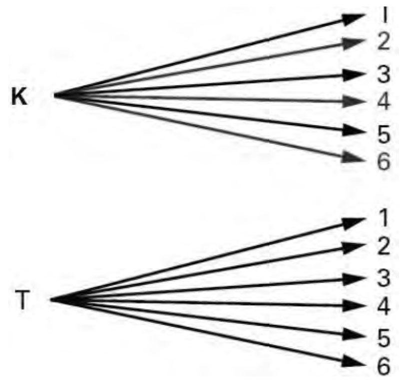
T

Note that you could have started with rolling the die instead—it does not matter, because they are independent events.

Step 2: Now include the possible outcomes of the second event, rolling the die. Draw arrows from each outcome of the first event to show the six possible outcomes of rolling a die. Label each arrow with its outcome: 1, 2, 3, 4, 5, or 6

Step 3: List the outcomes.

Tossing a Coin Rolling a Die List of Outcomes



(K, 1)
(K, 2)
(K, 3)
(K, 4)
(K, 5)
(K, 6)

(T, 1)
(T, 2)
(T, 3)
(T, 4)
(T, 5)
(T, 6)

There are 12 equally likely outcomes when you flip a coin and roll a die.

Step 4: To determine the probability of tossing Tashi Ta-gye and rolling a prime number, count the total number of possible outcomes and then count the number of outcomes that are Tashi Ta-gye and prime (T, prime).

Of 12 possible outcomes, 3 are Tashi Ta-gye and prime: (T, 2), (T, 3), and (T, 5).

$$P(T, \text{prime}) = \frac{3}{12}$$

- You can also determine the probability of two independent events by calculating. If two events are independent, you can multiply the probability of each event together to find the probability that both events will occur:

$$\text{Probability}(A, B) = \text{Probability}(A) \times \text{Probability}(B)$$

For example, to calculate the probability of Tashi Ta-gye and prime, determine the probability of each event and multiply:

$$P(T) = \frac{1}{2} \qquad P(\text{prime}) = \frac{3}{6}$$

$$P(T, \text{prime}) = P(T) \times P(\text{prime}) = \frac{1}{2} \times \frac{3}{6} = \frac{3}{12}$$

C. i) Suppose Pema had used a tree diagram to determine the probability of rolling two even numbers and the probability of rolling two odd numbers in **parts A and B**. What would his tree diagram have looked like?

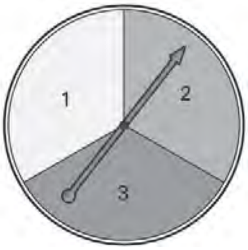
ii) If he had calculated the probabilities, what would his calculations have been?

D. How do the theoretical probabilities of (even, even) and (odd, odd) compare?

Examples

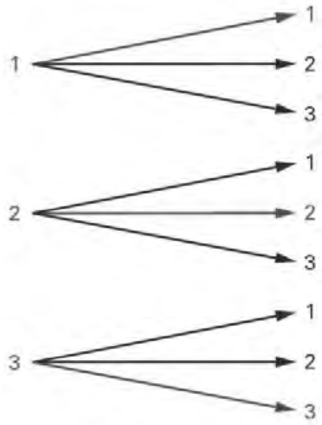
Example Determining Probability of Two Independent Events

Determine the probability of spinning a 1 and then a 3 in two consecutive spins.



Solution 1

First spin Second spin List of Outcomes



(1, 1)
(1, 2)
(1, 3)
(2, 1)
(2, 2)
(2, 3)
(3, 1)
(3, 2)
(3, 3)

1 of the 9 possible outcomes has a 1 on the first spin and a 3 on the second so

$$P(1, \text{then } 3) = \frac{1}{9}$$

Thinking

- I drew a tree diagram to list all the possible outcomes — there are 9 equally likely outcomes.
- From the diagram, I counted the number of outcomes that have a 1 on the first spin and a 3 on the second. There is only 1.
- I knew that if I had been looking for the probability of spinning a 3 and a 1 in either order — a 1 and then a 3 or a 3 and then a 1 — I would have considered both (3, 1) and (1, 3) and the probability would be $\frac{2}{9}$.

Solution 2

	1	2	3
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

There are 9 outcomes altogether and 1 of them is a 1 and then a 3 so

$P(1, \text{then } 3) = \frac{1}{9}$

Thinking

- I created an outcome chart and counted the total possible outcomes. Then I looked for outcomes that had a 1 and a 3 in that order.

Example Determining Probability of Two Independent Events [Continued]

Solution 3

$P(1) = \frac{1}{3}$ $P(3) = \frac{1}{3}$

$$P(1, \text{ then } 3) = P(1) \times P(3)$$
$$= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Thinking

- Each spin is independent of the other spin, so I knew that I could multiply the probabilities together.
- I knew the probability of spinning 1 was $\frac{1}{3}$ because there are 3 equal sections and one of them is 1. It's the same for spinning 3.

Practising and Applying

1. The spinner below is spun and the coin is tossed. Use an outcome chart to determine each probability.

a) P(4, K) **b)** P(even, T)

c) P(greater than 1, K) **d)** P(2 or 3, T)

2. a) Create a tree diagram to show the possible outcomes of tossing three coins.

b) What is the probability of getting exactly one Khorlo facing up?

c) What is the probability of getting exactly two Tashi Ta-gyes facing up?

d) What is the probability of not getting any Tashi Ta-gye facing up?

e) Create and solve your own problem about tossing three coins.

3. A bag contains 4 red balls and 5 blue balls. Determine the probability of drawing each of the following in two draws, assuming that the first ball is replaced each time.

a) a red ball, then a blue ball

b) a blue ball, then a blue ball

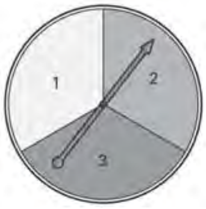
4. Jigme rolled a 12-sided die (with the numbers 1 to 12) twice. Calculate each probability.

a) two rolls of 8 in a row

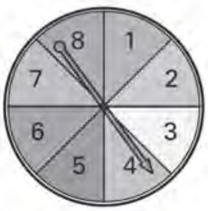
b) two odd numbers in a row

c) a number greater than 3 and then a number less than 6

5. Wangchuk plays two games with these spinners. In both cases, he spins each spinner twice.



Spinner for Game 1



Spinner for Game 2

Game 1: Using the first spinner, he scores 1 point if he does not spin a 1 in either spin.

Game 2: Using the second spinner, he scores 1 point if both his spins are less than 6.

Which game does he have a greater chance of winning? Explain.

6. A red die and a green die are rolled. Determine the probability of each.

a) the red die shows 6 and the green die shows 3

b) both dice show 1

c) the red die shows an even number and the green die shows an odd number

d) one die shows an even number and the other die shows an odd

e) the red die shows a 4 and the sum of the dice is greater than 7

7. In a deck of playing cards, there are four suits, with 13 cards in each suit.

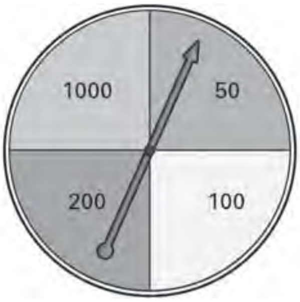
Calculate the probability of drawing each in two draws, if the first card is replaced.

a) the 9 of ♦ and then 10 of ♥

b) an ace and a 2 in any order

c) a club and then a heart

8. Singye plays a game by spinning this spinner twice. He wins if the sum is 1200. Use a tree diagram or outcome chart to determine the probability of Singye winning.



9. a) Design a spinner so that, when you toss a coin and spin the spinner, the probability of getting a Khorlo facing up and spinning a 6 is $\frac{1}{12}$.

b) Repeat **part a)** for P(K, 6) = $\frac{1}{20}$.

c) Repeat **part a)** for P(K, 6) = 0.

d) Repeat **part a)** for P(K, 6) = $\frac{1}{4}$.

10. You put 30 cards, numbered from 1 to 30, in a bag and draw two cards, one at a time, replacing the first one before drawing the second one. What is the probability of drawing two prime numbers in two draws?

11. A bag holds 3 red cubes and 2 blue ones. You draw a red cube and then a blue cube, without returning the red cube first. Why can you not determine the probability of these events by multiplying $\frac{3}{5} \times \frac{2}{5}$?

GAME: On a Roll

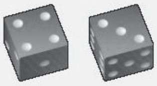
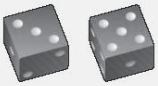
The goal of this game is to be the first player to make ten correct predictions.

- Each player creates a chart like the one shown below.
- Player 1 rolls two dice and calculates the sum of the numbers rolled.
- Each player then predicts whether the sum of the next roll will be greater than, less than, or equal to this sum and records the prediction in his or her chart.
- Player 2 then rolls the dice and calculates the sum. Each player scores 1 point for a correct prediction.
- Each player then predicts whether the sum of the next roll will be greater than, less than, or equal to this sum and records the prediction in his or her chart.
- Play continues in this way, with players taking turns rolling the dice and calculating the sum.
- The game continues until a player has 10 points.

For example, in the game below:

- After roll 1, Karma predicted that the sum of roll 2 would be greater than 7.
- After roll 2, he scored 1 point. His prediction was correct because the sum was 8. He then predicted that the sum of roll 3 would be equal to 8.
- After roll 3, he did not score because his prediction was incorrect. The sum was 2. He then predicted that the sum of roll 4 would be greater than 2.

Karma's Game Chart

Roll	Result of roll	Sum	My score	My prediction for next roll
1		7		greater
2		8	1	equal
3		2	1 + 0 = 1	greater

Play the game in groups of 2 to 4. Play it more than once.

Is there a strategy for winning? Explain.

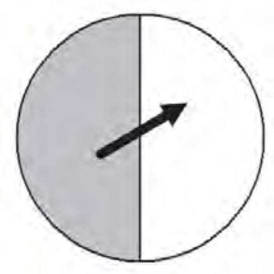
4.2.3 Randomness: Experimental Versus Theoretical Results

Try This

Maya conducted an experiment to determine the probability of spinning white and then grey in two spins of the spinner. She predicted she would spin white and then grey $\frac{1}{4}$, or 25% of the time.

Here are the results of her experiment.

1st spin	2nd spin	Frequency
White	Grey	10
Grey	White	3
White	White	3
Grey	Grey	4



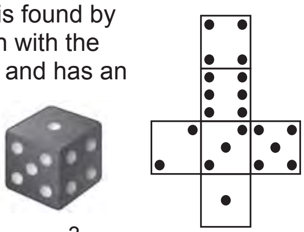
A. i) How do you think she determined her prediction of $\frac{1}{4}$, or 25%?

ii) In her experiment, did she spin white and then grey 25% of the time as predicted?

iii) Why do you think the she experimental results were different than expected?

- **Experimental probability** is determined by conducting an experiment or collecting data over a period of time. **Theoretical probability** is determined by a mathematical analysis of the situation. For example:

The theoretical probability of rolling an even number is found by comparing the number of faces on a die that are even with the total number of faces. This works because each face and has an equal chance of being rolled.



$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \qquad P(\text{even}) = \frac{3}{6} = 50\%$$

The experimental probability of rolling an even number is found by conducting an experiment where the die is rolled many times and the resulting data is used to determine the probability:

Event	Frequency
Even	58
Not even	42

$$P(\text{event}) = \frac{\text{number of times the event happens}}{\text{number of times the experiment is done}}$$

$$P(\text{even}) = \frac{58}{100} = 58\%$$

You will notice that, in the example above, the theoretical and experimental probabilities are different.

- The experimental and theoretical probabilities of the same event are not always the same. This is because of the element of chance, or **randomness**. However, the greater the number of trials, for example, by rolling a die or spinning a spinner more times, the more likely the experimental probability will approach the theoretical probability.

- Theoretical probability cannot be determined for some situations, like the probability of tossing a thumb tack and it landing point up or predicting the weather. In these situations, probability is determined by conducting experiments with many trials or by collecting a lot of data over time. That way, the probability is as close to theoretical as possible.

B. i) Explain why Maya's prediction based on theoretical probability and the results of her experiment were different in **part A**.

ii) What could she do to get experimental results that might more closely match her prediction?

Examples

Example Comparing Experimental and Theoretical Probability

Conduct a probability experiment with a deck of cards to determine the probability of drawing a club, returning it, and then drawing a red card.

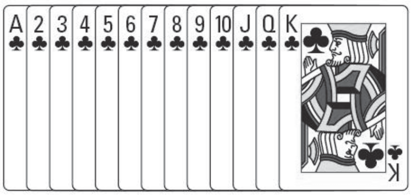
a) What is a reasonable prediction? Show your work.

b) Conduct an experiment to determine an experimental probability.

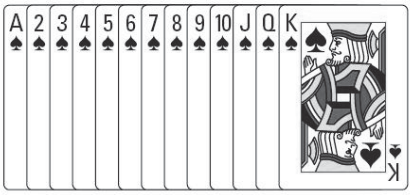
c) Did your experimental results match your prediction? Explain.

Here is a deck of 52 cards:

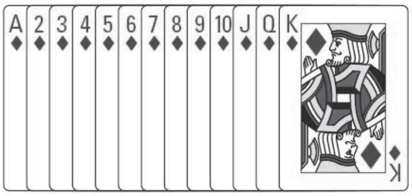
The suit of clubs (13 black cards)



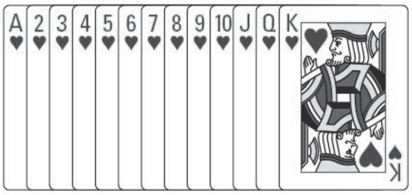
The suit of spades (13 black cards)



The suit of diamonds (13 red cards)



The suit of hearts (13 red cards)



Solution

a) P(club) = $\frac{1}{4}$, P(red) = $\frac{1}{2}$

P(club, red) = $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

I predict a club and then a red card $\frac{1}{8}$ or 12.5% of the time.

12.5% of 50 is 6.25 so I predict it will happen about 6 times in my experiment.

b) Experimental Results

Draw	Frequency
Club, red	4
Not club, red	46

P(club, red) = $\frac{4}{50}$ = 8%

c) My results of 4 for (club, red) did not match my prediction of 6.

Combined Experimental Results

Draw	Frequency
Club, red	26
Not club, red	174

P(club, red) = $\frac{26}{200}$ = 13%

Thinking

a) I used the theoretical probability to predict. I multiplied the probability of each event because the events are independent.

- In each deck of cards, there are 4 suits: clubs, spades, hearts, and diamonds, with an equal number of cards in each, so clubs make up $\frac{1}{4}$ of the deck. Two suits are red, so $\frac{1}{2}$ the deck is red.
- I decided to draw two cards 50 times in my experiment, so I used the theoretical probability to predict how many times I could expect to draw a club and then a red card.

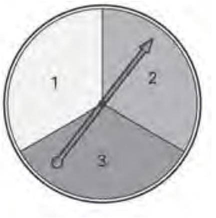
b) I drew a club and then a red card 4 out of 50 times in my experiment.

c) My results did not match my prediction because of the element of chance, which is greater in a small number of trials like 50.

- I decided to combine my data with some classmates' to increase the number of trials.
- The experimental probability of 13% from the greater trial size was closer to the theoretical probability of 12.5%.

Practising and Applying

1. Aum Yangki conducted an experiment to determine the experimental probability of spinning an odd number both times in a pair of spins using this spinner.



She spun the spinner 10 times. Here are her results.

	1	2	3	4	5
Spin 1	1	3	2	2 1	
Spin 2	2	1	2	1 3	

	6	7	8	9	10
Spin 1	1	2	3	1 3	
Spin 2	2	3	1	3 2	

[Continued]

1. [Cont'd] **a)** Calculate the theoretical probability of spinning (odd, odd) as a fraction and as a percentage.

b) Use the theoretical probability to predict the number of times she would expect to spin (odd, odd) in her experiment.

c) How does her experimental probability compare with the theoretical probability?

d) Can she expect this to happen every time she conducts the same experiment exactly the same way? Explain.

2. Samten calculated the theoretical probability of rolling two even numbers on two dice to be $\frac{1}{4}$ or 25%. He then conducted an experiment. Here are his results.

	1	**2**	**3**	**4**	**5**
Die 1	2	4 6		2	3
Die 2	5	1 6		4	3

	6	**7**	**8**	**9**	**10**
Die 1	6	2 5		3	1
Die 2	4	2 3		4	1

In his experiment, he rolled two even numbers 4 times out of 10, so his experimental probability was $\frac{2}{5}$ or 40%.

Which of the following is true? Justify your decision.

A. The theoretical probability was calculated incorrectly.

B. The recording of experimental results was done incorrectly.

C. Both probabilities are correct.

D. The experimental probability might have been closer to the theoretical probability if he had rolled the dice 100 times instead of 10 times.

3. Work in a group of three or four classmates. Together, design a probability experiment with coins or cards.

a) Decide on two independent events. For example:

- toss two coins and both coins land with Khorlo facing up

- draw a club, replace the card, and then draw a 6 from a deck of cards

b) Decide on the number of trials. For example, roll two dice 25 times.

c) Calculate the theoretical probability and then use it to make a prediction.

d) Have each group member conduct his or her own experiment and then compare all the individual experimental results with the prediction. What do you notice?

e) Combine the results of everyone in the group and compare the combined experimental results with the prediction. What do you notice?

4. Why is it sometimes a good idea to express theoretical and experimental probabilities as decimals or percents instead of as fractions?

5. Even though theoretical probability and experimental probability are not always the same, it is still best to use theoretical probability to make predictions. Explain.

4.2.4 Conducting a Simulation

Try This

Rinzin has kept track of his archery scores over the past year for 100 matches. He wants to use the data to make predictions about his future scores.

Score	Number of times
Under 10	23
10 to 15	52
Over 15	25

A. About what fraction of the time does he score each?

i) under 10 **ii)** from 10 to 15 **iii)** over 15

A **simulation** is a probability experiment that models an actual event using probability devices such as spinners, coins, cards, and dice. Simulations are used to make predictions about events that are impossible or difficult to do directly.

A simulation uses a probability device to represent the theoretical probability in order to determine an experimental probability for a situation. For example:

There are three runners competing in a race: Runners A, B, and C. Their performance against each other over a few years shows that Runner A wins about half the time, Runner B wins about one third of the time, and Runner C wins about one sixth of the time. This data about past performance is treated as "theoretical" probability, since it is based on data collected over a long period of time. The following simulation will predict the probability of Runner A winning the next race.
`

Step 1: Choose a device to model the probability:

A die would be suitable because its theoretical probabilities match the theoretical probabilities of the runners winning:

Each roll of the die represents the outcome of a race:
- Runner A winning can be modelled with a roll of 1, 2, or 3.
- Runner B winning can be modelled with a roll of 4 or 5.
- Runner C winning can be modelled with a roll of 6.

Other possible devices that could be used instead include:

- spinner in sixths: 3 sixths labelled A, 2 sixths labelled B, and 1 sixth labelled C
- bag with 6 cubes: 3 red cubes for A, 2 blue cubes for B, and 1 green cube for C

Step 2: Decide how many "races" or rolls of the die you will perform.

You might decide to roll the die 25 times so that there is enough data to determine a reasonably reliable experimental probability.

Step 3: Decide how you are going to collect and organize the data.

Step 4: Conduct the simulation.

Roll	Number of times	Frequency
1, 2, or 3	~~////~~ ~~////~~ ///	13
4 or 5	~~////~~ ///	8
6	////	4

You could use a chart with a row for each number rolled but you would have to add the rolls of 1, 2, and 3 and the rolls of 4 and 5.

Step 5: Use the data to determine the experimental probability.

To find the experimental probability of an event E, use this ratio:

$$P(E) = \frac{\text{number of times the event E happens}}{\text{number of times the experiment is done}}$$

$$P(A) = \frac{13}{25} \text{ or } 52\%$$

The probability that Runner A will win the next race is 52%.

B. Look back at Rinzin's archery data in **part A**. He wants to predict his chances of scoring over 15 in his next archery match by using a simulation. What probability device could he use? How would he use the device?

Examples

Example 1 Conducting a Simulation Using Trials

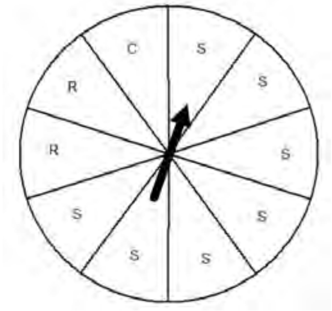
A meteorologist has collected weather data for a community in Thailand. The data set shows that it is sunny 70% of the time, rainy 20% of the time, and cloudy 10% of the time. This set of data can be treated as theoretical probability because it has been collected over a long period of time.

Conduct a simulation to predict the probability that the next five days will be sunny. Describe your simulation.

Solution

Step 1: Choose a device.

A spinner with 10 equal sections: 7 sections for sunny (S), 2 for rainy (R), and 1 for cloudy (C).



Step 2: Decide how many "periods of five days" or trials to use.

10 five-day periods should be enough. Each period is called a trial.

Steps 3 and 4: Decide on how to record the data. Then conduct the simulation.

	Sunny	Rainy	Cloudy
Trial 1	///		//
Trial 2	/	//	//
Trial 3	////	/	
Trial 4	//	//	/
Trial 5	$\boxed{\cancel{////}}$		
Trial 6	///	//	
Trial 7	//	/	//
Trial 8	////	/	
Trial 9	///	//	
Trial 10	//	/	//

Step 4: Determine the experimental probability.

$$P(SSSSS) = \frac{\text{\# of times all 5 spins were S}}{\text{total number of trials}}$$

$$= \frac{1}{10} \text{ or } 10\%$$

The probability that the next five days will be sunny is 10%.

Thinking

- I decided each spin would represent the weather for one day. Five spins in a row represent a period of five consecutive days.

- To simulate multiple events like this one, where I need to predict the weather for five days in a row and not just one day, I knew I would have to use trials, each trial representing five days in a row.

- You can't tell from the chart the order of the weather on the five days. For example, if I had wanted to predict the probability of three sunny days followed by two cloudy days, I would have had to collect the data differently.

- There was only 1 trial out of 10 where it was sunny five days in a row (S was spun five times in a row).

Example 2 Comparing Simulation Results to Theoretical Probability

Karma wants to know the probability that a family with three children will have two girls and one boy. The theoretical probability of a girl or boy being born is considered to be 0.5. This is based on data collected over a long period of time.

a) Conduct a simulation to determine an experimental probability of two girls and one boy being born in a family of three children.

b) Compare the experimental probability and the theoretical probability.

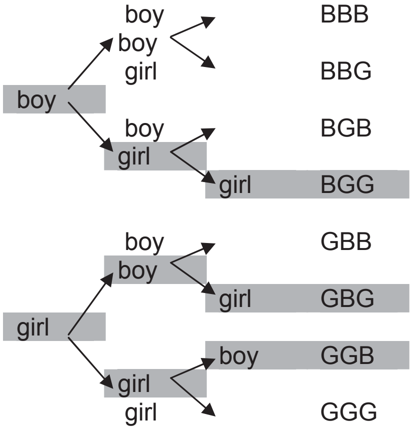
Solution

a) These are the results of my simulation:

Trial	**1**	**2 3**	**4 5**	**6 7**				**8**	**9 1**	**0**
Toss 1	B	G	G	B	B	B	B	B	B	G
Toss 2	G	B	B	G	B	G	B	G	G	B
Toss 3	B	B	G	B	G	G	B	B	G	G

A family of two girls and one boy would be represented by two Ks and one T, in any order. There were 4 trials out of 10 that resulted in two Ks and one T, so the experimental probability is $\frac{4}{10}$ or 0.4.

b) I used a tree diagram to determine the theoretical probability.



There are 8 possible outcomes and 3 of them are two girls and one boy so the theoretical probability is $\frac{3}{8}$ or 0.375.

The experimental and theoretical probabilities are not the same but they are close: $\frac{4}{10}$ vs. $\frac{3}{8}$

Thinking

a) I used a coin to represent the birth of each child — tossing Khorlo was a girl; tossing Tashi Ta-gye was a boy.

- I conducted ten trials. Each trial was three coin tosses representing three children being born in a family.

b) The tree diagram showed three outcomes with two girls and one boy: BGG, GBG, and GGB.

- I realized that $\frac{4}{10}$ and $\frac{3}{8}$ are as close as possible — with 10 trials, 4 out of 10 is as close as you can get to 3 out of 8. If I had done trials in a multiple of 8, it would have been possible for the probabilities to be the same. But even then, they could still be different because of the element of chance.

Practising and Applying

1. Which of these probability devices shown below could be used to conduct a simulation for each experimental probability described below? Explain.

a) The probability of three days of rain in a row, if there is a 50% chance of rain each day

b) The probability of drawing your name three times in a row from a box that contains eight students' names, including yours

c) The probability that Nado wins two of the next three games of Khuru against Leki, if Leki has a 1 in 6, or $\frac{1}{6}$ chance of beating Nado

d) The probability of scoring against a goalie if past performance shows a probability of scoring 25% of the time

2. Dawa had to complete a true/false quiz of four questions. He wondered about the probability of getting all four answers correct by guessing. He used a coin to conduct a simulation — Khorlo meant he got the question correct and Tashi Ta-gye meant he got it wrong. His results are shown below:

Trial number

1	2	3	4	5	6	7	8	9	10
K	K T		T	K	K	T	K	K	T
T	K	K	T	T	T T		T	K	K
K	T	T	T	T	K	T	K	K	T
T	T K		T	K	T	T	T	K	K

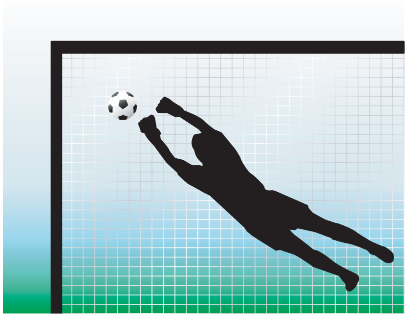
a) What is the experimental probability of getting all four questions correct?

b) Determine the theoretical probability. Compare the experimental and theoretical probabilities.

c) Why is it impossible for the theoretical and experimental probabilities to be equal in this situation, even if Dawa conducted thousands of trials?

3. Yan has a 1 in 3, or $\frac{1}{3}$ chance of beating Ping in a game of table tennis. Describe a simulation with a die that could be used to determine the experimental probability that Yan will win three of the next five games that he plays against Ping.

4. Rinzin plays on a football team. He scores on a penalty kick about 75% of the time. Describe a simulation you could use to determine the probability that he scores at least four times on his next five penalty kicks.



5. How do simulations combine ideas about experimental probability with theoretical probability?

4.2.5 EXPLORE: Designing a Simulation

Roshan and Yeshey are travelling from Paro to Thimphu in the morning. As they approach the airport, a Druk Air jet is about to land. They know they will likely be stopped by a guard and will have to wait until the plane lands before proceeding.

From experience they know the stop will take less than 5 min about 20% of the time, between 5 and 10 min about 50% of the time, and more than 10 minutes about 30% of the time.

The road beside Paro airport

A. Design a simulation to determine each experimental probability:

- that the stop will be less than 5 min
- that the stop will be between 5 min and 10 min
- that the stop will be more than 10 min

Step 1: Choose a probability device that models the situation.

Step 2: Decide how many trials you will need to do for meaningful results.

Step 3: Decide how you are going to record your experimental results.

Step 4: Conduct the simulation.

Step 5: Determine the experimental probabilities.

B. Compare your results with several other students' to determine a range of probabilities for each.

C. How long do you think Roshan and Yeshey will be stopped by the guard? Explain your thinking.

CONNECTIONS: Computer Simulations

Himalayan melting risk surveyed by Navin Singh, BBC News, Kathmandu

A new weather station has been installed on the longest Himalayan glacier in the Everest region of Nepal. It is designed to measure and collect data about the extent of warming in the Himalayas, one of the world's biggest deposits of ice and a key source of fresh water. There have been numerous reports of glacial retreats in the Himalayas over the years due to global warming. This is having an impact on the locate climate and the level of water in the rivers. This weather station is able to measure those changes.

Mount Jhomolhari in the Himalayas (Bhutan)

Data from weather stations such as this are used to design computer simulations that will help predict what will happen to the local climate and rivers if global warming continues.

These simulations suggest that spring flow in the rivers will increase over the next five decades. However, the time will come when there will be so few glaciers and so little snow in the Himalayas that the rivers could run dry in the dry season. "In some rivers, the flow may go down by as much as 90%," said hydrologist Syed Iqbal Hosnain, of the University of Calcutta, India, who modelled what would happen in snow-fed regional rivers.

1. Why do scientists conduct computer simulations?

2. Research to find out about other types of computer simulations.

UNIT 4 Revision

1. Create an appropriate graph for each.

a) Causes of Air Pollution

Cause	Percent
Industry	15
Vehicles	44
People	9
Fuel	20
Other	12

b) Blood Type

Blood type	O	A	AB	B
Number of people	661	616	53	121

c) Amount of Garbage Produced per Person

Year	Amount (kg)
1960	450
1970	540
1980	600
1990	700
2000	620

d) Distance Walked to School (km)

1.1	2.5	3.0	2.7	0.5
1.6	2.0	3.2	5.0	0.8
1.1	2.4	3.6	0.2	1.6
4.3	3.6	2.7	1.6	0.3

e) Height vs. Age of a Sample of Trees

Age	1	2	1	3	3
Height (cm)	32	45	28	57	50

Age	4	5	5	7	8
Height (cm)	60	66	72	95	99

2. Sonam claims that a broken line graph would be a good choice to represent this set of data. Do you agree or disagree? Explain.

Population of Bhutan by Age Group

Age group	Persons	Percent
0 – 14	209,959	33.1
15 – 64	395,278	62.3
65 +	29,745	4.7

3. a) Explain why a circle graph would be inappropriate for this set of data.

Country	Vehicles per 1000 people
USA	767
Australia	605
Italy	591
New Zealand	579
Canada	560
Japan	560
France	530
Germany	522

b) What type of graph could be used?

4. Construct a multiple broken line graph for this data.

Average Monthly Maximum and Minimum Temperatures in Bumthang (°C)

	Jan	Feb	Mar	Apr
Max	10.8	10.0	16.2	18.7
Min	–5.1	–1.4	3.5	3.9

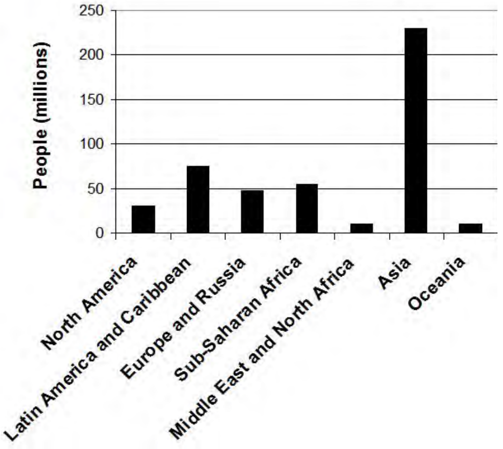
	May	Jun	Jul	Aug
Max	21.3	22.5	24.1	23.0
Min	9.5	13.5	10.9	13.7

	Sep	Oct	Nov	Dec
Max	21.6	19.5	16.1	12.3
Min	12.1	5.9	–0.5	–2.3

5. List three reasons you might draw an incorrect conclusion from a graph.

6. From the following graph, Dodo claims that North America has very few acres of forest ecosystem remaining. Is his claim valid? Explain.

People Living in Forest Ecosystems



7. a) Thinley rolled a die 5 times and each time it came up 6. The probability that he gets a 6 on the next roll is

A. less than $\frac{1}{6}$

B. greater than $\frac{1}{6}$

C. exactly $\frac{1}{6}$

b) Explain your choice in **part a)**.

8. How much greater is the probability of rolling a sum of 7 on two dice than rolling a sum of 10?

9. Pema drew a card from a deck of cards and tossed a coin. Determine the probability of each.

a) drawing a club and tossing Khorlo

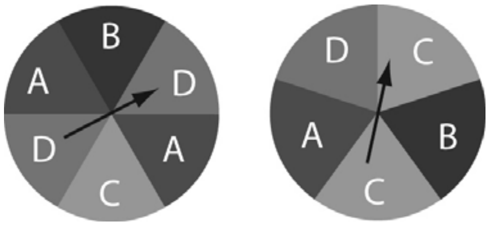
b) drawing a red card and tossing Tashi Ta-gye

c) drawing a 10 card and tossing Tashi Ta-gye

d) drawing the ace of spades and tossing Khorlo

10. Novin drew a card from a deck of cards, returned it to the pile, and then drew another card. What is the probability he drew two nines?

11. A game is played by spinning each spinner once. To win a prize the two spinners must stop on the same letter. Determine the probability of both spinners stopping on the same letter.



12. A bag contains 4 green, 5 red, and 6 blue balls. A ball is drawn, returned to the bag, and another ball is drawn. What is the probability of each?

a) drawing a red ball after drawing a green one

b) drawing a green ball after drawing a blue one

c) drawing a blue ball after drawing a blue one

d) drawing a red ball after drawing a red one

13. As the number of trials increases, how is the experimental probability of an event likely to relate to its theoretical probability? Explain.

A. The difference becomes greater.

B. The difference stays the same.

C. The difference becomes less.

14. When you conduct a simulation, is it possible that your experimental probability could be 0 even though the theoretical probability is $\frac{2}{5}$? Explain.

15. a) Design a simulation using a coin to determine the experimental probability that the next four people to walk in the door will be female. Assume that a woman is equally likely to walk in as a man.

b) Determine the theoretical probability.

16. A “best of seven game series” is a sequence of seven games where the first person to win four games is the winner. Lemo and Maya are equally skilled at playing a game.

Design a simulation using a die to determine the probability that a best of seven game series will end after exactly four games. That is, Lemo or Maya will win four games in a row in the first four games.

17. Nezarine wins one out of six races she runs. She has run enough times that this can be thought of as a theoretical probability.

a) Conduct a simulation to determine the probability that she will win the next two races she enters.

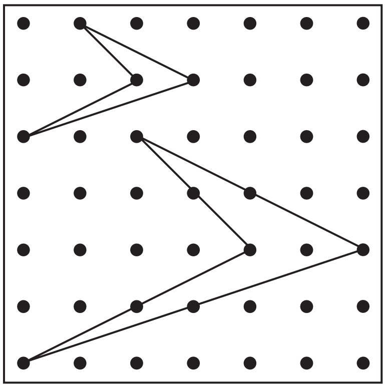
b) Assuming that her chance of winning a race is independent of whether she won or lost the previous race, calculate the probability that she will win the next two races she enters.

c) Compare the probabilities you found in **parts a) and b)** and comment on any similarities and differences.

UNIT 5 GEOMETRY

Getting Started

Use What You Know



A. i) Make a polygon on a geoboard. You could instead use square dot or grid paper. Your polygon should fit in one fourth of the total area you have to work with.

ii) Make the same shape again and double the length of each side.

iii) Describe the strategy you used to double the side lengths in your second shape.

B. Look at your two polygons.

i) Measure and compare the angles in the polygons. What do you notice?

ii) Look for parallel side lengths. What do you notice?

iii) Compare your results with a classmate's.

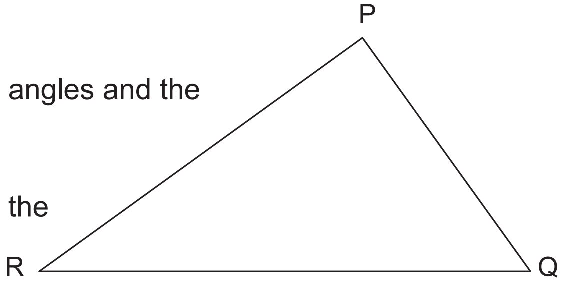
Skills You Will Need

1. a) Use a ruler and compass to copy ∆RPQ.

b) Use a protractor and a ruler to measure the angles and the lengths of the sides.

c) What is the sum of the angles?

d) Use the Pythagorean theorem to show how the side lengths are related.

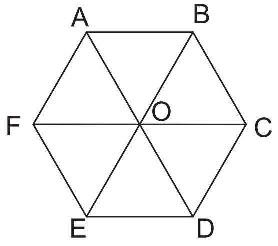


2. Find x in each equation. **a)** $\frac{9}{12} = \frac{x}{8}$ **b)** $\frac{12}{x} = \frac{30}{75}$ **c)** $\frac{x}{3.4} = \frac{11.3}{8.5}$

3. a) Name all sets of parallel line segments in this diagram.

b) Describe the symmetry in the diagram.

c) Explain how you know that $\angle AOB = \angle EOD$, $\angle OAB = \angle AOF$, $\angle BOC = \angle BED$, and $\angle COE + \angle DEO = 180^\circ$.



4. a) Plot the points A(2, 3), B(1, –2), C(–1, –2), and D(–6, 3) on a coordinate grid.

b) Connect the points and identify the quadrilateral.

c) Locate the smallest angle and measure it.

5. What rule does Lhamo seem to be using with this chart?

You say	3	10	16
Lhamo responds	–5	2	8

Chapter 1 Similarity and Congruence

5.1.1 EXPLORE: Unique Triangles

Dorji describes his triangular paddy field in a phone conversation with a friend. He wants to describe the shape of the field so that only one possible triangle fits his description.

He says the following:

- Two sides are 20 m long.
- One side is about 28 m long.
- The angle across from the long side is 90º.
- The two smaller angles are both 45º.

When there is only one possible shape that fits a description, the shape is called **unique**.

A. Dorji has given more information than is necessary to describe a unique triangle. Explain why Dorji only needed to make the first and last statements.

B. Draw a new triangle. Label its vertices A, B, and C. Measure and record two of its side lengths. Now try to draw another triangle, ∆DEF, that is different from ∆ABC but whose two side lengths are the same as BC.

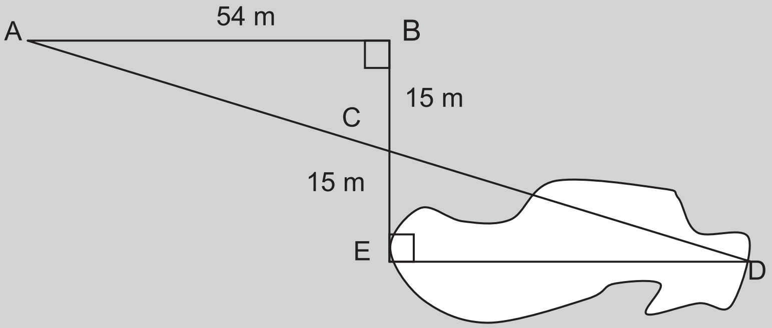
C. Repeat **part B** for each possibility listed below — draw a triangle and then measure and record only the information indicated. Then try to draw a different triangle that has the same measurements you recorded. Which possibilities below result in a unique triangle?

i) two angles
ii) three angles
iii) two sides and the angle between them
iv) two sides and an angle not between them
v) two angles and the side between them
vi) two angles and a side not between them
vii) three sides

5.1.2 Congruent Triangles

Try This

Seldon and Nima need to find the distance across a small lake (ED). They drew this sketch to help them plan a way of measuring the distance indirectly.

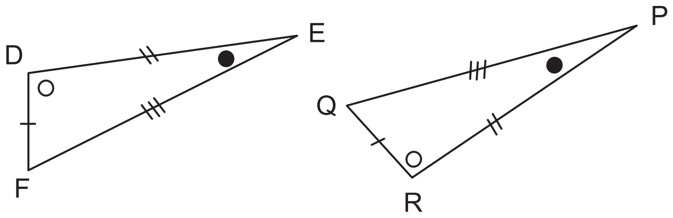


A. i) What information about ΔCDE in their sketch are they trying to find out?

ii) What information do they already know about ΔCDE and ΔCAB?

B. Is there enough information to know that ΔCDE and ΔCAB are the same shape and size? Explain.

When triangles are the same size and shape they are **congruent**.
The symbol ≅ is used to represent congruence. For example:



ΔDEF ≅ ΔRPQ because ∠D = ∠R, ∠E = ∠P, and ∠F = ∠Q
DE = RP, EF = PQ, and DF = RQ.

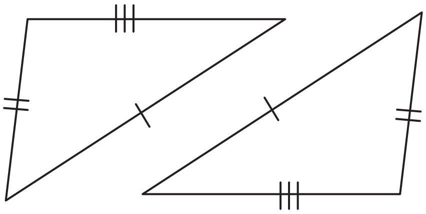
- Because the sum of the angles in any triangle is 180º, you only need to know two angles to figure out the third angle. So ∠F must equal ∠Q in the above triangles.

- When you say triangles are congruent, it is important that you put the vertices in the corresponding order. In the two triangles above, ∠D = ∠R so D and R are written in the same position (first) in the congruency statement Δ**D**EF ≅ Δ**R**PQ.

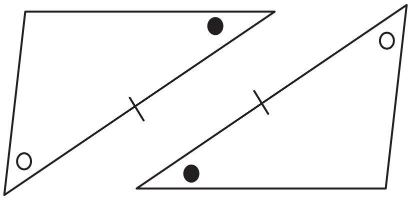
In the two congruent triangles on the previous page, ∆DEF and ∆RPQ, all angle measurements and all side lengths are known to be equal. However, it is possible to know if two triangles are congruent with less information.

There are four ways to know that any two triangles are congruent:

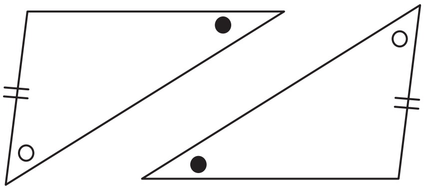
SSS (Side-Side-Side): If the lengths of the three sides of one triangle match the lengths of the three sides of the other, the triangles are congruent. You do not need to check the angles.



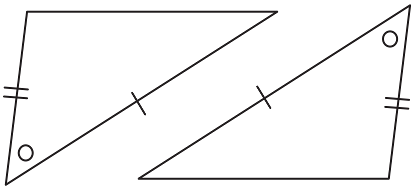
ASA (Angle-Side-Angle): If two angles and the length of the side contained by them of one triangle match two angles and the length of the side contained by them on the other triangle, the triangles are congruent. You do not need to check the other angle or sides. A side is contained by two angles when it is between the angles.



AAS (Angle-Angle-Side): If two angles and the length of a side not contained by them match two angles and the length of a corresponding side not contained by them on the other triangle, the triangles are congruent. You do not need to check the other angle or sides.



SAS (Side-Angle-Side): If two side lengths and the angle contained by them on one triangle match two side lengths and the angle contained by them on the other triangle, the triangles are congruent. You do not have to check the other angles or side.



Knowing three angles (AAA) or knowing two sides and an angle not contained by those sides (SSA) is not enough information to know if two triangles are congruent because those conditions do not result in a unique triangle.

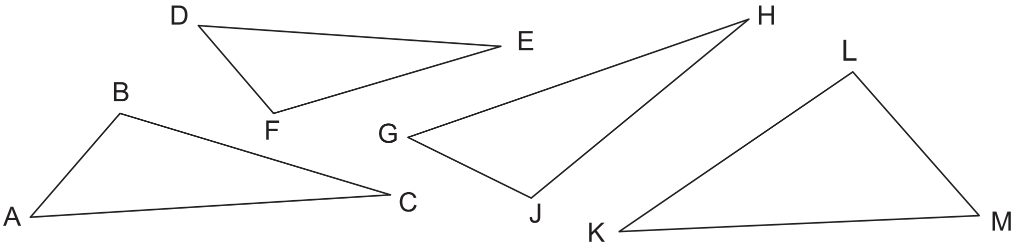
C. Which set of conditions for congruence, SSS, ASA, AAS, or SAS, tells Seldon and Nima that ΔCAB ≅ ΔCDE in **part B**?

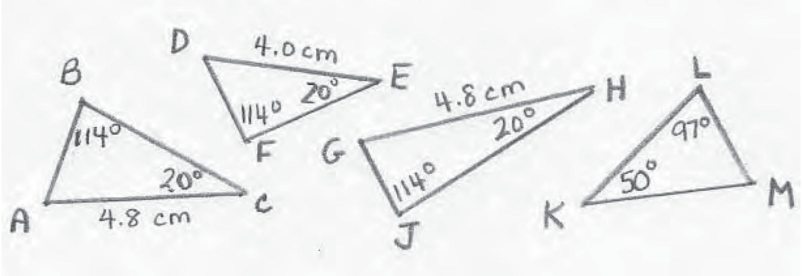
D. What is the distance across the lake? How do you know?

Examples

Example Establishing Congruence

Which of these triangles are congruent? Which are not?
How do you know?

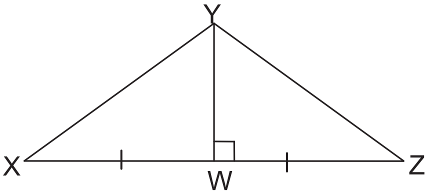


Solution	**Thinking**
ΔABC ≅ ΔGJH ∠B = ∠J = 114º ∠C = ∠H = 20º AC = GH = 4.8 cm I used AAS.	• I knew I didn't have to measure all the angles and sides so I measured the largest and smallest angles in each because they're easy to identify. 
ΔDFE is the same shape as both ΔABC and ΔGJH because they all have the same angle measurements, but ΔDFE is smaller, so it is not congruent.	• Three triangles had the same angles: ΔABC, ΔDFE, and ΔGHJ. To use AAS or ASA, I only had to measure one side in each, so I chose to measure the longest side in each. • I sketched the triangles to record my measurements. 
ΔLKM is neither the same shape nor the same size as ΔABC, ΔGJH, or ΔDFE, so it is not congruent.	• I noticed that the angles in ΔKLM didn't match the others, so ΔKLM can't be congruent to any others. • I changed the name of ΔGHJ to ΔGJH to match ΔABC when I wrote my congruency statement, ΔABC ≅ ΔGJH (but I could have changed ΔABC to ΔACB instead and written ΔACB ≅ ΔGHJ).

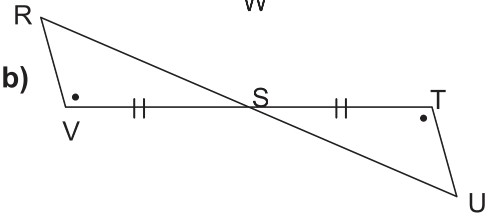
Practising and Applying

1. For each pair of triangles, indicate which conditions for congruence tell you they are congruent and write a congruency statement.

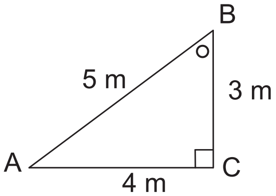
a)

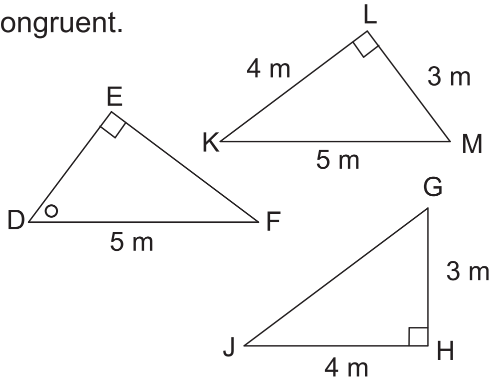


b)



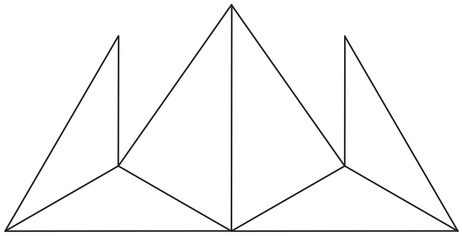
2. Consider ΔABC. For each triangle below, write a statement saying it is congruent to ΔABC. Tell how you know it is congruent.





3. For two right triangles, you only need to show that the hypotenuses and one other side in each triangle are equal to know they are congruent. Explain why.

4. Find the congruent triangles in this diagram. Explain how you know.

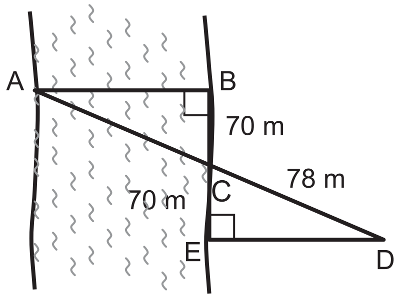


5. Dema uses congruent triangles to find the width of a river.

a) What conditions can she use to be sure the triangles are congruent?

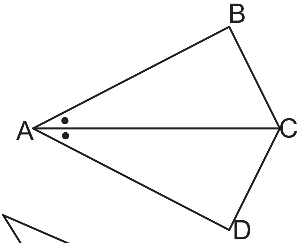
b) What is the length of AC?

c) Use the Pythagorean theorem to find the width of the river, AB.

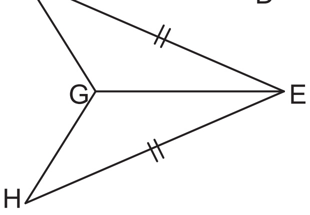


6. To establish that each pair of triangles is congruent, what further information do you need to know? Explain why. (List all the possibilities.)

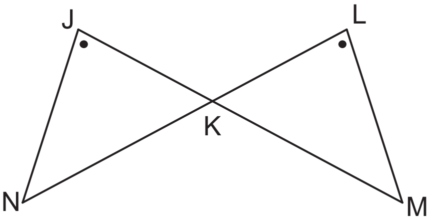
a)



b)



c)



7. Dechen measures every angle and side length of two triangles to be certain they are congruent. What advice would you give her?

5.1.3 Similar Triangles

Try This

David will be visiting Bhutan soon so he is consulting a map of the country (at right). The scale tells him that 1 cm on the map represents an actual distance of 50 km (or 5,000,000 cm). David wants to visit Thimphu, Phuntsholing, and Trashigang. He draws line segments between the places to represent the distance between them "as the crow flies." These line segments form a triangle.

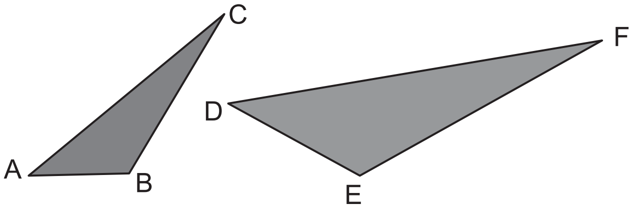
A. i) Measure the sides of the triangle on David's map.

ii) Use your measurements from **part i)** to estimate the actual distances between the three places. Explain how you estimated.

When corresponding side lengths in two triangles have the same ratio, the shapes are **similar**. The symbol ~ is used to show similarity. When two triangles are similar, their corresponding angles are equal. For example, for the two triangles below:

$$\Delta ABC \sim \Delta DEF \text{ because } \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

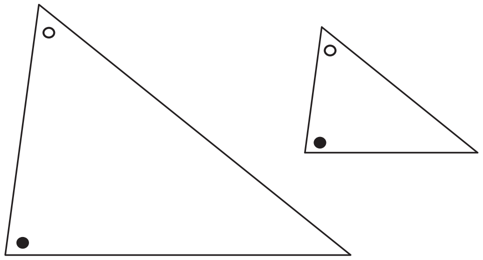
Therefore, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.



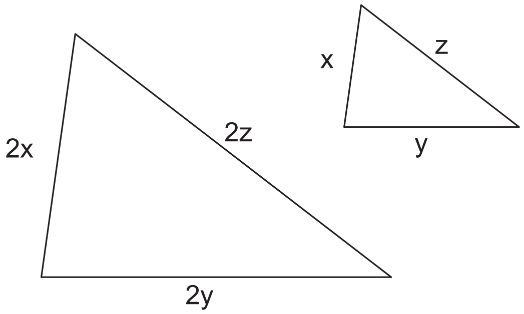
- The ratio between all corresponding sides of similar triangles is called the **scale factor**.
- Because the ratios of each pair of corresponding sides are equal, the ratios of the triangles' perimeters are also equal: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{\text{perimeter of } \Delta ABC}{\text{perimeter of } \Delta DEF}$
- Congruent triangles are similar triangles with a scale factor of 1.
- As with congruent triangles, when you name similar triangles using the vertex labels, it is important to keep the order of the corresponding vertices in mind. In the example above, $\angle A$ in ΔABC = $\angle D$ in the second triangle, $\angle B = \angle E$, and $\angle C = \angle F$, so the second triangle should be named ΔDEF when writing the similarity statement ΔABC ~ ΔDEF.

In the two similar triangles on the previous page, ∆ABC and ∆DEF, corresponding side lengths have the same ratio and all angle measurements are equal. However, it is possible to determine if two triangles are similar with less information. Any of the following sets of conditions establishes that two triangles are similar:

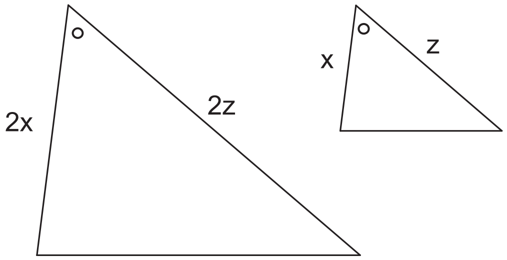
AAA (Angle-Angle-Angle): Corresponding angles are equal. You only need to measure two angles because the third angle can be calculated by subtracting the two known angles from 180º.



SSS (Side-Side-Side): All pairs of corresponding sides have the same ratio.



SAS (Side-Angle-Side): Two pairs of corresponding sides have the same ratio, and the angle contained by these sides is the same in both triangles.



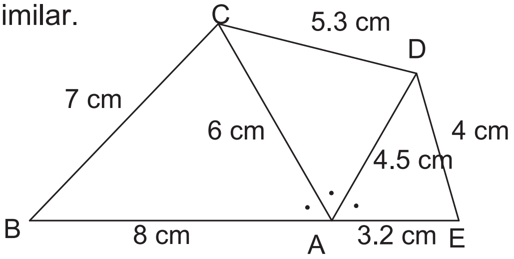
B. i) What is the scale factor that shows how each line segment David drew on the map in **part A** is related to the actual distance?

ii) How do you know that David's triangle is similar to the imaginary triangle that represents the actual distances between the three communities?

Examples

Example Identifying Similar Triangles

Identify at least two similar triangles in this diagram.
Explain how you know they are similar.



Solution 1

∆ADE ~ ∆ACD because all the corresponding angles are equal (AAA).

Thinking

- I worked with two classmates. We started with ∆ADE and ∆ACD because they looked similar.
- We already knew from the diagram that the angles at A were equal.
- Pema suggested that we only measure one more angle in each triangle. If two corresponding angles are equal, then the third angles would be equal too. We weren't sure which angles corresponded, so we measured all of them and found that ∠DCA = ∠EDA = 46° and ∠CDA = ∠DEA = 74°.

Solution 2

∆ABC ~ ∆ACD because two pairs of corresponding sides have the same ratio and the angle contained by these sides is the same in both triangles (SAS).

Since

∆ADE ~ ∆ACD

and

∆ABC ~ ∆ACD,

then

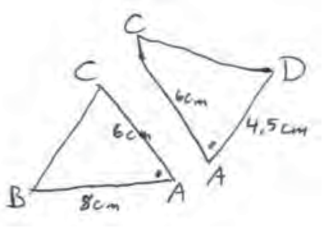
∆ADE ~ ∆ABC.

Therefore,

∆ABC ~ ∆ACD ~ ∆ADE

Thinking

- In my group, we compared ∆ABC and ∆ACD because they looked similar.
- Because we already knew ∠CAD = ∠BAC, we only needed to compare the sides in each triangle that contained those angles.
- I drew a sketch, separating the triangles, to help see which sides corresponded. They were



- AC (in ∆ACD) and AB
- AD and AC (in ∆ABC)

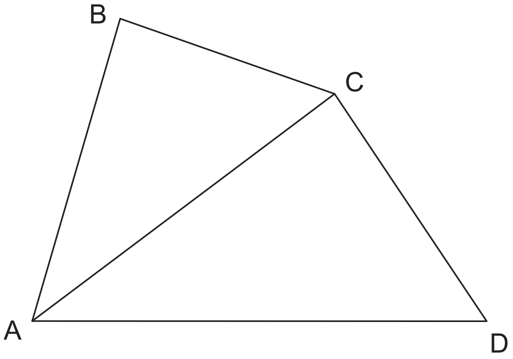
- We calculated the ratios and they were the same:

- AC (in ∆ACD) ÷ AB = 6 cm ÷ 8 cm = 0.75
- AD ÷ AC (in ∆ABC) = 4.5 cm ÷ 6 cm = 0.75

- Then, Karma found out from Pema’s group that ∆ADE ~ ∆ACD. That meant ∆ABC ~ ∆ADE. Then we realized that all three triangles were similar.

Practising and Applying

1. a) Measure the angles to show that the triangles are similar.



b) For each vertex in the smaller triangle, identify the corresponding vertex in the larger triangle.

c) Write a statement representing the similarity. Remember to write the corresponding vertices of the triangles in the same order.

d) Measure the sides to show that each pair of corresponding sides is in the same ratio. What is the ratio?

2. Which statements are always true?

a) Similar triangles are the same size.

b) Similar triangles are the same shape.

c) Corresponding angles in similar triangles are equal.

d) Corresponding sides in similar triangles are equal in length.

3. a) Construct ∆ABC with AB = 12 cm, BC = 9 cm and AC = 6 cm.

b) Construct ∆DEF with sides $\frac{2}{3}$ the length of the sides in ∆ABC.

c) Measure and compare the angles in the two triangles.

d) Find and compare the perimeters of the two triangles.

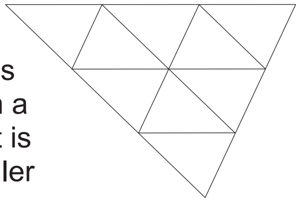
e) How do you know the triangles are similar?

4. Repeat **question 3** with your own original triangle and scale factor.

5. Each instruction below provides more information than necessary to tell if two triangles are similar. Explain.

a) find all three angles

b) find two pairs of corresponding angles and sides



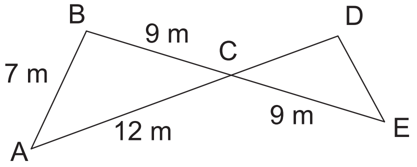
6. In this diagram, nine congruent triangles fit together to form a larger triangle that is similar to the smaller triangles.

a) How many different similar triangles can you find in the diagram?

b) Try this design with a different-shaped triangle. Do you think it can be done for any triangle?

c) Can you do it with a different number of smaller congruent triangles? Try it with two, three, and four congruent triangles.

7. Pema's friend says that ∆ABC ~∆EDC but Pema does not think there is enough information to be sure they are similar.



a) What other information would he need to be sure? Explain.

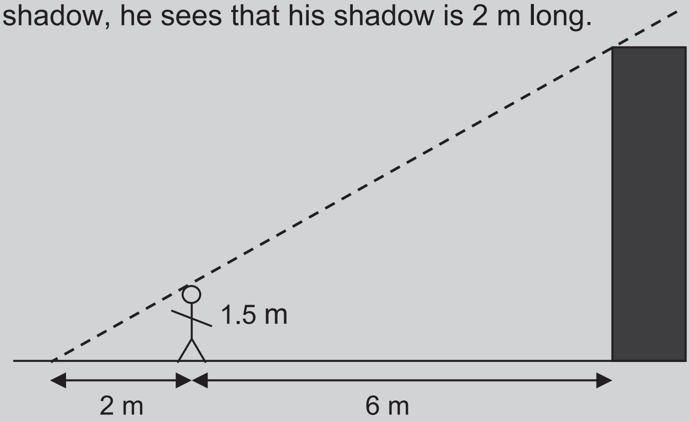
b) Compare your answer to **part a)** with at least two classmates' answers.

8. All congruent triangles are similar but not all similar triangles are congruent. How is this possible?

5.1.4 Solving Problems with Similarity

Try This

Roshan, who is 1.5 m tall, notices that a wall casts a shadow 8 m long. When he moves to the spot where the end of his shadow matches the end of the wall's shadow, he sees that his shadow is 2 m long.



A. i) Find two similar triangles in this sketch.

ii) How do you know they are similar?

B. When Roshan tells Pema about his height and the lengths of the shadows, Pema suggests that the wall is 4 times as tall as Roshan. Why would he say this?

When you know that two triangles are similar, you can use the ratio of the corresponding side lengths to find missing measurements. This allows you to measure objects that might be difficult or impossible to measure, like the height of the wall casting a shadow on Roshan above.

For example, these two triangles are similar because two pairs of corresponding angles are equal. Because the triangles are similar, their corresponding side lengths are in the same ratio. You can find the unknown side length, *x*, using this ratio and a proportion equation:

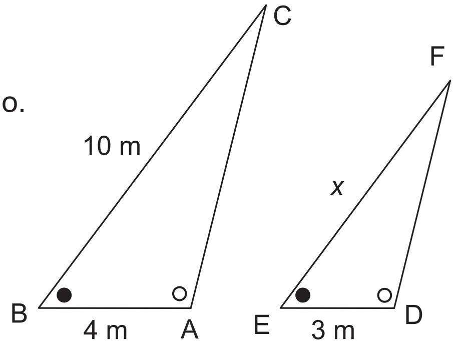
$$\frac{ED}{BA} = \frac{EF}{BC} \text{ so } \frac{3}{4} = \frac{x}{10}$$

$$3 \times 10 = 4x$$

$$30 = 4x$$

$$x = 30 \div 4$$

$$x = 7.5 \text{ m}$$



$\Delta ABC \sim \Delta DEF$

Be careful about which sides correspond to each other when you create a proportion equation.

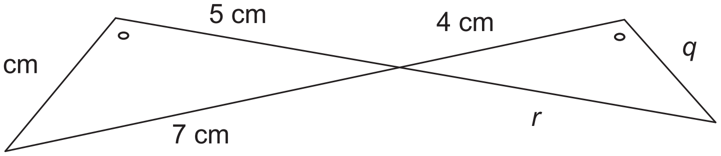
C. i) Write a proportion equation that compares equal ratios between corresponding sides of the two similar triangles in the Roshan's sketch from **parts A and B.**

ii) Solve this equation to find the height of the wall.

Examples

Example 1 Using Similar Triangles

In the given triangles, find the unknown lengths *q* and *r*. Show your work.



Solution 1

Finding *q*:

$$\frac{q}{3} = \frac{4}{5}$$
$$5q = 3 \times 4$$
$$5q = 12$$
$$q = 2.4$$

The length of *q* is 2.4 cm.

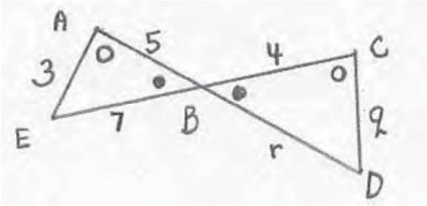
Finding *r* :

$$\frac{\text{perimeter } \triangle BCD}{\text{perimeter } \triangle BAE} = \frac{4}{5}$$
$$\frac{4 + 2.4 + r}{5 + 3 + 7} = \frac{4}{5}$$
$$\frac{6.4 + r}{15} = \frac{12}{15}$$
$$6.4 + r = 12$$
$$r = 5.6$$

The length of *r* is 5.6 cm.

Thinking

- I drew a sketch and labelled the vertices. Then I started checking to see if the triangles were similar.
- I knew ∠ ABE = ∠ CBD because they are vertically opposite. I marked these •.

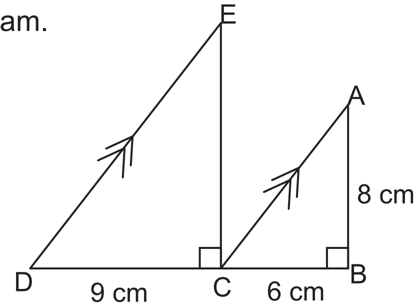


- Since two angles are the same, I knew △BAE ~ △BCD because of AAA.
- Then I looked for corresponding sides to figure out the ratio. AB (5) corresponds to CB (4), because they are both between angles marked ○ and •, so I knew the ratio of the side lengths was $\frac{4}{5}$.
- Then I looked for the corresponding side to *q* so I could write a proportion equation. AE (3) corresponds to CD (*q*), because both are between an unmarked angle and an angle marked ○.
- I set up a proportion equation using the ratios of the side lengths and found *q* using a common denominator.
- I knew the ratio of the perimeters was the same as the ratio of the corresponding sides so I used that to find *r* — I set up a proportion using the ratio of the perimeters and the ratio of the corresponding sides.

Solution 2	**Thinking**
The scale factor is $\frac{4}{5}$ or 0.8. $q = 0.8 \times 3$ cm $= 2.4$ cm $r = 0.8 \times 7$ cm $= 5.6$ cm	 • I knew $\angle ABE = \angle CBD$ because they are vertically opposite. I marked these with a black dot, •. • Since two angles were the same, I knew $\triangle BAE \sim \triangle BCD$ because of AAA. • I found the scale factor by comparing two known corresponding side lengths. AB (5) corresponds to CB (4) because they are between angles marked ○ and •. The scale factor was $\frac{4}{5}$. • I multiplied each unknown length in the larger triangle by the scale factor to find the corresponding length in the smaller triangle.

Example 2 Finding the Perimeter of a Similar Triangle

Find the perimeter of $\triangle CDE$ in the given diagram.
Show your work.

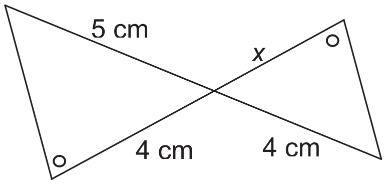


Solution	**Thinking**
$\angle DCE = \angle CBA$ $\angle CDE = \angle BCA$ $\triangle CDE \sim \triangle BCA$ (using AAA) $AC^2 = 6^2 + 8^2 \rightarrow AC = 10$ cm Perimeter $\triangle CBA = 6 + 8 + 10$ $= 24$ cm The scale factor is $\frac{9}{6} = 1.5$. Perimeter $\triangle CDE = 24 \text{ cm} \times 1.5$ $= 36$ cm	 • The triangles looked similar, but I checked by finding two pairs of equal angles. • I knew that the right angles were equal and that $\angle CDE = \angle BCA$ because DE and CA are parallel and DB forms a transversal. So $\triangle CDE \sim \triangle BCA$ (using AAA). • I used the Pythagorean theorem to find AC and then calculated the perimeter of $\triangle CBA$. • I determined the scale factor using corresponding sides DC (9 cm) and CB (6 cm). • I found the perimeter of $\triangle CDE$ using the perimeter of $\triangle CBA$ and the scale factor.

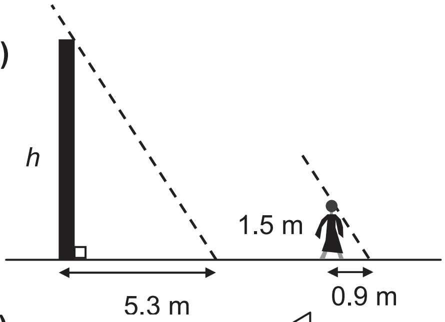
Practising and Applying

1. Find the unknown measurement in each pair of similar triangles.

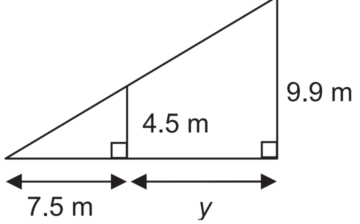
a)



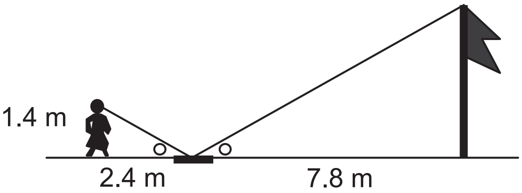
b)



c)



2. Devika places a mirror on the ground to help her measure the height of a flagpole. Use similar triangles to find the height of the flagpole. Show your work.



3. a) Draw any triangle. Locate the midpoint of each side. Join the midpoints to create a smaller triangle inside your original triangle.

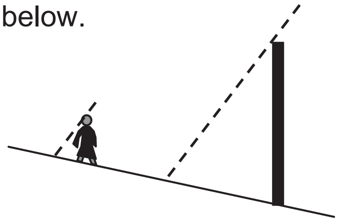
b) Compare the lengths of the sides in the two triangles. What do you notice? What does this mean about the inner and outer triangles?

c) Compare your results from **part b)** with some classmates to see if what you discovered is true for other triangles.

4. Hold a ruler vertically (or horizontally) with your arm extended straight in front of you. Close one eye and find a position in the room or outside so that the end edges of the ruler seem to match the top and bottom (or left and right sides) of an object or person. What measurements do you need in order to use similar triangles to estimate the height (width) of the object or person? Explain.

5. Rajesh notices that all the applications of similar triangles he has seen involve ground that is either horizontal or flat and even. The land he knows best is rarely horizontal or flat.

a) Explain why it is still possible to use shadows and similar triangles to measure the height of something when the ground is flat but not horizontal, as shown below.



b) Explain why it is not possible to use shadows and similar triangles to measure the height of something when the ground is uneven or hilly.

6. Create your own similar triangle problem for a classmate to solve.

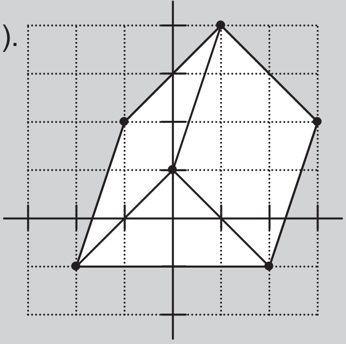
Chapter 2 Transformations

5.2.1 Translations

Try This

Mindu gave Dodo these instructions for drawing a triangular prism:

- Plot ∆ABC with vertices A(–2, –1), B(0, 1), and C(2, –1).
- Translate the triangle so that B', which is the image of point B, is at (1, 4).
- Draw a line segment from each vertex in ∆ABC to its image in ∆A'B'C'.

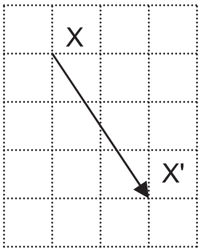


A. Use Mindu's steps to help you draw a rectangular prism, starting with a rectangle instead of a triangle.

B. How far did you translate your rectangle upwards? to the right?

When you slide a geometric shape from one place to another, the move is called a **translation**. The shape in its new position is called the translation **image**.

- The image of a point X is usually described as X', which is read as “X prime.” If X' is then translated, the image of X' is X'' and is read as “X double prime.”
- When a shape is translated, each point in the shape slides the same distance and in the same direction.
- The translation arrow from X to X' shown here represents a translation that is 2 units to the right and 3 units down. The distance right or left is always stated before the distance up or down.



- When you translate a shape, you only need to draw a translation arrow from one of the vertices to its image vertex.
- The translation of X to X' can be represented in two ways:
  - using the **translation rule** [2, –3]
  - using **mapping notation** $(x, y) \rightarrow (x + 2, y - 3)$
- [2, –3] in mapping notation is $(x, y) \rightarrow (x + 2, y - 3)$ because you add 2 to the *x*-coordinate and subtract 3 from the *y*-coordinate of each point to get the coordinates of the image point.
- The mapping notation $(x, y) \rightarrow (x + a, y + b)$ translates each point in a shape *a* units to the right and *b* units up.

C. i) Where could you draw a translation arrow in Mindu's diagram from **parts A and B**?

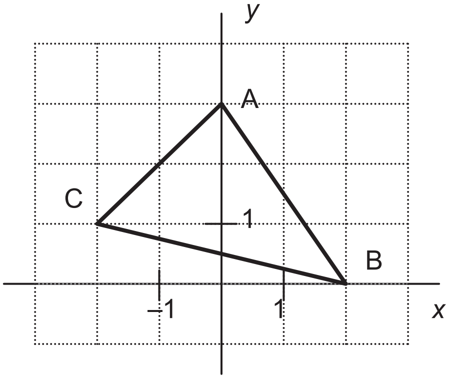
ii) Describe Mindu's translation with words and using two other representations.

D. Describe your translation from **part B** with words and using two other representations.

Examples

Example 1 Using Mapping Notation to Translate a Shape

Determine the coordinates of the vertices of ΔABC after it is translated with the mapping $(x, y) \rightarrow (x + 4, y - 2)$. Explain how you got your answer.



Solution

The vertices are:

A(0, 3)
B(2, 0)
C(–2, 1)

$$A(0, 3) \rightarrow A'(0 + 4, 3 - 2),\ \text{or } A'(4, 1)$$

$$B(2,0) \rightarrow B'(2 + 4, 0 - 2),\ \text{or } B'(6, -2)$$

$$C(-2, 1) \rightarrow C'(-2+ 4, 1 - 2),\ \text{or } C'(2, -1)$$

Thinking

- I wrote the coordinates of the vertices of ΔABC.
- I substituted the x- and y-coordinates from each point into the mapping notation expression, $(x + 4, y - 2)$.

Example 2 Representing a Translation

The vertices of ΔDEF are D(3, –1), E(2, –3), and F(1, –1).

The vertices of ΔD'E'F' are D'(–2, 6), E'(–3, 4), and F'(–4, 6).

Describe the translation that maps ΔDEF onto ΔD'E'F'.

Solution 1	**Thinking**
D(3, –1) → D'(–2, 6) horizontally: –2 – 3 = –5 vertically: 6 – (–1) = 7 Translation rule: [–5, 7] E(2, –3) → E'(–3, 4)	 • I chose to use point D and its image D' to figure out how ΔDEF was translated, because I knew each point in ΔDEF translates the same way. • I subtracted the corresponding *x*-values and the corresponding *y*-values to find the length of the horizontal slide and the vertical slide. (I made sure I subtracted the *y*-values in the same order as I did the *x*-values, D' – D.) • I described the translation using a translation rule.
Solution 2	**Thinking**
E(2, –3) → E'(–3, 4) horizontally: –3 – 2 = –5 vertically: 4 – (–3) = 7 Mapping notation: (*x*, *y*) → (*x* – 5, *y* + 7)	 • I chose point E and its image E' to figure out how ΔDEF was translated, because I knew each point in ΔDEF translates the same way. • I subtracted the *x*-value in E from the x-value in E' to find the length of the slide horizontally. • I subtracted the *y*-values in E from the *y*-value in E' to find the length of the slide vertically. • I described the translation using mapping notation.

Practising and Applying

1. What are the coordinates of the image for each point using the given mapping notation?

a) (2, 6) after (x, y) → (x + 5, y – 3)

b) (–2, 6) after (x, y) → (x + 5, y – 6)

c) (7, –3) after (x, y) → (x – 7, y + 3)

2. What are the coordinates of the image for each point using the given translation rule?

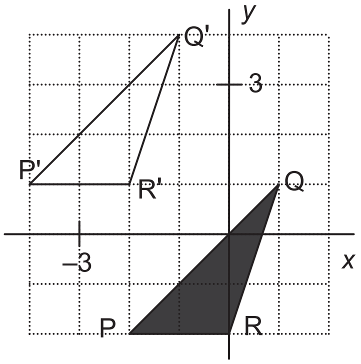
a) (3, –5) translated with [– 5, 3]

b) (0, 6) translated with [– 3, 0]

3. Describe the translation that maps PQR onto P'Q'R' in each way.

a) using a translation rule

b) using mapping notation



4. In **question 3**, which line segments are parallel in ΔPQR and ΔP'Q'R'?

5. Write the translation rule that maps each point onto its image.

a) A(0, 0) → A'(5, –3)

b) B(3, 1) → B'(9, 2)

c) C(3, 1) → C'(0, –5)

d) D(0, –3) → D'(–3, 2)

6. Represent each rule from **question 5** using mapping notation.

7. A rectangle has vertices A(–7, –2), B(–6, 0), C(–2, –2) and D(–3, –4).

a) ABCD maps onto A'B'C'D' with the translation rule [3, 2]. Write the coordinates of A', B', C', and D'.

b) A'B'C'D' maps onto A"B"C"D" with the translation rule [4, 1]. Write the coordinates of A", B", C", and D".

c) What single translation rule would map ABCD on to A"B"C"D"? How does this relate to the translation rules in **parts a) and b)**?

d) Write the translation rule that will map A"B"C"D" back on to ABCD.

e) What do you notice about the mappings in **parts c) and d)**?

8. a) Plot ∆ABC with vertices A(–2, –3), B(3, 3) and C(–1, 2).

b) Describe the translation that moves C(–1, 2) to C'(2, –4).

c) What are the coordinates of A' and B' using the same translation?

d) Is ∆ABC ≅ ∆A'B'C'? Explain.

e) Which side lengths are parallel?

f) Draw line segments to connect A with A', B with B', and C with C'. Which line segments are parallel?

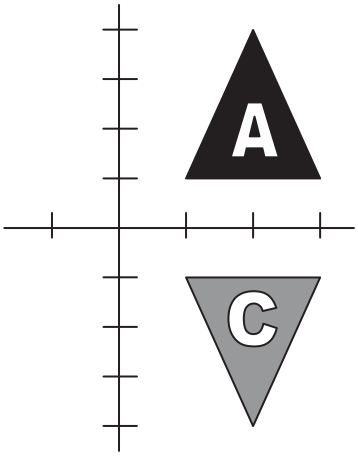
g) Compare the lengths of AA', BB', and CC'. What do you notice?

9. a) The points A(–1, –1) and B(1, –3) are the end points of a side of a square. Write a translation rule that would map these points onto the other vertices of the same square.

b) Explain why there is more than one translation rule for **part a)**.

c) Repeat **parts a) and b)** for the points A(0, 0) and B(1, 3).

10. Explain why triangle A cannot be translated onto triangle C.



11. Why is it possible to represent the translation of a shape with multiple vertices using just two numbers, for example, [3, 4]?

5.2.2 Reflections and Rotations

Try This

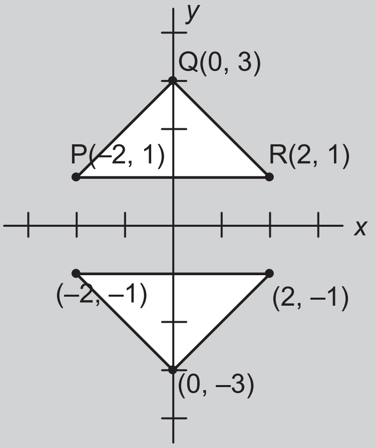
Two students are examining these two triangles. One triangle has vertices P(–2, 1), Q(0, 3), and R(2, 1). The other has vertices (–2, –1), (0, –3), and (2, –1).

A. Novin says that the bottom triangle is a reflection image of ∆PQR. Lobzang says that it is a rotation image. Both students are right. How is that possible?

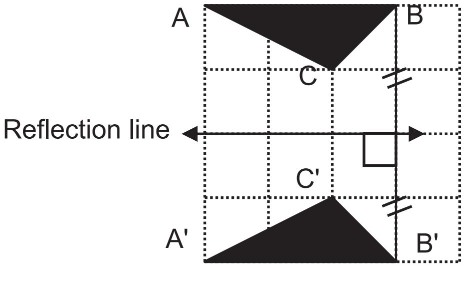
B. i) Novin says that the image of vertex Q is at (0, –3). How would Novin describe the images of the other two vertices?

ii) Lobzang agrees that the image of vertex Q is at (0, –3) but disagrees with Novin about the other two vertices. How would Lobzang describe the images of the other two vertices?

iii) What do you notice about your answers to **parts i) and ii)**?

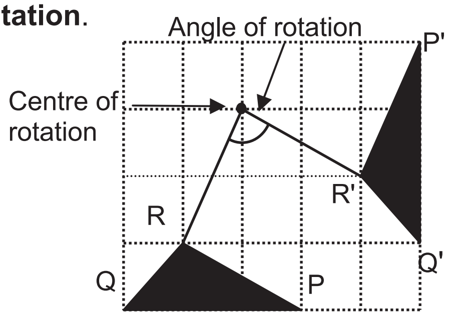


In a **reflection**, the **reflection line** acts like a mirror. If you draw a line segment to connect any point on the original shape with its image point, the reflection line will be the **perpendicular bisector** of that line segment.



When a shape is reflected, the **orientation** is reversed. For example, in the reflection above, the vertices in ∆ABC go clockwise and the corresponding vertices in ∆A'B'C' go counterclockwise.

In a **rotation**, an image is turned around a **centre of rotation** either clockwise (cw) or counterclockwise (ccw). If you draw line segments to the centre of rotation from any original point and from its image, they form the **angle of rotation**.



When a shape is rotated, the orientation stays the same because the vertices in the original triangle and the image go in the same direction. For example, the vertices in ∆QRP and ∆Q'R'P' go in the same direction.

Reflections and rotations can be represented with mapping notation. For example:

- A reflection in the *x*-axis is described with $(x, y) \rightarrow (x, -y)$. This is because the *x*-coordinate does not change, and the image point has the opposite *y*-coordinate.
- The mapping notation for a reflection in the *y*-axis is shown in **Example 1**.
- A 180º rotation around (0, 0) is described with $(x, y) \rightarrow (-x, -y)$. This is because the image points have the opposite *x*- and *y*-coordinates of the original.
- A 90º rotation cw around (0, 0) is described with $(x, y) \rightarrow (y, -x)$. This is because the coordinates of the image points are reversed and one of the coordinates changes its sign.
- The mapping notation for a 90° rotation cc around (0, 0) is shown in **Example 2**.

You will discover other reflection and rotation rules as you work through the exercises.

C. i) What mapping notation would Novin write to represent how he viewed the transformation as described in **part A**?

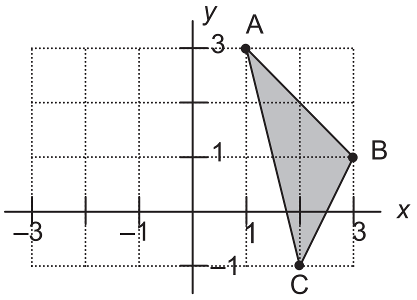
ii) What mapping notation would Lobzang write to represent his way of viewing the transformation?

Examples

Example 1 Determining the Mapping Notation for Reflections in the *Y*-axis

a) Determine the vertices of the image of △ABC after a reflection in the *y*-axis.

b) Compare the coordinates of each vertex in △ABC with the coordinates of its image vertex to determine the mapping notation that describes a reflection in the *y*-axis.

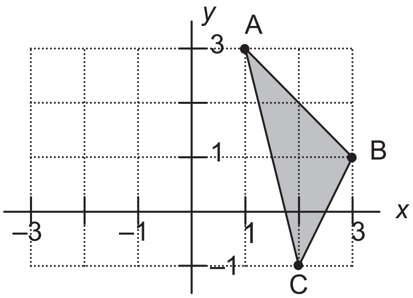


Solution	**Thinking**
a) The reflection results in these image vertices: A(1, 3) → A'(–1, 3) B(3, 1) → B'(–3, 1) C(2, –1) → C'(–2, –1)	 **a)** I found the coordinates of A' by counting the distance from A to the *y*-axis and then counting that same amount past the axis. • I did the same for B' and C'.
b) A reflection in the *y*-axis can be represented with the following mapping: $(x, y) \rightarrow (-x, y)$	**b)** I compared the coordinates of the vertices and their images to look for a pattern. • The *x*-values became their opposites but the *y*-values didn't change.

Example 2 Determining the Mapping Notation for a 90° ccw Rotation

a) ∆ABC is rotated 90º counterclockwise around (0, 0). What are the coordinates of ∆A'B'C'?

b) Compare the coordinates of each vertex in ∆ABC with the coordinates of its image vertex to find the mapping notation that describes a 90º counterclockwise rotation around (0, 0).



Solution

a) The rotation results in these image vertices:

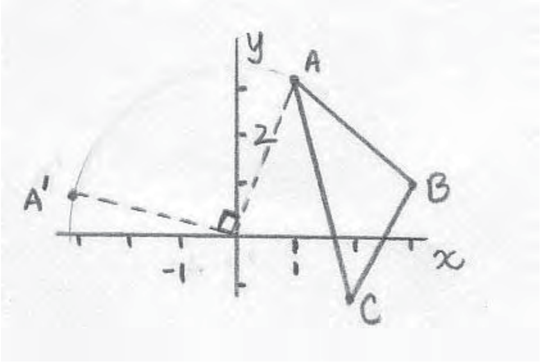
A(1, 3) → A'(–3, 1)
B(3, 1) → B'(–1, 3)
C(2, –1) → C'(1, 2)

b) A 90º rotation ccw around (0, 0) can be represented with the following mapping:

(*x*, *y*) → (–*y*, *x*)

Thinking

a) I found the coordinates of A' by first drawing a line segment from A to (0, 0).
I measured a 90° angle at the origin, and drew another line segment, the same length as the first one, from the origin to A'.



- I did the same for B' and C'.

b) I compared the coordinates of the vertices and their images to look for a pattern.

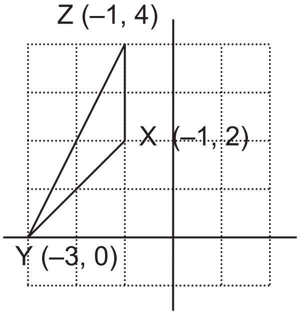
- The *x*- and *y*-values changed places, but the sign only changed when the *y*-coordinate moved into the *x*-coordinate position.

Practising and Applying

1. Determine the coordinates of the image vertices after each reflection of ∆XYZ.

a) a reflection in the *x*-axis

b) a reflection in the *y*-axis



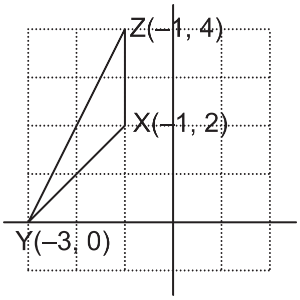
2. Determine the coordinates of the image vertices after each rotation of ∆XYZ below.

a) a rotation of 90º counterclockwise (ccw) about the origin

b) a rotation of 90º clockwise (cw) about the origin

c) a rotation of 180º ccw about the origin

d) a rotation of 180º cw about the origin



3. Look back at your work in **question 2**

a) For each part, are XY and X'Y' parallel? Explain.

b) What do you notice about the image vertices in **parts c) and d)**?

4. a) Graph ∆PQR, with vertices P(2, 1), Q(1, –2), and R(1, 4).

b) Graph the image of ∆PQR under the transformation (*x*, *y*) → (*x*, –*y*)

c) Describe the transformation.

d) Are the sides in the original triangle parallel to their corresponding sides in the image triangle? Explain.

e) How would the results be different with the mapping notation (*x*, *y*) → (–*x*, *y*)?

5. a) Graph ∆KLM, with K(–2, –1), L(–5, –2), and M(–3, 4).

b) Graph the image of ∆KLM under the transformation (x, y) → (–y, x)

c) Describe the transformation.

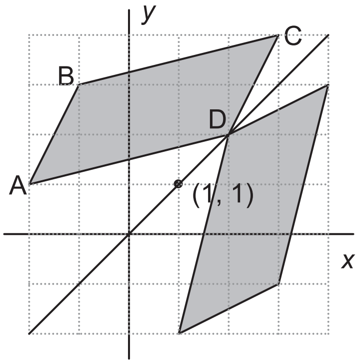
6. a) Graph ∆DEF, with D(1, 3), E(5, –2), and F(–1, 1).

b) Graph the image of ∆DEF under the transformation (x, y) → (–x, –y)

c) Describe the transformation.

d) How would the results be different if the mapping notation had been (*x*, *y*) → (–*y*, *x*)?

7. This graph shows ABCD reflected in the line that passes through the points (0, 0) and (1, 1).



a) Write the coordinates of the vertices A, B, C, and D and their image vertices.

b) Look for a pattern to help you describe the transformation using mapping notation.

8. How are the mapping descriptions of these the same and how are they different?

a) a rotation of 90° clockwise around the origin compared to a rotation of 180° clockwise around the origin

b) a reflection in the *x*-axis compared to a reflection in the diagonal line that passes through (0, 0) and (1, 1)

c) a reflection in the *y*-axis compared to a reflection in the diagonal line that passes through (0, 0) and (1, 1).

9. Look back at your work on reflections.

a) When a shape is reflected, is the image congruent, similar, both, or neither? Explain.

b) Describe the orientation of a reflection image compared to the original shape.

10. Look back at your work on rotations.

a) When a shape is rotated, is the image congruent, similar, both, or neither? Explain.

b) Describe the orientation of a rotation image compared to the original shape.

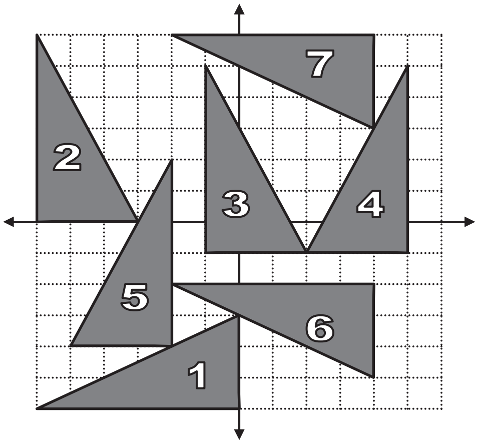
GAME: Shards

Shards is a game for two players.

- To set up, construct a coordinate grid from –6 to 6 on each axis.
- Player A plots and draws a right triangle no larger than 6 units measured horizontally or vertically.
- Player B rotates, reflects, or translates Player A’s triangle and draws its image, staying within the coordinate grid (*x*- and *y*-coordinates must be between –6 and 6) and not overlapping with Player A's triangle. Player B then describes the transformation.
- Players take turns transforming and drawing the triangle and describing their transformations until no more moves are possible without overlapping.
- The last player to move wins.

For example, in the game shown here,

- Player A drew triangle 1, with vertices at (0, –3), (0, –6), and (–6, –6).
- Player B then rotated triangle 1 about the origin 90º cw and drew triangle 2.
- Player A then translated triangle 2 using (*x*, *y*) → (*x* + 5, *y* – 1) and drew triangle 3.
- Player B then reflected triangle 3 across the vertical line *x* = 2 and drew triangle 4.
- Play continued until Player A drew triangle 7 and won the game, because there was no more space for Player B to draw a triangle.

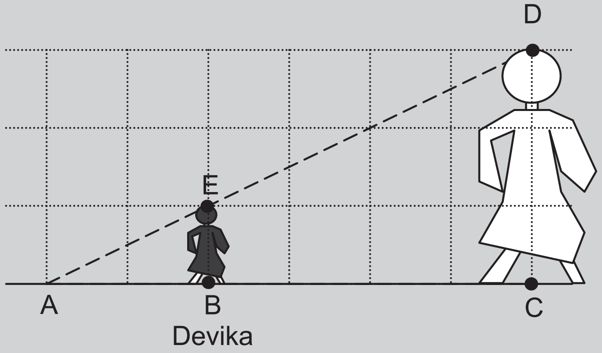


Play the game with a partner. As you play, indicate why you have chosen the transformation you did each time.

5.2.3 Dilatations

Try This

Devika is using shadows and similar triangles to find the height of a statue. She drew a sketch on grid paper to help visualize the similar triangles.



A. Suppose point A is the origin (0, 0) of a coordinate grid.

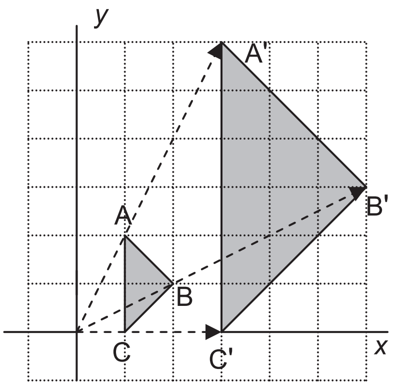
i) What are the coordinates of ΔBAE?

ii) What are the coordinates of ΔCAD?

B. Compare the coordinates of the corresponding vertices in the two similar triangles. What do you notice?

A **dilatation** is a transformation that enlarges or reduces a shape.

- To dilatate a shape, you multiply all the shape's side lengths by a **scale factor.** When the scale factor is greater than 1, the shape is enlarged. When the scale factor is positive and less than 1, the shape is reduced.
- In the diagram below, ΔABC was plotted on a coordinate grid and then enlarged to create the dilatation image ΔA'B'C'.



- Compare the coordinates of ΔABC and ΔA'B'C':

A(1, 2) → A'(3, 6)

B(2, 1) → B'(6, 3)

C(1, 0) → C'(3, 0)

The coordinates of ΔABC have been multiplied by 3, the scale factor, to give the coordinates of ΔA'B'C'. This occurs when the dilatation centre is at the origin, (0, 0).

- Notice that a straight line drawn from (0, 0) through each vertex in ΔABC also passes through its corresponding vertex in ΔA'B'C'. This occurs because the dilatation centre is at the origin, (0, 0).

- The mapping $(x, y) \rightarrow (ax, ay)$ represents a dilatation with a scale factor of a when the dilatation centre is at (0, 0).
- The mapping that dilatated ∆ABC to ∆A'B'C', as illustrated on the previous page, would be $(x, y) \rightarrow (3x, 3y)$ because the scale factor was 3.
- If the larger triangle had been the original shape and the smaller triangle the dilatation image, the scale factor would have been $\frac{1}{3}$, indicating a reduction. The mapping in this case would be $(x, y) \rightarrow (\frac{1}{3}x, \frac{1}{3}y)$.
- Dilatation images and their original shapes are similar triangles. The ratio of their corresponding side lengths is the dilatation scale factor and their corresponding angles are equal.

C. i) How do you know Devika's ∆CAD from **parts A and B** is similar to ∆BAE?
ii) How do you know ∆CAD is a dilatation image of ∆BAE with centre (0, 0)?
iii) Write the mapping for enlarging ∆BAE to ∆CAD.

Examples

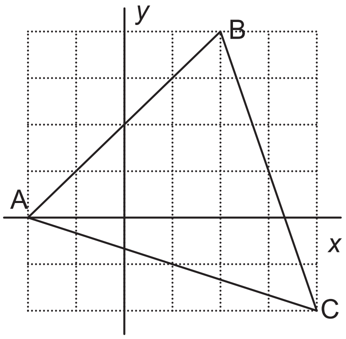
Example Performing a Dilatation

a) Graph ∆ABC with vertices A(–2, 0), B(2, 4), and C(4, –2) and graph its dilatation image using (0, 0) as the centre and a scale factor of $\frac{1}{4}$.

b) Compare the sides of ∆ABC with the corresponding sides of ∆A'B'C'. What do you notice?

Solution

a)



$(x, y) \rightarrow (\frac{1}{4}x, \frac{1}{4}y)$

$A(-2, 0) \rightarrow (\frac{1}{4} \times -2, \frac{1}{4} \times 0) \rightarrow A'(-\frac{1}{2}, 0)$

$B(2, 4) \rightarrow (\frac{1}{4} \times 2, \frac{1}{4} \times 4) \rightarrow B'(\frac{1}{2}, 1)$

$C(4, -2) \rightarrow (\frac{1}{4} \times 4, \frac{1}{4} \times -2) \rightarrow C'(1, -\frac{1}{2})$

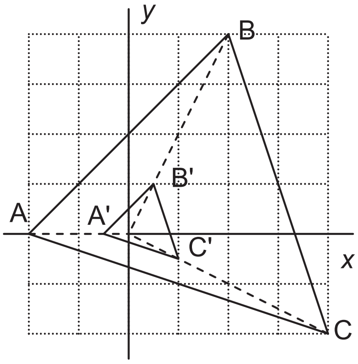
[Continued]

Thinking

a) I graphed ∆ABC.

- I knew I could use the mapping $(x, y) \rightarrow (\frac{1}{4}x, \frac{1}{4}y)$ because the centre of the dilatation was (0, 0) and the scale factor was $\frac{1}{4}$.
- The mapping helped me find the coordinates of the vertices of ∆A'B'C'.

Example Performing a Dilatation [Continued]

Solution	**Thinking**
	• I graphed ΔA'B'C' and then I drew dashed lines through pairs of corresponding vertices to check that the dilatation centre was (0, 0).
b) When I compared ΔABC and ΔA'B'C', I noticed that • each side of ΔABC is parallel with its corresponding side in ΔA'B'C' • the ratio of the lengths of the corresponding sides is the same as the dilatation scale factor: $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = \frac{1}{4}$	**b)** When I compared the side lengths, I looked at whether they were parallel and also at the ratio of their side lengths. • I realized ΔABC and ΔA'B'C' are similar, because the ratios of all the corresponding side lengths are equal.

Practising and Applying

1. Write the coordinates of the image triangle after a dilatation with centre (0, 0) and scale factor 2.

a) ΔPQR: P(1, –2), Q(3, 2), and R(5, 0)

b) ΔSTU: S(1, 1), T(–1, 1), and U(0, –2)

2. a) Dilatate rectangle WXYZ with vertices W(4, 4), X(6, 0), Y(0, –3), and Z(–2, 1) using the following mapping:

$(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$.

b) What is the scale factor? How do you know?

c) How can you check that the centre of the dilatation is (0, 0)?

d) Are the sides of the original rectangle parallel to their corresponding sides in the image? Explain why or why not.

3. Determine the scale factor for each dilatation centred at the origin.

a) J(2, –7) → J'(4, –14)

b) K(–12, 6) → K'(–4, 2)

c) L(3, 6) → L'(2, 4)

4. Passang dilatated ΔABC. The image was ΔA'B'C' with vertices A'(2, 4), B'(2, –2), and C'(–4, 0).

a) If the mapping were $(x, y) \rightarrow (2x, 2y)$, what would be the vertices of ΔABC?

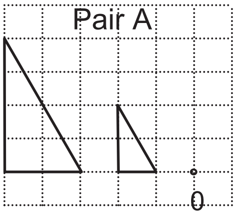
b) Find another possibility for ΔABC using a different mapping.

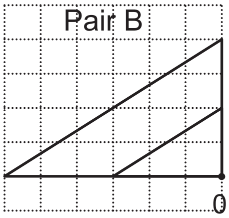
c) Why are there many answers for **part b)**?

5. Maya noticed that she can do dilatations on grid paper even if there are no *x*- and *y*-axes. She can choose any point on a grid to be the centre.

a) For each pair of triangles, explain how you can tell if they are similar.

b) For each pair, decide if they show a dilatation with centre 0 (the origin). Explain your decisions.

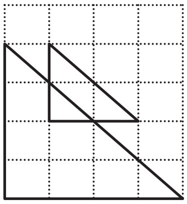




6. Devika performed a dilatation on a triangle but only two of the three vertices changed. What do you know about the position of Devika's triangle?

7. Explain why $(x, y) \rightarrow (2x, 3y)$ does not result in a dilatation.

8. Explain how you would find the centre for the dilatation shown here.



9. Look back at your work on dilatations.

a) When a shape is dilatated, is the image congruent, similar, both, or neither? Explain.

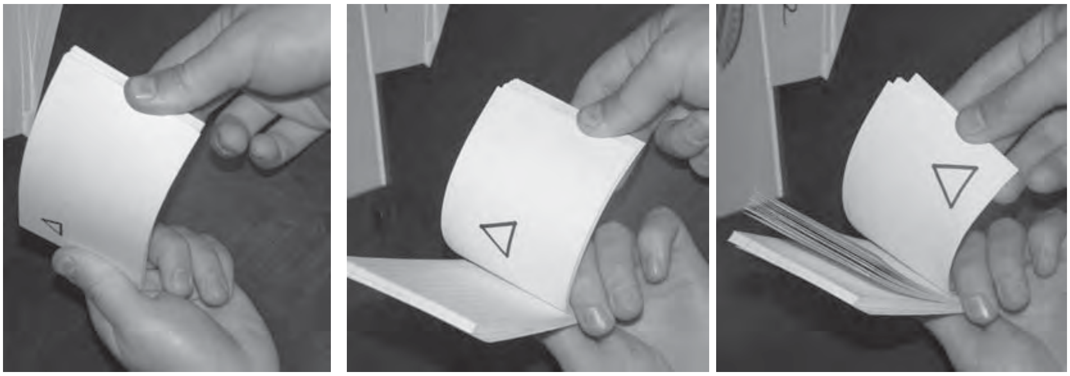
b) What does this tell you about the angles of a shape and its dilatation image?

10. All dilatations result in similarity but not all similar shapes are dilatations. Explain.

CONNECTIONS: Making an Animated Movie

Animated movies present a series of still images with a slight change from one to the next. When the images are shown in quick succession, this gives the illusion of movement and is called animation. You can make a flipbook that uses this technique to animate a very simple and short series of movements.

- Cut ten or more 8 cm by 12 cm rectangles out of stiff paper or light cardboard.
- On each piece, draw a stage of your planned movement. Each image will be a translation, rotation, or dilatation of the image before it.
- Staple the cards together to make a flipbook. Flip through the book quickly to view your animation.

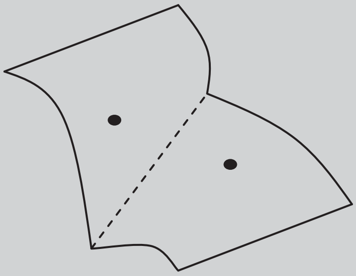


The triangle gets larger (dilatates) as it moves (translates) across the pages of the flipbook.

5.2.4 Combining Transformations

Try This

Drakpa drew two dots on a piece of paper and showed a friend how to fold the paper so that one dot was reflected onto the other dot. His friend then challenged Drakpa to do it again, but this time to reflect the dot onto the other dot using two folds.



A. i) Draw two dots on a piece of paper. Fold the paper to reflect one dot onto the other with just one fold. How does your fold line relate to an imaginary line that goes from one dot to the other?

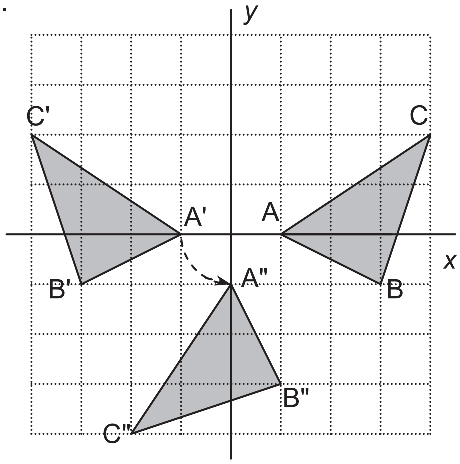
ii) How could you do it in two folds?

When you combine transformations by performing one transformation after another, the series of transformations is called a **composite transformation**.

In the following example, ∆ABC was reflected in the y-axis using the mapping $(x, y) \rightarrow (-x, y)$ to ∆A'B'C' and then rotated 90º ccw around (0, 0) using the mapping $(x, y) \rightarrow (-y, x)$ to result in ∆A"B"C".

The composite transformation can be described using a single mapping:
$(x, y) \rightarrow (-x, y)$ and then $(x, y) \rightarrow (-y, x)$ is the same as $(x, y) \rightarrow (-y, -x)$.

Note that, if ∆ABC had been rotated first and then reflected, the final image, ∆A"B"C", would be different.
$(x, y) \rightarrow (-y, x)$ and then $(x, y) \rightarrow (-x, y)$ is the same as $(x, y) \rightarrow (y, x)$.



A composite transformation can consist of two or more of the same type of transformation.

For example, if a triangle is reflected in the x-axis using $(x, y) \rightarrow (x, -y)$ and then reflected again, but this time in the y-axis using $(x, y) \rightarrow (-x, y)$, the composite transformation can be described with the single mapping $(x, y) \rightarrow (-x, -y)$.

B. How is what you did in **part A ii)** like a composite transformation?

C. Open up your paper and examine the two dots. Think of one as point A and the other as point A'. What other composite transformation could you use to map A onto A'?

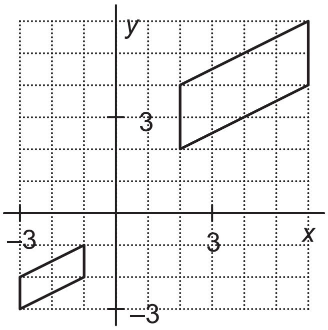
Examples

Example Describing a Composite Transformation

a) What two transformations could map the larger parallelogram onto the smaller parallelogram? Show your work.

b) Describe the composite transformation using a single mapping.

c) How do you know you could also describe it using four transformations?



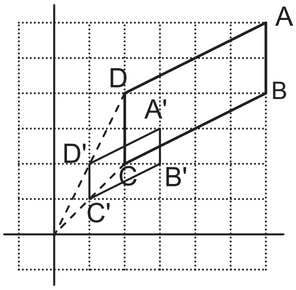
Solution 1

a) ABCD: A(6, 6), B(6, 4), C(2, 2), D(2, 4)

First Transformation

Dilatation with centre (0, 0) and scale factor $\frac{1}{2}$: $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$

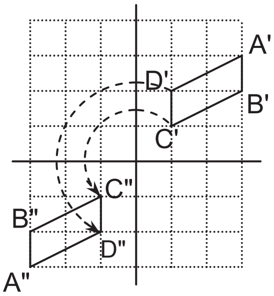
A'(3, 3)
B'(3, 2)
C'(1, 1)
D'(1, 2)



Second Transformation

180° rotation around (0, 0): $(x, y) \rightarrow (-x, -y)$

A"(–3, –3)
B"(–3, –2)
C"(–1, –1)
D"(–1, –2)



It could be a dilatation and then a rotation.

b) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y) \rightarrow (-x, -y)$

$(x, y) \rightarrow (-\frac{1}{2}x, -\frac{1}{2}y)$

[Continued]

Thinking

a) I knew one transformation had to be a dilatation because the image is similar but reduced.

- I measured and compared corresponding sides to find the scale factor, $\frac{1}{2}$.
- I drew lines from the origin through two of the corresponding vertices to check if the centre was (0, 0).
- After creating the dilatation image A'B'C'D' I noticed that I could rotate the image 180° around (0, 0) to map it onto the final image.

b) I combined the two mapping notations to create a single mapping. I knew that the coordinates were divided by 2 and also became their opposites.

Example Describing a Composite Transformation [Continued]

Solution **c)** You could follow the composite transformation in **part b)** with the translation [2, 1] and then the translation [–2, –1]	**Thinking** If you do a transformation and then one that's opposite to it, they cancel each other out and the shape ends up in the same place.
Solution 2 **a)** *Original vertices*: (6, 6), (6, 4), (2, 2), (2, 4) *First transformation* Dilatation with centre (0, 0) and scale factor $\frac{1}{2}$: $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ *Second transformation* Translation 4 units down and 4 units left: $(x, y) \rightarrow (x - 4, y - 4)$	**Thinking**  **a)** It was obvious that one transformation was a dilatation. • I knew the scale factor was $\frac{1}{2}$ because the sides of the small shape are half as big as the sides of the large shape. • I changed the vertices of the original shape to what they would be after the dilatation. Then I compared them to the vertices of the final image. I think that a [–4, –4] translation would slide the dilatation image onto the final image. original after $\frac{1}{2}$ dilatation final (6, 6) (3, 3) (–1, –1) (6, 4) (3, 2) (–1, –2) (2, 2) (1, 1) (–3, –3) (2, 4) (1, 2) (–3, –2)
b) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ and then $(x, y) \rightarrow (x - 4, y - 4)$ $= (x, y) \rightarrow (\frac{1}{2}x - 4, \frac{1}{2}y - 4)$	**b)** I combined the two mapping notations to create a single mapping. I knew that the coordinates were divided by 2 and then 4 was subtracted.
c) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ and then $(x, y) \rightarrow (x - 4, y - 4)$ $= (x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ and then $(x, y) \rightarrow (x - 1, y - 1)$ and then $(x, y) \rightarrow (x - 1, y - 1)$ and then $(x, y) \rightarrow (x - 2, y - 2)$	**c)** I broke the final translation $(x - 4, y - 4)$ into three separate translations $(x - 1, y - 1)$ and then $(x - 1, y - 1)$ and then $(x - 2, y - 2)$ to give me four transformations altogether: one dilatation and three translations.

Practising and Applying

1. The vertices of ∆XYZ are X(2, 2), Y(–1, 1), and Z(3, –1).

a) Where are the vertices of the final image if ∆XYZ is reflected in the *x*-axis and then translated [1, –2]?

b) Write each transformation in **part a)** in mapping notation.

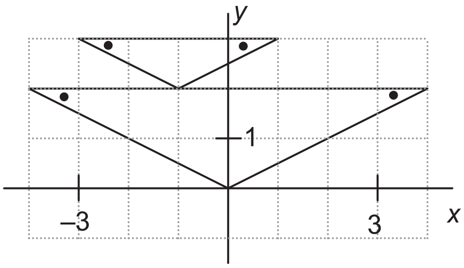
c) Write a single mapping for the composite transformation in **part b)**.

d) What would happen if you reversed the order of the transformations in **part a)**?

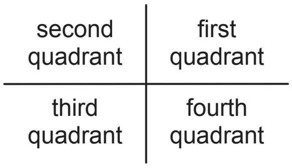
2. a) Describe a series of two transformations that would map the larger triangle onto the smaller triangle, shown below.

b) Write each transformation in mapping notation.

c) Write a single mapping for the composite transformation in **part b)**.



3. a) Draw any triangle in the second quadrant of a coordinate grid.

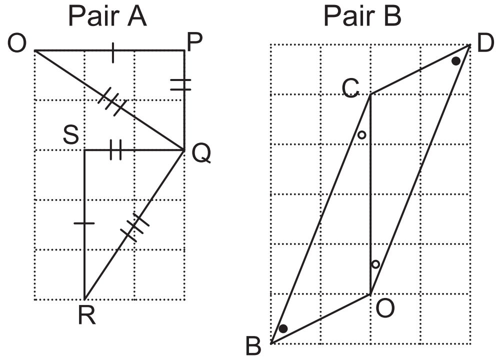


b) Perform four or more consecutive transformations so that at least one of the successive images is in each quadrant and the final image maps onto the original triangle. Use at least two different kinds of transformations.

4. a) How do you know the triangles in each pair below are congruent?

b) Describe a composite transformation that would map one triangle in each pair onto the other.

Note that vertex O on each grid below is the origin (0, 0) of a coordinate grid.


Pair A
Pair B

5. a) Plot any scalene triangle and transform it with the mapping $(x, y) \rightarrow (5 - x, y - 2)$

b) Describe a series of transformations that would map your triangle onto the image triangle.

c) Write each transformation in **part b)** in mapping notation.

6. Meto says that a reflection in the *x*-axis followed by a reflection in the *y*-axis gives the same final image as a single 180º rotation around the origin. Do you agree with Meto? Explain.

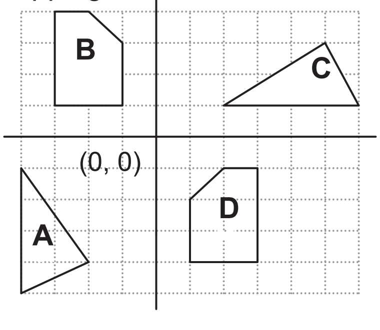
7. Describe a composite transformation for each. Use mapping notation.

a) mapping A onto C

b) mapping C onto A

c) mapping B onto D

d) mapping D onto B

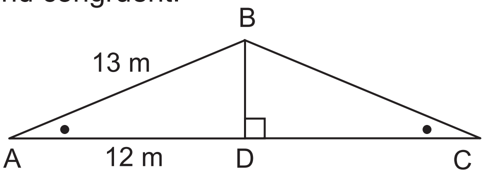


UNIT 5 Revision

1. a) Show that at least two different triangles, each with vertices P, Q, and R, can be constructed such that PQ = 9 cm, QR = 7 cm, and ∠P = 45°.

b) What does this tell you about whether it is enough to know two sides and one angle in each of two triangles if you want to be sure they are congruent?

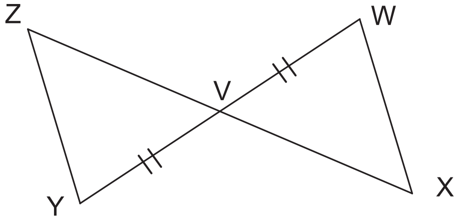
2. a) Identify the similar triangles and the congruent triangles in this diagram. Show how you know they are similar and congruent.



b) Find the length of side BC. Explain how you know.

3. Use several examples to show that triangles are congruent if all three sides are the same lengths.

4. What further information do you need to prove that ΔVZY ≅ ΔVXW? Explain why this information would help.

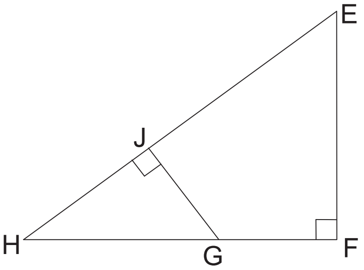


5. ΔPQR has vertices P(3, –1), Q(5, 0), and R(4, 7).

a) Locate the vertices of its image after a translation of [–3, 5].

b) Use mapping notation to represent the translation.

6. a) Identify the similar triangles. Explain how you know they are similar.



b) If GF = 3 cm, HJ = 4 cm, and HG = 5 cm, find the length of HE and JE. Show your work.

7. ΔKLM has vertices K(–3,1), L(3, –1), and M(–3, –7).

a) Locate the vertices of its image after ΔKLM is reflected in the *x*-axis.

b) Use mapping notation to represent the reflection in **part a)**.

c) Locate the vertices of its image after ΔKLM is reflected in the line that passes through (0, 0) and (1, 1).

d) Use mapping notation to represent the reflection in **part c)**.

e) How does the orientation of ΔKLM compare to the reflection image in **part a)**? In **part c)**? What do you notice?

8. ΔSTU has vertices S(–2, –1), T(–2, –4), and U(–4, –4).

a) Locate the vertices of the image after ΔSTU is rotated using (*x*, *y*) → (–*y*, *x*).

b) Locate the vertices of the image after ΔSTU is rotated using (*x*, *y*) → (*y*, –*x*).

c) Use words to describe each of the rotations in **part a**) **and part b)**.

d) How does the orientation of ΔSTU compare to the rotation image in **part a)**? in **part b)**? What do you notice?

9. a) Plot any triangle on a coordinate grid.

b) Dilatate the triangle using a dilatation centre of (0, 0) and a scale factor of 3.

c) Measure and compare the sides of the triangles to explain how you know the original triangle and its dilatation image are similar.

10. ∆ABC has vertices A(0, 4), B(6, 2), and C(3, –2).

a) Locate the vertices of the image after ∆ABC is dilatated with centre (0, 0) and a scale factor of $\frac{1}{2}$.

b) Use mapping notation to represent the above transformation.

c) Compare the orientations of ∆ABC and its image from this dilatation.

11. ∆DEF has vertices D(0, 2), E(5, 4), and F(2, –2). Describe the transformation that would give each image ∆D'E'F' below with these vertices.

a) D'(0, 2), E'(–5, 4), F'(–2, –2)

b) D'(0, 4), E'(10, 8), F'(4, –4)

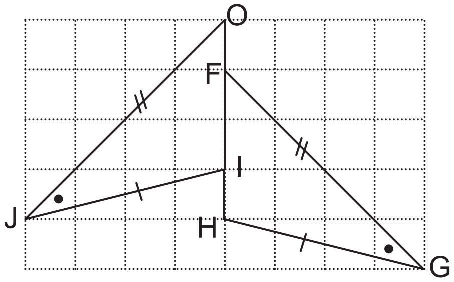
c) D'(0, –2), E'(–5, –4), F'(–2, 2)

d) D'(2, 0), E'(4, –5), F'(–2, –2)

e) D'(–3, 4), E'(2, 6), F'(–1, 0)

12. a) Describe how you know the triangles below are congruent.

b) Describe a transformation or composite transformation that would map one of the triangles onto the other. Note that vertex O on the grid below is the origin (0, 0) of a coordinate grid.



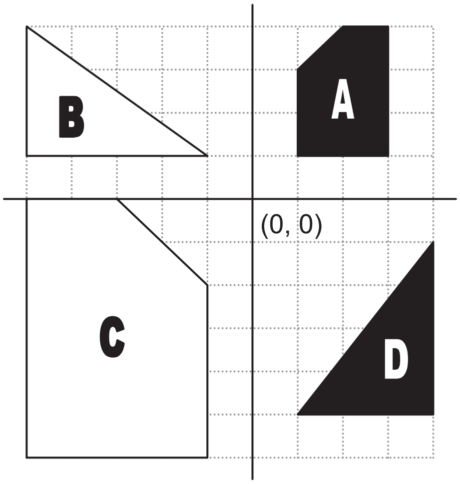
13. Describe a composite transformation for each. Use mapping notation.

a) mapping A onto C

b) mapping C onto A

c) mapping B onto D

d) mapping D onto B



14. For each kind of transformation, describe the relationship between the original shape and its image in terms of congruency, similarity, and orientation.

a) translation

b) reflection

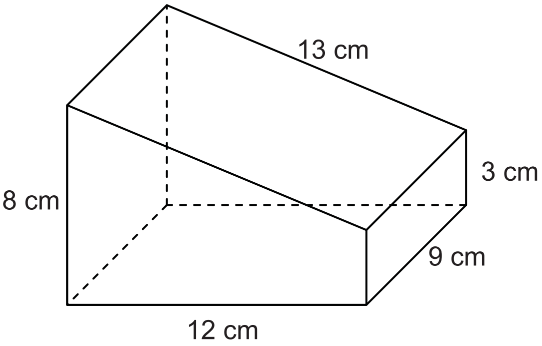
c) rotation

d) dilatation

UNIT 6 MEASUREMENT

Getting Started

Use What You Know



A. i) Name this 3-D shape.

ii) Name the shape of each face.

iii) Use the measurements of each face to determine its area.

iv) Suppose you doubled each measurement. How would that affect the area of each face? Why?

B. Imagine you have wrapped your textbook in paper.

i) What type of prism is your wrapped textbook?

ii) Name the shape of each face of the prism.

iii) Draw a sketch of the prism or its net. Label the dimensions of the faces.

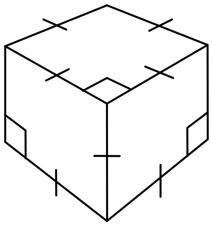
iv) Calculate the area of each face.

v) Compare your results with a classmate's.

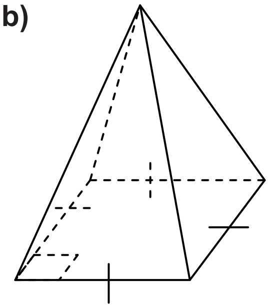
Skills You Will Need

1. Name each 3-D shape.

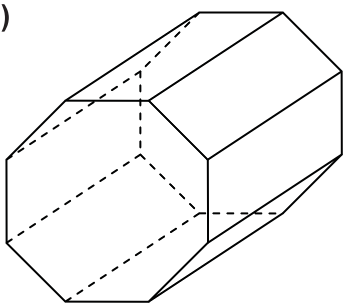
a)



b)

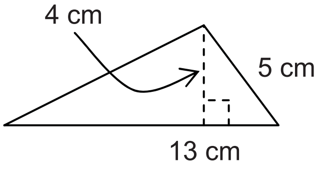


c)

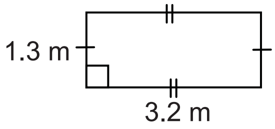


2. Determine the area of each 2-D shape.

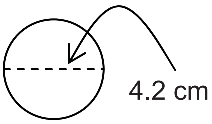
a)



b)



c)

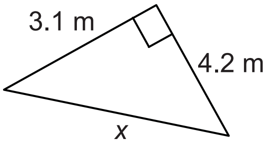


3. Calculate.

a) 5^3 **b)** 2.6^3 **c)** x, if $x^3 = 49$

4. Find x.

a)



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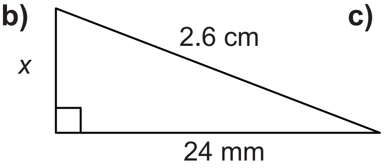

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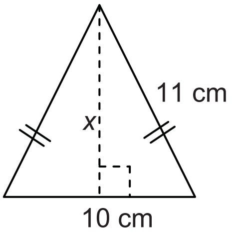
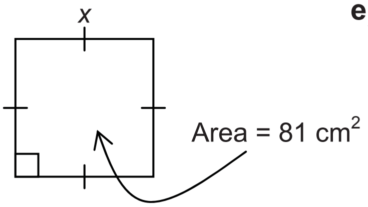
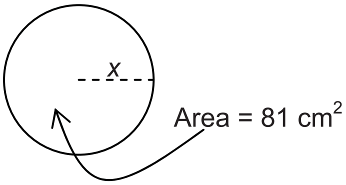


d)

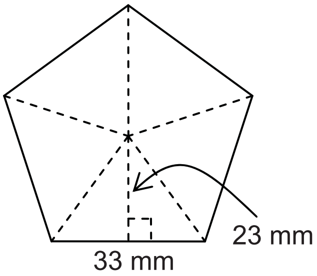


e)

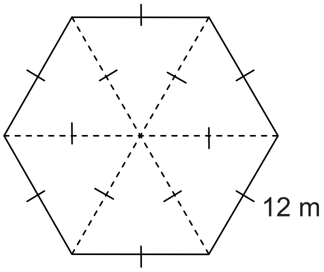


5. Determine the area of each regular polygon.

a)



b)



6. a) Determine the circumference of a circle with radius 8 cm.

b) Determine the diameter of a circle with circumference 15 m.

7. Fill in the blanks to make equivalency statements.

a) 2.4 kg = ___ g

b) 230 mL = ___ L

c) 3 m^2 = ___ cm^2

d) 5 mL = ___ cm^3

e) 4.3 L = ___ cm^3

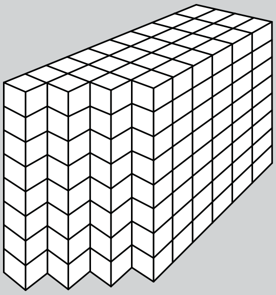
Chapter 1 Volume and Capacity

6.1.1 Volume of Prisms and Cylinders

Try This

Kado used a set of 1 cm^3 cubes to build a structure as shown.

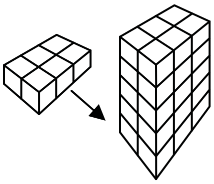
A. i) How many cubes did Dodo use for the bottom layer?
ii) How many layers are there?
iii) How could you figure out the total number of cubes in the structure using your answers to **parts i) and ii)**?
iv) How many cubes did he use for the whole structure?



Shapes that have length, width, and height are called three-dimensional, or 3-D. The **volume** of a 3-D shape is a measurement of the amount of space it occupies.

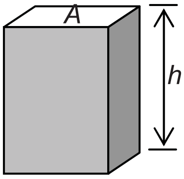
- A **prism** is a 3-D shape with two opposite, parallel, and congruent faces, which are polygons. The lateral faces are always rectangles. The two opposite congruent faces are the **bases** of the prism and the rectangular faces form the **lateral surface**. Prisms are named according to the shape of their bases.
- To find the volume of a prism, you determine the area of its base and then multiply by the height of the prism:

Volume of any prism = Area of base × height, or $V = Ah$



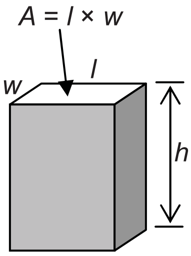
- To understand how this formula works, imagine a layer of 6 cubes, as shown here to the right. If 5 of these layers are stacked on top of each other, the number of cubes in the stack will be 6 × 5 = 30 because there are 5 layers, each with 6 cubes.

- It does not matter what the shape of each layer is. As long as each layer is the same, the total number of cubes in the stack can be calculated by multiplying the number of cubes in the base layer by the number of layers.



- In the prism at right, the area of the base (A) is like the base layer of the structure made of cubes above, and the height of the prism (h) is like the number of layers. So, the volume is found by multiplying the area of the base by the height.

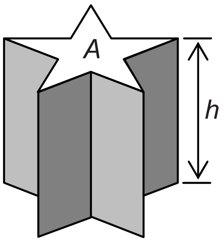
- For a rectangular prism, the formula $V = Ah$ yields the same result as a method you may have used before to find the volume, which is multiplying the three dimensions, $l \times w \times h$ or lwh. When you multiply the length by the width, you get the area of the base, which you then multiply by the height.



Volume of a rectangular prism = Area of base × height
= length × width × height = lwh

The formula $V = Ah$ applies to any prism.

For example, the volume of the star-based prism at the right is the area of the star base, A, multiplied by the prism's height, h. If the area of the base is 55 cm^2 and the height h is 10 cm, then the volume of the prism is $Ah = 55 \times 10 = 550$ cm^3.

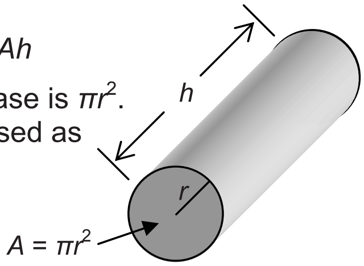


- A **cylinder** is not a prism because its base is a circle and not a polygon. However, it is like a prism because it has two opposite congruent faces (or bases) and the volume is calculated in the same way.

Volume of a cylinder = *Area of base* × *height*, or $V = Ah$

Because the base is always a circle, the area of the base is πr^2. That means the formula for the volume can be expressed as

$$\boxed{\text{Volume of a cylinder} = \pi r^2 \times h = \pi r^2 h}$$

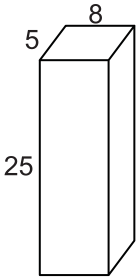


- Volume is often measured in cubic centimetres (cm^3) and cubic metres (m^3).

- 1 cm^3 is equivalent to the space occupied by a 1 cm × 1 cm × 1 cm cube.
- 1 m^3 is equivalent to the space occupied by a 1 m × 1 m × 1 m cube.

- For a hollow shape, such as a container, the **capacity** of the shape is the volume of material that the shape would hold. Capacity is usually expressed in millilitres (mL) and litres (L). To estimate the capacity of a container, assuming the walls of the container are so thin that they are not considered, you can use its volume.

For example, a rectangular prism container has an 8 cm × 5 cm base and a height of 25 cm. To estimate its capacity, you can follow these steps:



Find the volume: $V = Ah$

$= 8 \text{ cm} \times 5 \text{ cm} \times 25 \text{ cm}$

$= 1000 \text{ cm}^3$

Find the capacity: $1 \text{ cm}^3 = 1 \text{ mL}$

so, $1000 \text{ cm}^3 = 1000 \text{ mL}$

The capacity could also be described as 1 L because 1000 mL = 1 L.

- If the container were full of water, the mass of the water it held could be estimated at 1000 g or 1 kg (because 1 mL of water has a mass of 1 g).

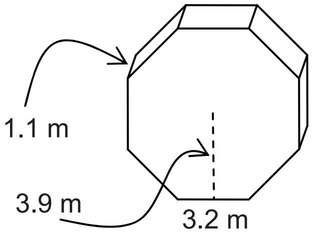
B. i) How do you know Dodo's structure in **part A** is a prism?

ii) How was finding the total number of cubes in Dodo's structure like finding its volume?

Examples

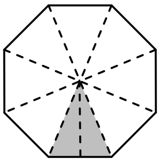
Example 1 Finding the Volume of an Octagon-Based Prism

Determine the volume of this prism, which has a height of 1.1 m. Its base is a regular octagon. The apothem (the distance from the midpoint of a side to the centre of the base) is 3.9 m and the sides of the octagon base are 3.2 m.



Solution

Divide the octagon base into 8 congruent triangles, each with base 3.2 m and height 3.9 m.



Area of each triangle:

$A = \frac{b \times h}{2} = \frac{3.2 \times 3.9}{2} = 6.24 \text{ m}^2$

Area of prism base:

$A = 8 \times 6.24 = 49.92 \text{ m}^2$

Volume of prism:

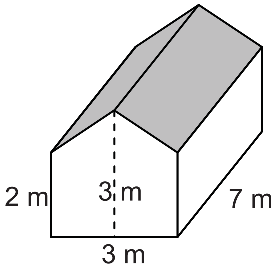
$V = Ah = 49.92 \times 1.1 \approx 54.91 \text{ m}^3$

Thinking

- To find the area of the base, I divided it into 8 congruent triangles because I knew the formula for the area of a triangle.
- I combined these triangles to get the area of the octagon base.
- I used the formula for the volume of a prism.

Example 2 Finding the Volume of a Pentagon-Based Prism

Find the volume of this building.

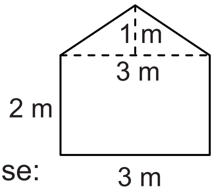


Solution

Area of pentagon base:

Area of triangle:

$A = \frac{b \times h}{2} = \frac{3 \times 1}{2} = 1.5 \text{ m}^2$



Area of rectangle:

$A = l \times w$

$= 3 \times 2 = 6 \text{ m}^2$

Area of pentagon base:

$A = 6 + 1.5 = 7.5 \text{ m}^2$

Volume of prism:

$V = Ah = 7.5 \times 7 = 52.5 \text{ m}^3$

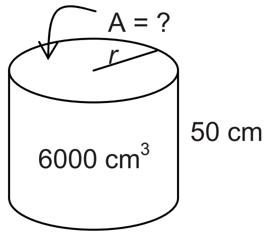
Thinking

- To find the area of the base, I divided it into a rectangle and a triangle, because I knew the formulas for the areas of those shapes.
- I found the two areas and then combined them to find the area of the base, which is the end face of the building.
- To find the volume, I multiplied the area of the base by the height, which, in this case, is the length of the building.

Example 3 Using Volume to Estimate Capacity and Mass

The mass of a cylindrical container is 7.2 kg when it is full of water and 1200 g when it is empty. It is 50 cm tall.

a) What is its diameter? **b)** What is its capacity in litres?

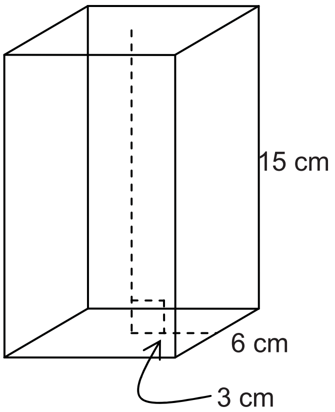
Solution	**Thinking**
a) *Volume of cylinder, using mass*: The mass of the water is the difference between the mass of the container when it is full of water and the mass of the container when empty: 7.2 kg = 7200 g 7200 g – 1200 g = 6000 g 6000 g of water has a volume of 6000 cm^3 so the volume of the container is 6000 cm^3.	 **a)** I knew that the mass of the water wouldn't help me find the diameter, but the volume would, so I converted the mass to grams because I knew 1 g of water had a volume of 1 cm^3.
Area of cylinder base, using volume: $V = Ah$ $6000 = A \times 50$ $\frac{6000}{50} = A$ $A = 120\text{ cm}^2$  	• I used the volume formula to find the area of the base circle.
Radius of cylinder, using area: $A = \pi r^2$ $120 = \pi r^2$ $\frac{120}{\pi} = r^2$ $r^2 \approx 38.20$ $r \approx \sqrt{38.2} \approx 6.18\text{ cm}$	• I used the formula for the area of the circle to find the radius.
Diameter of cylinder, using radius: $d = 2r$ $d \approx 2 \times 6.18 \approx 12.36\text{ cm}$ The diameter is approximately 12.36 cm.	• I knew that the diameter was twice the radius.
b) *Capacity of cylinder, using volume*: 6000 cm^3 = 6000 mL 6000 mL = 6 L The container has a capacity of 6 L.	**b)** I knew that 1 cm^3 = 1 mL so that 1000 mL = 1 L.

Practising and Applying

For each question, show your work.

1. Determine the volume of each prism. Each base is a regular polygon.

a)



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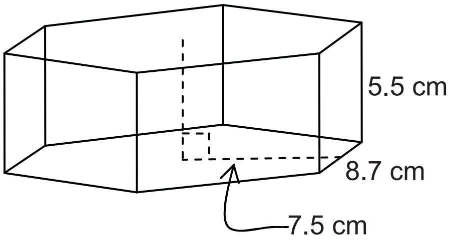

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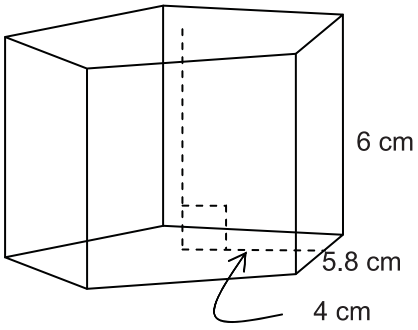
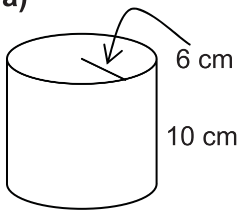


2. Determine the volume of each cylinder.

a)



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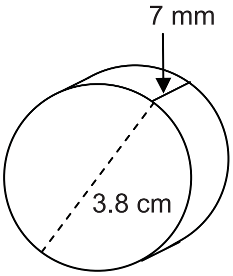
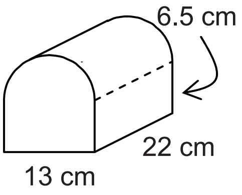
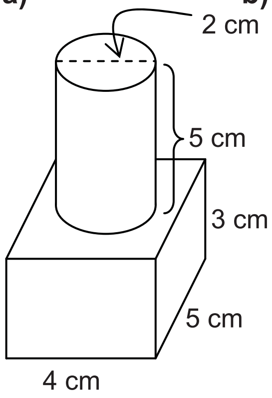


3. Terchu used this model to approximate the volume of a loaf of bread. Estimate the volume.



4. Determine the volume of each shape.

a)



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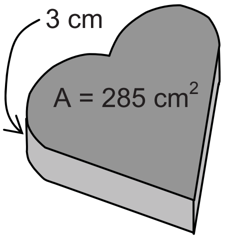
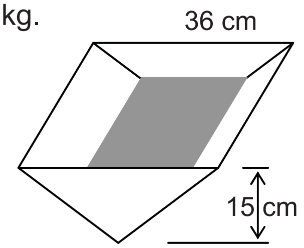


5. a) What is the height of a cylinder with diameter 11 cm and volume 450 cm^3?

b) What is the radius of a cylinder with height 54 cm and a capacity of 18 L?

6. This feeding trough is filled with water for animals to drink. When the trough is empty its mass is 3.7 kg. When it is full of water the mass is 32.6 kg.

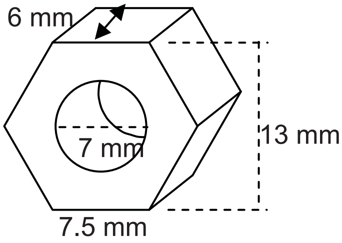


a) What is the capacity of the trough in litres and in millilitres?

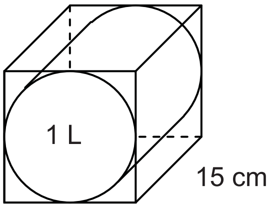
b) About how long is the trough?

7. Calculate the volume of plastic that is needed to make a 5 m pipe with outside diameter 3 cm and inside diameter 2.5 cm.

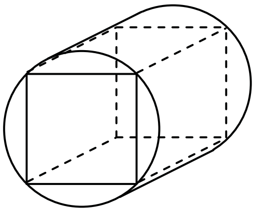
8. Calculate the volume of steel that would be needed to make 1000 of these regular-hexagon based nuts with dimensions as shown below.



9. What is the volume of the smallest square-based prism container that would hold this 1 L cylinder with height 15 cm?



10. What is the volume of the smallest cylindrical container that would hold a 1 L cube?



11. Explain how you could quadruple the volume of a prism by changing its dimensions. (Quadruple means to increase by a factor of 4.) Find more than one way. For each way, explain how it works.

GAME: Bean Counting

Bean Counting is a game for the whole class to play, with everyone working in small groups.

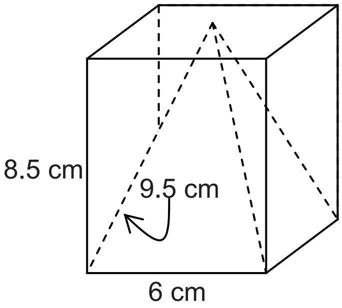
You will need rulers and a large glass jar full of dried beans. The goal is to estimate the number of beans in the jar.

- Each group
  - measures the height and diameter of the jar
  - takes a small handful of beans from the jar
  - uses the sample of beans and measurements of the jar to estimate how many beans there are in the full jar
- The jar is then filled again and the beans are counted.
- The group whose estimate is closest to the actual number of beans wins the game.

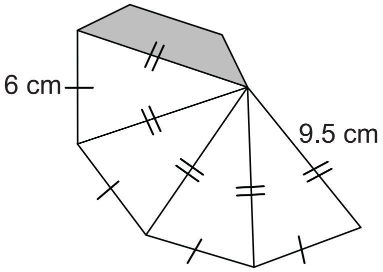
6.1.2 EXPLORE: Comparing Pyramid and Prism Capacities

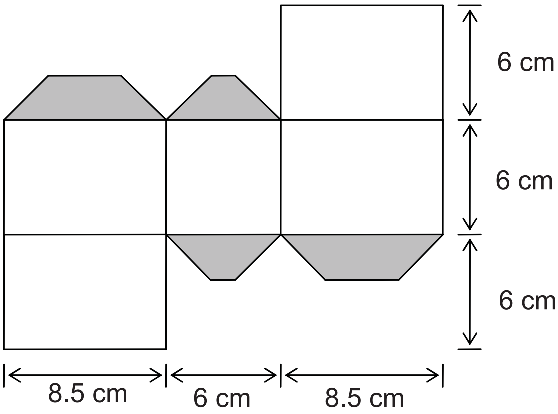
A square-based pyramid with a 6 cm × 6 cm base and 9.5 cm slanted edges fits perfectly inside a 6 cm × 6 cm × 8.5 cm square-based prism.

You can use stiff paper or cardboard to construct nets as described below for the pyramid and prism. All measurements are in centimetres. These nets can be cut, folded, and glued (or taped) to form their 3-D shapes.



Notice that each net is designed to have an open base, and that there are extra flaps (the shaded areas) for gluing or taping the faces together.





The 3-D shapes you make from these nets will be used below to determine volume relationships between prisms and pyramids.

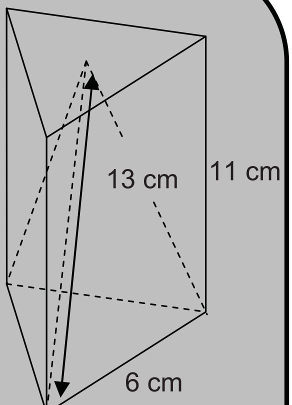
A. Construct the 3-D shapes described above.

i) Determine the volume of the prism.

ii) Fill the pyramid with rice or dried beans and then pour the beans into the prism. About how many times must you do this to fill the prism?

iii) Use your answers from **parts i) and ii)** to estimate the volume of the pyramid.

B. Design nets for the prism and pyramid at right. For both shapes, the triangular base is equilateral with 6 cm sides and the height is 11 cm. The slanted edge of the pyramid is 13 cm. Repeat **part A i), ii), and iii)** for this pair of 3-D shapes.



C. How does the volume of a pyramid relate to the volume of a prism with the same base and height?

6.1.3 Volume of Pyramids and Cones

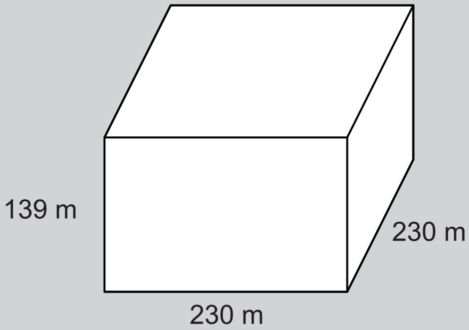
Try This

The Great Pyramid in Giza, Egypt is about 139 m tall with a square base that is about 230 m by 230 m. It was built using blocks of stone and then covered with a casing to make the faces smooth. The casing has since eroded away.

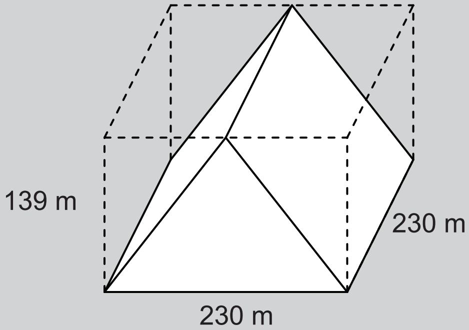
Imagine building a pyramid this size with cube blocks that have a volume of 1 m^3. The blocks can be cut if necessary.

The Great Pyramid in Giza

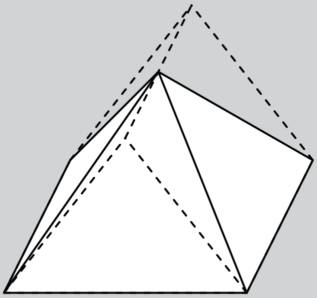
A. i) How many 1 m^3 blocks would you need to make a rectangular prism structure with a 230 m × 230 m base and a height of 139 m?



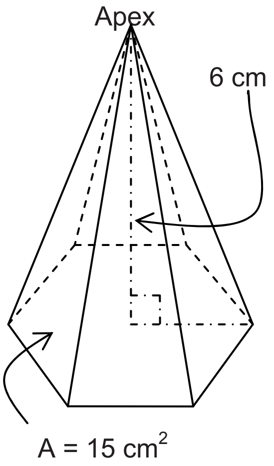
ii) How many 1 m^3 blocks would you need to make a triangle-based prism structure with the same height and base as the rectangular prism in **part i)**?



iii) Compare your answers to **parts i) and ii)** to help you estimate the number of blocks you would need to rebuild the pyramid in Giza. Explain how you estimated the number of blocks.



- A **pyramid** has a polygon base and a lateral surface that consists of triangles. The triangle faces all meet at a common vertex, the **apex**. A pyramid is named according to the shape of its base. For example, the hexagon-based pyramid shown here has seven faces: a regular hexagon base and six isosceles triangles that meet at the apex.



The volume of a pyramid is exactly one third the volume of a prism with the same base and height. That means you can use the formula for the volume of a prism, $V = Ah$, and the factor $\frac{1}{3}$ to create a formula for the volume of a pyramid:

$$\textit{Volume of a pyramid} = \frac{1}{3} \times \textit{Area of base} \times \textit{height}$$

$$V = \frac{1}{3}Ah$$

For example, to find the volume of the hexagon-based pyramid shown above:

$$V = \frac{1}{3}Ah$$

$$= \frac{1}{3} \times 15 \times 6 = 30 \text{ cm}^3$$

- **Cones**, which are not pyramids because of their curved lateral surface and circular base, are similar to pyramids in some ways: A cone has one base and a an apex like a pyramid and its volume is calculated in the same way as a pyramid.

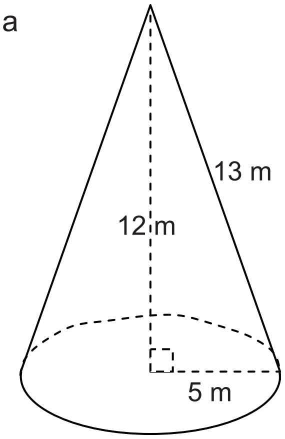
$$\textit{Volume of a cone} = \frac{1}{3} \times \textit{Area of base} \times \textit{height}$$

$$V = \frac{1}{3}Ah$$

This formula also makes sense if you compare a cone and a cylinder with the same base and height. The volume of the cone is exactly $\frac{1}{3}$ the volume of the cylinder.

Because the base of a cone is always a circle, you can replace A with the formula for the area of a circle, πr^2:

$$\textit{Volume of a cone} = \frac{1}{3} \times \pi r^2 \times h = \frac{1}{3}\pi r^2 h$$

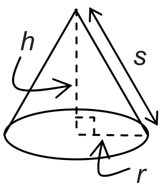


For example, to find the volume of the cone to the right:

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \pi \times 5^2 \times 12 = 100\pi \text{ cm}^3 \approx 314.16 \text{ cm}^3$$

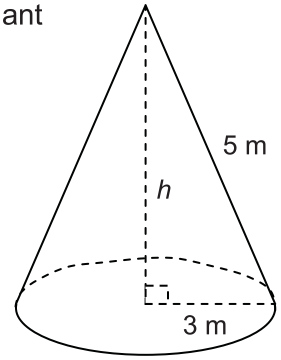
- For cones and pyramids you have to be careful about the height measurement. You may be given a **slant height**, *s*, but you need the actual height, *h*, to be able to calculate the volume.



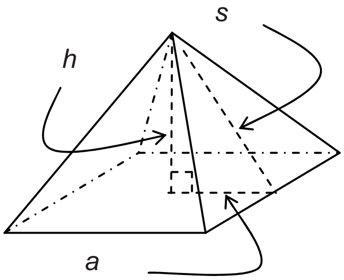
To determine the actual height of the cone given the slant height, you use *s* and the radius *r* in the Pythagorean theorem, $s^2 = r^2 + h^2$, and solve for *h*. This makes sense because the slant height is the hypotenuse of a right triangle.

For example, to find the height of a cone with slant height 5 m and radius 3 m:

$$\begin{aligned} s^2 &= r^2 + h^2 \\ 5^2 &= 3^2 + h^2 \\ 25 &= 9 + h \\ h^2 &= 16 \\ h &= 4 \text{ m} \end{aligned}$$

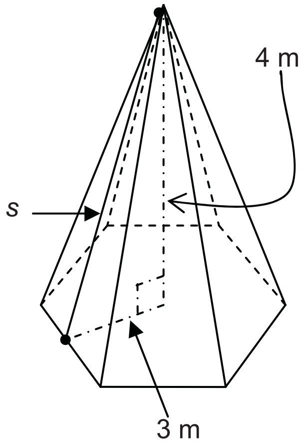


In the case of a pyramid with a regular polygon base, you can find the slant height, *s*, given the height *h* and the **apothem**, *a* (the distance from the centre of the base to the midpoint of one of the edges of the base), using the Pythagorean theorem: $s^2 = a^2 + h^2$.



For example, to find the slant height of the pyramid with height 4 m and apothem 3 m:

$$\begin{aligned} s^2 &= a^2 + h^2 \\ s^2 &= 3^2 + 4^2 \\ s^2 &= 25 \\ s &= 5 \text{ m} \end{aligned}$$

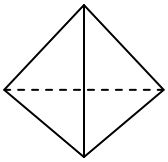


B. Use the formula for the volume of a pyramid to find the volume of the pyramid in Giza in cubic metres. How does it compare with your estimate in **part A iii)**?

Examples

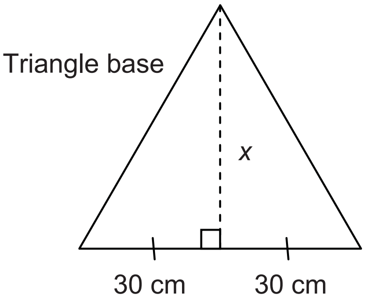
Example 1 Determining the Volume of a Triangle-Based Pyramid

A regular tetrahedron is a triangle-based pyramid whose faces are four equilateral triangles. In this tetrahedron, each edge is 60 cm long and the height of the pyramid is 49 cm. Determine the volume of this tetrahedron.



Solution

Area of triangle base:



Height of triangle base (x):

$$30^2 + x^2 = 60^2$$
$$900 + x^2 = 3600$$
$$x^2 = 2700$$
$$x = \sqrt{2700}$$
$$x \approx 51.96$$

Area of triangle base (height is x):

$$A = \frac{b \times x}{2} = \frac{60 \times 51.96}{2} \approx 1558.85 \text{ cm}^2$$

Volume of tetrahedron:

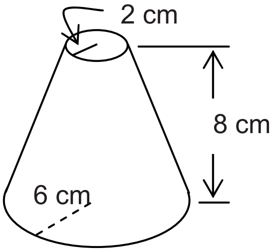
$$V = \frac{1}{3}Ah$$
$$\approx \frac{1}{3} \times 1558.85 \times 49$$
$$\approx 25{,}461.15 \text{ cm}^3$$

Thinking

- The formula for the volume of a pyramid involves its height and the area of its base. The height was given so I needed to find the area of the base.
- I knew the base length of the triangle base (60 cm) and needed the height to find its area.
- I used x for the height of the triangle base instead of h so I wouldn't confuse it with the height of the pyramid.
- I used the Pythagorean theorem to find the height of the triangle base.
- Once I had determined the height of the triangle, I could find its area.
- I used the formula for the volume of a pyramid.

Example 2 Determining the Volume of a Truncated Cone

Cutting off one of the vertices of a shape is called truncation. Determine the volume of this truncated cone. It is 8 cm tall. The radius of the base is 6 cm and the radius of the top circular face is 2 cm.

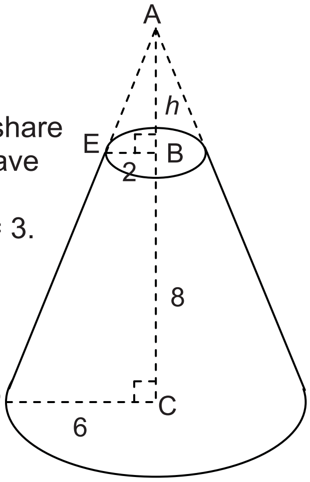


[Continued]

Example 2 Determining the Volume of a Truncated Cone [Continued]

Solution

Height of each cone, using similar triangles:



$\triangle ACD \sim \triangle ABE$ because they share an angle at A and they both have right angles (AAA). The scale factor is 3 because DC ÷ EB = 3.

$$\begin{aligned} 3 \times AB &= AC \\ 3h &= 8 + h \\ 3h - h &= 8 + h - h \\ 2h &= 8 \\ h &= 4 \end{aligned}$$

The height of the small cone is 4 cm.

The height of the large cone is 8 + 4 = 12 cm.

Volume of large and small cones:

Large cone: $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 6^2 \times 12 \approx 452.39 \text{ cm}^3$

Small cone: $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 2^2 \times 4 \approx 16.76 \text{ cm}^3$

Volume of truncated cone:

$V_{\text{large cone}} - V_{\text{small cone}} = 452.39 - 16.76 = 435.63 \text{ cm}^3$

Thinking

- I visualized the small cone that was cut off the top.
- I planned to subtract the volume of the small cone from the volume of the large cone.
- I wasn't given the height of the small cone, but I was able to use similar triangles to figure it out.
- I determined the volumes of the small and large cones and subtracted to find the volume of the truncated cone.
- I could have used π to report the volume exactly:

$144\pi \text{ cm}^3 - 5\frac{1}{3}\pi \text{ cm}^3$

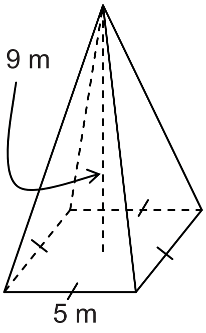
$= 138\frac{2}{3}\pi \text{ cm}^3$

Practising and Applying

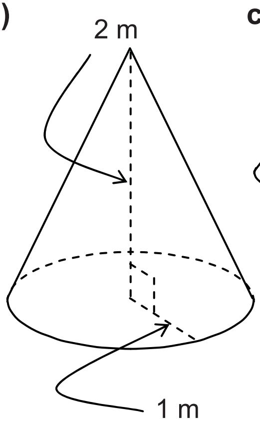
For each question, show your work.

1. Determine the volume of each shape. The bases of **a)** and **d)** are regular polygons.

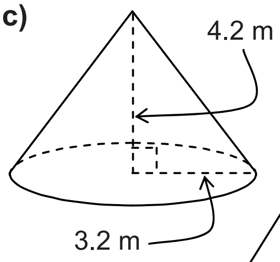
a)



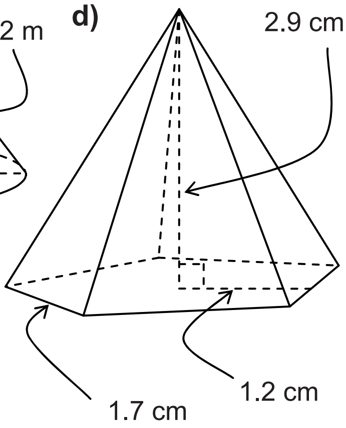
b)



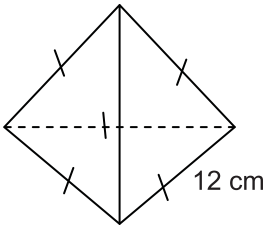
c)



d)



2. This triangle-based pyramid is a regular tetrahedron. The length of the sides of each triangular face is 12 cm.



a) Determine the area of the base triangle.

b) The height of the pyramid is 9.8 cm. What is the volume of the pyramid?

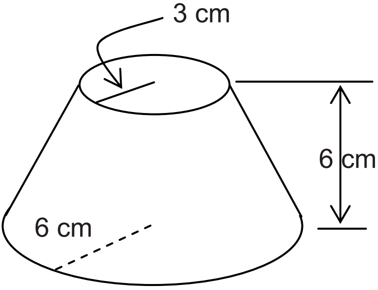
3. If you were to string a rope from the top of the Great Pyramid in Giza to the centre of a base edge, it would represent the slant height. How long would the rope be? (See the **Try This** at the beginning of the lesson.)

4. The pyramid on top of a clock tower is 120 cm tall and has a square base with 3 m sides. What is the volume of this pyramid in cubic metres?

5. a) What is the height of a cone with a base diameter of 11 cm and a volume of 450 cm^3?

b) What is the radius of a cone with height 54 cm and a capacity of 18 L?

6. This truncated cone was made by cutting a small cone off the top off a large cone with a 6 cm base radius.



a) What was the height of the large cone?

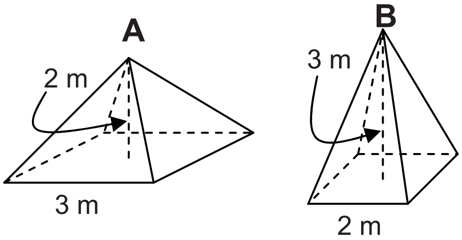
b) What was the height of the small cone that was cut off the top?

c) What was the volume of the large cone?

d) What was the volume of the small cone that was cut off the top?

e) What is the volume of the truncated cone?

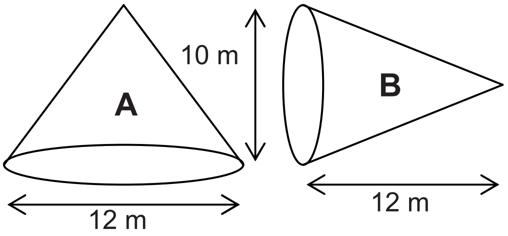
7. a) Determine the volume of each square-based pyramid.



b) Both pyramids have dimensions 2 m by 3 m. Why are their volumes different?

c) Determine the ratio of the volume of pyramid A to the volume of pyramid B, $V_A : V_B$.

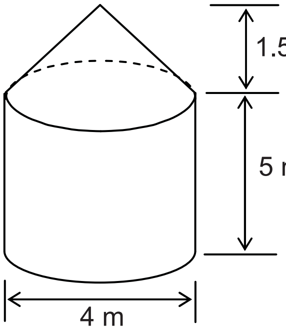
8. a) Determine the volume of each cone.



b) Both cones have dimensions 10 m by 12 m. Why are their volumes different?

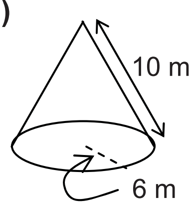
c) Determine the ratio of the volume of cone A to the volume of cone B, $V_A : V_B$.

9. What is the volume of grain that could be stored in this silo?

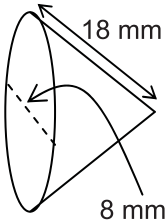


10. Determine the volume of each shape. Note that for **d)**, 5.8 cm refers to the length of the edge, not the slant height, and the bases of **c)** and **d)** are regular polygons.

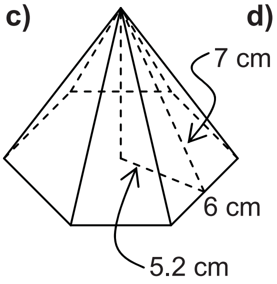
a)



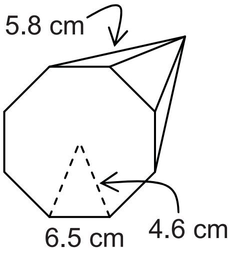
b)



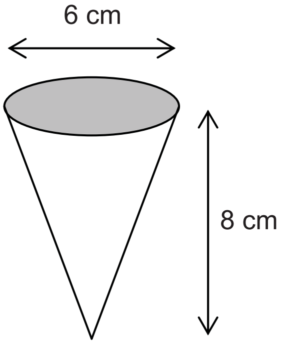
c)



d)



11. Design two smaller cone-shaped cups that would have a combined capacity equivalent to the capacity of this cup.

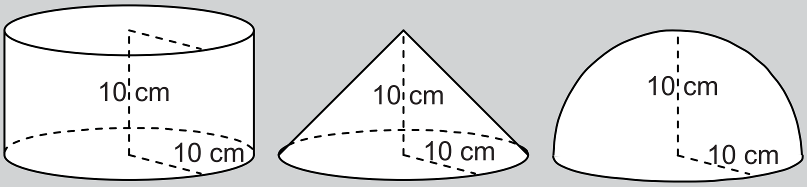


12. By just visually comparing a pyramid and a prism with the same base and height, it is obvious that the volume of a pyramid is less than half of the volume of the prism. Explain.

6.1.4 Volume of Spheres and Composite Shapes

Try This

Dawa asked for help organizing his wooden blocks in order of size. He did not know what to do with the three shown below. He noticed they were the same height and covered the same area on the table, but they looked different in size.



A. i) Determine the volume of the cylindrical shape.

ii) Determine the volume of the cone.

B. i) Is the volume of the third shape larger or smaller than the volume of the cylinder? Is it larger or smaller than the volume of the cone?

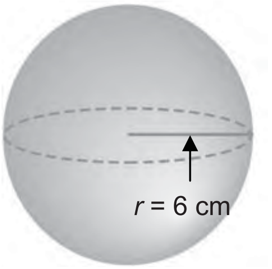
ii) Use your answers to **part A** to estimate the volume of the third shape.

- In mathematics, the shape of a ball is usually called a **sphere** (from the Greek word for ball). The volume of a sphere can be calculated using this formula:

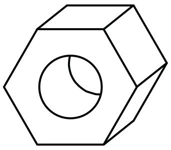
$$\text{Volume of a sphere} = \frac{4}{3} \times \pi \times r^3 = \frac{4}{3}\pi r^3$$

The sphere shown to the right has a radius of 6 cm, so its volume is

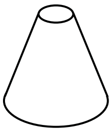
$$V = \frac{4}{3}\pi \times 6^3 = 288\pi \text{ cm}^3 \approx 904.78 \text{ cm}^3$$



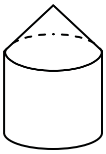
- Shapes that are constructed by combining or separating shapes can be called **composite shapes**. A hemisphere (half of a sphere) is an example of a 3-D composite shape. Other examples include the following:



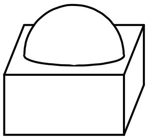
A nut is a hexagon-based prism with a cylinder cut out.



A truncated cone is made by cutting off the top of a cone.



This structure is made of a cylinder and a cone.



This structure is a rectangular prism with a hemisphere on top.

- To refer to the different parts of a 3-D shape, you can use subscripts. For the hexagonal nut shown above, the volume may be written and calculated as

$$V_{\text{nut}} = V_{\text{prism}} - V_{\text{cylinder}}$$

C. i) Determine the volume of Dawa's hemisphere from **part B**.

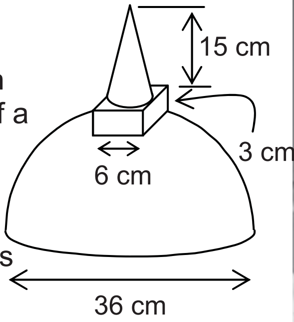
ii) How does its volume compare with the volume of the other two shapes?

Examples

Example Measuring the Volume of a Composite 3-D Shape

Buthri plans to build a model of the Swayambhunath Stupa in the Kathmandu valley. Her initial rough design is shown here. It consists of a cone, a square-based prism, and a hemisphere.

Estimate the amount of clay in litres that she would need in litres to complete her model.



Solution

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times (6 \div 2)^2 \times 15$$
$$= \frac{1}{3} \times \pi \times 9 \times 15 \approx 141.37 \text{ cm}^3$$

$$V_{\text{prism}} = Ah$$
$$= (6 \times 6) \times 3 = 108 \text{ cm}^3$$

$$V_{\text{whole sphere}} = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times (36 \div 2)^3$$
$$= \frac{4}{3} \times \pi \times 18^3$$
$$\approx 24{,}429.02 \text{ cm}^3$$

$$V_{\text{hemisphere}} = 24{,}429.02 \div 2$$
$$\approx 12{,}214.51 \text{ cm}^3$$

$$V = V_{\text{cone}} + V_{\text{prism}} + V_{\text{hemisphere}}$$
$$= 141.37 + 108 + 12{,}214.51$$
$$= 12{,}463.88 \text{ cm}^3$$

Convert volume to capacity:

$$12{,}463.88 \text{ cm}^3 = 12{,}463.88 \text{ mL}$$
$$\approx 12.46 \text{ L}$$

She would need about 12 L of clay.

Thinking

- To use the formula for the volume of the cone, I needed the radius, which was half the diameter (6 ÷ 2).
- I could have reported the cone's volume exactly as 45π cm^3 instead of 141.37 cm^3.
- For the prism's volume, I found the area of the base and multiplied by the height.
- I used the volume formula for a sphere and divided by two for the hemisphere. I divided the diameter by 2 because the formula uses the radius.
- I could have reported the sphere's volume exactly as 7776π cm^3 instead of 24,429.02 cm^3 and the hemisphere's volume as 3888π cm^3 instead of as 12,214.51 cm^3
- I added up the parts to get the volume of the model.
- I converted to millilitres using 1 cm^3 = 1 mL and then to litres using 1000 mL = 1 L.
- I rounded to the nearest litre because I only needed an estimate.

Practising and Applying

For each question, show your work.

1. Determine the volume of each.

a) a sphere with radius 1 m

b) a sphere with radius 2 m

c) a sphere with diameter 6 m

d) a sphere with diameter 8 m

2. a) Complete the chart below for the spheres in **question 1**. Express each volume exactly using π. The first one has been done for you.

Sphere	Radius (m)	Volume (m^3)
a)	1	$1\frac{1}{3}\pi$
b)	2	
c)	3	
d)	4	

b) By what factor does the volume of each sphere: sphere **b)**, sphere **c)**, and sphere **d)**, compare to the volume of sphere **a)**?

c) Describe the relationship between the factors in **part b)** and the radius of the spheres.

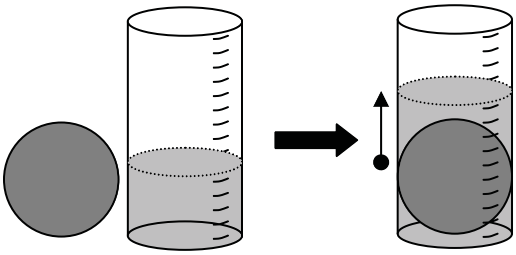
d) Use the relationship to predict the volume of a sphere with a radius of 5 m.

3. What happens to the volume of a sphere when you multiply its radius by a factor of n?

4. The planets are roughly spherical. The equatorial diameter of Earth is about 12,800 km. The equatorial diameter of Jupiter is about 143,000 km. How many times larger in volume is Jupiter than Earth?

5. Yangchen has a metal ball 5 cm in diameter. If it were melted and reformed as a cylinder 5 cm in diameter, how tall would the cylinder be? Explain.

6. In cooking and baking, the volume of some ingredients is sometimes measured by determining how much water is displaced when immersed in water in a measuring cup.



a) How much water would a ball with diameter 11 cm displace?

b) If the cylindrical measuring cup has a radius of 6 cm, determine the change in depth of the water when the ball is immersed.

7. Estimate the capacity in litres of this gas storage tank with a diameter of 90 cm and a total length of 3.4 m. It is roughly a cylinder with a hemisphere at each end.

(Hint: Use the radius of one of the hemispheres to figure out the length of the cylindrical part.)

8. The spherical cover of a lamp has a mass of 780 g and a diameter of 25 cm. If you were to use it to carry water, what would its mass be when it is full of water? Express your answer in kilograms.

9. A football with diameter 22.28 cm fits tightly inside a cube.

a) What is the volume of the ball?

b) What is the volume of the cube?

c) Approximately how many times larger is the cube than the ball?

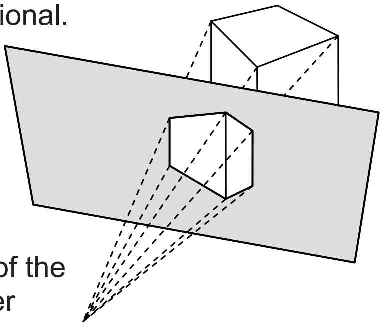
10. A sphere with radius 12 cm fits tightly inside a cylinder. Explain why the volume of the sphere is $\frac{2}{3}$ the capacity of the cylinder.

CONNECTIONS: Perspective

When painters and other artists represent 3-D objects on 2-D surfaces, they use perspective to make the objects appear three-dimensional.

For example, in this unit, there are many diagrams and photographs of 3-D objects, but they are all represented on 2-D surfaces, the pages of the book.

The following will help you get a sense of how perspective works.



1. Look through glass at a cube and trace the edges of the cube with your finger on the glass. Now draw on paper the shape that you traced with your finger.

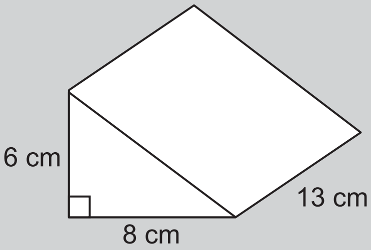
2. The shape you see depends on the orientation of the object in relation to your eye. That is, if you turn the object, it looks different. How can you position the cube so that its outline (the outermost edges) appears as a hexagon? as a square?

3. If you try the same thing with a cylinder, what different shapes can you form with its outline as you turn it behind the glass?

6.2.1 Surface Area of Prisms

Try This

Maya is making a model of this right triangle-based prism.



A. Find the length of the hypotenuse in the triangular base. Explain your method.

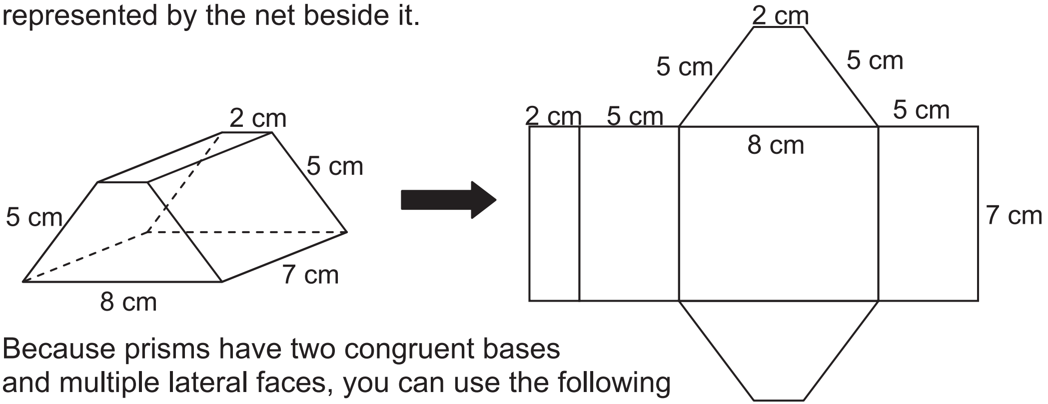
B. Sketch all the faces of the prism. Label your sketch with all of the dimensions.

The total **surface area** of a 3-D shape is determined by adding together the areas of all its surfaces.

Some other ways of thinking about surface area include the following:

- the area of the paper you would need to wrap or cover the entire shape
- the area of the net of the shape

For example, the total surface area of the trapezoid-based prism below can be represented by the net beside it.



Because prisms have two congruent bases and multiple lateral faces, you can use the following formula to find the total surface area:

SA_{prism} = 2 × Area of the base + Combined area of the lateral faces

You may notice that, in the net above, the combined area of the lateral faces is a large rectangle 7 cm (the height of the prism) by 20 cm (the perimeter of the base of the prism). Because this is true for any prism, you can modify the formula:

$SA_{\text{prism}} = 2 \times \text{Area of the base} + h \times \text{Perimeter of the base} = 2A + hP$

The units of a surface area measurement are two-dimensional units and therefore should always be in square units such as cm^2 and m^2 because area measures involve multiplying two dimensions.

C. Determine the total surface area of Maya's model prism in **part A**.

Examples

Example 1 Determining the Surface Area of a Rectangular Prism

Druk Air snacks are served in a rectangular box that measures 62 mm by 113 mm by 138 mm.

a) Determine the total surface area of the box in cm^2.

b) Explain why the total surface area is less than the area of cardboard needed to make the box.

a) Solution 1

Top or bottom face:

$A = bh = 13.8 \times 11.3$

$= 155.94\ \text{cm}^2$

Back or front face:

$A = bh = 13.8 \times 6.2$

$= 85.56\ \text{cm}^2$

Left or right face:

$A = bh = 11.3 \times 6.2$

$= 70.06\ \text{cm}^2$

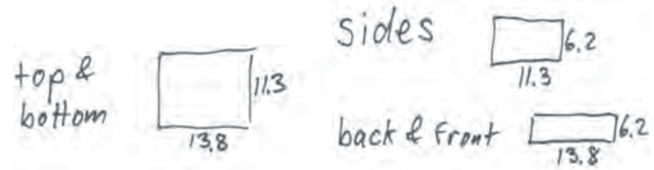
Total surface area:

$(155.94 + 85.56 + 70.06) \times 2$

$= 623.12\ \text{cm}^2$

Thinking

- I converted mm to cm because I needed to find the area in square centimetres.
- I sketched and labelled one face for each pair of dimensions and then determined the area of each.



- I calculated the sum of these areas and then multiplied by 2 because there were two of each.

a) Solution 2

SA = 2*lw* + 2*wh* + 2*lh*

Dimensions:

length: 138 mm = 13.8 cm

width: 113 mm = 11.3 cm

height: 62 mm = 6.2 cm

Using the formula:

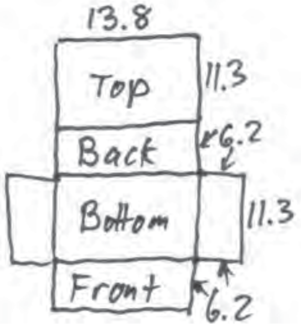
$SA = 2(13.8)(11.3) + 2(11.3)(6.2) + 2(13.8)(6.2)$

$= 311.88 + 140.12 + 171.12$

$= 623.12\ \text{cm}^2$

Thinking

- I used the formula I learned last year for the SA of a rectangular prism.
- I sketched a net to check that I included all the faces.
- It didn't matter which dimensions were length, width, and height. I just had to make sure I used them the same way each time.



b) Solution

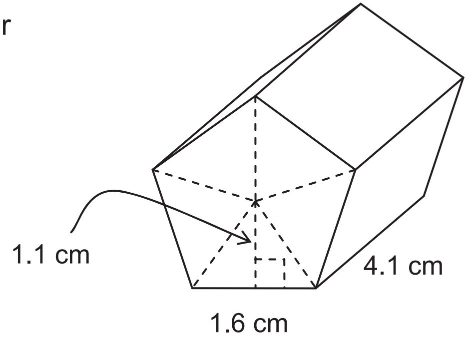
To construct the box, you need extra pieces to help attach the faces. These extra pieces are not part of the total surface area.

Thinking

I noticed that there are more pieces in the unfolded box than in the net. These extra pieces are used to fasten the parts together.

Example 2 Determining the Surface Area of a Pentagon-Based Prism

Determine the total surface area of this regular pentagon-based prism.



Solution 1

Area of prism base:

Divide it into five congruent triangles, find the area of one triangle, and multiply by 5.

$$\begin{aligned} A_{\text{base}} &= (1.6 \times 1.1 \div 2) \times 5 \\ &= 4.4 \text{ cm}^2 \end{aligned}$$

Perimeter of prism base:

Multiply the width of one rectangular face by 5.

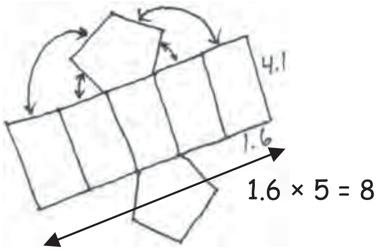
$P_{\text{base}} = 1.6 \times 5 = 8$ cm

Total surface area of prism:

$$\begin{aligned} SA &= 2A_{\text{base}} + h \times P_{\text{base}} \\ &= 2 \times 4.4 + 4.1 \times 8 \\ &= 41.6 \text{ cm}^2 \end{aligned}$$

Thinking

- To use the $P = 2A + hP$ formula for the surface area of a prism, I needed to find the area and perimeter of the base. The height of the prism was given, 4.1 cm.
- I visualized a net of the prism to help me figure out the dimensions I needed for the perimeter of the base.



Solution 2

Area of prism base:

Divide it into five congruent triangles, find the area of one triangle, and multiply by 5.

$$\begin{aligned} A_{\text{base}} &= (1.6 \times 1.1 \div 2) \times 5 \\ &= 4.4 \text{ cm}^2 \end{aligned}$$

Area of one rectangle face:

$A_{\text{rectangle}} = 1.6 \times 4.1 = 6.56 \text{ cm}^2$

Total surface area of prism:

$$\begin{aligned} SA &= 2A_{\text{base}} + A_{\text{lateral faces}} \\ &= 2 \times 4.4 + 5 \times 6.56 \\ &= 41.6 \text{ cm}^2 \end{aligned}$$

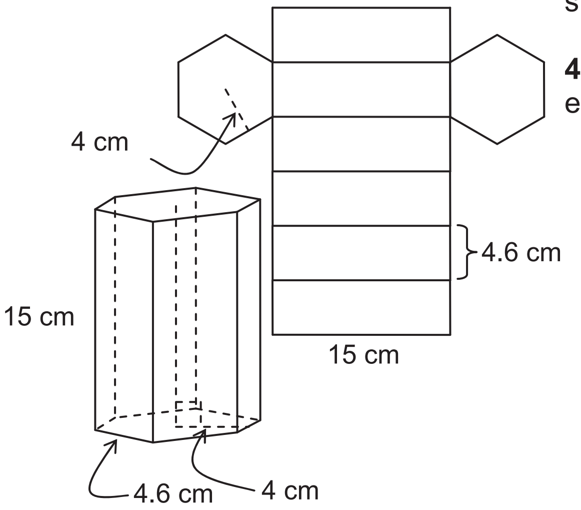
Thinking

- I knew the prism had two congruent pentagon bases and five congruent rectangle faces.
- To use the formula for the surface area of a prism, $SA = 2A_{\text{base}} + A_{\text{lateral faces}}$, I needed to find the area of one base and one lateral rectangle face.
- To find the area of one rectangle face, I multiplied the base side length, 1.6, by the height of the prism, 4.1.

Practising and Applying

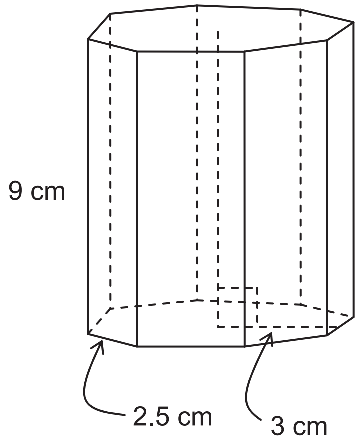
For each question, show your work.

1. The net of a regular hexagon-based prism is shown. Determine the total surface area of the prism.

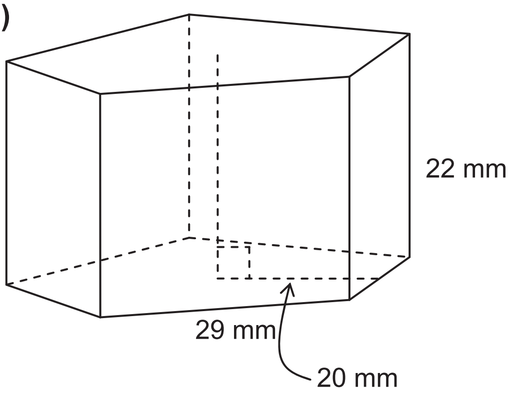


2. Determine the total surface area of each prism with a regular polygon base.

a)

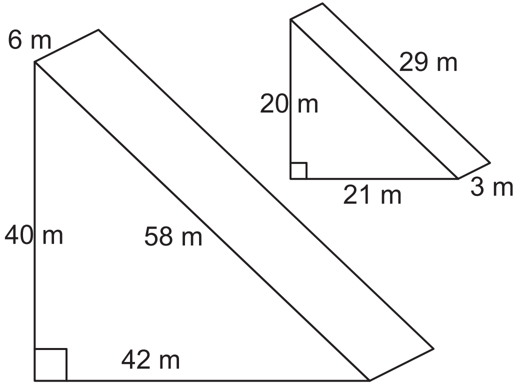


b)



3. Sketch a net for the prism in **question 2, part a)**. Use the net to explain your method for finding the total surface area.

4. a) Determine the total surface area of each right triangle-based prism.

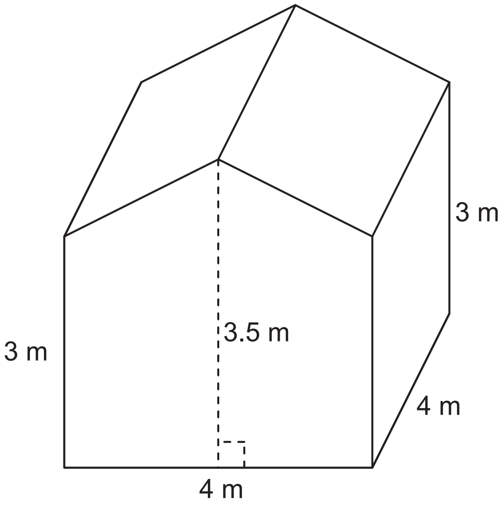


b) What is the ratio of the side lengths of the two prisms?

c) What is the ratio of their total surface areas?

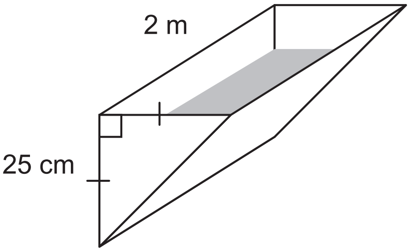
d) Why are the answers to **part b)** and **part c)** not equal?

5. a) Determine the total surface area of this tent, including the floor, in order to estimate the amount of canvas needed to construct it.
b) What is the volume of the tent?

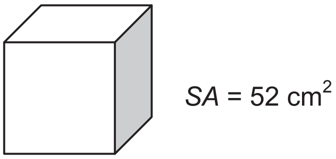


6. Dorji needs to cover a building with a metal roof 12 m by 12 m (including overhang). If he makes the peak of the roof 1.5 m higher than the edges, about how many square metres of metal roofing will he need?

7. How many square centimetres of metal would be needed to make this water trough?

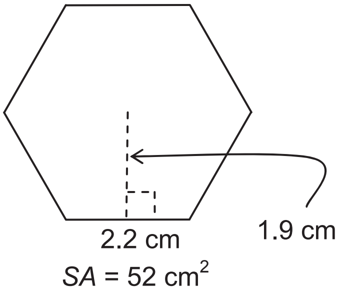


8. a) What is the side length of one face of a cube with total surface area 52 cm^2?

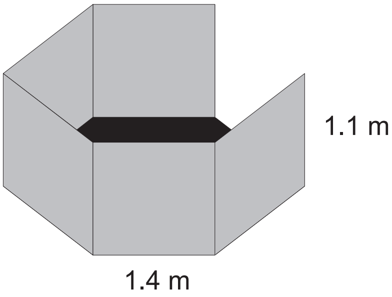


b) Design another rectangular prism (not a cube) with total surface area 52 cm^2.

c) What is the height of a prism with this regular hexagon base and with total surface area 52 cm^2?



9. Seldon plans to paint the outside walls of this regular hexagon-based prism traffic control booth. She needs to know the total surface area in order to know how much paint to buy. Determine the total surface area of the walls that need to be painted.



Note that one side of the booth is missing, as it is used as an entrance.

10. a) How can you determine the total surface area of a cube from its volume?

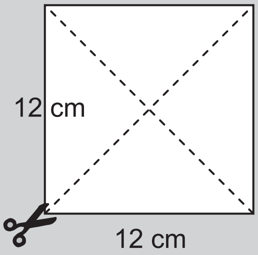
b) Can you determine the total surface area of other prisms from their volumes? Explain.

6.2.2 Surface Area of Pyramids

Try This

Tshewang describes a method for constructing a pyramid:

- Cut out a 12 cm × 12 cm square.
- Crease the diagonals (fold and then unfold).
- Cut along one crease from a corner to the centre.
- Overlap two of the triangles to fold it into a pyramid.



A. i) What is the area of Tshewang's square?

ii) What is the area of each triangle?

B. i) What is the shape of the base of the pyramid?

ii) What are the dimensions of the base?

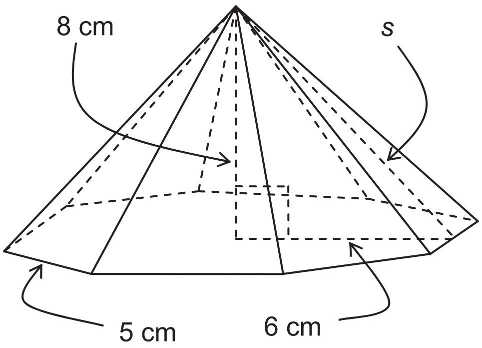
The total **surface area** of a pyramid is found in the same way as for a prism, by adding together the areas of all the faces.

- A pyramid has a polygon base and triangle lateral faces. In the case of a **right pyramid** (where the pyramid's vertex is centred directly over the base) with a regular polygon base, the lateral triangle faces are congruent isosceles triangles. You can use the following formula to find the total surface area:

$$SA_{\text{pyramid}} = \text{Area of the base} + \text{Combined area of the lateral faces}$$

- You will often have to calculate certain dimensions in order to determine the area of some of the faces. The Pythagorean theorem will help in most of these cases.

For example, for the regular octagon-based pyramid below, to find the area of the one of the congruent lateral triangle faces, you need the slant height of the pyramid, *s*, to be able to use the formula for the area of a triangle. The slant height of the pyramid is also the height of one of the triangle faces.



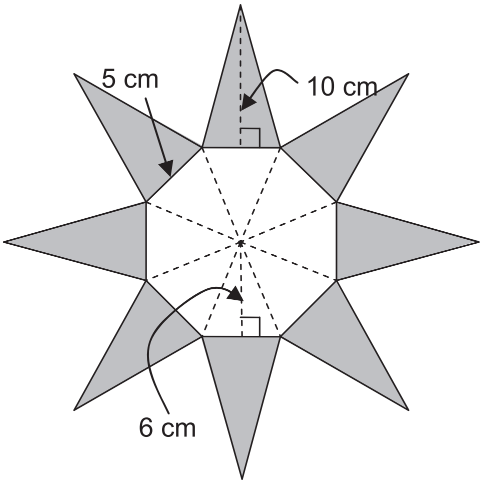
The slant height can be determined by applying the Pythagorean theorem to the triangle formed by the apothem, the height of the pyramid, and the slant height.

$$s^2 = 6^2 + 8^2$$
$$s^2 = 36 + 64$$
$$s^2 = 100$$
$$s = 10 \text{ cm}$$

- To help visualize the faces of a pyramid in order to determine their areas, it is sometimes helpful to sketch a net.

For example, the net below of the regular octagon-based pyramid from the previous page shows that the total surface area consists of a regular octagon base and eight congruent isosceles triangles (shaded), which form the lateral faces. To find the total surface area of this pyramid using the formula below, you need to find the area the base and the area of the lateral faces.

$$SA_{\text{pyramid}} = \text{Area of the base} + \text{Combined area of the lateral faces}$$



- To find the area of the octagon base, divide it into eight congruent isosceles triangles, find the area of one, and then multiply by 8:

$$\begin{aligned}\text{Area of one triangle in base} &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 5 \times 6 \\ &= 15 \text{ cm}^2\end{aligned}$$

Area of base = 8 × 15 = 120 cm^2

- To find the combined area of the lateral triangle faces, find the area of one, and then multiply by 8:

$$\begin{aligned}\text{Area of one lateral triangle face} &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 5 \times 10 \\ &= 25 \text{ cm}^2\end{aligned}$$

Area of lateral faces = 8 × 25 = 200 cm^2

SA of octagon-based pyramid = 120 + 200 = 320 cm^2

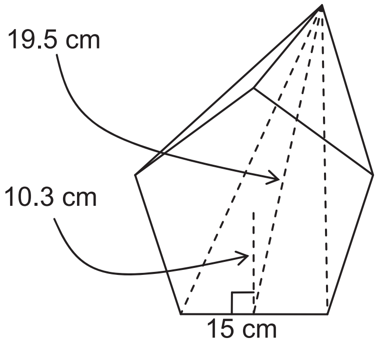
C. What is the surface area of the lateral faces of the pyramid from **part B**?

D. Suppose Tshewang's pyramid were a closed shape with a base. What would be the total surface area of the pyramid?

Examples

Example 1 Determining Surface Area of a Pentagon-Based Pyramid

Determine the total surface area of this regular pentagon-based pyramid.



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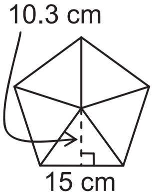
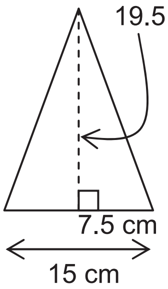
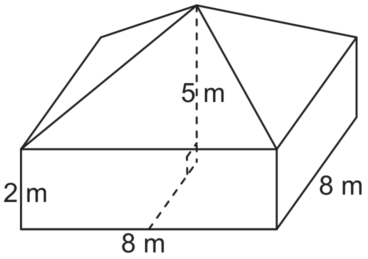
Example 1 Determining Surface Area of a Pentagon-Based Pyramid [Cont'd]

Solution	**Thinking**
SA = Area of the base + Combined area of the lateral faces	 • I knew I had to find the area of the prism base and the area of the triangle faces.
Area of the prism base:  Area of one triangle in the base: $A = \frac{bh}{2} = \frac{15 \times 10.3}{2}$ $= 77.25 \text{ cm}^2$ Area of prism base: $5 \times 77.25 = 386.25 \text{ cm}^2$	• I divided the prism base into five congruent triangles—each triangle had a height of 10.3 cm and a base of 15 cm. • I calculated the area of one triangle and then multiplied by 5 for the area of the prism base.
Combined area of lateral faces:  Area of one triangle face: $A = \frac{bh}{2} = \frac{19.5 \times 15}{2}$ $= 146.25 \text{ cm}^2$ Combined area of lateral faces: $5 \times 146.25 = 731.25 \text{ cm}^2$	• To find the area of the lateral triangle faces, I knew I only had to find the area of one triangle because they are all congruent. • I calculated the area of one triangle face and then multiplied by 5 to find the combined area of the triangle faces.
Total surface area of pyramid: $SA = 386.25 + 731.25 = 1117.5 \text{ cm}^2$	• I added the area of the base and area of the lateral faces to find the total surface area.

Example 2 Determining Surface Area of a Composite Shape

Determine the number of square metres of canvas needed to make this tent. The tent has no floor.



Solution

The amount of canvas needed is the combined areas of the lateral faces of the prism and the lateral faces of the pyramid.

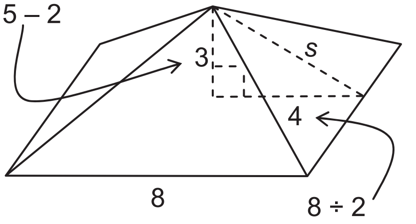
Combined area of lateral faces of prism:

Height of prism × Perimeter of base

$2 \times (4 \times 8) = 64 \text{ m}^2$

Combined area of lateral faces of pyramid:

Use the Pythagorean theorem to find slant height of the pyramid, which is the height of one triangle face.



$$s^2 = 3^2 + 4^2$$
$$= 9 + 16$$
$$= 25$$
$$s = 5 \text{ m}$$

Area of one triangle face (height is s):

$$A = \frac{b \times s}{2} = \frac{8 \times 5}{2} = 20 \text{ m}^2$$

Combined area of lateral faces:

$4 \times 20 = 80 \text{ m}^2$

Total amount of canvas needed:

$64 + 80 = 144 \text{ m}^2$

Thinking

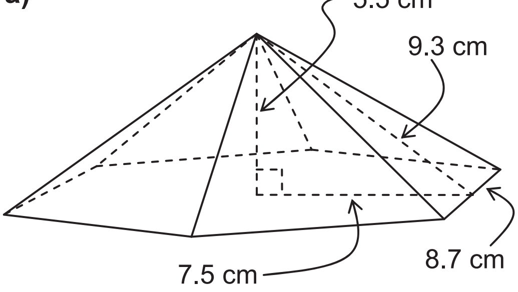
- I knew the tent was a pyramid with an open base on top of a square-based prism with open bases.
- To find the area of the prism's lateral faces, I multiplied the prism height by the perimeter of the base.
- To find the area of one triangle face, I needed to find the height, which was the slant height of the pyramid.
- Because the slant height was the hypotenuse of a right triangle, I was able to use Pythagoras to find it.
- I calculated the area of one face and then multiplied by 4 to find the combined area of the triangle faces, because the triangles are congruent.
- I added the areas of the lateral surfaces.

Practising and Applying

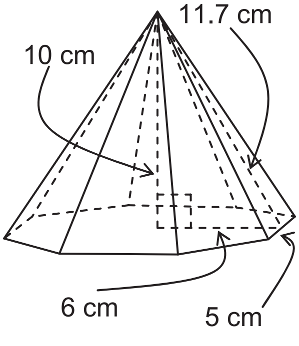
For each question, show your work.

1. Determine the total surface area of each pyramid with a regular polygon base.

a)



b)



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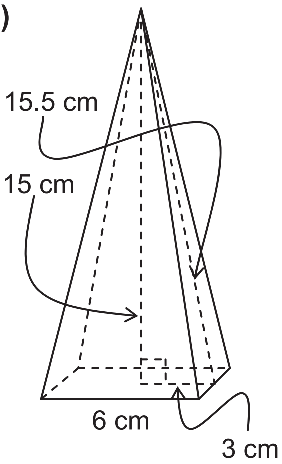
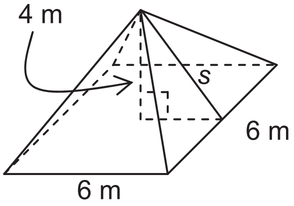
1. [Cont'd] **c)**



2. Sketch a net for the prism in **question 1, part a)**. Use the net to explain your method for finding the total surface area.

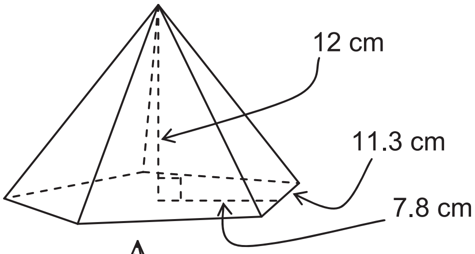
3. a) Determine the slant height, *s*, of this square-based pyramid.



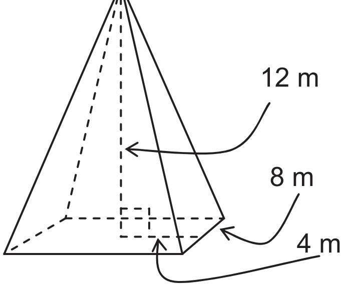
b) Determine the total surface area.

4. Determine the total surface area of each pyramid with a regular polygon base.

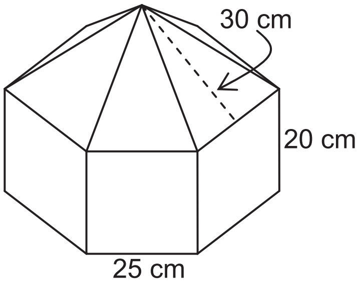
a)



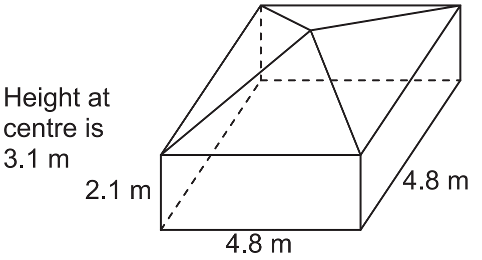
b)



5. Maya wants to paint the exterior of a sculpture with a regular hexagon base. The base itself does not need to be painted. Determine the number of square centimetres that needs to be painted.



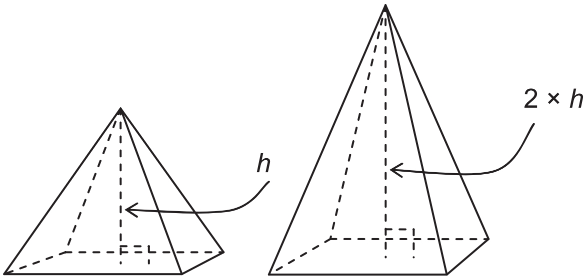
6. Gembo was asked to paint a house with the dimensions shown below. One 4 L can of roof paint covers 24 m^2 and one 4 L can of wall paint covers 28 m^2. How many cans of each type of paint will he need?



7. If you double the height of a pyramid without changing the base,

a) does the total surface area double? Explain.

b) does the volume double? Why?

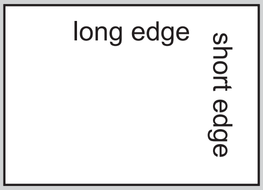


8. Explain why the combined area of the lateral faces of a pyramid must be greater than the area of the base.

6.2.3 Surface Area of Cylinders

Try This

Write "long edge" and "short edge" along the edges of a rectangular piece of paper 20 cm by 25 cm. You will curl this paper two different ways to make cylinders with open bases.



A. Curl the paper so that the two short edges touch.

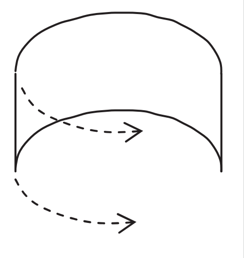
i) Which edge of the paper is the height of the cylinder?

ii) Which edge is the circumference of the cylinder?

B. Now curl the paper so that the two long edges touch.

i) Which edge of the paper is the height of the cylinder?

ii) Which edge is the circumference of the cylinder?



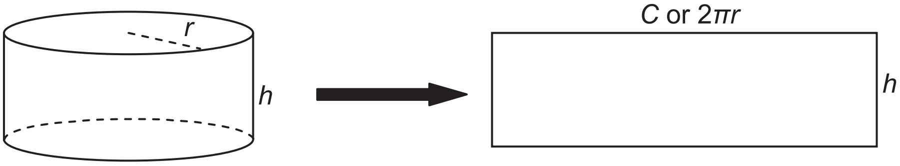
The total surface area of a cylinder is found by combining the area of its two circular bases and the area of its curved lateral surface:

$$SA_{\text{cylinder}} = 2 \times \text{Area of the base} + \text{Area of curved lateral surface}$$

Because the base is a circle, the formula for the area of the circle, $A = \pi r^2$, can be substituted into the formula:

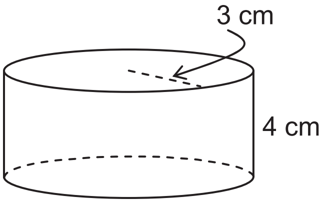
$$SA_{\text{cylinder}} = 2\pi r^2 + \text{Area of curved lateral surface}$$

The shape of the curved lateral surface is a rectangle when it is flattened. Its length is the height of the cylinder, h, and its width is the circumference of the circular base, C.



The formula for the circumference of a circle, $C = 2\pi r$, can be substituted into the surface area formula to modify it further:

$$\boxed{SA_{\text{cylinder}} = 2\pi r^2 + 2\pi r \times h = 2\pi r^2 + 2\pi rh}$$



All you need to find the total surface area of a cylinder is the radius of the base and the height of the cylinder.

For example, for the cylinder at right:

$SA = 2\pi r^2 + 2\pi rh = 2\pi(3)^2 + 2\pi(3)(4)$

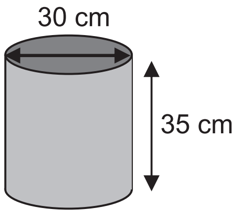
$= 18\pi + 24\pi = 42\pi \text{ cm}^2 \approx 131.95 \text{ cm}^2$

C. What is the surface area of each cylinder you made in **part A** and **part B** (not including the open bases)?

Examples

Example 1 Using the Formula for the Surface Area of a Cylinder

Determine the total surface area of this kerosene tank.



Solution

Dimensions:

$r = 15$ cm ($r = d \div 2 = 30 \div 2 = 15$)

$h = 35$ cm

Using the SA formula:

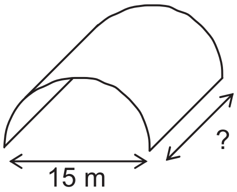
$$
\begin{aligned}
SA &= 2\pi r^2 + 2\pi rh \\
&= 2 \times \pi \times 15^2 + 2 \times \pi \times 15 \times 35 \\
&\approx 4712.39 \text{ cm}^2
\end{aligned}
$$

Thinking

- The formula that I used for the surface area of a cylinder uses the radius and the height.
- The diameter was given instead of the radius so I had to divide it by 2 to get the radius.
- I could have reported the surface area as 1500π cm^2 instead of as 4712.39 cm^2.

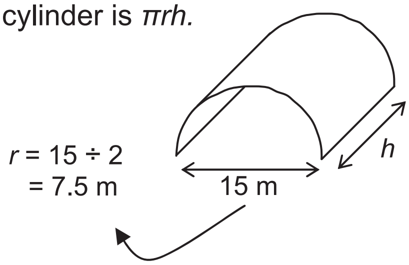
Example 2 Using the Surface Area Formula to Find the Height

A roof is made of half a cylinder with open bases. Its surface area is 630 m^2. Find the length of the roof.



Solution 1

$SA_{\text{whole cylinder}} = 2\pi r^2 + 2\pi rh$ so the area of the lateral surface of half a cylinder is πrh.



Height of half cylinder:

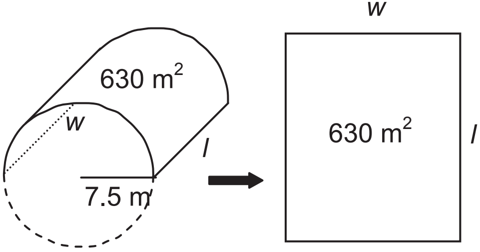
$$
\begin{aligned}
\pi rh &= 630 \\
\pi \times 7.5 \times h &= 630 \\
23.56 \times h &\approx 630 \\
h &= 630 \div 23.56 \\
h &\approx 26.74 \text{ m}
\end{aligned}
$$

The length of the roof is about 27 m.

Thinking

- Because the half cylinder had open bases, I used only the part of the surface area formula that applied to the lateral surface, $2\pi rh$, and then I divided it in half.
- I knew the height of the half cylinder was the length of the roof.
- I made an equation using the formula for the area of the lateral surface and the area of the roof that was given, 630 cm^2.
- Then I solved for h.
- I rounded to the nearest whole metre.

Solution 2



Width of rectangle (half the circumference of circular base):

$C = 2\pi r = 2\pi(7.5) \approx 47.12$ m

$w = C \div 2 \approx 23.56$ m

Length of rectangle:

$$\begin{aligned} A &= l \times w \\ 630\text{ m}^2 &= l \times 23.56 \\ l &= 630 \div 23.56 \\ &= 26.74\text{ m} \end{aligned}$$

The roof is about 27 m long.

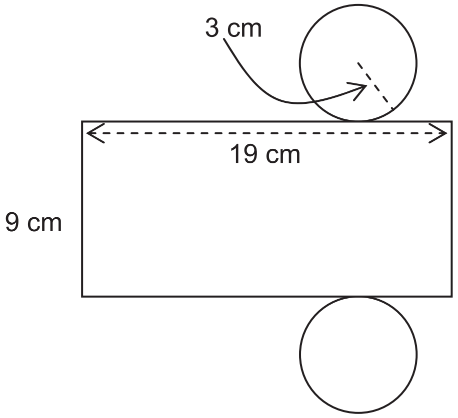
Thinking

- I thought of the curved lateral surface of the half cylinder as a rectangle.
- I knew I could use the formula for the area of a rectangle to find the length because I knew the area and I was able to figure out the width.
- The width of the rectangle was half the circumference of the cylinder. I used the radius to find the circumference and divided by 2 to find the width.
- I created an equation using the formula for the area of a rectangle. I substituted in the area and the width of the rectangle and then solved for the length.
- I rounded to the nearest whole metre.

Practising and Applying

For each question, show your work.

1. This net is used to form a cylinder.



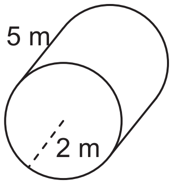
a) What is the height of the cylinder?

b) What is the circumference of each base circle?

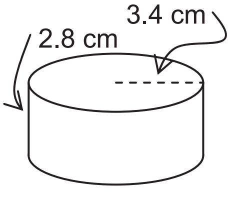
c) What is the area of each base?

2. Determine the total surface area of each cylinder.

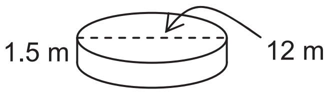
a)



b)



c)



3. A 20 cm by 25 cm rectangular piece of paper can be curved to make the lateral surface of a cylinder.

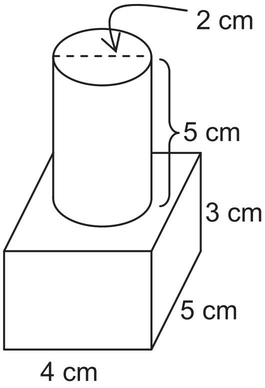
a) What is the area of the circular base if the paper is curved so that the 20 cm edges meet?

b) What is the area of the circular base if the paper is curved so that the 25 cm edges meet?

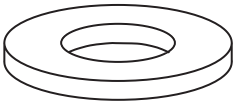
4. A cylindrical letterbox is 40 cm high with a diameter of 20 cm. Find the total surface area to estimate the amount of sheet metal it would take to construct the letterbox.

5. A prayer wheel has a radius of 23 cm and the area of its lateral surface is 9,500 cm^2. (Assume the wheel does not have a circular piece on the top or bottom.) What is the height of the prayer wheel?

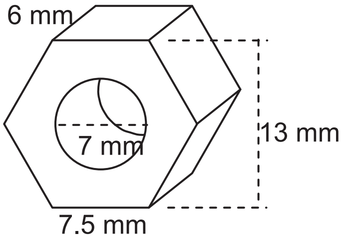
6. Determine the total surface area of the composite shape below.



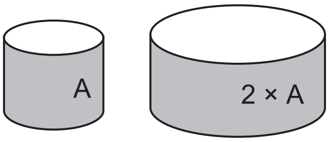
7. Determine the total surface area and volume of a cylindrical ring with outside diameter 22 mm, inside diameter 11 mm, and depth 2 mm.



8. A nut in the shape of a regular hexagon-based prism is 13 mm across with a 7 mm diameter hole, 7.5 mm sides, and a depth of 6 mm. The surface is plated with zinc to resist rust. Determine the number of square millimetres of zinc plating needed to coat one nut.



9. If the height of a cylinder remains constant but the area of the lateral surface is doubled, what will happen to the area of each circular base? Use an example to explain.

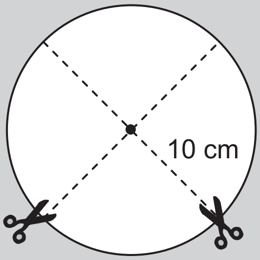


6.2.4 Surface Area of Cones

Try This

Tshewang describes a method for constructing a cone:

- Cut out a circle with diameter 20 cm.
- Draw two perpendicular diameters on your circle.
- Cut along two adjacent radii (half a diameter) to remove one of the quarters.
- Bend the remaining part of the circle until the two radii where you cut are touching each other and a cone is formed.



A. i) What is the area of Tshewang's circle?

ii) What is the area of the remaining sector after you cut out one quarter?

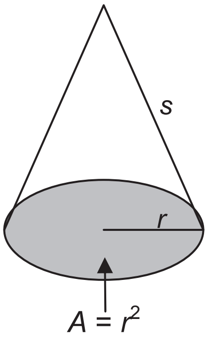
B. i) What is the shape of the base of the cone?

ii) How is the circumference of the base related to the circumference of the original circle? What is the circumference of the base?

iii) How is the radius of the base related to the radius of the original circle? What is the radius?

The total surface area of a cone is composed of a circular base and a partial circle, or **sector**, which forms the curved lateral surface:

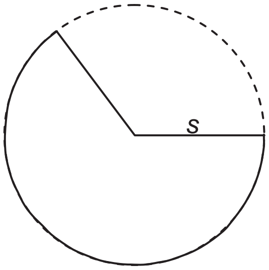
SA = Area of the base + Area of curved lateral surface



Since the base of the cone is a circle, the formula for the area of a circle can be substituted into the formula:

$SA = \pi r^2$ + Area of curved lateral surface

To make the formula easier to use, you can substitute an expression for the area of the curved lateral surface into the formula. It is helpful to visualize the curved lateral surface as a sector of a full circle. The slant height of the cone, s, is the radius of that full circle.



To find the area of the sector, you need to figure out what fraction of the area of the full circle is taken up by the sector. That fraction is the same as the sector's fraction of the circumference of the full circle:

$$\frac{\text{circumference of sector}}{\text{circumference of full circle}} = \frac{\text{area of sector}}{\text{area of full circle}}$$

This makes sense if, for example, you think of a sector that is half a circle. Its circumference will be half the circumference of the full circle and its area will be half the area of the full circle.

Since the circumference of the sector fits exactly onto the circumference of the base of the cone, you know it is $2\pi r$. You also know that the circumference of the full circle is $2\pi s$ (because its radius is s).

So, $\dfrac{\text{circumference of sector}}{\text{circumference of full circle}} = \dfrac{2\pi r}{2\pi s} = \dfrac{r}{s}$ and $\dfrac{\text{area of sector}}{\text{area of full circle}} = \dfrac{r}{s}$.

If the area of the full circle is πs^2, then the area of the sector is $\dfrac{r}{s} \times \pi s^2 = \pi rs$.

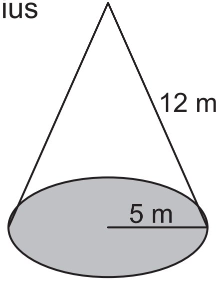
The area of the sector, πrs, which is the curved lateral surface, can then be substituted into the formula.

$$\boxed{SA_{\text{cone}} = \pi r^2 + \text{Area of curved lateral surface} = \pi r^2 + \pi rs}$$

All you need to find the total surface area of a cone is the radius of the base and the slant height of the cone.

For example, for the cone shown to the right:

$$\begin{aligned} SA &= \pi r^2 + \pi rs \\ &= \pi(5)^2 + \pi(5)(12) \\ &= 85\pi \text{ m}^2 \\ &\approx 267.04 \text{ m}^2 \end{aligned}$$

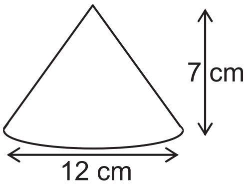


C. Assume that Tshewang's cone from **part B** has a base. What is the total surface area of his cone?

Examples

Example 1 Determining Surface Area of a Cone Given Height and Diameter

Determine the total surface area of a cone with diameter 12 cm and height 7 cm.

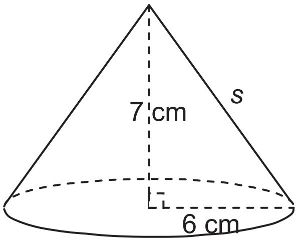


Solution

Radius:

d = 12 cm, so r = 6 cm

Slant height, using Pythagoras:



$$\begin{aligned} s^2 &= 7^2 + 6^2 \\ s^2 &= 85 \\ s &= \sqrt{85} \\ s &\approx 9.22 \end{aligned}$$

Thinking

- To use the surface area formula $SA = \pi r^2 + \pi rs$, I needed the radius and slant height.
- The radius was half the diameter.
- The slant height was the hypotenuse of a right triangle with legs that consist of the cone's height and the radius of the base. I used Pythagoras to find the slant height.

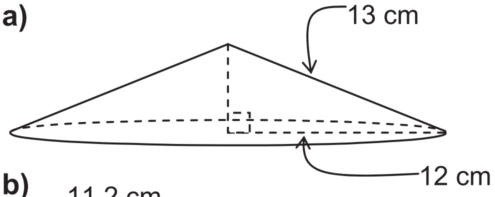
Solution	**Thinking**
$SA = \pi r^2 + \pi rs$ $= \pi r(r + s)$ $= \pi \times 6(6 + 9.22)$ $\approx 286.89\text{ cm}^2$ Total surface area ≈ 286.89 cm^2	• I rearranged the formula by finding a common factor for both terms, πr, because I thought it might be easier to work with that way.

Practising and Applying

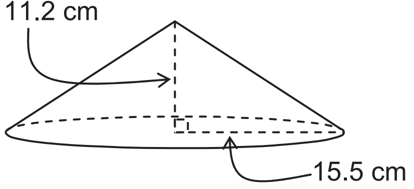
For each question, show your work.

1. Determine the total surface area.

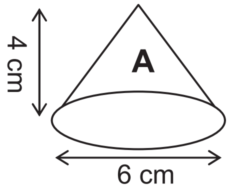
a)

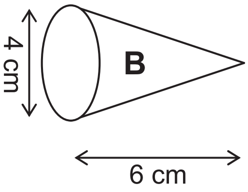


b)



2. Predict which cone has the greater total surface area. Calculate to check.





3. A cone shaped paper cup has a diameter of 7 cm and a depth of 12 cm.

a) What is the area of paper needed to make the cup?

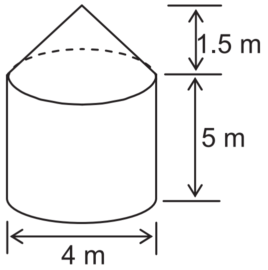
b) What is the capacity of the cup?

4. a) How much paper is needed to make a cone-shaped cup with a diameter of 6 cm and depth of 4 cm?

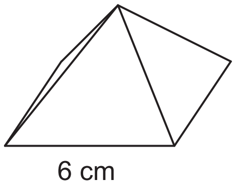
b) Terchu decides to see if two paper cups, each with a diameter of 3 cm and a depth of 4 cm, require the same amount of material as the single cup in **part a)**. What will he find out? Explain.

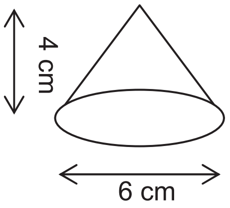
c) Terchu now wants to see if two cone-shaped paper cups, each with a diameter of 6 cm and a depth of 2 cm, require the same amount of material as the single cup in **part a)**. What will he find out? Explain.

5. How many square metres of sheet metal are needed to make this grain storage silo? (The floor is not included.)



6. a) Predict which shape below has the greater total surface area: a pyramid with a 6 cm by 6 cm base and height of 4 cm, or a cone with a base diameter of 6 cm and a height of 4 cm.





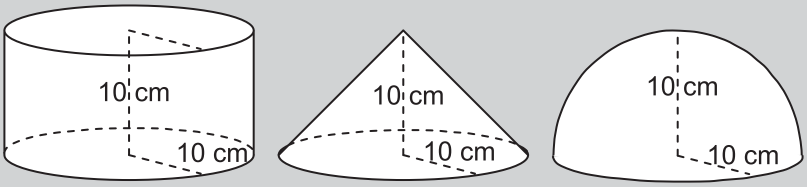
b) Calculate the total surface areas to check.

7. Use an example to help explain why the lateral surface of a cone must be greater than the area of its base.

6.2.5 Surface Area of Spheres

Try This

Dawa wants to paint his wooden blocks. He is wondering how the total surface areas of these blocks compare.



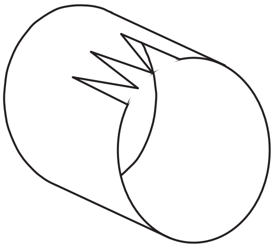
A. Determine the total surface areas of the cylinder and the cone.

B. i) Is the total surface area of the hemisphere larger or smaller than that of the cylinder? Is it larger or smaller than that of the cone? Explain.

ii) Use your answers to **part A** to estimate the total surface area of the third shape.

- The total surface area of a sphere is the same as the area of the lateral surface of a cylinder with the same diameter and a height equal to its diameter.

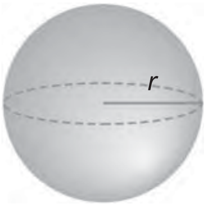
For example, in the photograph, a football (diameter 22 cm) is wrapped with a piece of paper representing the lateral surface of a cylinder (diameter 22 cm and height 22 cm). The area of the paper will have the same area as the total surface area of the sphere.



To show this is true, you can cut small triangles out of the cylinder and use these triangles to fill in the parts not covered, as you wrap the cylinder tightly around the ball.

- This relationship between the total surface area of a sphere and the lateral surface of a cylinder will help you understand the formula for the surface area of a sphere.

You know that the area of the lateral surface of a cylinder is $2\pi r \times h$. If the height is the same as the diameter then the height can be expressed as $2r$ and the area of the lateral surface is $2\pi r \times 2r$ or $4\pi r^2$. If the total surface area of the sphere has the same area, then the total surface area of a sphere must also be $4\pi r^2$.



$$\boxed{SA_{\text{sphere}} = 4\pi r^2}$$

For example, for a ball with a radius of 11 cm:

$$\begin{aligned} SA &= 4\pi r^2 \\ &= 4\pi(11)^2 \\ &= 484\pi \text{ cm}^2 \\ &\approx 1520.53 \text{ cm}^2 \end{aligned}$$

C. i) Determine the total surface area of Dawa's hemisphere from **part B**.

ii) How does its total surface area compare with the total surface areas of the other two shapes?

Examples

Example Determining the Surface Area of a Sphere

Compare the total surface areas of an official size football and a cube that the ball could fit inside. An official size football has a circumference of 69 cm.

Solution

Total surface area of ball:

Radius, using circumference

$C = \pi d$

If $C = 69$ cm, then $d = 21.96$ cm

If $d = 21.96$, then $r = 10.98$ cm

Total surface area of ball

$$SA = 4\pi r^2 = 4\pi(10.98)^2$$
$$\approx 1515.47 \text{ cm}^2$$

Total surface area of cube:

Side length, s, of each face is equal to the sphere's diameter, 21.96 cm.

Area of each face is s^2 so the total surface area of the cube is $6s^2$

$$SA = 6s^2 = 6 \times 21.96^2$$
$$\approx 2894.34 \text{ cm}^2$$

Compare total surface areas:

SA_{sphere} is 1515.47 cm^2

SA_{cube} is 2894.34 cm^2

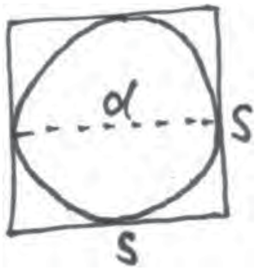
$2894.34 - 1515.47 = 1378.87 \text{ cm}^2$

$2894.34 \div 1515.47 \approx 1.91$

The cube's total surface area is about 1400 cm^2 greater than the ball's, which is about 1.91 times greater.

Thinking

- I needed to find the radius of the ball to be able to use the surface area formula.
- I used the formula for the circumference of a circle to find the diameter and then used the diameter to find the radius.
- I used the radius and the SA formula to find the total surface area of a sphere.
- I knew the diameter of the sphere was the same as the side length of one of the faces of the cube.



- The instructions didn't say how to compare the total surface areas, so I did it both ways I know:

- finding the difference
- finding the ratio

Practising and Applying

For each question, show your work.

1. Determine the total surface area of a sphere with each dimension.

a) radius 6 cm

b) diameter 18 m

c) radius 7.2 mm

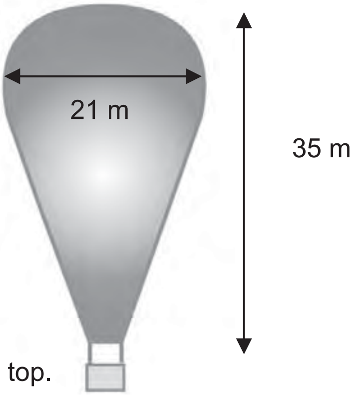
d) circumference 1 m

2. Traditionally, the Inuit of northern Canada lived in igloos, which are built from blocks of snow and have the shape of a hemisphere. These structures lose less heat compared to a structure shaped like a prism because of their minimal surface area. What is the total surface area of an igloo with diameter 4.5 m? Do not include the floor.

3. a) A farmer keeps grain in a silo with a hemisphere roof. Calculate the number of square metres that would need to be painted on a silo if the diameter was 3.8 m and the total height to the top of the hemisphere was 9.3 m.

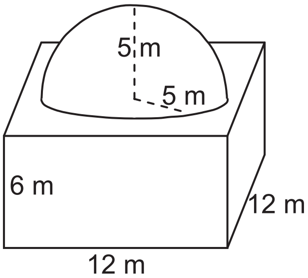
b) Estimate the volume of grain that this silo could hold.

4. Estimate the number of square metres of material needed to make this hot air balloon.



Hint:
Assume the balloon is an upside down cone with a hemisphere on top.

5. A new science centre is built with dimensions as shown. Determine the number of square metres of surface area that would need to be painted.



6. Determine the total surface area of this propane storage tank with diameter 90 cm and length 3.4 m. It is cylindrical with hemispheric ends.

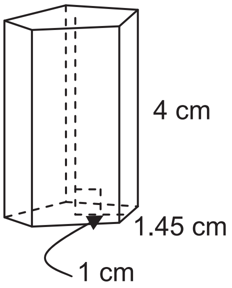
7. Describe the effect on the total surface area if you triple the diameter of a sphere. Explain using an example.

UNIT 6 Revision

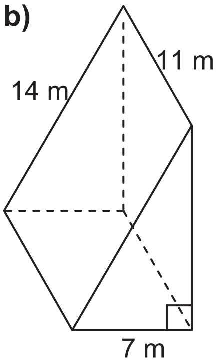
For each question, show your work.

1. Determine the volume of each shape. **Part a)** has a regular pentagon base.

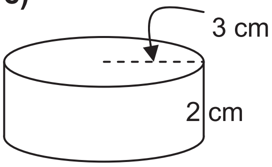
a)



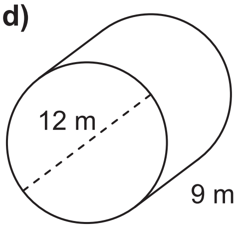
b)



c)



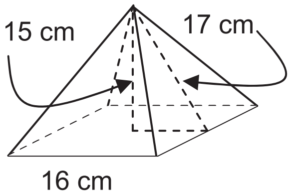
d)



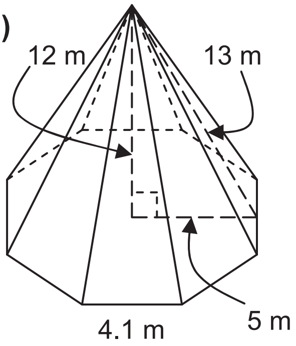
2. A cylindrical container has a diameter of 23 cm and a height of 14 cm. If the container's mass is 1300 g when empty, what will be its mass when it is full of water?

3. Determine the volume of each shape with a regular polygon base.

a)

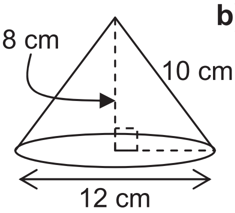


b)

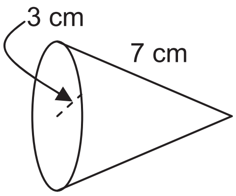


4. Determine the volume of each cone.

a)



b)



5. What is the volume of a sphere with diameter 28 mm?

6. Estimate the amount of ice cream in this ice cream cone in millilitres. The height of the cone is about 15 cm and the diameter of the top is about 7 cm.

7. Determine the total surface area of each shape in **question 1**.

8. Determine the total surface area of each pyramid in **question 3**.

9. Determine the total surface area of each cone in **question 4**.

10. The equatorial diameter of a model of the Earth is 25.6 cm. Find the total surface area of this model.

11. Which do you think is easier? Why?

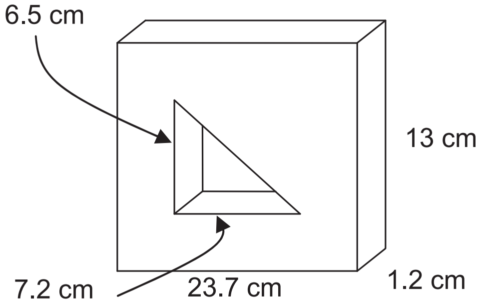
- Calculating the volume of a triangle-based prism
- Calculating the total surface area of a triangle-based prism

12. Determine the height of each.

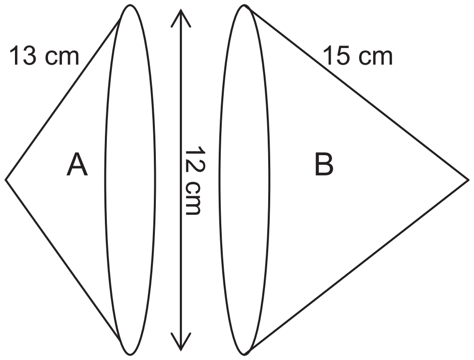
a) A square-based pyramid with a 12 m × 12 m base and volume 1440 m^3

b) A cylinder with radius 2.5 cm and total surface area 140 cm^2

13. This sculpture is a rectangular prism with a triangular hole in it. The triangle is a right-angled triangle. Determine the sculpture's volume and total surface area.



14. Two open cones (with no bases) each have a radius of 12 cm. Their slant heights are 13 cm and 15 cm.



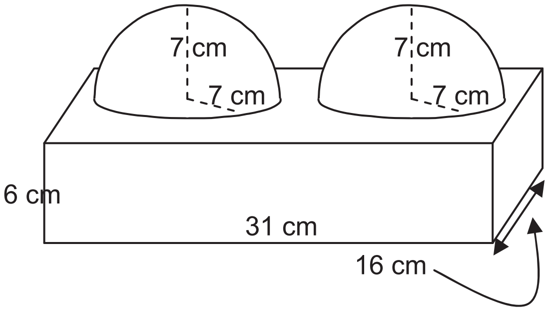
a) Calculate the total surface area of each.

b) What is the ratio of their slant heights, A ÷ B, as a decimal?

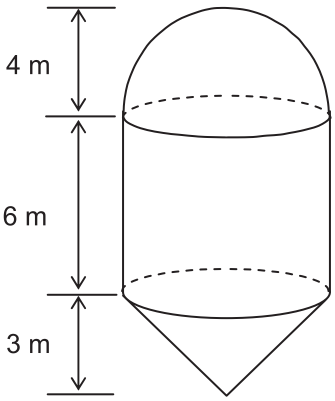
c) What is the ratio of their total surface areas, A ÷ B, as a decimal?

d) Explain the relationship between the two ratios from **part b)** and **part c)**.

15. Calculate the volume and total surface area of this model of a building. The radius of each hemisphere is 7 cm.



16. a) What is the volume of this rice storage container? The diameter is 8 m.



b) What is its total surface area?

17. Karma plans to build a model of the Stupa of the World Peace Pagoda.

a) What shapes would you suggest Karma use to estimate the total surface area and volume?

b) What would Karma need to measure on the Pagoda to design his model?

UNIT 7 COMMERCIAL MATHEMATICS

Getting Started

Use What You Know

FOR SALE

Gho for Nu 1000

Shoes for Nu 1500

A. Deki's father buys shoes and a gho. What percent of the total cost is each item?

i) the shoes **ii)** the gho

B. The selling price for the shoes is 5% greater than the store's cost price. The selling price for the gho is also 5% greater than the store's cost price.

i) Estimate the store's cost price for each item.

ii) Which pair of calculations below could you use to calculate the store's cost for the shoes and gho? Explain.

1500 × 1.05	1500 – 5%	1500 ÷ 1.05
1000 × 1.05	1000 – 5%	1000 ÷ 1.05

iii) Are the calculations you chose in **part ii)** equivalent to multiplying each selling price by 95%? Explain.

iv) Calculate the store's cost price for each item.

C. Deki's father spent 20% of his monthly income on the gho and the shoes. How much does he earn each month?

D. Suppose the store wants to make a 12% gain over its cost price on each item instead of the 5% gain it is making now. How much would they have to charge for each?

E. Use the original prices of Nu 1500 for shoes and Nu 1000 for the gho. The store decides to discount the price of the shoes. What percent would the discount have to be for the price of the shoes to be less than the price of the gho?

Skills You Will Need

Round all answers to the nearest ngultrum.

1. Calculate the discounted price for each.

a) Nu 250 price reduced by 15% **b)** Nu 325 price reduced by 8%

2. Calculate the selling price for each.

a) reduced by 20% to Nu 360 **b)** reduced by 15% to Nu 420

3. Calculate each commission.

a) 2% on sales of Nu 12,560 **b)** 3% on sales of Nu 23,220

4. Calculate the interest for each for one year.

a) 3% simple interest on Nu 13,320 **b)** 2% simple interest on Nu 8270

Chapter 1 Household Finances

7.1.1 Income and Expenditures

Try This

Tshering is saving money to buy a car that costs Nu 265,000. The bank will only lend him 75% of the cost of the car. That means he must save 25% of the cost. He earns Nu 10,300 a month and he spends Nu 9500 a month for his family's needs.

A. i) What percentage of his salary can he save each month?

ii) How long will it take him to save enough to pay 25% of the car's price?

- **Income** describes money you receive or earn that is available for you to spend. It can be received in different ways:

- salary and wages from employment
- commissions on sales
- allowances and bonuses from an employer
- rental income from land or buildings
- dividend income from shares invested in a company
- interest income from bank deposits
- proceeds from sales of goods and commodities
- gifts
- other sources

For example, Kinley earns a salary of Nu 9100 a month as well as an allowance of 35% of her pay. She also receives Nu 500 a year in interest income on her savings. Her annual income is calculated by adding the 12 months' salary, the allowance, and the interest income:

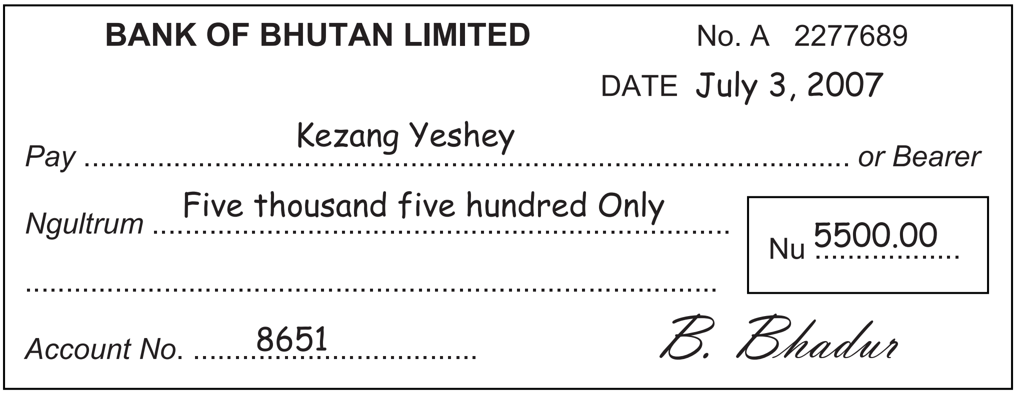
$12 \times \text{Nu } 9100 + 12 \times (0.35 \times \text{Nu } 9100) + \text{Nu } 500 = \text{Nu } 147{,}920$

- **Expenditures** or **expenses** are the money you need to spend for your everyday life. Some of the expenditures are for basic necessities in life such as food, clothing, shelter, while others are for comfort and recreational purposes such as car, TV, movies, etc.

- It is better if your expenditures are less than your income. That way you can save some of your money. One benefit of saving is that it allows you to have money you might need later on. Another benefit is that savings can grow if you invest them, for example, in a bank account that pays interest, so that you end up with more money than you started with.

- Many people keep money in the bank and issue cheques to pay for their expenses. A cheque is paid to a particular person or business. It is serially numbered, dated and the amount of money the cheque pays is written in both words and numbers.

For example, Bal Bhadur made out this cheque for Nu 5500 to pay Kezang Yeshey for his monthly rent.



BANK OF BHUTAN LIMITED No. A 2277689

DATE July 3, 2007

Pay Kezang Yeshey *or Bearer*

Ngultrum Five thousand five hundred Only

Nu 5500.00

Account No. 8651

B. Bhadur

B. From Tshering's point of view, is the money he will use to buy the car in **part A** an expenditure or income?

Examples

Example 1 Calculating Total Income

Pema's sources of annual income are shown on the right.

a) What is his average monthly income?

b) Estimate his average weekly income.

Salary	Nu 192,000
Rental income	Nu 60,000
Bank interest income	Nu 15,200

Solution

a) 192,000 + 60,000 + 15,200

= 267,200

267,200 ÷ 12 = 22,266.666…

His average monthly income is Nu 22,266.67.

b) 22,266.67 ÷ 4

≈ 22,000 ÷ 4

= 5500

His average weekly income is about Nu 5500.

Thinking

a) I knew that average monthly income is what his monthly income would be if his annual income were spread evenly over 12 months, so I added up his annual income from all sources and divided by 12.

b) I only needed an estimate for weekly income so I rounded the monthly income, 22,266.67 to 22,000, and then divided it by 4 (because there are about 4 weeks in a month).

Example 2 Calculating Interest and Dividend Income

Karma's family earned 5% interest this year on a bank investment of Nu 15,200. They also own Nu 22,000 worth of shares of a company that paid a 10% annual dividend. A dividend is a fee paid to all shareholders of the company, often once a year. How much more is the dividend income than the interest income?

[Continued]

Example 2 Calculating Interest and Dividend Income [Continued]

Solution	**Thinking**
10% of 22,000 vs. 5% of 15,200 10% of 22,000 is 2200 10% of 15,200 is 1520 so 5% is 1520 ÷ 2 = 760 2200 – 760 = 1440 They earned Nu 1440 more on dividend than interest income.	 I calculated 10% of 22,000 and 5% of 15,200 and compared results: • Finding 10% was easy to calculate mentally — I just divided 22,000 by 10. • I calculated the interest income, 5%, using mental math by taking half of 10% of 15,200.

Example 3 Calculating Expenditures

Sonam's family incurs these expenditures (in Nu) each month:

Rent	Food	Clothing	Car loan, gasoline	TV, phone, electricity	Other
5500	4200	400	2500	1000	1500

a) What is the total monthly expense of the family?

b) How much annual income does the family need if they want to save 10% of their income?

a) Solution	**Thinking**
5500 + 4200 + 400 + 2500 + 1000 + 1500 = 15,100 The monthly expense is Nu 15,100.	 I found the sum of all their expenses for the month.
b) Solution 1 $0.9x = 15,100$ $x = 15,100 \div 0.9 = 16,777.777\ldots$ They need a monthly income of about Nu 16,777. $12 \times 16,777.77... = 201,333.333\ldots$ They need an annual income of about Nu 201,333.	 • I wrote and solved an equation to find what their monthly income had to be—I let x represent monthly income and I knew that 90% (0.9) of x had to be 15,100 for them to save 10%.
b) Solution 2 $15,100 \times 12 = 181,200$ $\frac{90}{100} = \frac{181,200}{x} \rightarrow 90x = 18,120,000$ $x = 18,1200,00 \div 90$ $x = 201,333.333\ldots$ They need an annual income of about Nu 201,333.	 • I knew that if they needed Nu 15,100 a month, they'd need 12 times that amount for a year. • I created and solved a proportion with x representing the annual income.

Practising and Applying

1. Calculate the monthly and weekly incomes for each annual income.

a) Nu 110,000 **b)** Nu 80,000

c) Nu 142,000 **d)** Nu 156,000

2. Four families each set a goal to save a certain percentage of their monthly income. Calculate the savings goals, in ngultrums, for each family below.

a) Monthly income: Nu 8600
Savings goal: 4%

b) Monthly income: Nu 12,000
Savings goal: 8%

c) Monthly income: Nu 14,500
Savings goal: 10%

d) Monthly income: Nu 13,750
Savings goal: 9%

3. a) Predict which of these interest amounts is greatest without calculating. Explain your thinking.

i) 4.5% on Nu 12,200

ii) 3.8% on Nu 14,500

iii) 4.75% on Nu 22,100

iv) 4.95% on Nu 19,750

b) Calculate each interest amount to check your prediction.

c) Was your prediction correct? If not, explain why you might have been wrong.

4. Calculate each dividend income.

a) 9% on Nu 23,300

b) 11% on Nu 34,450

c) 12% on Nu 27,800

5. Four families' monthly expenditures and goals for savings are listed below. For each family, calculate how much income they need each month to cover their expenditures and meet their savings goals.

a) Expenditures: Nu 13,200
Savings on income: 10%

b) Expenditures: Nu 14,500
Savings on income: 12%

c) Expenditures: Nu 9200
Savings on income: 6%

d) Expenditures: Nu 8700
Savings on income: 4%

6. Four families calculated the percentage of their monthly income they spent on food. Calculate the monthly income for each family to the nearest ngultrum.

a) 27% on food costing Nu 2500

b) 33% on food costing Nu 2100

c) 29% on food costing Nu 2250

d) 31% on food costing Nu 2450

7. Write a cheque for Nu 5800 to an imaginary landlord, K. Dorji, to pay for the rent on your home for one month.

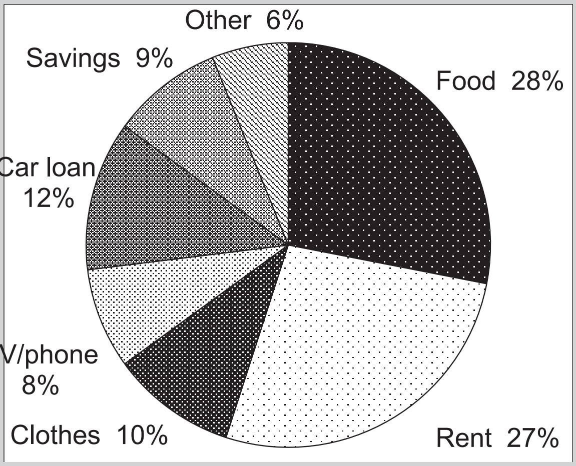
8. A family rents out two properties and earns Nu 4500 a month for each one. They borrowed money from a bank to buy the properties. The loan was for Nu 300,000 at an interest rate of 12% each year. They also pay insurance of Nu 15,000 a year for the two properties. What is their profit on the rental properties in the first year?

9. Why would a family want to keep track of both income and expenditures?

7.1.2 Budgets

Try This

This circle graph describes how Sonam's family uses its income.



A. Do they use more or less than 50% of their income for food and rent? Explain.

B. Use the graph to complete each sentence below.

i) They use about 15% for ____.

ii) They use about the same amount for____ as for _____.

iii) They use about $\frac{1}{3}$ of their income for ______.

A **budget** is a plan for spending your available funds. People use budgets to make sure that they do not get themselves into financial difficulty by spending more money than they have.

• A family or personal budget should consider expenditures in a variety of areas:

- necessities like rent, food, school supplies, clothing, and furniture
- recreation and improving the quality of life
- giving (to friends, relatives)
- savings

• Some people use weekly budgets; others use monthly budgets.

• If certain expenses are annual, monthly, or weekly, budgets have to account for the appropriate proportion of those expenses. For example, if you give to a certain charity once a year, you might account for $\frac{1}{12}$ of the amount each month to make sure you have enough in the month you want to donate.

• A budget does not have to be in a particular form, but many people use charts or computer spreadsheets. Spreadsheets are useful since they do the calculations for you; a small change can be made to one piece of information and any other calculations that depend on it are changed by the spreadsheet program.
Here is one possible chart for a monthly budget:

Income	**Salary**	**Allowance**					**Total**
	8400	1000					9400
Expenses	**Rent**	**Food**	**Loan**	**Clothing**	**TV/elect/ phone**	**Savings**	**Total**
	3000	3000	2000	200	800	400	9400

- Some people allocate their budget by percentages instead of ngultrum values. They decide that they want to spend no more than certain a percentage on different expenditure categories. An example is shown below.

Rent	Food	Bank loan	Clothing	Furniture/TV/ phone/power	Recreation	Other
30%	35%	10%	5%	10%	5%	5%

- Many people manage their money through bank accounts. The bank keeps detailed records of all transactions. An example is shown below.

Date	Particulars	Amount withdrawn	Amount deposited	Balance
June 1	Balance			4100.00
June 9	Cash		600.00	4700.00
June 12	Cheque	2200.00		2500.00
June 17	Cheque	1000.00		1500.00

- You might decide to use a similar chart to keep track of your own expenses that are not related to bank transactions.

Date	Particulars	Spent	Received	Balance
November 1	From mother		1000.00	1000.00
November 3	Snacks	100.00		900.00
…..				

C. Look back at the circle graph in **parts A and B.** What would be the advantages and disadvantages of using a circle graph to display information about a personal budget?

Examples

Example 1 Filling out a Budget Chart

Thinley's mother gave Thinley a budget chart to keep track of his spending.

Income source	**From ____**	**From ____**	**From ____**	**…**	**Total**
Amount					
Expenses	**For ______**	**For ______**	**For ______**	**….**	**Total**
Amount					

She had given him Nu 2000 to last for three months. He would need to pay Nu 250 once for school supplies, Nu 100 for soccer each month, and Nu 300 for toiletries once. Fill in the budget chart as Thinley should to show his income and expenses for one month.

[Continued]

Example 1 Filling out a Budget Chart [Continued]

Solution

Monthly income amount: 2000 ÷ 3 = 666.66…

Income source	**From mother**			**Total**
Amount	667			667
Expenses	**For school supplies**	**For soccer**	**For toiletries**	**Total**
Amount	83	100	100	283

250 ÷ 3 = 83.3

300 ÷ 3 = 100

Thinking

For those items (income and expenses) that happened only once in the three months, I divided by 3 to get a monthly amount.

Example 2 Creating a Budget to Meet Specifications

Kinzang earns a total of Nu 10,500 each month. How many ngultrums can he spend on each category?

Rent: 29%
Food: 31%
Household expenses: 15%
Savings: 10%
Other: 15%

Solution

Savings:

10% × 10,500 = 1050

Household expenses:

15% × 10,500 = 1575

Other:

15% × 10,500 = 1575

Rent:

0.29 × 10,500 = 3045

Food:

0.31 × 10,500 = 3255

Thinking

- Finding 10% is the same as dividing by 10 and I did that mentally.
- To find 15% mentally, I found 10% and then added on half of 10% (because 10% + 5% = 15%): 10% of 10,500 = 1050 and half of 1050 is 525. So 1050 + 525 = 1575.
- For rent and food, I changed 29% and 31% into decimals and then multiplied using a calculator.
- I checked my rent and food calculations by estimating that they would each be just less than one third of 10,500, which is 3500, and they were.

Practising and Applying

Use a template like this budget chart for the questions on the next page.

Income source	**From ____**	**From ____**	**From ____**	**…**	**Total**
Amount					
Expense	**For _____**	**For _____**	**For _____**	**…**	**Total**
Amount					

1. Fill out a monthly budget form including these items for each person

a) Sonam, a Class IX student

Spending money from parents for two months: Nu 1800

New clothes for two months: Nu 500

School supplies for two months: Nu 200

Recreation for two months: Nu 100

Snacks each month: Nu 300

Gift to give mother once: Nu 300

b) Dechen's father

Salary: Nu 10,200

Allowance: Nu 3500

Rent: Nu 5800

Food: Nu 3100

TV, phone, etc: Nu 700

Interest from bank: Nu 200

Clothing, toys: Nu 200

Money for Dechen: Nu 500

2. How much money will Sonam and Dechen's father each have left over after all of their expenditures in **question 1?**

3. Kinley earns a total of Nu 9500 from all sources each month. How much can he spend on each category if he spends the percentages listed below?

Rent: 30%

Food: 30%

Household expenses: 15%

Recreation: 5%

Savings: 8%

Other: 12%

4. Ugyen earns Nu 12,200 each month from all sources. Prepare a reasonable budget for Ugyen.

5. a) Why is a budget especially useful for someone who is not earning a lot of money?

b) Why might it also be useful for a person who has a comfortable income?

6. You are planning a personal budget.

a) What categories would you use? Why?

b) How could your budget be useful?

GAME: Lucky Shopper

Smart Shopping is a game that will allow you to practise mathematical skills.

- Play the game with a partner. Use one die.
- In each round, roll the die three times to get the three digits of the price of an item and then roll twice more to get the two digits of a percentage discount. You can use the first three rolls in any order to create the price and the second two numbers in any order to get the discount.
- Calculate a sale price using the price and the discount.
- The player whose sale price is lowest wins the round and gets 1 point.
- The first player to get 10 points wins the game.

For example, Player A rolls 1, 2, and 4 and then rolls 2 and 2, and Player B rolls 4, 6, and 2 and then 5 and 3. Player A calculates a discount of 22% on Nu 124 and Player B calculates a discount of 53% on Nu 246.

Player A's price: Nu 97   Player B's price: Nu 116

Player A wins since Nu 97 < Nu 116.

Chapter 2 Taxes

7.2.1 Reporting Income and Taxes

Try This

Yangki earns Nu 262,000 each year and pays Nu 10,080 for personal income tax.

A. What percentage of Yangki's annual income does she pay in taxes?

You must keep track of all the income you earn or receive in order to determine how much you must pay the government in **PIT** (**personal income tax**). The amount that you owe is a percentage of your income. The percentage depends on the level of your income. Generally, the more money you make, the more tax you pay. In 2016, the percentages for each tax slab were as shown below in the chart. The amount of tax payable is calculated once a year when citizens report income to the government.

Annual Income Slab	2005 PIT Rate
Up to Nu 200,000	0%
Nu 200,001 to 250,000	0% on the first Nu 200,000 10% on the rest
Nu 250,001 to 500,000	0% on the first Nu 200,000 10% on the next Nu 50,000 15% on the rest
Nu 500,001 to 1,000,000	0% on the first Nu 200,000 10% on the next Nu 50,000 15% on the next Nu 250,000 20% on the rest
More than Nu 1,000,000	0% on the first Nu 200,000 10% on the next Nu 50,000 15% on the next Nu 250,000 20% on the next Nu 500,000 25% on the rest

- In Bhutan, personal income tax is charged on six types of income: salary, rental income, dividend income, cash crop income and income from other sources (For example, commissions, leave encasements and other benefits).
- Some other income is taxable but not at the same rates as above. For example, if some of your income is based on owning a property that is rented out, the government recognizes that there are expenses associated with maintaining that property and takes that into account by applying a lower tax rate.

Taxes are often withheld or deducted before your employer pays you. **TDS** stands for **tax deducted at source**. This is done to make sure that the government has a constant inflow of money and to make sure people do not get into a situation where they have not saved enough money to pay their taxes. TDS is withheld on monthly income.

- TDS rates are different for different sources of income. For example, there is a 5% TDS rate for interest income, a 10% TDS rate for dividend income and a 5% TDS rate for rental income.
- There are different kinds of income for tax purposes:

- **Gross income**: Total income earned or received in a particular category, for example, gross salary income or gross rental income.
- **Adjusted gross income**: Gross income adjusted for certain deductions.
- **Net taxable income**: Adjusted gross income less additional allowable deductions (see **lesson 7.2.2**). This is the amount used to determine how much tax should be paid.

- Personal tax rates are different from business tax rates.

B. Why does it make sense that the percentage Yangki pays in taxes, calculated in **part A**, is not one of the percentages listed in the tax chart (0%, 10%, 15%, 20%, or 25%)?

Examples

Example 1 Calculating Tax on Net Taxable Income

a) What would be the tax owing (or tax payable) for each net taxable income? Use the PIT chart on **page 274** for the tax rates.

i) Nu 180,000 **ii)** Nu 260,000 **ii)** Nu 530,000

b) What percentage, to the nearest tenth of a percent, of the net taxable income is the tax owing?

Solution

a) i) 180,000 is less than 200,000, which is in the first tax slab.

Total tax owing is Nu 0.

ii) 260,000 is between 250,001 and 500,000, which is in the third tax slab.

260,000 – 200,000 = 60,000

10% of 50,000 = 0.10 × 50,000 = Nu 5,000

15% of 10,000 = 0.15 × 10,000 = Nu 1,500

Total tax owing is Nu 6,500.

iii) 530,000 is between 500,001 and 1,000,000, which is in the fourth tax slab.

530,000 – 200,000 = 330,000

10% of 50,000 = 0.10 × 50,000 = Nu 5,000

15% of 250,000 = 0.15 × 250,000 = Nu 37,500

20% of 30,000 = 0.20 × 30,000 = Nu 6,000

Total tax owing is Nu 48,500. [Continued]

Thinking

- For each amount, I first found the tax slab it was in.
- I then subtracted Nu 200,000 from the income because the first Nu 200,000 in all tax slabs is not taxed.
- The last thing I did was apply the percentages shown in the chart for each slab to the different parts of the remaining income and add the amounts.

Example 1 Calculating Tax on Net Taxable Income [Continued]

Solution	Thinking
b) i) $\frac{0}{180,000} = 0$ The tax is 0% of the income. **ii)** $\frac{6,500}{260,000} \approx 0.025$ The tax is about 2.5% of the income. **iii)** $\frac{48,500}{530,000} \approx 0.0915$ The tax is about 9.15% of the income.	For each amount, I divided the tax paid by the income to get a decimal and then rewrote the decimal as a percentage. That told me what percentage of the net taxable income was tax.

Example 2 Calculating TDS

Calculate the TDS (tax deducted at source) for each of these monthly income amounts:

a) interest income of Nu 5680

b) dividend income of Nu 2800

c) rental income of Nu 100,500

d) salary of Nu 11,500 (at a rate of about 1.7%)

Solution	Thinking
a) 5% of Nu 5680 5680 ÷ 10 = 568, 568 ÷ 2 = 284 TDS is Nu 284. **b)** 10% of Nu 2800 2800 ÷ 10 = 280 TDS is Nu 280. **c)** 5% of Nu 100,500 100,500 ÷ 10 = 10050 10050 ÷ 2 = 5025 TDS is Nu 5025. **d)** 1.7% of 11,500 .017 [×] 11500 [=] 195.5 TDS is Nu 195.50.	 · I remembered the different percentages for deductions at source for **parts a) to c)** and the rate for **part d)** was given: 10% for dividend income, 5% for rental income, 5% for interest income, and 1.7% for salary. · I multiplied each income amount by the right percentage. I tried to do as many calculations mentally as I could. **a)** To calculate 5% mentally, I divided by 10 and then took half because 5% is half 10%. **b)** To calculate 10% mentally, I divided by 10. **c)** To calculate 5% mentally, I divided by 10 and then took half because 5% is half 10%. **d)** For 1.7%, I used my calculator. I knew that 1.7% was 0.017 as a decimal.

Practising and Applying

1. When you read the tax chart on the tax form you complete for the government, the information is presented somewhat differently than what you saw at the beginning of this lesson on **page 274**.

Income Status (Nu)	Rate (%)	Allocation of Taxable Income (Nu)
Up to 200,000	0	0
200,001 to 250,000	10	0 + 10% of (net taxable income – 200,000)
250,001 to 500,000	15	5,000 + 15% of (net taxable income – 250,000)
500,001 to 1,000,000	20	42,500 + 20% of (net taxable income – 500,000)
1,000,001 and above	25	142,500 + 25% of (net taxable income – 1,000,000)

a) How do you know that this is the same information as in the previous chart on **page 274**, even though it looks different?

b) How might the presentation in this chart be easier to use than the information in the previous chart?

c) How does the presentation in the previous chart do a better job of explaining how the tax system works?

2. Calculate the amount of tax owing for each amount of net taxable income.

a) Nu 650,000

b) Nu 300,000

c) Nu 122,000

d) Nu 490,000

3. Calculate the amount of TDS for each monthly income amount.

a) salary of Nu 9,500 at about 0.8%

b) rental income of Nu 87,000

c) dividend income of Nu 12,500

d) interest income of Nu 7,540

4. Bishnu earned Nu 247,000 this year.
a) How do you know the percentage of his income he pays in taxes will be less than 10% before calculating it?

b) How could you have predicted that the percentage in **part a)** would be less than 5%?

5. Nu 198 was deducted at source each month from Ugyen's pay, but he still owes Nu 24 at the end of the year. What was his net taxable income for the year?

6. Why do you think the tax percentage rates increase for higher incomes?

7. Does someone with more income always pay more tax than a person with less income? Justify your answer. Use an example to support your justification.

7.2.2 Income Deductions

Try This

Two men have an income of Nu 18,000 a month each. One earns his income by selling fruit from his orchard. The other earns his income from a government salary.

A. Why might it be fair for the man with the orchard to keep more of his income (pay less tax) than the other man, even though their incomes are the same?

The government recognizes that certain expenditures people make are valuable or essential for the economy. To encourage this, the government allows **deductions** on income. These amounts are deducted, or subtracted from gross income to result in net taxable income, which is then used to calculate tax owing.

- The word deduction has two meanings. It is used to describe both an amount that is subtracted from your income to pay taxes (**lesson 7.2.1**) and an amount that is subtracted from your income to lower your net taxable income so you pay less tax.
- Examples of allowable deductions are described here. The full amount can be deducted from gross annual income unless otherwise indicated.

Allowable Annual Deductions (2016)

Contributions to a pension and provident fund (amounts paid in old age once a person has retired, currently mostly paid by government workers) and group insurance schemes	**20%** of rental income, in addition to the full costs of interest paid on loans to purchase the rental property, and property insurance premiums, and local property taxes associated with that property
Nu 10,000 worth of dividends from company shares as well as the interest paid on loans taken out to purchase those shares	Donations, **up to 5%** of adjusted gross income, to approved relief fund Natural Calamities, Preservations/Promotion of Religion and Culture fund, or Promotion of Sports, Educational, and Science fund
30% of income from cash crops or other sources	Life insurance premiums with an insurance company in Bhutan
Nu 10,000 of interest income, for example, on savings in a bank account	Education allowance: - **Up to Nu 5,000** per student in government institutions - **Up to Nu 150,000** per student for private education within/outside Bhutan

- The government income tax form has various steps to follow involving different calculations. The deductions listed above are subtracted at various points throughout the form. The final result is a reduction in the amount of tax owing because net taxable income is always less than gross income.
- You can never take a deduction greater than the amount you are subtracting from. For example, if you earn Nu 9,000 in interest income, you can only deduct Nu 9,000 even though a deduction of Nu 10,000 is allowable for amounts Nu 10,000 or greater. And if you earn Nu 12,000 in interest, you can only deduct Nu 10,000 of it and you must pay tax on the other Nu 2000 interest income.

B. About how much more tax would the government worker in **part A** pay than the orchard owner if the rest of their deductions are the same?

Examples

Example Calculating a deduction

Tandin was completing his income tax form. Which deductions can he apply to his gross annual income based on the information below?

- annual rental income of Nu 72,000
- interest of Nu 12,000 paid each month on a loan taken out to purchase rental property
- annual dividend at a rate of 25% on an investment worth Nu 50,000
- interest income at a per annum rate of 6% on a bank account of Nu 75,000
- life insurance premium of Nu 10,000 paid yearly to a company in Bhutan
- two children in government school

Solution	**Thinking**
Rental income: 20% × 72,000 = 14,400 He can deduct Nu 14,400.	 • 20% of rental income can be deducted.
Rental loan: 12 × 12,000 = 144,000 He can deduct Nu 144,000.	• All of the interest on a loan to purchase rental property can be deducted. I multiplied by 12 to get the annual interest on the loan.
Dividend: 25% × 50,000 = 12,500 He can deduct Nu 10,000 of it.	• Nu 10,000 worth of dividends can be deducted. I multiplied the investment amount by 25% to get the actual dividend amount.
Bank interest: 6% × 75,000 = 4500 He can deduct Nu 4500.	• I was able to deduct all of the bank interest income because it was less than Nu 10,000. I knew that "per annum" interest meant yearly.
Life insurance: He can deduct Nu 10,000.	• I deducted the entire life insurance premium because it was from a Bhutanese company.
Education allowance: 2 × 5,000 = 10,000	• There is a Nu 5,000 allowance for each child attending a government school so for two children it would be twice that amount.
Total deduction: 14,400 + 144,000 + 10,000 + 4500 + 10,000 + 10,000 = 192,900 He can deduct a total of Nu 192,900.	

Practising and Applying

1. Calculate the allowable deduction for each rental property.

a) Annual rental income: Nu 250,000

Interest on loan to purchase rental property: Nu 14,500 each month

Property insurance annual premium: Nu 16,000

Local taxes on property: Nu 1450

b) Annual rental income: Nu 180,200

Property insurance annual premium: Nu 12,000

Local taxes on property: Nu 980

2. Calculate the allowable deduction for the annual interest earned on each bank account. Note that the abbreviation p.a. means per annum, or yearly.

a) Nu 16,200 earning interest at a rate of 5% p.a.

b) Nu 28,400 earning interest at a rate of 4.5% p.a.

c) Nu 62,000 earning interest at a rate of 4.75% p.a.

d) Nu 252,200 earning interest at a rate of 5.1% p.a.

3. What is the maximum amount of money you can invest at each interest rate before you might have to pay tax on the interest income?

a) rate of 4.5% p.a.

b) rate of 5% p.a.

c) rate of 4% p.a.

d) rate of 6% p.a.

4. Calculate the allowable deduction for each dividend.

a) rate of 30% for shares worth Nu 41,000

b) rate of 18% for shares worth Nu 26,500

c) rate of 22% for shares worth Nu 19,500

5. What is the maximum income you can earn in dividends at each rate before you might have to pay tax on the income from the dividend?

a) rate of 25%

b) rate of 21%

c) rate of 30%

d) rate of 18%

6. A family has three children in government school. What is the minimum monthly income they would have to earn before they might have to pay taxes? Explain.

7. You can invest Nu 80,000 in shares that pay a dividend at a rate of 25% or you can purchase property that yields a rental income of Nu 15,000 a year. Which option results in greater income after deductions have been applied? Explain.

7.2.3 EXPLORE: Income Tax Rates

In **lesson 7.2.1** on **page 274**, the current income tax rates for different tax slabs were shown. Suppose the rates were increased. How would it affect the amount of money available for citizens to spend? How would it affect the amount of money the government has to spend on its citizens?

A. Suppose the rates were increased by 2% on each tax slab except the first one. Determine the *additional* tax that would be owing for each income.

i) Nu 100,000 **ii)** Nu 200,000 **iii)** Nu 400,000
iv) Nu 800,000 **v)** Nu 1,600,000

B. For each income in **part A**, express the increased tax rate as a percentage of the original rate. For example, no change in the tax would be 100%. Round to the nearest whole number percentage.

C. Which tax slab was most affected by a 2% increase in the tax rates? Explain.

D. Suppose the rates were doubled (multiplied by 2) for each tax slab. Determine the *additional* tax that would be owing for each income listed in **part A**.

E. When the rate was doubled, did the tax owing also double? Explain.

F. Changes in the tax rate can greatly affect the amount citizens pay in taxes and, as a result, the amount they have to spend. Changes in the rate also affect the amount the government collects through taxes and, as a result, the amount it has to spend on its citizens. What factors do you think the government should take into account when deciding tax rates?

CONNECTIONS: Taxation Around the World

Income tax rates vary considerably around the world. For example, income tax rates in Bhutan range from 0% to 15%, in Canada they range from 15% to 29%, and in Denmark they range from 41% to 60%. Countries use the income from taxes to provide services for its citizens such as education, health care, and child care.

1. Research to find out typical income tax rates (low and high end rates) and the range of tax rates in other countries.

2. Does the income tax rate (low and high end rates) and range in Bhutan seem to be typical? Find a way to graph the data in order to analyse it.

1. Calculate the annual income for each monthly or weekly income amount.

a) monthly income: Nu 10,500

b) weekly income: Nu 1990

c) weekly income: Nu 2260

d) monthly income: Nu 8800

2. Calculate the interest income on each savings amount in a bank account earning the interest rate given. Each rate is per annum.

a) rate: 4.25% amount: Nu 22,000

b) rate: 4.75% amount: Nu 18,600

c) rate: 5.25% amount: Nu 28,300

d) rate: 5.5% amount: Nu 14,600

3. Calculate the dividend income on each investment at the interest rate given. Each rate is per annum.

a) rate: 25% amount: Nu 24,000

b) rate: 30% amount: Nu 19,500

c) rate: 32.5% amount: Nu 32,640

d) rate: 17.5% amount: Nu 15,050

4. Several families were asked about the amount they spend and the percentage of their income they save each month. What was the monthly income for each family? Round to the nearest ngultrum.

a) saved: 8% spent: Nu 8680

b) saved: 9% spent: Nu 9550

c) saved: 11% spent: Nu 14,230

d) saved: 12% spent: Nu 12,310

5. Show what each cheque would look like:

a) Nu 2450 to L. Dorji

b) Nu 31,200 to G. Tshering

6. Use the budget template at the bottom of **page 272** in **lesson 7.1.2**. For each person below, fill in the appropriate amounts where they belong.

a) Sonam, a Class XI student
Spending money from parents for four months: Nu 3000
Clothing for four months: Nu 1000
School supplies for four months: Nu 250
Recreation for four months: Nu 300
Snacks each month: Nu 200

b) U. Pem
Monthly salary: Nu 12,800
Monthly rent: Nu 4800
Monthly food: Nu 4200
Monthly household expenses: Nu 800
Monthly clothing: Nu 300
Monthly loan payment: Nu 1500
Other monthly expenses: Nu 500

7. What percentage of his or her income did each person in **question 6** have left over for savings? Round to the nearest tenth of a percent.

8. What are the most important reasons for keeping a budget?

9. Pema's family decided to use these percentages of their Nu 9640 monthly income on the following expenditures. How much can they spend on each?

a) Rent: 25%

b) Food: 27%

c) Savings: 15%

d) Donations: 5%

e) Other, including taxes: 28%

10. a) Deki spends 28% of her monthly salary on rent. Her rent is Nu 5300. What is her monthly salary? Round to the nearest ngultrum.

b) How would your answer to **part a)** change if her rent was Nu 5700?

c) How would your answer to **part a)** change if her Nu 5300 rent was 30% of her monthly salary?

11. Calculate the tax owing for each net taxable income. Consult the chart on **page 274** in **lesson 7.2.1**.

a) Nu 235,500

b) Nu 425,000

c) Nu 595,000

d) Nu 1,470,000

12. Calculate the tax that would be deducted at source for each of these income amounts.

a) dividend income of Nu 12,300

b) interest income of Nu 780

c) rental income of Nu 120,000

d) salary income of Nu 11,300 at 1.6%

13. Why are the tax rates different percentages for different tax slabs?

14. Why are some taxes deducted at source?

15. Ugyen earned Nu 13,200 in dividend income.

a) What percentage of this dividend income is taxable?

b) If the dividend rate was 27%, how much did he invest?

16. Which deduction is greater, A or B?

A: a deduction on rental income of Nu 220,000 with a related loan interest payment of Nu 32,000, and local property taxes of Nu 1200

B: an education allowance deduction for two children; one attending private school and another in government school.

17. Calculate the allowable tax deduction for each.

a) interest rate of 5% p.a. on Nu 14,200

b) dividend rate of 25% p.a. on Nu 36,600

GLOSSARY

Instructional Terms

calculate: Figure out the number that answers a question; compute

clarify: Make a statement easier to understand; provide an example

classify: Put things into groups according to a rule and label the groups; organize into categories

compare: Look at two or more objects or numbers and identify how they are the same and how they are different (e.g., compare the numbers 6.5 and 5.6; compare the size of the students' feet; compare two shapes)

conclude: Judge or decide after reflection or after considering data

construct: Make or build a model; draw an accurate geometric shape (e.g., use a ruler and a protractor to construct an angle); the term construct is sometimes reserved for drawings that use a compass and straight edge only

create: Make your own example or problem

describe: Tell, draw, or write about what something is or what something looks like; tell about a process in a step-by-step way

determine: Decide with certainty as a result of calculation, experiment, or exploration

draw: 1. Show something in diagram form **2.** Pull or select an object (e.g., draw a card from the deck; draw a tile from the bag)

estimate: Use your knowledge to make a sensible decision about an amount; make a reasonable guess (e.g., estimate how long it takes to walk from your home to school; estimate how many leaves are on a tree; estimate 3210 ÷ 789)

evaluate: 1. Determine whether something makes sense; judge **2.** Calculate the value as a number (e.g., evaluate the expression $m^2 + 3$ for $m = 5$)

explain: Tell what you did; show your mathematical thinking at every stage; show how you know

explore: Investigate a problem by questioning, brainstorming, and trying new ideas

extend: 1. In patterning, continue the pattern **2.** In problem solving, create a new problem that applies the idea of the original problem further

justify: Give convincing reasons for a prediction, an estimate, or a solution; tell why you think your answer is correct

measure: Use a tool to describe an object or determine an amount (e.g., use a ruler to measure a height or distance; use a protractor to measure an angle; use balance scales to measure mass; use a measuring cup to measure capacity; use a stopwatch to measure elapsed time)

model: Show or demonstrate an idea using objects, pictures, words, and/or numbers (e.g., model addition of integers using red and blue counters, model a relationship using an equation)

predict: Use what you know to work out what is going to happen (e.g., predict the tenth number in the number pattern 1, 2, 4, 7, …)

reason: Develop ideas and relate them to the purpose of the task and to each other; analyse relevant information to show understanding

relate: Describe how two or more objects, drawings, ideas, or numbers are similar

represent: Show information or an idea in a different way (e.g., draw a graph of an equation; make a model from a word description; create an expression to model a situation)

show your work: Record all calculations, drawings, numbers, words, or symbols that make up the solution

simplify: Write a number or expression in a simpler form (e.g., combining like terms of a polynomial, writing an equivalent fraction with a smaller numerator and denominator)

sketch: Make a rough drawing not necessarily to scale, often to help with visualization and problem solving (e.g., sketch a picture of the field with given dimensions)

solve: 1. Develop and carry out a process for finding an answer to a problem **2.** to find the value of a variable in an equation or inequality

sort: Separate a set of objects, drawings, ideas, or numbers according to an attribute (e.g., sort 2-D shapes by the number of sides)

validate: Check an idea by showing that it works

verify: Work out an answer or solution again, usually in another way; show evidence of; check a result

visualize: Form a picture in your head of what something is like (e.g., visualize the number 6 as 2 rows of 3 dots as on a die; visualize the equation $y = x$ as a diagonal line at a 45º angle)

Definitions

A

acute angle: An angle less than 90º

acute triangle: A triangle in which all interior angles are acute angles

algebraic equation: An equation that includes algebraic expressions and an equal sign (e.g., $3x + 5 = 8$)

algebraic expression: A combination of one or more variables; it may include numbers and operation signs (e.g., $8x + 2y^2 – 9$)

algorithm: A specific set of instructions or a procedure for finding the solution to a problem or the answer to a calculation

altitude: The line segment that represents the height of a 2-D or 3-D shape; also the length of that line segment; for a triangle, it is a segment from a vertex perpendicular to the opposite side; for a cone or pyramid, it is a segment from the vertex perpendicular to the base

angle: A geometric figure formed by two rays with a common end point, or vertex

angle bisector: A line that separates an angle into two equal parts

angle of rotation: The angle through which a shape has moved after a rotation

apex: The highest point or vertex of a cone or pyramid when resting on its base

apothem: The perpendicular distance from the centre of a regular polygon to the midpoint of one of its sides

arc: A section of the circumference of a circle that lies between two ends of a chord (each chord creates two arcs); the length of this section of the circumference. See *circle*

area: The measure of the surface of a 2-D shape, expressed in terms of the number of square units needed to cover the shape; the number of square units needed to cover a surface

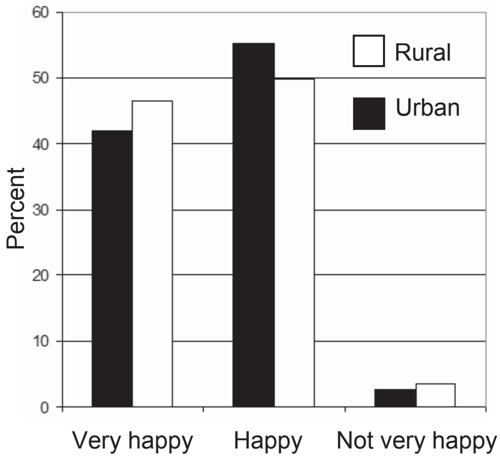
average: In common use, average is the same as mean See *measure of central tendency*

axis: A line drawn for reference when locating points in a coordinate system

B

bar graph: A graph that compares the sizes of bars that represent each category in a set of data; a double bar graph (shown below) compares two aspects of a category as well as the categories themselves

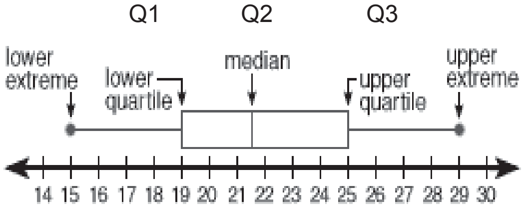
State of Happiness



base: 1. The face(s) that determines the name of a prism or pyramid
2. In a 2-D shape, the line segment(s) that is perpendicular to the height
3. The number that is repeatedly multiplied in a power (e.g., in the power 5^3, 5 is the base)

binomial: A polynomial with two terms (e.g., $4x – 7y$ and $5x^2 + 3$ are binomials)

box and whisker plot: A graph that uses the median (Q2) and extremes as well as the lower and upper quartiles (Q1 and Q3) to organize data into four groups, or quartiles, that each contain equal numbers of data values



broken line graph: See *line graph*

C

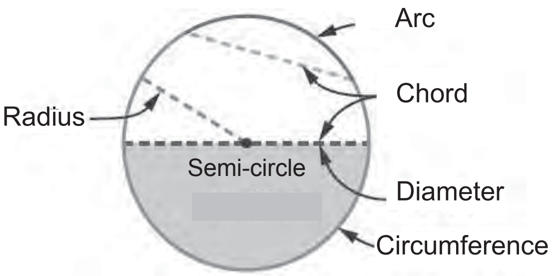
capacity: The amount that a container can hold. Common units are millilitres (mL) or litres (L). Capacity can be related to volume, 1 mL = 1 cm^3

centre of dilatation: See *dilatation*

centre of rotation: A fixed point around which the points in a shape rotate in a clockwise (cw) or counter-clockwise (ccw) direction; the centre of rotation may be inside or outside the shape

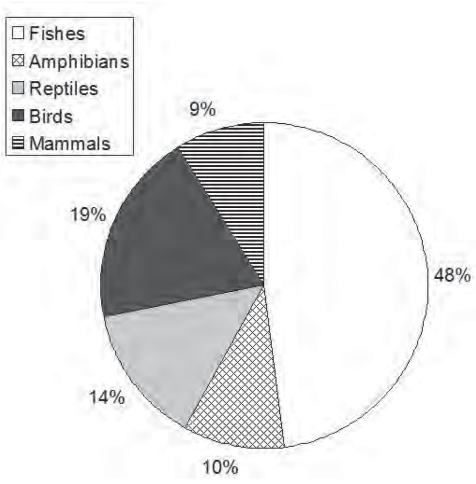
chord: A line segment connecting any two points on a curve. See *circle*

circle: The set of all the points in a plane that are the same distance, called the radius (*r*), from a fixed point called the centre



circle graph: A graph that shows how a complete set of data is broken into categories, each represented by a section of a circle

Known Vertebrate Species



circumference: 1. The boundary of a circle **2.** the length of the boundary of a circle calculated using the formula $C = 2\pi r$, where r is the radius, or $C = \pi d$, where d is the diameter. See *circle*

coefficient: The number or constant by which a variable is multiplied (e.g., in the term $3z$, the numerical coefficient of z is 3; in the term by^2, b is the literal coefficient of y^2)

common denominator: A common multiple of the denominators of two or more fractions (e.g., 12 is a common denominator of $\frac{1}{2}$ and $\frac{1}{3}$)

common factor: A number that divides into two or more other numbers with no remainder

common multiple: A number that is a multiple of two or more given numbers (e.g., 12, 24, and 36 are common multiples of 4 and 6)

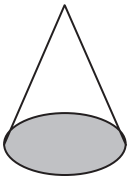
complementary angles: Two angles whose sum is 90º

composite number: A number with more than two factors (e.g., 12 is a composite number with factors 1, 2, 3, 4, 6, and 12)

composite shape: A 2-D or 3-D shape that is made up of multiple simple 2-D or 3-D shapes

composite transformation: A transformation described by two or more other transformations

cone: A 3-D shape that has a circular base and a curved surface from the boundary of the base to a vertex, or apex



congruence: The property shared by geometric shapes that are identical in shape and size

congruent: Identical in size and shape. The symbol ≅ means "is congruent to", as in $\Delta ABC \cong \Delta DEF$

continuous data: A set of data with no gaps, represented by a solid line or curve on a graph

coordinate plane: See *x-y-plane*

coordinates: A set of numbers used to define a position. In the *x*-*y*-plane, coordinates are in the form of ordered pairs (x, y)

correlation: A description of the relationship between two variables

correlation coefficient: A measure of how well the points in a scatter plot fit an algebraic model. A value close to 1 or –1 indicates a good fit; a value close to 0 indicates a poor fit

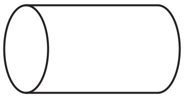
corresponding angles: Matching angles that are formed by a transversal and two parallel lines

cube: 1. To raise a number to a power, or exponent of 3 **2.** (Geometry) A polyhedron that has 6 congruent square faces

cube root: One of three equal factors of a number. (e.g., the cube root of 8 is 2 because $2^3 = 8$)

cuboid: See *rectangular prism*

cylinder: A 3-D shape with two congruent, parallel, planar, circular faces joined by one curved surface



D

data: Information gathered in a survey, in an experiment, or by observing (e.g., data can be in words like a list of students' names, in numbers like quiz marks, or in pictures like drawings of favourite pets). The word data is plural, not singular

decagon: A ten-sided polygon

degree of a polynomial: The greatest exponent that appears in any term of a single-variable polynomial. The greatest sum of exponents in any term of a multi-variable polynomial (e.g., $3x + 2xy$ has degree 2 because of the term xy)

denominator: The number in a fraction that represents the number of parts in a set, or the number of parts the whole has been divided into (e.g., in $\frac{4}{5}$, the denominator is 5, or fifths).
See *numerator*

dependent event: An event affected by the outcome of another event

dependent variable: In an algebraic relation, a variable whose values are determined by assigning values to the other (independent) variable. Often represented by y and plotted on the vertical axis

diagonal: A line segment joining two vertices of a polygon that are not next to each other

diameter: **1.** A line segment that joins two points on a circle and passes through the centre **2.** the length of the line segment described in **1**. See *circle*

difference: The result of a subtraction (e.g., in 45 – 5 = 40, the difference is 40). See *subtrahend* and *minuend*

dilatation: A transformation that enlarges or reduces a figure by a scale factor. Lines joining corresponding points on the original and transformed shapes meet at the centre of dilatation

dimension: The size or measure of an object (e.g., the width and length of a rectangle are its dimensions)

discrete data: Data that consists of a set of isolated points, sometimes represented by a dashed line or curve on a graph

distributive property: The product of a number and a sum is equal to the sum of the products: $a(b + c) = ab + ac$

dividend: The number that is being divided (e.g., in 45 ÷ 5 = 9, the dividend is 45). See *divisor* and *quotient*

divisibility rule: A way to determine whether one number is a factor of another number without actually dividing (e.g., a number is divisible by 3 if the sum of the digits is divisible by 3)

divisor: The number by which a number is being divided
(e.g., in 45 ÷ 5 = 9, the divisor is 5).
See *dividend* and *quotient*

double bar graph: See *bar graph*

E

element: One of the objects or numbers belonging to a set or pattern

equation: A mathematical statement in which the value on the left side of the equal sign is the same as the value on the right side of the equal sign
(e.g., the equation $5n + 4 = 39$ means that 4 more than the product of 5 and a number equals 39)

equidistant: The same distance
(e.g., all points on the circumference of a circle are equidistant from the centre)

equilateral triangle: A triangle with three sides of equal length (and with all angles equal and 60º)

equivalent equations: Equations that have exactly the same solution

equivalent fractions: Fractions that represent the same part of a whole or set (e.g., $\frac{2}{4}$ is equivalent to $\frac{1}{2}$)

event: A set of outcomes for a probability experiment (e.g., if you roll a die with the numbers 1 to 6, the event of rolling an even number has the outcomes 2, 4, or 6); a subset of the sample space

expand: Write the full product of an algebraic expression (e.g., $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 4x + 4$)

expanded form: A way of writing a number that shows the value of each digit as a power of 10 (e.g., 1209 in expanded form is $1 \times 10^3 + 2 \times 10^2 + 9 \times 1$)

experimental probability: The observed probability of an event based on data from an experiment, calculated using the following expression:

$$\frac{\text{Number of times the event happens}}{\text{Number of times the experiment is done}}$$

exponent: A superscript in mathematics that denotes repeated multiplication (e.g., 4^3 means $4 \times 4 \times 4$ and the exponent is 3); sometimes referred to as a power or an index

exponential relation: A relation between two variables that can be represented by an exponential function (e.g., $y = 2^x – 5$)

expression: See *algebraic expression*

extrapolate: To estimate a value that is beyond the range of given data by following a pattern or trend

extreme: The greatest and least values in a set of data

F

factor or factorise: To express a number or algebraic expression as the product of two or more numbers or algebraic expressions. The numbers or algebraic expressions in such a product are also called factors (e.g., $a^2 – b^2$ can be factored as $(a + b)(a – b)$ and the factors are $(a + b)$ and $(a – b)$;
24 can be factored as 8×3 or $2 \times 2 \times 2 \times 3$; 1, 2, 3, 4, 6, 8, 12, and 24 are all factors of 24)

favourable outcome: A desired result when calculating a probability (e.g., that a spinner will stop on green instead of red)

finite differences: In a table of values where the x-coordinates are evenly spaced, the first differences are the differences between consecutive y-coordinates. The second differences are the differences between consecutive first differences, and so on. For a linear function the first differences are constant, while for a quadratic function they are not. The second differences are constant for a quadratic function

first-degree equation: An equation in which the exponent of the variable is 1. (e.g., $3x + 1 = 8$)

first-degree polynomial: A polynomial in which the exponent of the variable is 1. (e.g., $8x – 17$)
See *degree of a polynomial*

first differences: See *finite differences*

flip: See *reflection*

formula: A general rule or principle stated in mathematical language

G

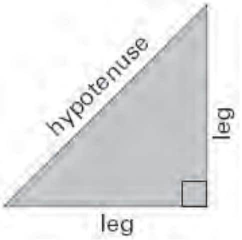
greatest common factor (GCF): The greatest whole number that divides into two or more other whole numbers with no remainder (e.g., 4 is the greatest common factor of 8 and 12)

H

height: The perpendicular distance from the base of a geometric shape to its highest point

hexagon: A six-sided polygon

hypotenuse: The side opposite the right angle in a right triangle



I

image: A new shape that is created when a shape undergoes a transformation

improper fraction: A fraction in which the numerator is greater than or equal to the denominator (e.g., $\frac{5}{4}$ and $\frac{6}{6}$)

independent event: An event that is unaffected by the outcome of another event

independent variable: In an algebraic relation, a variable whose values are chosen and upon which the values of other variables depend. Often represented by x and plotted on the horizontal axis

integers: The set of whole numbers and their opposites (…, –2, –1, 0, 1, 2, …)

intercept: The distance from the origin of the x-y-plane to a point at which the graph meets or crosses the x- or y-axis (the x-intercept or y-intercept); the value of the y- or x-coordinate where the graph meets or crosses the axis

interior angle: One of the angles inside a polygon (e.g., a square has four interior angles)

interpolate: To estimate a value between two given elements of data by following a pattern or trend

intersection: In geometry, the point or points that are common to two or more shapes. (e.g., two lines that intersect have one point in common)

inverse operation: An operation that undoes another operation (e.g., addition is the inverse of subtraction)

irrational number: A number that cannot be written as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$ (e.g., π, $\sqrt{5}$)

isosceles triangle: A triangle with two sides of equal length

L

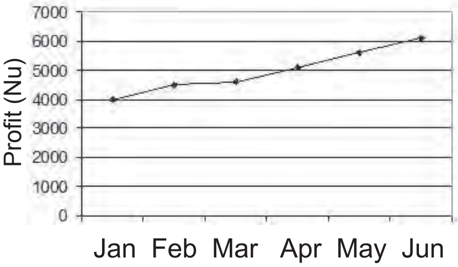
lateral surface: The surface of a prism, cylinder, pyramid, or cone that does not include the base(s)

least common multiple (LCM): The smallest whole number that has two or more given numbers as factors (e.g., 12 is the least common multiple of 4 and 6)

like terms: Terms of a polynomial that have the same variables and exponents but may have different coefficients (e.g., in $3x^2 + 2x + 6x + 5$, $2x$ and $6x$ are like terms)

line graph: A graph of a set of points showing one variable (often time) on the horizontal axis and another variable on the vertical axis, with each point joined to the next point by a dashed (discrete data) or solid (continuous data) line segment. Sometimes called a broken line graph

Profit from January to June, 2005



line of best fit: The straight line that best describes the relationship between two variables in a scatter plot of data

line segment: Part of a line, consisting of two end points and all points between

linear equation: An equation of degree 1 that represents a linear relation between two variables, typically in the form $y = mx + b$

linear relation: A relation between two variables that appears as a straight line when graphed. Represented by a first-degree equation involving two variables

linear system: A set of two or more equations that represent linear relations between the same two variables

M

mapping notation: A way to show a transformation rule by showing an original point and its image, connected by an arrow
(e.g., $(x, y) \rightarrow (x + 2, y + 3)$)

mathematical model: A mathematical description (such as a diagram, graph, table of values, formula, equation, physical or computer model) of a situation

maximum: The greatest value taken by a dependent variable

mean: The sum of a set of numbers divided by the number of elements in the set; often called the average

measure: The indication, in standard units, of the size of something. For geometric shapes, there are measures of length, area, volume, and angle

measure of central tendency: A value, usually single, that can be used to represent a set of data. See *mean*, *median*, and *mode*

median: 1. (Geometry) A line that joins a vertex of a triangle to the midpoint of the opposite side **2.** (Statistics) The middle number of a set of data arranged in order. If there is an even number of numbers in the set, the median is the mean of the two middle numbers

metre (m): A unit of measurement for length (e.g., 1 m is about the distance from a doorknob to the floor); 1000 mm = 1 m; 100 cm = 1 m; 1000 m = 1 km

midpoint: The point that divides the line segment into two equal parts

minimum: The least value taken by a dependent variable

minuend: The number that is being subtracted from (e.g., in 45 – 5 = 40, the minuend is 45). See *subtrahend* and *difference*

mixed number: A number made up of a whole number and a fraction
(e.g., $5\frac{1}{7}$)

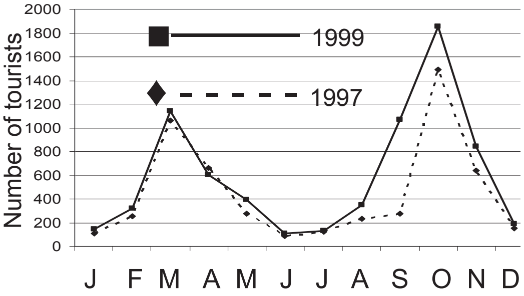
mode: The piece(s) of data that occurs most often in a set of data; there can be more than one mode or there might be no mode

monomial: A polynomial made up of one term (e.g., $6x^2$ and $-13y$ are monomials)

multiple: The product of a whole number and any other whole number (e.g., when you multiply 10 by the whole numbers 0 to 4, you get the multiples 0, 10, 20, 30, and 40)

multiple line graph: A graph used to compare the trends in two or more variables, usually over time on the same grid

Number of Tourists Arriving in Bhutan

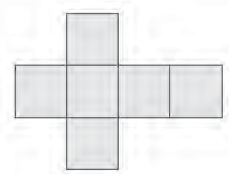


N

natural number: One of the counting numbers: 1, 2, 3, …

negative correlation: In a relationship between variables, as one variable increases, the other variable decreases

net: A 2-D pattern you can fold to create a 3-D shape; this is a net for a cube:



nonlinear relation: A relationship between two variables that does not fit a straight line when graphed on a coordinate system

numerator: The number in a fraction that shows the number of parts of a given size the fraction represents

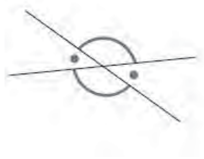
(e.g., in $\frac{4}{5}$, the numerator is 5)

See *denominator*

numerical coefficient: See *coefficient*

O

opposite angles: Non-adjacent angles that are formed by two intersecting lines as indicated by the symbols below:



opposite integers: Two integers that are the same distance away from zero in opposite directions (e.g., 6 and –6 are opposite integers)

obtuse triangle: A triangle in which one of the angles is an obtuse angle, that is, an angle greater than 90º and less than 180º

order of operations: Rules describing the sequence to use when evaluating an expression:

1 Evaluate within brackets

2 Calculate exponents and square roots

3 Divide then multiply

4 Add and subtract from left to right

ordered pair: A pair of numbers in which the order is important. The coordinates of a point in the *x*-*y*-plane form an ordered pair. (e.g., the ordered pairs (3, 5) and (5, 3) represent different points)

orientation: The direction around a shape when you name the vertices in order, clockwise or counter-clockwise

origin: The intersection of the axes in a coordinate system represented by the ordered pair (0, 0)

outcome: A single event that is proposed as the result of a probability experiment

P

parabola: An open curve shaped like the graph of $y = x^2$; the graph of a quadratic relation

parallel: Always the same distance apart (e.g., railway tracks are parallel to each other)

parallel lines: Lines in the same plane that never meet

parallelogram: A quadrilateral with pairs of opposite sides that are parallel

percent: A special ratio that compares a number to 100 using the symbol %

perfect square: A whole number whose square root is a whole number

perimeter: 1. The boundary of a 2-D shape **2.** The length of the boundary; the circumference of a circle is a special perimeter

period: 1. The block of repeating digits in a repeating decimal number (e.g., in 0.345345..., the period is 345)
2. a set of 3 digits in a number (e.g., in the number 3,458,675, the periods are 458 and 675

perpendicular: At a right angle (e.g., the base of a triangle is perpendicular to the height of the triangle)

***π* (pi):** The value of the circumference of any circle divided by its diameter; it is an irrational number with a value of 3.141592654 … or about 3.14, rounded to two decimal places

plane: A flat two-dimensional surface

polygon: A closed 2-D shape formed by three or more line segments in a plane. Examples include triangles, quadrilaterals, hexagons, and decagons

polyhedron: A 3-D shape that has faces which are polygons

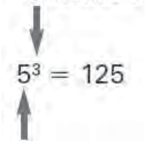
polynomial: An algebraic expression consisting of one or more terms with variables raised to whole-number powers, usually of the form $a + bx + cx^2 + \ldots$, where a, b, c, … are numbers

positive correlation: In a relationship between variables, as one variable increases, the other variable also increases

possible outcome: A single result that can occur in a probability experiment (e.g., when tossing a coin, getting Tashi Ta-gye is a possible outcome)

power: A numerical expression that shows repeated multiplication (e.g., the power 5^3 is a shorter way of writing $5 \times 5 \times 5$). A power has a base and an exponent: the exponent tells the number of equal factors there are in a power. Sometimes the exponent is also called the power

3 is the exponent of the power



5 is the base of the power

power law: The exponent of a power raised to a power is the product of the exponents: $(a^r)^s = a^{rs}$

power of a product law: The power of a product equals the product of the powers: $(ab)^r = a^r b^r$

power of a quotient law: The power of a quotient equals the quotient of the powers: $(\frac{a}{b})^r = \frac{a^r}{b^r}$ or $(a \div b)^r = a^r \div b^r$

prime factorisation: The representation of a composite number as the product of its prime factors (e.g., the prime factorisation of 24 is $2 \times 2 \times 2 \times 3$, or $2^3 \times 3$); usually, the prime numbers are written in order from least to greatest

prime number: A number with exactly two factors, 1 and itself (e.g., 17 is a prime number since its only factors are 1 and 17)

principal square root: The positive square root of a number (e.g., 2 and –2 are square roots of 4 but only 2 is the principal square root)

prism: A polyhedron with two parallel and opposite congruent bases; the other faces are parallelograms. The shape of the base of the prism determines the name of the prism (e.g., pentagon-based prism)

probability: A number from 0 (will never happen) to 1 (certain to happen) that represents how likely it is that an event will happen

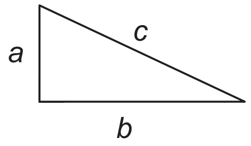
product law: If you multiply two powers with the same base, add the exponents to get the exponent of the product: $a^r \times a^s = a^{r+s}$

proper fraction: A fraction in which the denominator is greater than the numerator (e.g., $\frac{1}{7}$, $\frac{4}{5}$, $\frac{29}{40}$)

proportion: An equation of two equivalent ratios (e.g., $\frac{1}{2} = \frac{x}{50}$)

pyramid: A polyhedron with a polygon for a base; the other faces are triangles that meet at a single vertex (the apex)

Pythagorean theorem: The square of the length of the hypotenuse of a right triangle (the longest side) is equal to the sum of the squares of the lengths of the other two sides: $a^2 + b^2 = c^2$



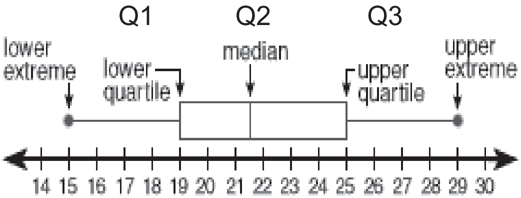
Q

quadrant: One of the four areas into which the x-axis and y-axis divide an x-y-coordinate system

quadratic relation: A relation between two variables that can be represented by the equation $y = ax^2 + bx + c$

quadrilateral: A four-sided polygon

quartile: 1. One of three points (Q1 or lower quartile, Q2 or median, and Q3 or upper quartile) that divide a set of data into four equal parts **2.** One of four groups of a set of data that each contain an equal number of data values



quotient: The result of dividing one number by another (e.g., in 45 ÷ 5 = 9, the quotient is 9).
See *divisor* and *dividend*

quotient law: If you divide a power by another power with the same base, you subtract the exponents to get the exponent of the quotient: $a^r \div a^s = a^{r-s}$

R

radius (plural is radii)**:** A line segment that joins the centre of a circle to any point on its circumference; the length of this line segment. See *circle*

range: The difference between the extremes (minimum and maximum) of a set of data

rate: A comparison of two quantities measured in different units; unlike ratios, rates include units (e.g., 45 km/h)

ratio: A number or quantity compared with another, expressed in symbols as $a{:}b$ or $\frac{a}{b}$

rational number: A number that can be expressed as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$

ray: Part of a line that starts at an end point and extends indefinitely in one direction

real number: Any number that can be represented by a point on a number line. Every real number is either rational or irrational

reciprocal: The multiplier of a number that gives 1 as the result (e.g., $\frac{1}{2}$ is the reciprocal of 2). The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ (a, $b \neq 0$). The negative reciprocal is the multiplier of a number that gives –1 as the result

rectangle: A parallelogram in which all interior angles are right angles

reflection: A transformation of a shape that produces a mirror image of the shape with respect to a line, which is called the reflection line; also called a flip

reflection line: See *reflection*

regular polygon: A polygon with all sides and all angles congruent

relation: A property that connects two sets of numbers or two variables; a relation can be expressed mathematically as a table of values, a graph, or an equation

repeating decimal: A decimal in which a block of one or more digits eventually repeats in a pattern (e.g., 0.124444...; 0.252525252…; 990.142857142857…)

rhombus: A parallelogram with all sides congruent; a square is a special rhombus that has interior right angles

right pyramid: A pyramid whose lateral faces are all congruent isosceles triangles

right triangle: A triangle with one right angle

rise: The vertical distance between two points. See *run*

rotation: A transformation in which each point in a shape moves about a fixed point (the centre of rotation) through the angle of rotation

run: The horizontal distance between two points. See *rise*

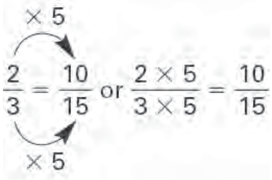
S

sample space: All possible outcomes in an experiment or probability situation (e.g., the sample space for rolling a die is 6 possible outcomes: 1, 2, 3, 4, 5, 6)

scale: The ratio between the size of an object in a drawing and the size of the actual object

scale drawing: A drawing that is the same shape as an actual object and whose size is determined by the scale

scale factor: 1. (Numeration) A number by which you can multiply or divide each term in a ratio to get the equivalent terms in another ratio; it can be a whole number or a decimal. (e.g., the scale factor below is 5)

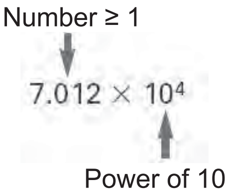


2. (Geometry) If two triangles are similar, corresponding side lengths relate by the same ratio, or scale factor

scalene triangle: A triangle with no congruent sides

scatter plot: A graph of isolated points on a coordinate grid. The graph can be used to try to determine a relationship between the two variables that are being graphed

scientific notation: A way of writing a number as a decimal greater than or equal to 1 and less than 10, multiplied by a power of 10 (e.g., 70,120 is written as 7.012×10^4)



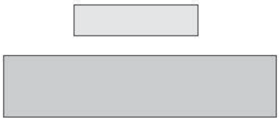
second-degree polynomial: A polynomial for which the sum of powers of at least one variable is 2 for at least one of the terms (e.g., $-3x^2 - 2x + 5$, $6y^2$, and $3xy + 5x$ are all second-degree polynomials)

second differences: See *finite differences*

sector (of a circle)**:** Part of a circle bounded by two radii and part of the circumference. See *circle*

sequence: A list of things that are in a logical order or that follow a predictable pattern (e.g., 1, 3, 5, 7, 9, … shows the odd numbers in order)

similar: Identical in shape, but not necessarily the same size; the symbol for similarity is ~, as in ΔABC ~ ΔDEF; all congruent shapes are similar but not all similar shapes are congruent



These rectangles are similar

similar triangles: Triangles in which the pairs of corresponding sides are proportional

similarity: The property shared by geometric shapes that are identical in shape, but not necessarily the same size

simulation: A probability experiment that models an event (e.g., repeatedly flipping four coins to find the probability that all four children in a family are girls)

slant height: The distance from the apex of a circular cone to the edge of its base; the perpendicular distance from the apex of a right pyramid to the midpoint of one of the edges of its base

slide: See *translation*

slope: A measure of the steepness of a line, expressed as the rise (vertical distance) divided by the run (horizontal distance) between any two points on the line

slope and *y*-intercept form: The equation of a straight line written as $y = mx + b$, where m is the slope of the line and b is the y-intercept. Sometimes expressed as $y = mx + c$

solution: 1. The complete answer to a problem **2.** The values of variables that make an equation or inequality true (e.g., in the equation $5n + 4 = 39$, the solution is $n = 7$ because $5(7) + 4 = 39$)

solution of a system of linear equations: The point of intersection of the graphs of two or more relations

speed: The rate at which a moving object changes position with time (e.g., a sprinter who runs 100 m in 10 s has an average speed of 100 m/10 s or 10 m/s)

sphere: A 3-D shape in which every point on the surface is the same distance from a single point, the centre

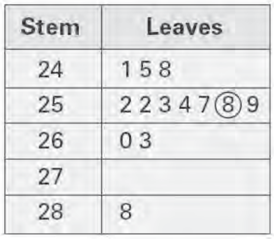
square: 1. (Geometry) A rectangle with equal sides **2.** (Algebra) To multiply a number or expression by itself

square number: A whole number that is a perfect square of another whole number (e.g., 1 is the square of 1, 4 is the square of 2, 9 is the square of 3, and so on)

square root: A number that multiplies by itself to result in another number. (e.g., The square roots of 49 are 7 ($7 \times 7 = 49$) and –7 ($(-7) \times (-7) = 49$)). In symbols, $\sqrt{49} = 7$.)
See *principal square root*

standard form (of a linear relation)**:** The equation of a straight line when written as $Ax + By = C$. Sometimes written as $ax + by = c$

stem and leaf plot: An organization of numerical data into categories or intervals based on place values; the most significant digits are the stems and the least significant digits are the leaves (e.g., the circled leaf in this stem-and-leaf plot represents the number 258)



Stem	Leaves
24	1 5 8
25	2 2 3 4 7 ⑧ 9
26	0 3
27	
28	8

straight angle: An angle that measures 180º

subtrahend: The number that is being subtracted (e.g., in 45 – 5 = 40, the subtrahend is 5). See *minuend* and *difference*

supplementary angles: Two angles whose sum is 180º

surface area: The sum of the areas of all the faces and curved surfaces of a 3-D shape

symmetry: 1. Line or reflectional symmetry: when a 2-D shape is folded or reflected across a line (the reflection line), the two sides of the shape match **2.** Plane symmetry: when two halves of a 3-D shape are reflections of each in the plane of symmetry **3.** Turn or rotational symmetry: when a 2-D or 3-D shape, when rotated, resembles the original shape

system of linear equations: A set of two or more linear equations in two or more variables

T

table of values: An arrangement of numerical values, usually in vertical and horizontal columns, that represents a relationship between two variables

term: 1. Part of an algebraic expression, separated from the rest by addition or subtraction signs
2. Each number or item in a sequence (e.g., in the sequence 1, 3, 5, 7, …, the third term is 5)

terminating decimal: A decimal that is complete after a certain number of digits (e.g., $\frac{29}{40}$ = 0.725)

tessellation: An arrangement of congruent 2-D shapes that will cover a plane (in all directions), without gaps or overlapping. See *tiling*

theoretical probability: How likely an event is to occur, expressed as a number from 0 (will never happen) to 1 (certain to happen); calculated using the expression:

$$\frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

(e.g., *P*(roll a 4 on a six-sided die) = $\frac{1}{6}$)

tiling: An arrangement of 2-D shapes that will cover a plane (in all directions) without gaps or overlapping.
See *tessellation*

transformation: Changing a shape according to a rule; transformations include translations, rotations, reflections, and dilatations

translation: A transformation of a shape in which each point moves the same distance and in the same direction; also called a *slide*

translation rule: A rule that determines or describes the effect of a transformation on any shape

transversal: A line that intersects two or more lines

trapezium: See *trapezoid*

trapezoid: A quadrilateral in which one pair of opposite sides are parallel

tree diagram: A way to record and count all combinations of events (e.g., the tree diagram below shows all the three-digit numbers that can be made from the digits 1, 2, and 3, if 1 must be the first digit and each digit is used only once)

trend: A pattern of general direction or movement, often for a variable that is measured against time; represented by the line or curve of best fit in a scatter plot

triangle: A polygon with three sides

trinomial: A polynomial that has three terms (e.g., $4x - 7y + 2z$ and $5x^2 + 2x - 3$ are trinomials)

truncate: To cut off a vertex of a 3-D shape (e.g., a truncated cone)

U

unique triangle: When only one triangle can be created from a given description

V

variable: A letter or symbol, such as *a*, *b*, *x*, or *n*, that represents a number (e.g., in the formula for the area of a rectangle, $A = l \times w$, the variables *A*, *l*, and *w* represent the area, length, and width of the rectangle)

vertex (plural is vertices)**:** The point at the corner of an angle or shape where two or more sides or edges meet (e.g., a cube has eight vertices, a triangle has three vertices, an angle has one vertex)

vertex of a parabola: The point where a parabola intersects its axis of symmetry; the maximum or minimum point

volume: The amount of space occupied by an object

W

whole numbers: The set of counting numbers that begins at 0 and continues forever; 0, 1, 2, 3, …

X

x*-axis*: See *x-y-plane*

***x*-intercept:** See *intercept*

***x-y*-plane:** A coordinate system based on the intersection of two perpendicular lines called axes. The *x*-axis is the horizontal axis and the *y*-axis is the vertical axis. The origin is the point of intersection of the two axes

Y

***y*-axis:** See *x-y-plane*

***y*-intercept:** See *intercept*

Z

zero principle: Two opposite integers or algebraic terms, when added, give a sum of zero (e.g., (+1) + (–1) = 0 and $-2x^2 + 2x^2 = 0$)

Financial Definitions

adjusted gross income: Gross income adjusted for certain deductions. See *gross income*

budget: A plan for spending your available funds

deductions: Amounts that are deducted, or subtracted, from gross income to result in net taxable income, which is then used to calculate tax owing. See *gross income* and *net taxable income*

expenditures: Amounts you need to spend for your everyday life. Some are necessities, like food, clothing, shelter, and taxes; others are things that make your life more pleasant, like a car

gross Income: Total income earned or received in a particular category (e.g., gross salary income or gross rental income)

income: Money that you receive or earn that is available for you to spend. It can be received in different ways, such as salary and wages from employment, commissions on sales, or allowances and bonuses. See *gross income*, *adjusted gross income*, and *net taxable income*

net taxable income: Adjusted gross income less additional allowable. See *gross income*

PIT (**personal income tax**)**:** How much you must pay the government. The amount that you owe is a percentage of your income. The percentage depends on the level of your income

TDS (**tax deducted at source**)**:** Taxes that are withheld or deducted your income before you get paid by your employer

MEASUREMENT REFERENCE

Measurement Abbreviations and Symbols

Time second minute hour	s min h		**Capacity** millilitre litre kilolitre	mL L kL
Length millimetre centimetre metre kilometre	mm cm m km		**Volume** cubic centimetre cubic metre	cm^3 m^3
Mass milligram gram kilogram ton	mg g kg t		**Area** square centimetre square metre hectare (10,000 m^2) square kilometre	cm^2 m^2 ha km^2

Metric Prefixes

Prefix	milli × 0.001	centi × 0.01	deci × 0.1	unit 1	deka × 10	hecto × 100	kilo × 1000
Example	*milli*metre mm	*centi*metre cm	*deci*metre dm	metre m	*deka*metre dam	*hecto*metre hm	*kilo*metre km

Measurement Formulas

Perimeter
rectangle $P = 2(l + w)$
square $P = 4s$
regular polygon $P = ns$ (n is number of sides)

Circumference
circle $C = \pi d$ or $C = 2\pi r$

Area
rectangle $A = lw$
square $A = s^2$
parallelogram $A = bh$
triangle $A = \frac{1}{2}bh$
circle $A = \pi r^2$

Volume
rectangular prism $V = lwh$
any prism $V = Ah$ (A is the area of the base)
cylinder $V = \pi r^2 h$
cone $V = \frac{1}{3}\pi r^2 h$
pyramid $V = \frac{1}{3}Ah$ (A is the area of the base)
sphere $V = \frac{4}{3}\pi r^3$

Surface Area
rectangular prism $SA = 2(lw + wh + lh)$
any prism $SA = 2A + hP$ (A is the area of base and P is the perimeter of the base)
cylinder $SA = 2\pi r^2 + \pi dh$
cone $SA = \pi r^2 + \pi rs$ (s is the slant height of the cone)
pyramid $SA = A$ + Area of lateral faces (A is the area of the base)
sphere $SA = 4\pi r^2$

ANSWERS

UNIT 1 NUMBER AND OPERATIONS pp. 1–38

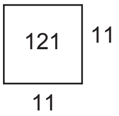
Getting Started—Skills You Will Need p. 1

1. a) 0.001 **b)** 0.05
c) 300 **d)** 0.4

2. a) –8 **b)** –16 **c)** 9

3. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289

4. *Sample response*:



5. 12 and 13

6. *Sample response*: the last digit of 1.414 × 1.414 must be 6 because 4 × 4 = 16

1.1.1 Introducing the Exponent Laws p. 4

1. a) 6 **b)** 15 **c)** 7

2. A, C, and D

3. a) *Sample response*: $5^4 \times 5^4$
b) *Sample response*: $5^{10} \div 5^2$
c) 25^4
d) *Sample response*: $25^2 \times 25^2$
e) *Sample response*: $25^{10} \div 25^6$

4. a) 2^6
b) *Sample response*: $2^4 \times 2^2$
c) *Sample response*: $2^2 \times 2^2 \times 2^2$

5. a) 2^1 **b)** 2^2 **c)** 2^7

6. $3^4 = 3 \times 3 \times 3 \times 3$ and $9^2 = 9 \times 9$, since $9 = 3 \times 3$, then $(3 \times 3) \times (3 \times 3) = 9 \times 9$

7. 7^2, since $7^2 \times 7^2 = 7^4$

8. a) No, since $a + b = 11$ and you cannot add two even numbers and get 11.
b) 5 pairs: 1 and 10, 2 and 9, 3 and 8, 4 and 7, 5 and 6

9. Add the exponents (4, 7, and 19) to get 5^{30}, because the base is 5 in each power.

10. When the bases are the same, you can subtract the exponent of the divisor from the exponent of the dividend; *sample response*: $5^6 \div 5^4 = 5^{6-4} = 5^2$

1.1.2 The Power Law of Exponents p. 7

1. a) 5^{15} **b)** 13^{36} **c)** 9^{25} **d)** 2^{21}

2. a) $b = 6$ **b)** $b = 12$ **c)** $b = 9$

3. -2^{30}, $(-2)^8$, $(2^7)^2$, 8^5, $(2^4)^4$, $(4^3)^3$

4. a) *Sample response*: $m = 3$ and $n = 1$
b) *Sample response*: $m = 3$ and $n = 2$

5. $n = 2$

6. *Sample responses*:
a) $(8^2)^5$ **b)** $(2 \times 4)^{10}$ **c)** $(16 \div 2)^{10}$

7. 10^6

8. a) 10^2; $5^2 \times 2^2$
b) 12^2; *sample response*: $4^2 \times 3^2$
c) 40^2; *sample response*: $8^2 \times 5^2$

9. *n* must be even since you have to be able to divide it by 2 using the power law to create a perfect square.

10. *Sample response*:
- power law: $5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{4 \times 3} = 5^{12}$
- product law: $5^8 \times 2^8 = 10^8$
- quotient law: $20^8 \div 2^8 = 10^8$

1.1.3 Negative and Zero Exponents p. 10

1. a) 1 **b)** $\frac{1}{25}$ **c)** $\frac{1}{25}$ **d)** 1

2. a) $\frac{1}{9}$ and $\frac{1}{9}$ **b)** 9 and 9

c) Since $9 = 3^2$, whatever power 9 is raised to should be doubled if 3 is raised to that power.

3. 9

4. a) *Sample response*: $a = 3$, $b = -3$
b) *Sample response*: $a = 3$, $b = -2$
c) *Sample response*: $a = 2$, $b = 49$
d) *Sample response*: $a = 0$, $b = 0$

5. a) $\frac{1}{2} = 2^{-1}$ and $\frac{1}{3} = 3^{-1}$ **b)** $\frac{1}{6}$

c) The values are equal since $\frac{1}{2} \times \frac{1}{3} = 2^{-1} \times 3^{-1} = (2 \times 3)^{-1} = 6^{-1}$, using the power of a product law.

6. Five possibilities: 0, 1, 2, $\frac{1}{2}$, and $\frac{1}{4}$

7. a) 3^{-5} since it is a positive fraction and $(-5)^3$ is a negative number
b) $(-5)^4$ since it is the same as 5^4, which is a positive whole number but 4^{-5} is a fraction less than 1.
c) $(-9)^2$ since it is the same as 9^2 which is 81, but -9^2 is -81.
d) equal since they both represent the opposite of 9^3

8. a) if n is odd, the value is negative; if n is even, the value is positive
b) If the greater value is odd, the result is negative and therefore less; *sample response*: $(-6)^3 < (-6)^2$

9. $\left(\frac{3}{10}\right)^{-2} = \frac{1}{\left(\frac{3}{10}\right)^2} = 1 \div \frac{9}{100} = 1 \times \frac{100}{9} = \frac{100}{9}$ and $\frac{100}{9} = \left(\frac{10}{3}\right)^2$

10. *Sample response*: A number divided by itself is equal to 1 and $a^b \div a^b = a^0$, so $a^0 = 1$.

1.1.4 Fractional Exponents p. 12

1. a) 12 **b)** 5 **c)** $\frac{1}{8}$ **d)** 64

2. a) 6 **b)** 36

c) *Sample response*: $\frac{2}{4} = \frac{1}{2}$ and the $\frac{1}{2}$ power means the square root

3. a) 63^{24} **b)** 30^{31} **c)** 118^{13}

4. Using the power law, $(49^{\frac{1}{2}})^3 = 49^{\frac{3}{2}}$ and $(49^3)^{\frac{1}{2}} = 49^{\frac{3}{2}}$; since $7 = 49^{\frac{1}{2}}$, then $7^3 = (49^{\frac{1}{2}})^3 = 49^{\frac{3}{2}}$

5. a) *Sample response*: 3, 6, 9, 12, 15

b) *Sample response*: $(5^3)^{\frac{1}{3}} = 5^1 = 5$ and $(5^6)^{\frac{1}{3}} = 5^2 = 25$

c) No; when you multiply $100 \times \frac{1}{3}$ using the power law, you do not get a whole number exponent, since 100 is not a multiple of 3.

6. a) 4 **b)** 0 **c)** 3

7. The base stays the same and you take half the value of the exponent to get the exponent of the square root; *sample response*: $\sqrt{5^8} = (5^8)^{\frac{1}{2}} = 5^4$

1.2.1 Scientific Notation with Large Numbers p. 17

1. a) 2×10^6 **b)** 4.357893389×10^9
c) 3.6×10^{13} **d)** 2.04783895×10^7

2. 1.6×10^{30}

3. 1.4×10^9

4. a) *Sample response*: $423{,}572 \times 10^2$; $4{,}235{,}720 \times 10^1$; $42{,}357.2 \times 10^3$
b) *Sample response*: No, since the multiplier was never between 1 and 10.

5. a) $(2 \times 10^6) \times (7 \times 10^{13}) > 10^{20}$ because $(2 \times 10^6) \times (7 \times 10^{13}) = 14 \times 10^{19} = 1.4 \times 10^{20}$; $10^{20} = 1 \times 10^{20}$ and $1.4 > 1$
b) $(2 \times 10^4)^3 > 250$ billion because 250 billion $= 250 \times 10^9 = 2.5 \times 10^2 \times 10^9 = 2.5 \times 10^{11}$; $(2 \times 10^4)^3 = 2^3 \times 10^{12}$ and $10^{12} > 10^{11}$

5. c) $375 \times 10^6 > 3.5893 \times 10^8$ because $375 \times 10^6 = 3.75 \times 10^8$ and $3.75 > 3.5893$

6. a) 3750×10^{36} **b)** 3.750×10^{39}

7. *Sample response*: $46.6 \times 10^9 \times 68.1 \approx 50 \times 70 \times 10^9 = 3500 \times 10^9 = 3.5 \times 10^{12}$

8. *Sample response for a 15-year-old*: $15 \times 365 \times 24 \times 60 \times 12 \approx 94{,}608{,}000$ or 9.4608×10^7

9. The parrot's; *sample response*: $45 \times 365 \times 24 \times 60 \times 550 > 75 \times 365 \times 24 \times 60 \times 70$ because $45 \times 550 > 75 \times 70$.

10. a) $N = 2{,}000$
b) No; $500 \times 7.5 \times 10^5 = 5 \times 10^2 \times 7.5 \times 10^5 = 3.75 \times 10^8$ which is much greater than 3.4×10^7

11. The greatest value for the first number is $9.9\ldots \times 10^9$, which is almost 10^{10}, the least value for the second number is $1.0 \times 10^{10} = 10^{10}$, so the first value is always less.

12. *Sample response*: $6{,}536{,}211{,}569 \approx 6{,}000{,}000{,}000 = 6 \times 10^9$ and the multiplier for $6{,}536{,}211{,}569$ must be ≥ 1 and < 10, so $6{,}536{,}211{,}569 = 6.536211569 \times 10^9$.

CONNECTIONS: The Richter Scale p. 18

1. The increase in magnitude from 5.7 to 8.1, which is an increase of 2.4 ($8.1 - 5.7 = 2.4$), means the 1897 earthquake was $10^{2.4} \approx 251.19$ times stronger.

1.2.2 Scientific Notation with Small Numbers p. 22

1. a) 7×10^{-5} **b)** 1.34893×10^{-3}
c) 8.8×10^{-14} **d)** 3.561587×10^5

2. a) 4.50000^{-01} **b)** 7.39400^{-02}

3. a) 10,584,100
b) the ones digit should be 4 and not 0 because the product of the ones digit of the factors is 4 because $4 \times 6 = 24$

4. a) i) $\frac{1}{3}$ **ii)** $\frac{1}{2}$
b) The first one is not exact since the 3s keep repeating, but the second likely is since it ends in a lot of zeros.

5. 2.1×10^{-4}

6. 9.2567×10^{10}

7. a) 2.64×10^{-7} km/day
b) 9.636 cm/year

8. 350 min (moves about 0.16 m/h)

9. a) Yes, if $m \times n \geq 10$
b) Yes, if $1 \leq m \times n < 10$
c) No, since the least possible values are $1 \times 10^p \times 1 \times 10^q$ and the exponent is $p + q$

1.3.1 Estimation with Rational Numbers p. 25

1. a) *Sample response*: about 200
b) *Sample response*: about 7
c) *Sample response*: about 180
d) *Sample response*: about 2 million

2. *Sample response*: about 200

3. *Sample response*:
a) about 6 days; high because I estimated it would take twice as long with half as many people, but there are more than half as many people.
b) *Sample response*: They might not have used $\frac{1}{2}$ to estimate the fraction of people.

4. a) *Sample response*: about 720,000
b) 604,800; *sample response*: high since I used 10 instead 7 (days/week) even though I used 20 instead of 24 (h/day).

5. *Sample response*: about 1750

6. a) *Sample response*:
$-7 \times \left(\frac{2}{3}\right)^4 \approx -7 \times \left(\frac{1}{2}\right)^2 \approx -2$

or $-7 \times \left(\frac{2}{3}\right)^4 \approx -7 \times (0.4)^2 \approx -1.2$

b) *Sample response*:
$7.06 \div 0.3 \approx 7 \div \frac{1}{3} = 21$
or $7.06 \div 0.3 \approx 7 \times 3.3 \approx 21 + 2 = 23$
c) *Sample response*:
$0.25 \times 465 \approx \frac{1}{4} \times 480 = 120$
or $0.25 \times 465 \approx \frac{1}{4} \times 464 = 116$
d) *Sample response*:
$1078 \times 512 \approx 1000 \times 500 = 500{,}000$
or $1078 \times 512 \approx 1100 \times 500 = 550{,}000$

7. *Sample response*: comparing the distance from Venus to Earth with the distance from Saturn to Earth

1.3.2 Order of Operations p. 28

1. a) 4.3×5.7 **b)** $5.8 \div 3.6$
c) $-6.7 \div 3.2$ **d)** $(2.3)^3$

2. a) 32.5 **b)** $8\frac{1}{9}$
c) –3.8 **d)** 0.128

3. A, C, and D

4. a) *Sample response*: no brackets needed
b) *Sample response*:
$\left(\frac{2}{3}\right)^{-1} - \left(1 - \frac{2}{3}\right) \times 3 = \frac{1}{2}$
c) *Sample response*:
$\frac{1}{4} \times \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{4} \div \frac{1}{4} = 1$
d) *Sample response*:
$\frac{2^3}{5} - 2 \times 3 \div (4 + 1) = \frac{2}{5}$

5. a) *Sample response*:
$1 \times \left(\frac{2}{3} + 3\right) \times \left(4 + \frac{2}{5}\right) = \frac{242}{15}$

5. b) *Sample response*: no brackets needed

6. No; adding is commutative so the order in which you add the numbers doesn't affect the answer; *sample response*: $[-2 + 3] + (-4) + 5 + (-6) = -2 + 3 + [(-4) + 5 + (-6)] = -4$

7. Yes; *sample response*:
$4 - (\frac{1}{2} - 3) + 6 \neq (4 - \frac{1}{2}) - 3 + 6$

8. The expression in the brackets does not have to be calculated since $a^0 = 1$ and the expression $\neq 0$ since it's the sum of positive numbers.

9. Yes; total area is $2.5^2 + 3.7^2 = 19.94$ m^2 not $(2.5 + 3.7)^2 = 38.44$ m^2.

10. Some expressions change value if operations are performed in a different order, so without rules people could get different answers for the same question; *sample response*: $20 \div 4 \times 5 = 25$ but if you multiply first, it would be 1.

1.3.3 Square Roots p. 31

1. a) 6 **b)** 10 **c)** 17 **d)** 80

2. B since 30 × 30 = 900

3. a) 82 **b)** 820

4. 0.4444… = $\frac{4}{9}$ and the square root is $\frac{2}{3}$, which is a rational number

5. a) 70 × 70 = 4900, which is close to 4823
b) less **c)** 69.4 m

6. a) i) equal; *sample response*: both are about 5.3
ii) equal; *sample response*: both are about 17.3
iii) If you break a number into factors, you can multiply the square roots of the factors to calculate the square root of the number.
b) *Using the product of powers law*:
$(20 \times 4)^{\frac{1}{2}} = 20^{\frac{1}{2}} \times 4^{\frac{1}{2}}$
b) *By calculating*:
$\sqrt{20 \times 4} = 8.94427191$; $\sqrt{20} \times \sqrt{4} = 4.472135955 \times 2 = 8.94427191$

7. Since $8 = 4 \times 2$, $\sqrt{8} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2}$ and $\sqrt{2}$ is not rational, so $2 \times \sqrt{2}$ cannot be rational.

8. All numbers have a positive square root and a negative square root, in this case, 29.4 and –29.4.

9. a) i) 4.5 s **ii)** 14.2 s **iii)** 100.6 s
b) number of seconds cannot be negative

10. If you square any whole number greater than 1, the result is a greater whole number. So the square root of any whole number must be less than the number.

11. a) $9 \times 7 \times 6 \times 9 \times 7 \times 6 = 142{,}884$
b) $9216 = 9 \times 1024 = 9 \times 2^{10}$, so $\sqrt{9216} = 3 \times 2^5 = 96$

12. a) 4.6216×10^4, so about 200
b) 62.6147×10^4, so about 800

13. *Sample response*: $39{,}417 \approx 40{,}000 = 4 \times 10{,}000$, so $\sqrt{39{,}417} \approx \sqrt{4} \times \sqrt{10{,}000} = 2 \times 100 = 200$

1.3.5 Representing Real Numbers p. 37

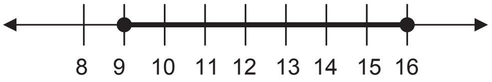
1. A, B, C, and F:
A. repeating decimals are rational
B. terminating decimals are rational
C. repeating decimals are rational
F. $3\pi^0 = 3$ which is rational

2. estimates: $\frac{88}{7}$, 3.14×4, $3.14 \times \sqrt{16}$; each used an estimate for π

3. A, B, D, and F:
A. adding 0.8 + 0.4 = 1.2 which is a terminating decimal which is rational
B. subtracting two fractions gives a fraction and fractions are rational
D. dividing a number (other than 0) by itself gives 1 which is a rational number
F. adding a terminating and repeating decimal gives a repeating decimal which is rational

4. a) *Sample response*: 4.5
b) *Sample response*: $\sqrt{17}$
c) *Sample response*: $\frac{5}{2}$
d) *Sample response*: $\pi - 5.7$

5. a) 9 is least and 16 is greatest
b)

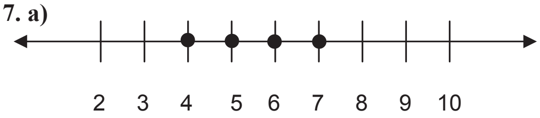


c) You would put solid circles only at each integer from 9 to 16 inclusive

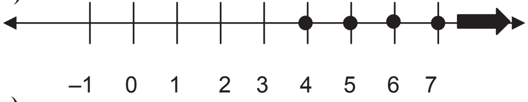
1.3.5 Representing Real Numbers [Continued] p. 37

6. a) F; the answer is 0 which is rational
b) T; adding a rational to an irrational usually results in an irrational
c) T; the answer is 1 which is rational

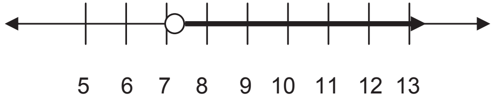
7. a)



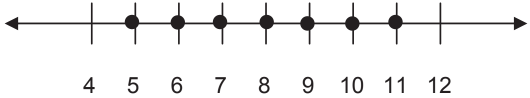
b)



c)



d)



8. a) $9 \le n \le 6\sqrt{3}$, n is a real number **b)** $\sqrt{40} \le n < 13$, n is a real number

9. $-\frac{7}{9} = -\frac{14}{18}$ and $-\frac{8}{9} = -\frac{16}{18}$, so $-\frac{15}{18}$ is between $-\frac{14}{18}$ $(-\frac{7}{9})$ and $-\frac{16}{18}$ $(-\frac{8}{9})$

UNIT 1 Revision p. 38

1. a) 8^{29} **b)** 5^{19} **c)** $2^3 \times 3^8$

2. a) 10,000 **b)** 100,000

3. a) 4 **b)** 16 **c)** 0

4. $-\frac{2}{3}$

5. a) $\frac{1}{16}$ **b)** 196 **c)** 600,000

6. Since a and b are between 1 and 10, N is greater because its power of ten has a greater exponent.

7. a) 2,340,000
b) $2.34000^{06} = 2.34 \times 10^6 = 2{,}340{,}000$; *sample response*: 2.34×10^6 means you move the digit 2 six places to the left so 2.34 becomes 2,340,000

8. a) 3.518×10^{-7}
b) Yes; $0.0003518 = 3.518 \times 10^{-4} > 4 \times 10^{-7}$ because $10^{-4} > 10^{-7}$ and 3.518 and 4 are both ≥ 1 and < 10

9. *Sample response*: $1 \times -7 = -7$; $\frac{4}{5}$ of –7.5 is –6

10. No; *sample response*: a 15-year-old is about 9,000,000 min old ($15 \times 365 \times 24 \times 60 \approx 15 \times 400 \times 25 \times 60 = 9{,}000{,}000$)

11. a) 158 **b)** 16.3

12. a) two solutions, a positive and negative square root
b) no solutions because there is no square root for a negative number

13. The positive square root since you cannot have a negative thickness.

14. a) *Sample response*: about 29
b) *Sample response*: about 0.9
c) *Sample response*: about 2500
d) *Sample response*: about 48,000

15. $1764 = 9 \times 196 = 3 \times 3 \times 14 \times 14$, so $\sqrt{3 \times 3 \times 14 \times 14} = 3 \times 14 = 42$.

16. It is between about 0.6 and 0.7 (or between –0.6 and –0.7)

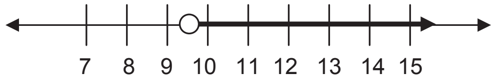
17. C and E:
- C is a repeating decimal, which is rational
- E is $\frac{3}{7}$, which is rational

18. a) True; *sample response*:
$\sqrt{2} \times \sqrt{2} = 2$, which is rational
b) False; *sample response*:
0.020406081012… is non–terminating and never repeats

19.
a)

sample response: rationals: 3.5, 3.6, 3.7; irrationals: $\sqrt{2} + 2.1$, $\sqrt{2} + 2.2$, $\sqrt{2} + 2.3$

b)



sample response: rationals: 13, 205, 327; irrationals: 4π, 5π, 6π

UNIT 2 POLYNOMIALS pp. 39–79

Getting Started—Skills You Will Need p. 40

1. a) –7 **b)** –7 **c)** +6 **d)** +4

2. a) $6x + 12$ **b)** $20x – 8$
c) $–6 + 3x$ **d)** $–8x + 20$

3. a) 1.89 square units
b) 4.41 square units
c) 1.62 square units
d) 3.14 or π square units

4. a) 6 units **b)** 6.28 or 2π units **c)** 9 units

5. a) –1 **b)** 19 **c)** –4.5 **d)** –2.4

2.1.1 Interpreting Polynomials pp. 44–45

1. a) binomial **b)** trinomial
c) monomial **d)** binomial
e) binomial **f)** monomial

2. a) 1 **b)** –2 **c)** 2.3 **d)** $\frac{3}{4}$

3. B, C, and D

4. a) 3 **b)** 4 **c)** 2 **d)** 1

5. *Sample response*: $x^2 – 3xy – 3$; $2y + (– 3)$

6. a) 184 **b)** 4 **c)** 93 **d)** –6

7. a) $3t$ and $7t$; $10t – 3t^2$
b) $–2n$ and $–7n$; $3 + 8m – 9n$
c) $3p$ and $–17p$; $4q$ and $–2q$; $–14p + 2q$
d) There are no like terms.

8. *Sample responses*:
a) $3x^3$ **b)** $2m + 4$
c) $p^3 – 5p^2 + p$ **d)** $2m + 1$
e) $–\frac{1}{2}(a^2 + b^2)$ when $a = 2$ and $b = –2$

9. *Sample response*: $2m – 3$, $2m^2 – 3$, $–3 + 2m$, $–3 + 2m^2$, $–3m – 2$, $–3m^2 – 2$, $2 – 3m$, $2 – 3m^2$, $2 – m$, $2 – m^2$, $–3 – m$, $–3 – m^2$

2.1.1 Interpreting Polynomials [Continued] p. 45

10. $6l + 4s$

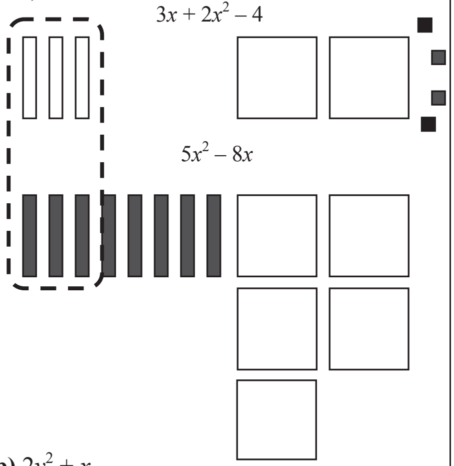
11. Any two of $2r$ (diameter), πr^2 (area), and $2\pi r$ (circumference)

12. *Sample responses*:
a) $80b + 85c$ to describe the cost of b kg of beef and c kg of chicken; or,
$85c – 80b$ to describe the difference in cost between b kg of beef and c kg of chicken
b) $2.25n$ to describe the time (hours) for n trips from Thimphu to Punakha; or,
$154r$ to describe the distance (km) for r round trips between Thimphu and Punakha
c) $0.7e + 0.3p$ to calculate the final mark
d) e^3 to describe the volume; or,
$6e^2$ to describe the total surface area

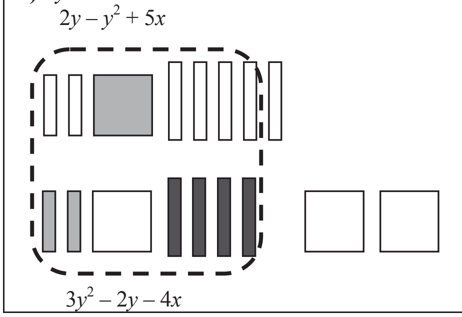
13. *Sample response*: Same: binomials, variable s, worth 8 if $s = 1$; Different: one is degree 1 and one is degree 2

2.1.2 Adding and Subtracting Polynomials p. 52

1. a) $–5x + 7x^2 – 4$



b) $2y^2 + x$



2. a) $–4x – x^2 – 4$
b) $4m^2 – 2m^3 + 7.5m + 2$
c) $7y^3 – y^2 – 11x + y$

3. $(y^2 – y + 2x + 1)$ and $(–y^2 – y – x – x^2 – 2)$; $x – 2y – 1 – x^2$

4. a) $–y + y^2 – 3x$ **b)** $4y^2 –x^2 – y$

5. a) $–6y – 2y^2 – 7x$ **b)** $x^2 – 4y + 2y^2$

6. a) $–2m + m^3 – 8t$ **b)** $–m^2 – 2m – 24$

7. a) $11x + 4y – 2y^2$
b) $–5k – 5h – 6y^2$
c) $4m^2 – 2m + 7r – 8$

8. $(–2y^2 – 3x – 2) – (–2y^2 – x) = –x – 2$

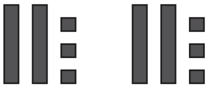
9. a) $16x + 12$ **b)** $4x + 15$

10. *Sample response*: When you subtract, you sometimes have to decide how to subtract depending on what you're subtracting; with addition, you always just combine like terms.

2.2.1 Multiplying a Polynomial by a Monomial p. 56

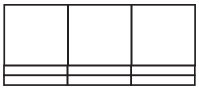
1. a) $2(-2x^2 + 1)$ **b)** $4(3x)$ **c)** $4(5x - 3)$

2. a)



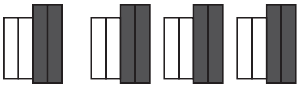
$2(-2x - 3) = -4x - 6$

b)



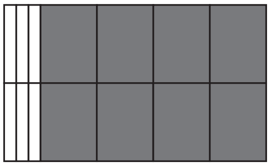
$3y(y + 2) = 3y^2 + 6y$

c)



$4(2y - 2x) = 8y - 8x$

d)



$2x(3 - 4y) = 6x - 8xy$

3. a) $-3x^2 -2xy$ **b)** $24 - 6t + 18t^2$
c) $2jk^2 - 6jk$ **d)** $15m + 5m^3 - 10r$

4. a) $6x$ **b)** $8x - 2$

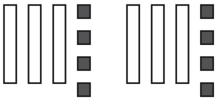
5. a) $6000x + 6050y$
b) $6000x + (6050 \times 2x) = 18{,}100x$
c) $6000x + (6050 \times 0.5x) = 9025x$

6. *Sample responses*:
a) $3(2t^2 - t + 5)$ **b)** $2(4xy - 5y^2 + 3y)$
c) $2(7x - 8x^2)$ **d)** $4(2 - 4x + 3y)$

7. a) $50 - 2c$ **b)** $10x^2 - 14xy - 6y$
c) $15y - 20x - 4y^2$ **d)** $2xy - 3x^2y + 4xy^2$

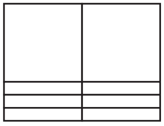
8. A monomial × a monomial = monomial, $3x(2x) = 6x^2$, but a monomial × a binomial = binomial, since it results in a sum of two unlike terms, $3x(2 + x) = 6x + 3x^2$.

9. *Sample responses*:
a)



$2(3x - 4) = 6x - 8$

b)



$2x(x + 3) = 2x^2 + 6x$

2.2.2 Multiplying a Binomial by a Binomial pp. 60–61

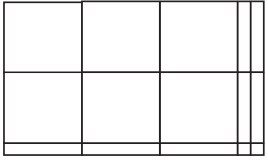
1. A, since $(2x - 1)(x + 1) = 2x^2 + 2x - 1x - 1$

2. The width is $2x + 3$ and the height is $x - 2$. The area is the product of the two dimensions.

3. $(2x - 4)(3x + 1)$;
$(2x - 4)(3x + 1)$ uses 24 tiles: six x^2-tiles, two x-tiles, twelve $-x$-tiles, four -1-tiles
$(2x + 1)(3x - 4)$ uses 21 tiles: six x^2-tiles, three x-tiles, eight $-x$-tiles, four -1-tiles

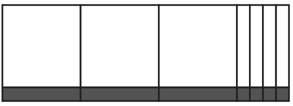
4. a) $(2x + 1)$ and $(y + 1)$
b) $(x + 1)$ and $(x - 2)$
c) y and $(3y - 1)$

5. a) $6x^2 + 7x + 2$



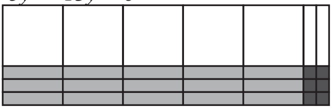
$(2x + 1)(3x + 2)$

b) $3x^2 + x - 4$



$(3x + 4)(x - 1)$

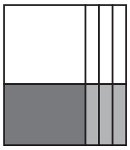
c) $5y^2 - 13y - 6$



$(y - 3)(5y + 2)$

2.2.2 Multiplying a Binomial by a Binomial [Continued] pp. 60–61

5. d) $x^2 - xy + 3x - 3y$

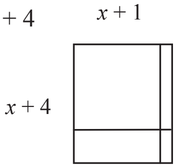


$(x + 3)(x - y)$

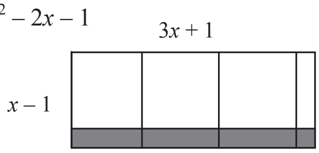
e) $6 - 3x + 2y - xy$

$(2 - x)(3 + y)$

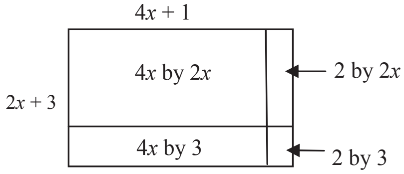
6. a) $x^2 + 5x + 4$



b) $3x^2 - 2x - 1$

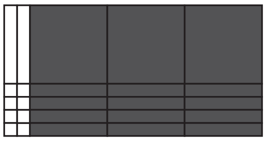


c) $8x^2 + 16x + 6$



7. a) $3x^2 + 7x + 2$ **b)** $4x^2 + 2x - 12$
c) $x^2 + 8x + 12$ **d)** $\pi(4x^2 + 24x + 36)$

8. a) $8 - 10x - 3x^2$



$(2 - 3x)(4 + x)$

b) 11 **c)** 11 **d)** 1
e) If you multiply 11 by 1, you get 11, just as $(2 - 3x)(4 + x) = 11$ for $x = -3$.

9. $6x(x + 2)$ or $3x(2x + 4)$ or $2x(3x + 6)$ or $x(6x + 12)$

10. *Sample response*: $(2x + 1)(x - 4)$ or $(x - 2)(2x + 3)$ or $(3x + 2)(2x + 1)$ or $(2x + 1)(x + 4)$ or $(x + 2)(4x + 1)$

11. The original x^2 is made up of a rectangle $(x + 1)(x - 1)$ and 1 more, so $(x + 1)(x - 1) = x^2 - 1$.

12. Both times you make a rectangle and calculate the area. When you multiply a binomial by a monomial there are only two kinds of tiles, but there could be 3 or 4 kinds of tiles when you multiply two binomials.

2.2.3 Multiplying Polynomials Symbolically pp. 64–65

1. a) $15x^2 + 26x + 8$ **b)** $24y^2 + 22y - 10$
c) $-6xy - 24y - 8x - 32$ **d)** $6y^2 + 34y + 48$

2. a) 3 **b)** 2 **c)** 4 **d)** 2

3. a) $5y^2 + 24y + 29$ **b)** $-18x - 8xy + 3y + 18$

4. a) $35 + 13s - 12s^2$
b) $-112 = 35 + 13(-3) - 12(-3)^2$
c) 16 **d)** –7
e) If you multiply the answers to **4 c) and d)**, you get the same answer as **4 b)**:
$16 \times (-7) = -112$

5. $8n^2 + 20n + 12$

6. The value of ▲ is 4.

7. a) $2x^2 + 7x - 4$
b) $(16 + 2\pi)x^2 + (16 + 2\pi)x + 4 + \frac{\pi}{2}$
c) $6y^2 + 11y - 10$

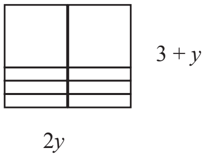
8. $4ax$

9. a) Calculate $40^2 + 160 + 4 = 1764$
b) Calculate $80^2 - 160 + 1 = 6241$
c) Calculate $(50 + 3)^2 - (50 - 3)^2 = 600$

10. To calculate 32×43, multiply each part of 32 (30 and 2) by each part of 43 (40 and 3): $(30 \times 40) + (30 \times 3) + (2 \times 40) + (2 \times 3) = 1376$.
For $(3x + 2)(4x + 3)$, you multiply each part of $3x + 2$ by each part of $4x + 3$:
$(3x)(4x) + 3x(3) + 2(4x) + (2)(3) = 12x^2 + 17x + 6$.

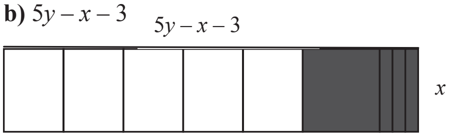
2.3.1 Dividing a Polynomial by a Monomial p. 69

1. a) $3 + y$



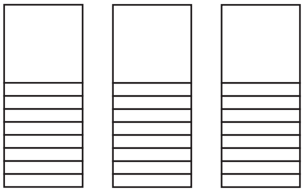
$(6y + 2y^2) \div 2y$

b) $5y - x - 3$



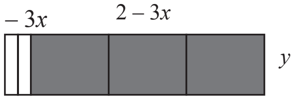
$(5xy - 3x - x^2) \div x$

c) $x^2 + 8x$



$(3x^2 + 24x) \div 3$

1. d) $2 - 3x$



$(2y - 3xy) \div y$

2.a) $3m^2 - mn$ **b)** $2s^2 + 1 - 5t$
c) $4m^2 - 2m$

3. a) $2k + 3m$; 5; –20; –4; if you divide –20 by –4 you get 5
b) $2k - 2$; –6; 36; –6; if you divide 36 by –6, the quotient is –6

4. a) $(4y^2 + 6) \div 2 = 2y^2 + 3$
b) $(-4xy - 6y) \div 2y = -2x - 3$
c) $(2x^2 - 3x) \div x = 2x - 3$

5. *Sample response*: $(6y - 4) \div 2$; $(3xy - 2x) \div x$; $(9y - 6) \div 3$; $(3x^2y - 2x^2) \div x^2$

6. To divide, think of a missing factor in the related multiplication;
sample response:
$(2xy + y^2) \div y \rightarrow y \times ? = 2xy + y^2$

2.3.3 Dividing a Polynomial by a Binomial p. 75

1. a) $2x + 3$ **b)** $3x - 4$
c) $2x + 5$ **d)** $4x + 9$

2. a) $2x^2 - 1$ **b)** $4x - 2$
c) $x^3 + 1$ **d)** $4 - 3x^2$

3. a) –1, –1, and 1 **b)** –24, 4, and –6
c) 0, –2, and 0 **d)** 6, 6, and 1
Each time, I noticed that when I divided the first two values, the last value was the quotient.

4. *Sample response*: $4x^2 - 1$ and $2x - 1$; $2x^2 - 5x - 3$ and $x - 3$; $2x^3 + 3x^2 + x$ and $x^2 + x$

5. a) $x + 1$ **b)** $4x - 12$

6. *Sample response*: $(10x^2 + 3y) \div y$ is not a polynomial since $\frac{10x^2}{y}$ is not a polynomial.

7. Two; when you divide polynomials with exponents, you subtract the exponents using the exponent quotient law; *sample response*: $4x^4 \div 2x^2 = 2x^2$

CONNECTIONS: Using Number Patterns to Factor p. 77

1. The factors are $(x + 2)$ and $(x + 3)$.

2. The factors are $(2x + 1)$ and $(2x + 1)$.

3. The factors are $(x + 1)$ and $(x – 1)$.

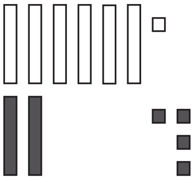
4. The factors are $(x + 1)$ and $(x – 2)$.

5. The factors are $(3x – 1)$ and $(3x – 1)$.

UNIT 2 Revision pp. 78–79

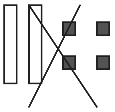
1. a) degree 1; binomial; like terms are $3x$ and $4x$ as well as $–2y$ and $6y$;
Sample response: the perimeter of a triangle with dimensions $6y$, $4x$, and $3x – 2y$
b) degree 2; monomial; no like terms;
Sample response: the area of a rectangle with dimensions of 16 and x^2

2. a) $4x – 3$



$(–2x – 4) + (6x + 1)$

b) $x – 2$



$(2x – 4) – (x – 2)$

c) $–8x – 5$
$(–2x – 4) – (6x + 1) = (–2x – 4) + (–6x – 1)$

$= –8x – 5$

3. a) $–2x^2 + x + 8 – y$
b) $x^2 – 4x + 5$
c) $4x – 7x^2 + 8x^3 + y^2$

4. Because a polynomial is not in its simplest form to model until you have combined like terms.

5. a) $6y – 8y^2$ **b)** $8 – 12x$
c) $–6xy – 24x$ **d)** $–15x + 6x^2$

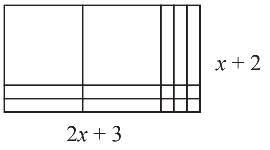
6. a) 6, –9, and –54 **b)** 4, 8, and 32
c) 12, 7, and 84 **d)** 9, 6, and 54
I noticed that multiplying the evaluated factors is equal to multiplying the factors and then evaluating the expression.

7. a) $y(x – 2y)$ **b)** $x(–3x – 2)$
c) $2(x^2 + 2xy – x)$

8. a) $12x^2 – 6x$ **b)** $9\pi y^2$

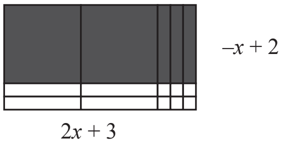
9. a) $(x + 5)(x + 1) = x^2 + 6x + 5$
b) $(2x + 3)(x – 2) = 2x^2 – x – 6$

10. a) $2x^2 + 7x + 6$



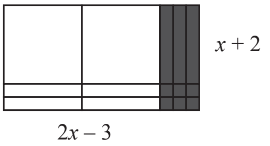
$(2x + 3)(x + 2)$

b) $–2x^2 + x + 6$



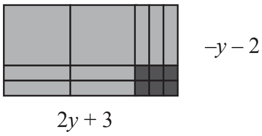
$(2x + 3)(–x + 2)$

c) $2x^2 + x – 6$



$(2x – 3)(x + 2)$

10. d) $-2y^2 - 7y - 6$



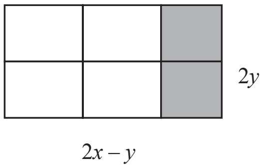
$(2y + 3)(-y - 2)$

11. $(2x - 3)(3x + 1)$ or $(x + 3)(4x + 1)$

12. The factors are degree 1 × degree 2 because of the exponent product law (1 + 2 = 3).

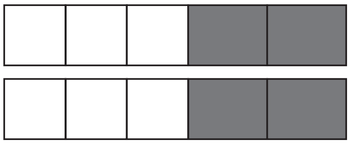
13. a) $(70 + 1)^2 = 70^2 + (2 \times 70) + 1 = 4900 + 140 + 1 = 5041$
b) $(71^2 - 69^2) = (71 + 69)(71 - 69) = 140 \times 2 = 280$

14. a) $2x - y$



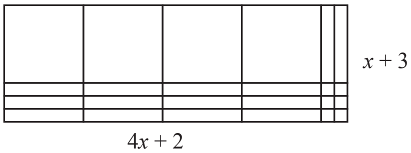
$(4xy - 2y^2) \div 2y$

14. b) $3y^2 - 2xy$



$(6y^2 - 4xy) \div 2$

c) $4x + 2$



$(4x^2 + 14x + 6) \div (x + 3)$

15. $(4x^2 + 2x - 2) \div (2x - 1)$ or $(4x^2 + 2x - 2) \div (2x + 2)$

16. a) $x^2 + 3x$
b) $-3x + 1$

17. a) $(3x - 1)(3x + 2) = 9x^2 + 3x - 2$
b) $(x + 3)(x - 3) = x^2 - 9$

UNIT 3 LINEAR RELATIONS AND EQUATIONS pp. 81–134

Getting Started — Skills You Will Need p. 81

1. a) 9 **b)** –8 **c)** 25 **d)** 1

2. a)

x	y
1	**–3**
2.5	0
3	**1**
4.5	**4**

2. b) 2 **c)** –2

3. a) a = –1 **b)** $x = -\frac{2}{3}$

4. $A = \pi (10)^2 \approx 314 \text{ cm}^2$
$C = 2\pi(10) \approx 63 \text{ cm}$

3.1.1 Patterns and Relations in Tables pp. 85–86

1. a), b), and c)

Figure	a) Perimeter	b) White triangles	c) Small triangles
1	3	1	1
2	6	3	4
3	9	6	9
4	12	10	16

1. d)
part a) is linear; first differences are all 3
part b) is quadratic; first differences are different, but second differences are all 1
part c) is quadratic; first differences are different, but second differences are all 2

2. Quadratic; second differences are 2.

3.1.1 Patterns and Relations in Tables [Continued] pp. 85–86

3. a) E **b)** L **c)** N **d)** N

4. a)

b)

5. a), b), and c)

Radius	**a)** Circum.	**b)** Area
0	0.00	0.00
1	6.28	3.14
2	12.57	12.57
3	18.85	28.27
4	25.13	50.27
c)	linear	quadratic

c) Circumference: linear since the formula has r only to the first power and first differences are constant (6.28)
Area: quadratic since the formula has r raised to the second power and second differences are constant and not zero (6.28)

6. a) 10
b) 20 (10 + 6 + 3 + 1)
c) 21
d) Quadratic

7. Linear; 1st differences are constant

8. Quadratic; 2nd differences are all 160

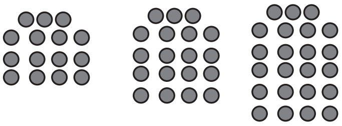
9. a) *Sample response*:

x	Linear	Quad.	Exp.
1	6	3	1
2	8	6	3
3	10	11	9
4	12	18	27

b) Linear; first differences are equal and second differences are zero
Quadratic; second differences are equal but not zero
Exponential; each term is the product of the previous term and a constant factor

3.1.2 Scatter Plots of Discrete and Continuous Data pp. 91–92

1. a)

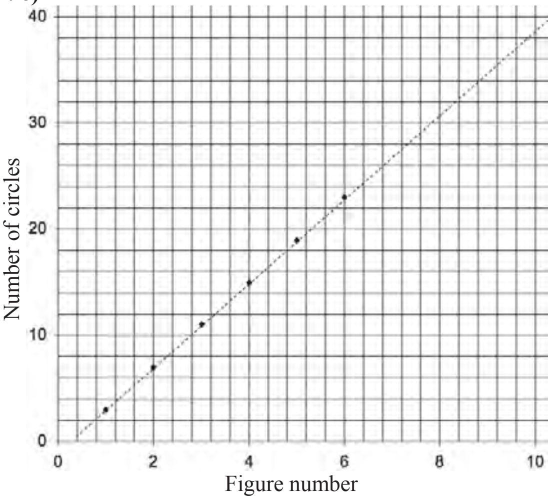


b)

Figure number	Number of circles
1	3
2	7
3	11
4	**15**
5	**19**
6	**23**

c) Discrete; does not make sense to have a fractional figure number
d) Dashed line

1. e)



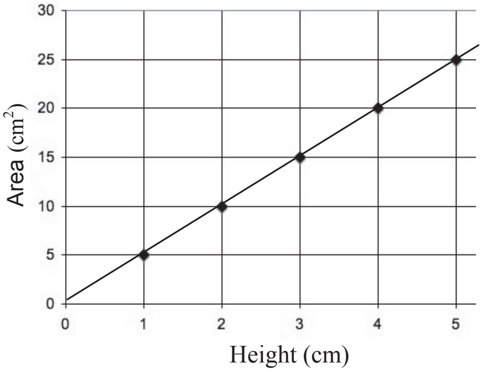
f) 39

2. a) and b)

Height (cm)	Area (cm^2)
1	5
1.5	7.5
2	10
3	15
4.75	23.75
5	25

c) Continuous; rectangles with a width of 5 cm can be any height
d) Solid line

e)



3. a) Day number is discrete

b) 15 trees on 7th day; 21 trees on 10th day

4. a) Time is continuous.
b) *Sample response*:
- How long did it take to drain 24,000 L? (120 min)
- How much water remained after 222 min? (35,000 L)

c) The pool is empty at 400 min and the volume of water in the pool cannot be negative.

3.1.2 Scatter Plots of Discrete and Continuous Data [Continued] pp. 91–92

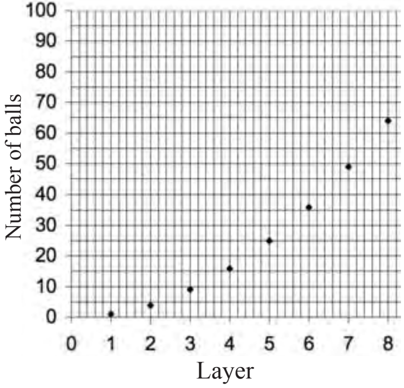
5. a)

Layer number	Number of balls
1	1
2	4
3	9
4	**16**
5	**25**
6	**36**
7	**49**
8	**64**

b) Discrete; no fraction layer numbers

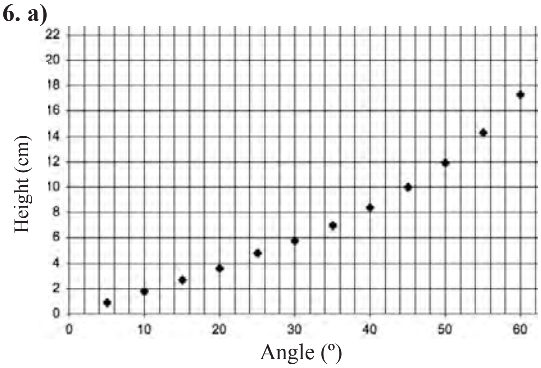
c) Dashed; data is discrete

d) *Sample response*:



The points do not lie on a line.

6. a)



b) *Sample response*:
A smooth curve; it looked like a line for a while, but then it started curving upward

c) Solid; an angle measure between those plotted is possible

7. Yes, the data points follow a curved pattern, so she was right to use a curve; No, the data is discrete so she should not have used a solid curve

CONNECTIONS: Half-Life p. 93

1. 128 g; 64 g; 32 g

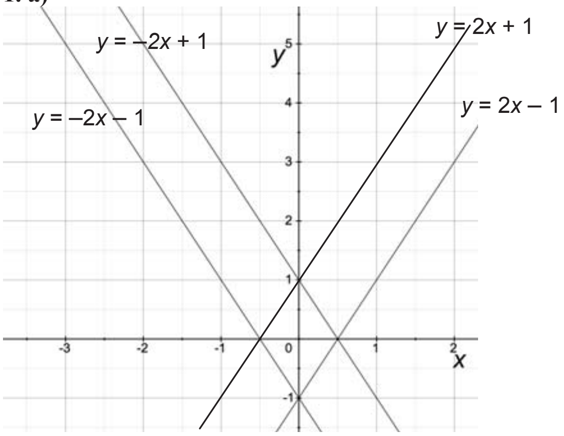
2. a) $\frac{1}{2}$; $\frac{1}{4}$; $\frac{1}{8}$ **b)** $\frac{1}{2^1}$; $\frac{1}{2^2}$; $\frac{1}{2^3}$

2. c) $\frac{1}{2^{10}}$ or $\frac{1}{1024}$; $\frac{1}{2^1}$ or $\frac{1}{2}$ after 1 day; $\frac{1}{2^2}$ or $\frac{1}{4}$ after 2 days; $\frac{1}{2^3}$ or $\frac{1}{8}$ after 3 days; so $\frac{1}{2^{10}}$ or $\frac{1}{1024}$ after 10 days

3. Exponential; the values of successive terms are found by multiplying by a constant ($\frac{1}{2}$)

3.1.4 Graphs of Linear and Non-Linear Relations p. 98

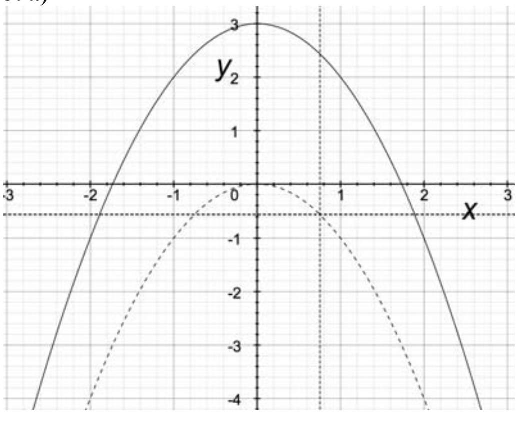
1. a)



b) They are all straight-line graphs.
c) $y = -2x - 1$ and $y = -2x + 1$ both have a slope of –2;
$y = 2x + 1$ and $y = 2x - 1$ both have a slope of 2

2. a) A; it's U-shaped with a vertical line of symmetry
b) C; it's a curve that is almost horizontal on the left and almost vertical on the right

3. a)

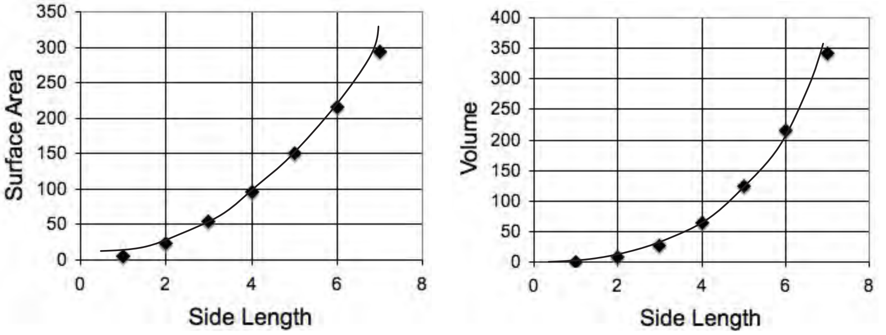


b) *Sample response*: Both are U-shaped with a vertical line of symmetry, both open downward, and each has a maximum point.
c) *Sample response*: One goes through the origin, but not the other.

4. A line; circumference is $C = 2\pi r$ and r is raised to the first power; also, circumference is a measure of length and length is linear

3.1.4 Graphs of Linear and Non-Linear Relations [Continued] p. 99

5. a) **b)**

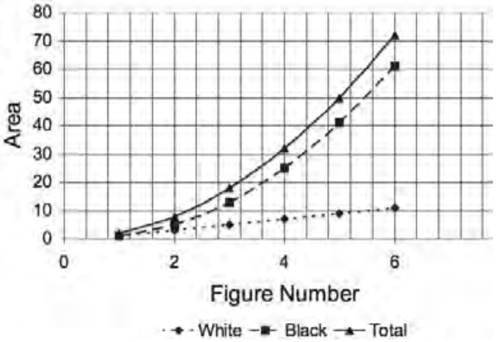


c) Neither graph has points that fall along a straight line.

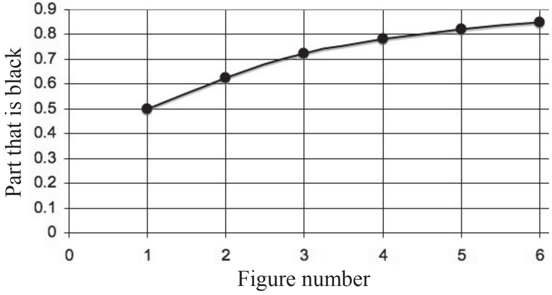
6. a)

Figure number	White area	Black area	Area of big square	Part of big square that is black
1	1	1	2	0.500
2	3	5	8	0.6250
3	5	13	18	0.7222
4	7	25	32	0.7813
5	9	41	50	0.82
6	11	61	72	0.8472

b) For columns 2, 3, and 4:



b) For last column



6. c)

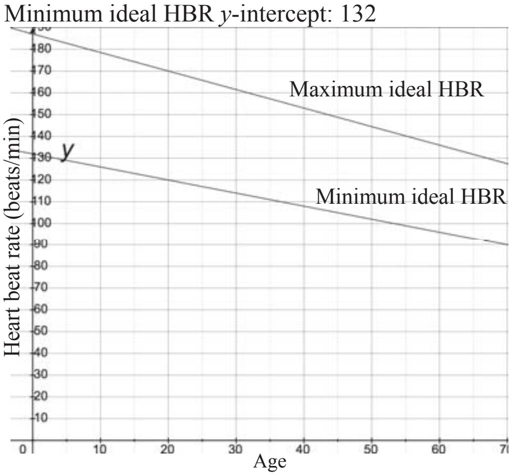
White area	Linear; because it's a straight line
Black area	Quadratic; because it looks like half a parabola
Big square	Quadratic; because it looks like half a parabola
Part of big square that is black	None; it's not straight, a parabola, or an exponential curve

3.2.1 The Meaning of Slope and *Y*-Intercept pp. 101–102

1. a) Cooling rate (in °C/min)
b) Temperature is falling
c) Starting temperature
d) Graph would be parallel and cross the *y*-axis at a higher point

2. a) The cost changes at a constant rate.
b) Cost per minute above base amount
c) Base amount

3. a) Maximum ideal HBR formula is $t = 0.85(220 - a)$, which is a linear expression
b) 187.
c) and d) Minimum ideal HBR formula is $t = 0.60(220 - a)$, which is a linear expression
Minimum ideal HBR *y*-intercept: 132

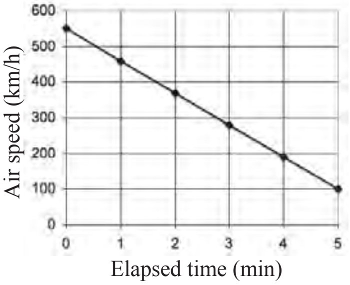


e) different slopes and *y*-intercepts; both are straight lines

4. a)

Elapsed time (min)	Airspeed (km/h)
0	550
1	**460**
2	**370**
3	**280**
4	**190**
5	100

4. b) A constant rate of change means a straight line graph.
c)



d) slope = –90; *y*-intercept = 550
e) *y*-intercept is starting airspeed, or 550 km/h; slope is change in speed per minute, so divide rise by run, e.g., –450 ÷ 5 = –90

5. a) A straight line, same *y*-intercept, shallower slope
b) Slope would be steeper because rise stays the same but run is reduced

6. a) Each time you add a square, the number of sticks increases by a constant amount, 3.
b) 3; for each increase of 1 in the figure number (run), there is an increase of 3 in the number of squares (rise)
c) 1; Figure 0 would have 1 stick

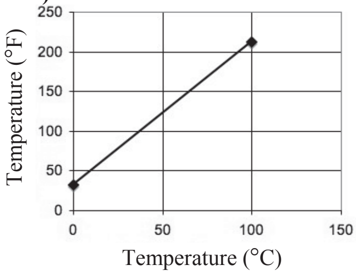
7. a) 2π
b) $r = 0$; the circumference of a circle with radius 0 cm is 0 cm

8. a) Perimeter is a measure of length and length is linear
b) 2π **c)** 4

3.2.1 The Meaning of Slope and *Y*-Intercept [Continued] pp. 101–102

9. If you don't, you get the opposite value for slope

10. a)



10. b) Locate 30°C on the horizontal axis and then find the corresponding point on vertical axis.

c) °F equivalent of 0°C

d) $\frac{9}{5}$ or 1.8

e) For every increase of 9°F, there is an increase of 1°C.

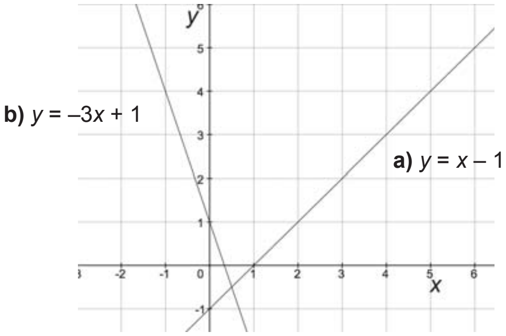
3.2.3 Slope and *Y*-Intercept Form pp. 106–107

1. a) m = 2, b = 0; $y = 2x$

b) m = $\frac{-4}{3}$, b = –2; $y = \frac{4}{3}x - 2$

c) m = $\frac{4}{3}$, b = 2; $y = \frac{-4}{3}x + 2$

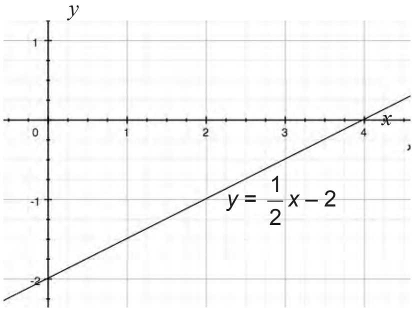
2.

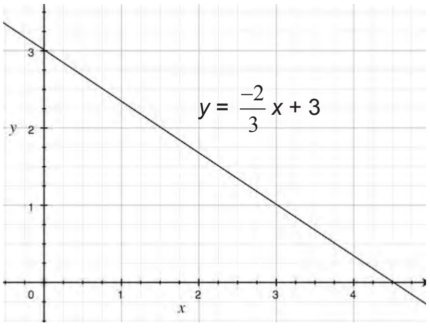


3. a) C; B

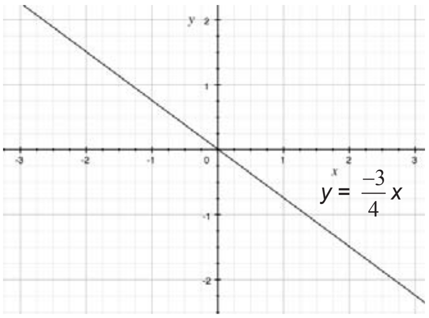
b) A; D

4. a) **b)**

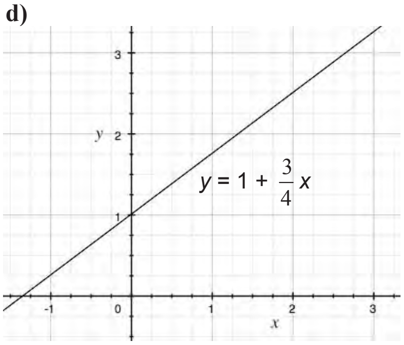




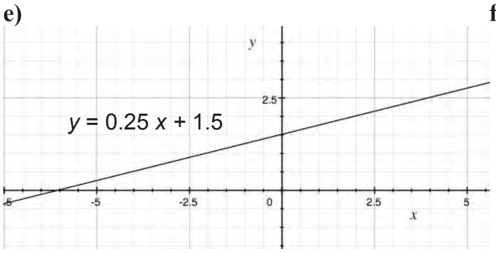
4. c)



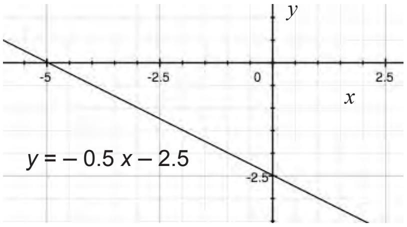
d)



e)

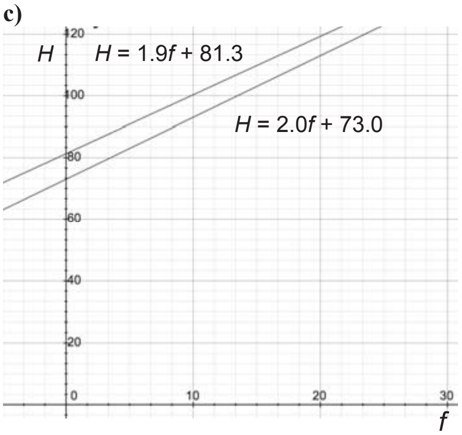


f)



5. a) 81.3; 73 **b)** 1.9; 2

c)



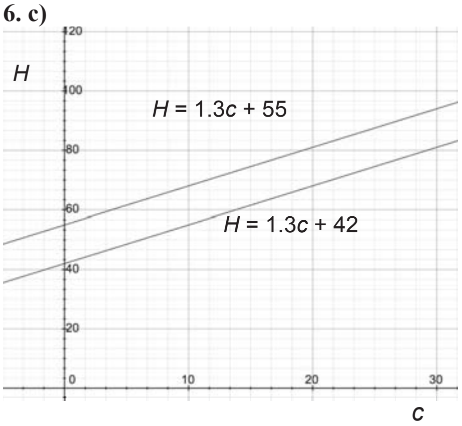
d) Nobody would have a femur of length 0 cm.

6. a) Because both equations are in the form $y = mx + b$, the slope (m) is 1.3 for each

b) 13; because both equations are in the form $y = mx + b$, 13 is the difference between the y-intercepts (b)

3.2.3 Slope and *Y*-Intercept Form [Continued] pp. 106–107

6. c)



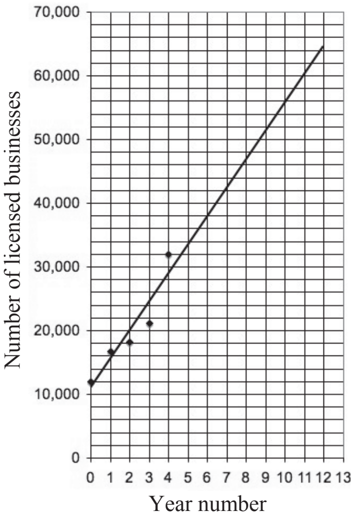
7. If you begin with the *y*-intercept, you can use the slope to locate a second point and then use a ruler to connect them. If you start with the slope, you don't know where to place the ruler.

8. Plot the *y*-intercept, move right a distance equal to the run and up (positive slope) or down (negative slope) a distance equal to the rise and plot the point, join the two points with a straight line

3.2.4 The Line of Best Fit pp. 111–112

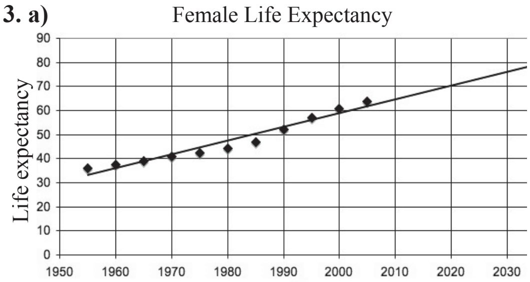
1. a) Strong positive correlation
b) About 10.4%; since a line was used instead of a curve, population for future years may be underestimated
c) About 14%
d) about 7.8%; about 8.7%

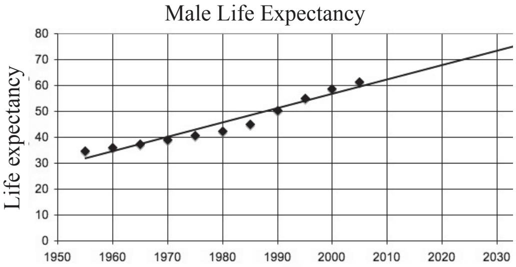
2. a)



b) *Sample response*: $y = 4700x + 11{,}000$
c) *Sample response*: 67,400
d) *Sample response*: I would prefer more data points to be more confident because it could have been non-linear.

3. a)





b)

Year	Female	Male
2010	65	62
2030	76	74

c)

Year	Female	Male
1992	54	52
2002	60	58

4. a) I **b)** IV **c)** II **d)** III

5. Kinley's graph

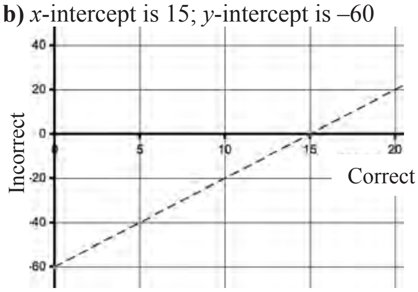
6. a) when data points fall along a line or close to a line
b) Use a ruler to draw a line that passes through as many data points as possible and make sure there are an equal number of points above as below the line.
c) Use the line to interpolate or extrapolate coordinates for points not plotted

3.2.5 Standard Form pp. 116–117

1. a) (0, 3) **b)** (2, 0) **c)** $m = \frac{-3}{2}$ **d)** $y = \frac{-3}{2}x + 3$

2. a) c is number of correct answers and i is number of incorrect answers; so add $4c$ points for correct answers and subtract i points for incorrect answers, which is $4c - i$ points; final score is 60 points so $4c - i = 60$

b) x-intercept is 15; y-intercept is –60

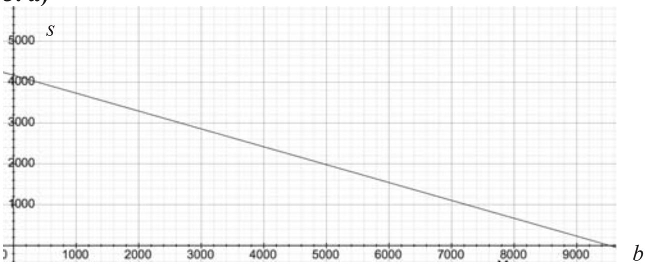


c) x-intercept means 15 correct answers with 0 errors; y-intercept means –60 errors and 0 correct answers; a y-intercept of –60 is not possible because you cannot have a negative number of answers

d) 40

3.2.5 **Standard Form** [Continued] **pp. 116–117**

3. a)



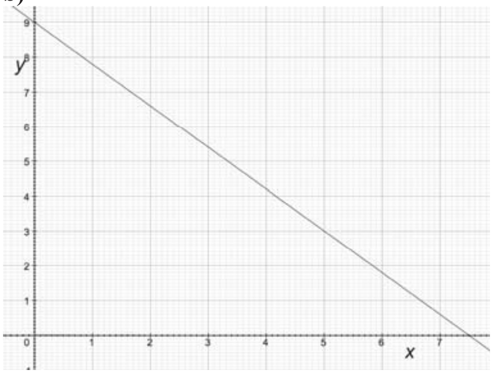
b) *Sample response*:

At 4.2% (*b*)	At 9.6% (*s*)
0	4200
9450	0
5000	2000
2600	3000

c) Cannot invest a negative amount

4. a) $600x + 500y = 4500$, x is the number of hours at Nu 600 and y is number at Nu 500

b)



c) *Sample response*:

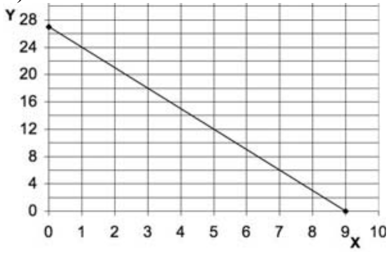
at Nu 600	at Nu 500
0	9
7.5	0
5	3

d) Cannot work a negative number of hours

5. a) $y = -\frac{2}{3}x + 4$ **b)** $y = \frac{4}{5}x - 4$ **c)** $y = -\frac{5}{2}x + \frac{5}{2}$ **d)** $y = \frac{10}{3}x - 5$

6. a) $3x + y = 27$

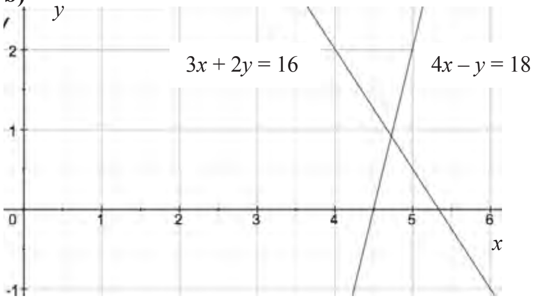
b)



c) *Sample response*: (9, 0), (5, 12), (0, 27)

7. a) $3x + 2y = 16$ and $4x - y = 18$

b)



c) (4, 2)

8. a) $y = \frac{3}{2}x + 2$ and $\frac{3}{2}x - y = -2$

b) *Sample response*:
(0, 2), (2, 5), (–2, –1), (4, 8)

c) d) Same graph as $\frac{3}{2}x - y = -2$; $\frac{3}{2}x - y = -2$ and $3x - 2y = -4$ are equivalent equations

9. Substitute 0 for y and solve equation to determine x-intercept, plot that value; substitute 0 for x and solve equation to determine y-intercept, plot that value; join points with a straight line

10. Determine x- and y-intercepts, calculate slope using the coordinates of intercepts, use slope and y-intercept to write equation in $y = mx + b$ form. Or, rearrange the equation algebraically.

3.3.1 Solving Linear Equations Algebraically pp. 121–122

1. a) $3x + 2 = 5$ **b)** $x + 1 = 2x$
c) $-2x = -x - 3$ **d)** $-2x = x - 3$

2. Subtract 4 from each side, then divide each side by 6.

3. a) $x = 4$ **b)** $a = 5$
c) $y = -3$ **d)** $x = 2$
e) $x = 6.9$ **f)** $x = 40$

4. a) $5w - 60 = 180$ **b)** $w = 48$

5. a) $250 + 50p = 1030$ **b)** $p = 15.6$ **c)** 15

6. a) $c = 3f + 2$ **b)** 16
c) $100 = 3f + 2; f = \frac{98}{3} = 32\frac{2}{3}$, but a fractional figure number is not possible.

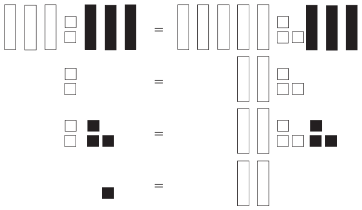
7. a) $x = 3$ **b)** $x = -2$
c) $x = -1$ **d)** $x = -9$
e) $x = -2$ **f)** $x = -8$

8. a) $3c = 100 - 40$ **b)** $c = 20$

9. a) $\frac{4}{5}$ **b)** –10 **c)** 6 **d)** 15

10. –40°F = –40°C

11. a)



b)

$$\begin{aligned} 3x + 2 &= 5x + 3 \\ -3x + 3x + 2 &= -3x + 5x + 3 \\ 2 &= 2x + 3 \\ 2 - 3 &= 2x + 3 - 3 \\ -1 &= 2x \\ x &= -\frac{1}{2} \end{aligned}$$

3.3.2 Solving Linear Inequalities p. 125

1. a) $x < 3$ **b)** $x \leq 3$
c) $2 > x$ **d)** $-2 \geq x$
e) $x < 1$ **f)** $2 \geq x$

2. a) $400 + 50t < 1000$
b) $t < 12$, so less than 12 min

3. a) $100,000 - 150t < 40,000$
b) $t > 400$, so more than 400 min

4. a) $a < 2$ **b)** $b < 7$
c) $x \leq 2$ **d)** $x < -2$
e) $x > 1$ **f)** $x < 2$

5. a) Figures 14 and greater
b) $4n - 3 > 50$

5. c) $n > 13\frac{1}{4}$, which means $n > 13$ or $n \geq 14$

6. a) $p = 2c + 1$, where p represents the number of pieces and c the number of cuts
b) $20 > 2c + 1$
c) $c < 9.5$, which means $c \leq 9$ or $c < 10$

7. 69 L

8. You can use inverse operations to solve both equations and inequalities.
Linear equations have a single solution, but inequalities usually have multiple solutions.

3.3.3 Solving Linear Equations Graphically p. 128

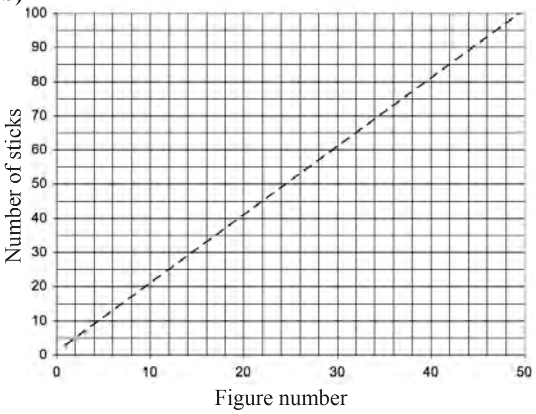
1. $x = 12$

2. a) $2x + 3 = 25$
b) Unless coordinates are on labelled grid lines, you often gave to estimate between increments on the horizontal and vertical axes scales.

3. Day 8

4. $2x - 3 = 14 \rightarrow 14 = 2x - 3$, locate 14 on the y-axis and then find the corresponding point on the x-axis (6).

5. a) $s = 2n + 1$, where s is the number of sticks and n is the figure number.
b)



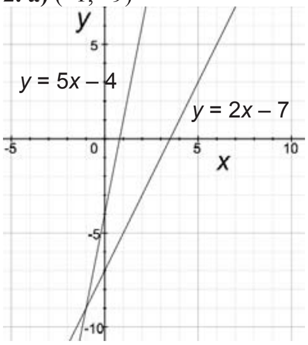
c) Locate 97 on the s-axis and then locate the corresponding point on the n-axis (48)
d) $s = 48$
e) When you use the graph you often estimate the position of points on the axes between scale increments. The larger the scale, the more likely you have to estimate.

6. You find the x-coordinate on the graph that goes with the given y-coordinate to get the solution. Or, you find the y-coordinate on the graph that goes with the given x-coordinate.

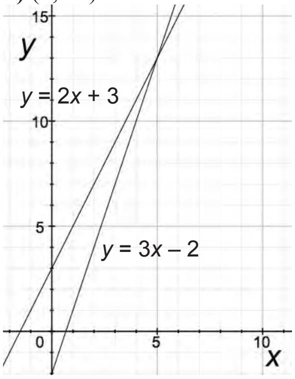
3.3.4 Solving a System of Linear Equations p. 131

1. a) (2, 3) **b)** (2, –1)

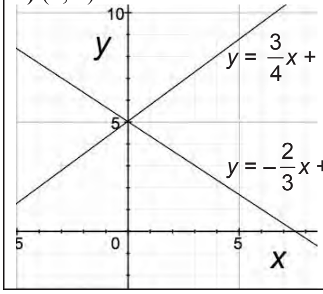
2. a) (–1, –9)



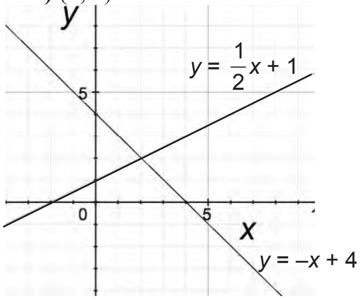
b) (5, 13)



c) (0, 5)



2. d) (2, 2)



3. Using *a* for the hours worked at Nu 600 and *b* for the hours worked at Nu 500:
a) 600*a* + 500*b* = 4500
b) *a* + *b* = 8
c) (5, 3)

4. Using *m* to represent the mass of the vehicle and fuel and *f* to represent the volume of fuel:
a) *m* = 1295 + 0.737*f*
b) *m* = 1290 + 0.820*f*
c) Approximately (60, 1340)

5. Alike: the solution involved determining the coordinate of a point on the graph of a linear relation
Different: to solve a linear equation you determine one coordinate of a point on the graph when you know the other coordinate, to solve a linear system, you determine both coordinates

UNIT 3 Revision pp. 132–134

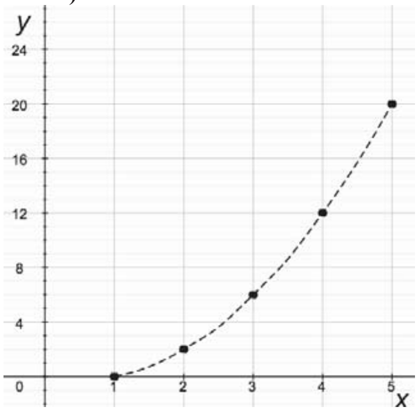
1. a) Quadratic; second differences are equal and not zero
b) Exponential; the ratios of the first differences to the term numbers are equal
c) Linear; the first differences are equal

2. **a)** A **b)** C **d)** B

UNIT 3 Revision [Continued] **pp. 132–134**

3. a) 0, 2, 6, 12, 20

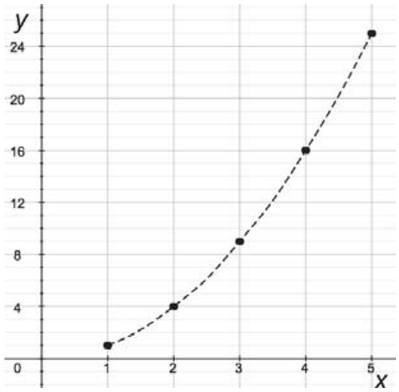
b) A curve fits the data better.



c) Dashed; there are no fractional figure numbers

4. a) 1, 4, 9, 16, 25

b)



c) Cannot see a full U shape

d) Show that the second differences are equal and not zero

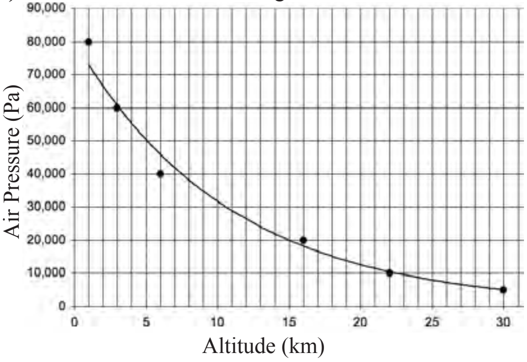
5. 56

6. a) x-coordinates are not equally-spaced

b) Continuous

c)

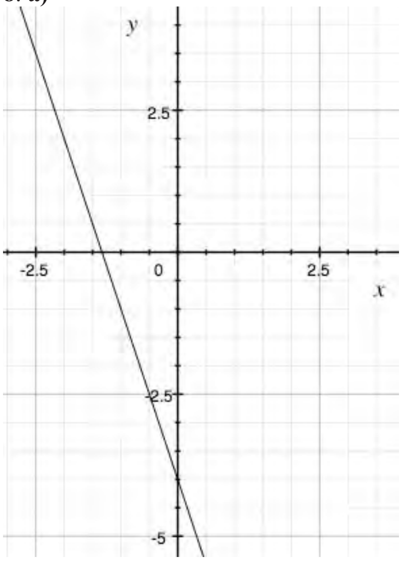
Air Pressure Against Altitude



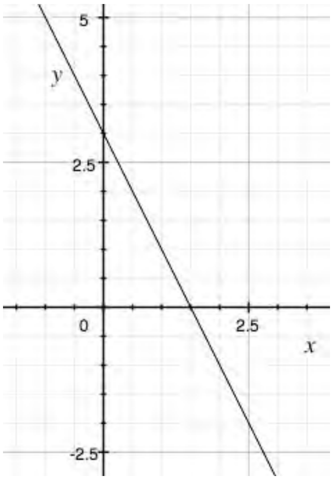
d) Looks exponential; it seems like the graph is becoming parallel to the x-axis

7. A: $y = -2x - 5$ **B:** $y = -2x + 5$ **C:** $y = \frac{2}{3}x + 5$

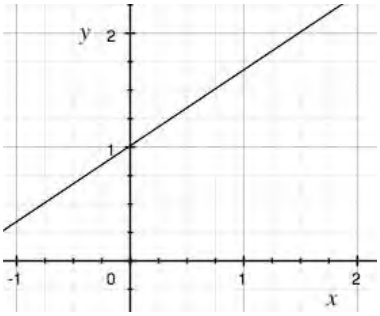
8. a)



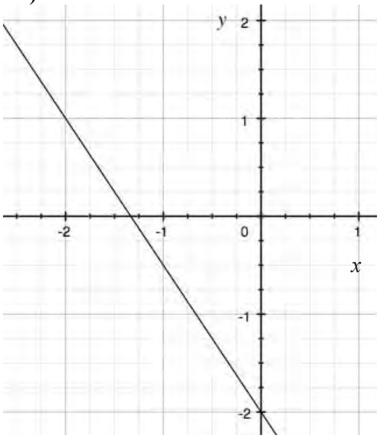
b)



c)

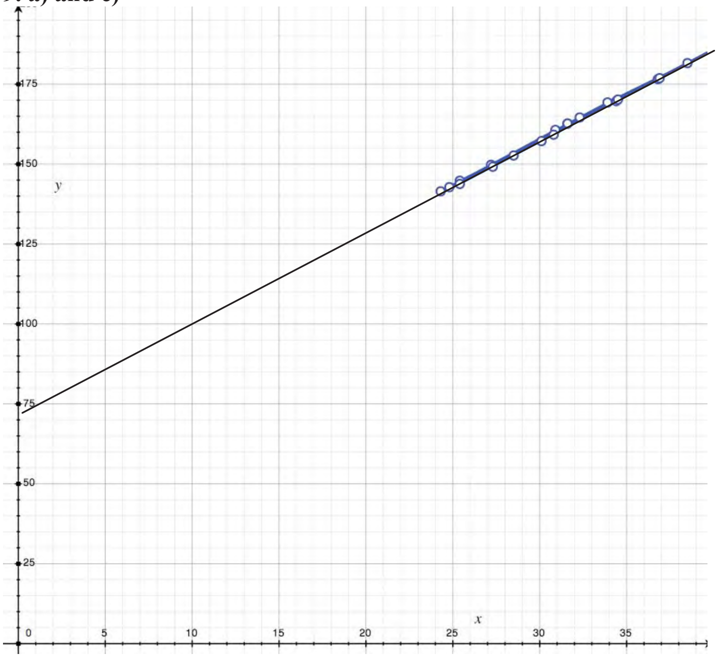


d)



UNIT 3 Revision [Continued] **pp. 132–134**

9. a) and c)



b) Strong positive correlation

d) About 152 cm; about 192 cm

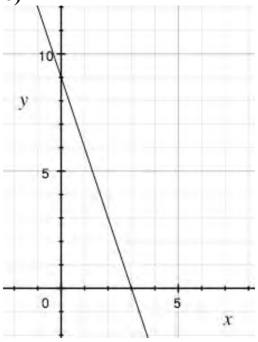
10. a) $y = 2.85x + 71.9$

b) 151.7 cm; 191.6 cm

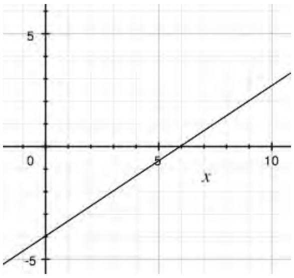
11. a) A: 3 B: 6 C: 10 D: –3

b) A: 9 B: 4 C: 2 D: 4

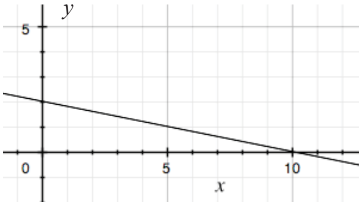
c) A



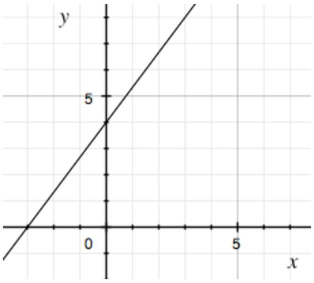
B



C



D

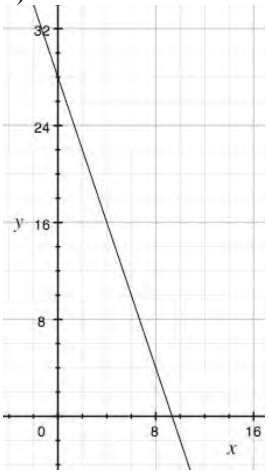


12. a) A: –3 B: $\frac{2}{3}$ C: $-\frac{1}{5}$ D: $\frac{4}{3}$

b) A: $y = -3x + 9$ B: $y = \frac{2}{3}x - 4$ C: $y = -\frac{1}{5}x + 2$ D: $y = \frac{4}{3}x + 4$

13. a) $1500m + 500t = 14{,}000$ **c)** (0, 28), (1, 25), (2, 22), (3, 19)

b)

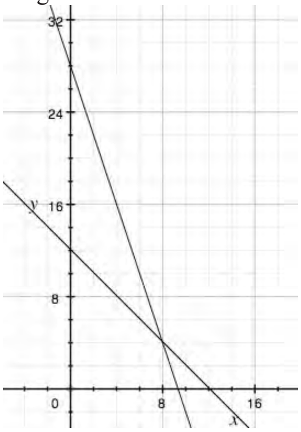


14. a) $a = 7$ **b)** $x = 6$ **c)** $y = \frac{7}{3}$ **d)** $x = 2$ **e)** $b = 20$ **f)** $x = -36$

15. a) $y < 2$ **b)** $6 \geq y$ **c)** $5 \leq a$ **d)** $n > -3$

UNIT 3 Revision [Continued] pp. 132–134

16. **a)** $m + t = 12$; $1500m + 500t = 14,000$ **b)** 8 magazine and 4 technical art



UNIT 4 DATA AND PROBABILITY pp. 135–188

Getting Started — Skills You Will Need p. 136

1. a) *Sample response*: There are more fishes than birds, mammals, reptiles, or amphibians. There are about five times as many fishes as mammals or amphibians. There are about twice as many birds as amphibians or mammals.
b) *Sample response*: You can see at a glance what fraction of the whole each species represents; you can also see the relative sizes of the different species.

2. a) fishes
b) amphibians
c) about twice as many

3. a) part b
b) part c

4. a) 8 mm **b)** between 2 and 3 pm
c) 5 mm, 1.5 mm **d)** 5 pm

5. a) 0.6 **b)** 0.1 **c)** 0.5
d) 0.5 **e)** 1.0 **f)** 0.9

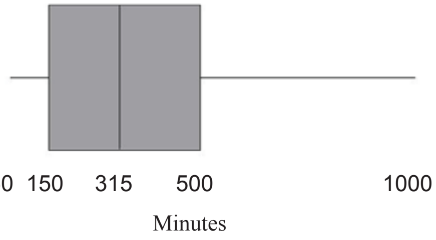
4.1.1 Constructing Familiar Data Displays pp. 142–143

Note that all box plots include the median in the calculation of Q1 and Q3.

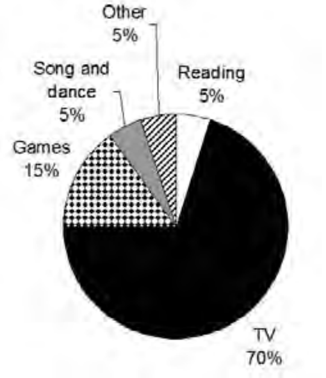
1. a)

10	00			
9				
8				
7	50			
6	00	50		
5	00	00		
4	00	20		
3	00	30	50	
2	00	50	85	
1	50	50		
0	60	75	90	90

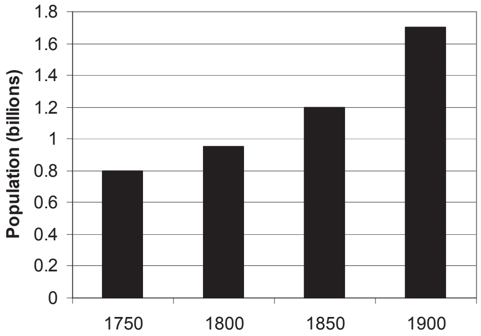
b)



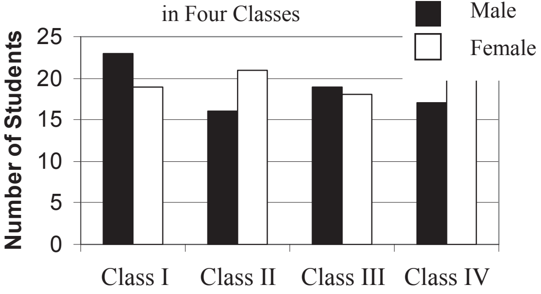
2. Percent of Time Teens in Thimphu Spend on Leisure Activities



3. Population of the Earth Between 1750 and 1900

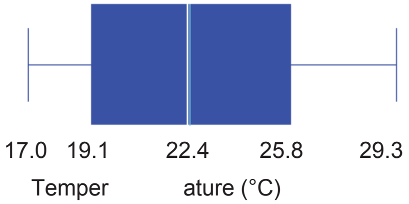


4. Number of Students by Gender in Four Classes

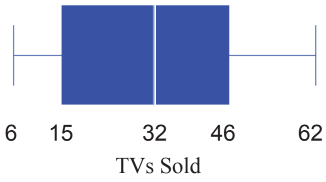


5. a) 22.4 **b)** 12.3 **c)** Q1 = 19.1, Q3 = 25.8

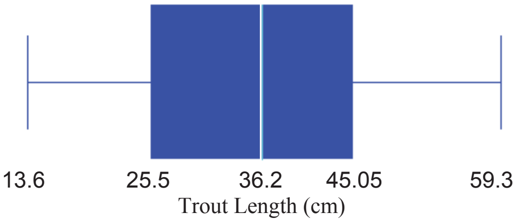
d)



6. a)

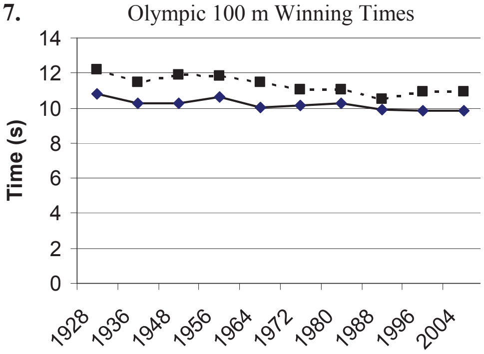


b)

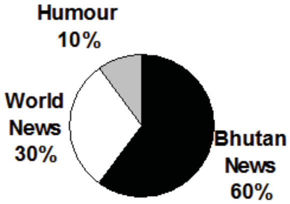


4.1.1 Constructing Familiar Data Displays [Continued] pp. 142–143

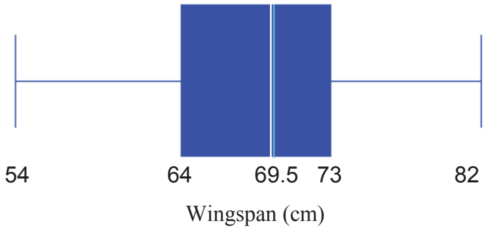
7.



8. a) Amount of Space Devoted to Various Sections of the *Bhutan Times*



b) *Sample response*:



4.1.2 Using Graphs to Compare and Organize Data pp. 147–149

1. a) B, D **b)** B, C, D **c)** A, B

2. a) *Sample response*: Swimming is the most popular and cycling the least popular. This circle graph lets you see how each sport in the survey compares with each other in terms of its popularity. A bar graph could also be used.
b) *Sample response*: Chocolate milk sales decrease through the week while white milk sales increase. This double bar graph lets you see how chocolate milk sales compare with white milk sales. A multiple line graph could also be used to represent the data.
c) *Sample response*: The majority of families have more children than pets. This bar graph lets you compare the number of children with the number of pets in families. A circle graph could be used.

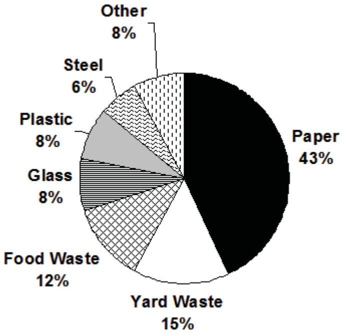
2. d) The students scored between 43 and 100 on the test and the median mark was about 67. This box plot lets you see how the data is clustered around the median mark. No other graph could be used.

3. a) *Sample response*: A circle graph because it compares parts of a whole.
b) *Sample response*: A bar graph because it would be easier to make a scale and plot the data than to calculate the percentages to make a circle graph.
c) *Sample response*: A box and whisker plot because it will show the median, range, quartiles, and spread of the data.

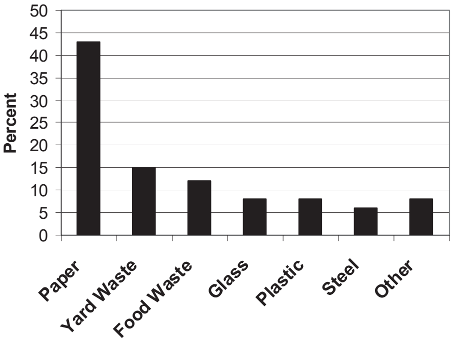
4. a) A box and whisker plot because the middle 50% of the data is easy to see.
b) *Sample response*: A double bar graph because there are discrete categories and it is easy to compare the heights of the bars.

4. c) *Sample response*: A box and whisker plot because it shows the median and you can compare the spread of the data to the median.
d) *Sample response*: A box and whisker plot because it shows the median, extreme values, and the spread of the data.
e) *Sample response*: A bar graph because you can compare building heights easily.

5. a) Type of Waste



Type of Waste



b) *Sample response*: The bar graph because you only have to compare the heights of the bars

6. a) *Sample response*: Compare the number of children in Bhutan to the number of adults at 5 year intervals starting in 1960
b) *Sample response*: Compare the favourite foods of people in your class
c) *Sample response*: Compare the median age of all the students in your school to the range of their ages
d) *Sample response*: Compare the percent of each type of crop grown in Bhutan

4.1.3 Using Graphs to Examine Change pp. 152–153

1. a) A, C, D **b)** A, B **c)** A, C, D

2. a) *Sample response*: As time increases so does the population. This scatter plot lets you see if there is a relationship between time and population. A broken line graph could also be used.
b) *Sample response*: Cell phone use is increasing faster among females than among males. This multiple broken line graph lets you see how cell phone use is changing over time in two different groups. A scatter plot could also be used with different types of dots for the male and female data.

3. a) *Sample response*: A scatter plot to show whether a relationship exists between arm length and height
b) *Sample response*: A multiple broken line graph to compare exports with imports and compare how both these quantities change over this time period
c) *Sample response*: A broken line graph to show how the number of tourist arrivals has changed over this time period.

4. a) *Sample response*:
Karma used two box plots on the same scale to compare the range, median, extremes, and spread of the data between men and women.
Lobzang used a multiple broken line graph to compare the trends in the heights for men and women over time.
b) The broken line graph is better at showing change over time. The box plot is better for looking at data spread.

4.1.4 Misleading Graphs p. 158

1. a) *Sample response*: The intervals on the horizontal axis are different, so it looks like the number of accidents increases until age 23 and then decreases, but that is not true because the three bars between 16 and 23 should be one bar showing about 42,000 accidents.

b) *Sample response*: The difference between the years is shown by the difference in the heights of the bars, but the pump on the right is also wider, so it looks like it represents even more than it should**.**

c) *Sample response*: The scale on the vertical axis is inconsistent, it doubles each time. It seems like the third bar is 1.5 times the height of the second but it really should be twice the height.

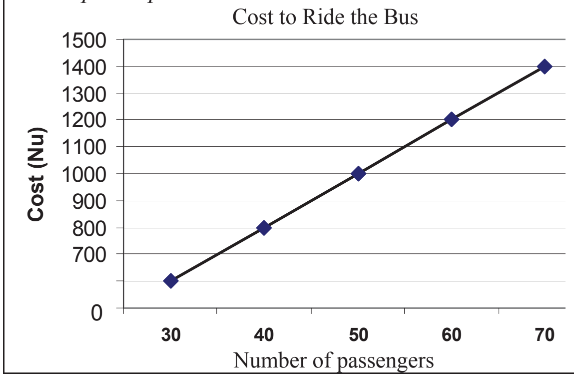
2. *Sample response*: wrong use of scale, misplaced zero point, wrong choice of graph type

3. a) True **b)** False **c)** False

4. The graph does not start at zero, so the length of the bars from 0 to 20 is not accounted for. That means a bar that appears to be three times as long as another really isn't so you can't just compare the length of the bars without checking the numbers on the scale.

5. *Sample response*: The vertical scale on the first graph goes much higher than the greatest plotted temperature, so the temperatures appear closer together than they really are.

6. *Sample response*:



Because an axis break was not used, the first two increments along the vertical scale look like they each have a value of Nu 350 but each really has a value of Nu 100. This might mislead the reader to think the cost for 30 passengers is Nu 350 instead of Nu 600.

4.1.5 Drawing Conclusions From Graphs pp. 162–163

1. **a)** *Sample response*: 8 L, 13 L
b) *Sample response*: 65 km, 90 km
c) *Sample response*:
- The faster you drive, the more gas you use over the same distance.
- They travelled at different speeds over the 100 km.

2. a) *Sample response*: Most teens spend the majority of time at school and sleeping
b) about 3 h **c)** homework **d)** about 29%

3. a) about 65 km
b) No; a box and whisker plot only shows the spread of the data and not the frequency of the actual data
c) *Sample response*: The median distance to work is about 15 km. The closest distance is 1 km and the farthest is 66 km.

4. a) About 4800 **b)** About 15 ^{o}C
c) No; *Sample response*: No; 200 ^{o}C is well beyond the last data point and you cannot be certain that the trend will continue. Also, the number of bacteria appears to reach zero before 100 ^{o}C if the trend continues.

4. d) *Sample response*: There is a relationship between the number of bacteria and temperature — as the temperature increases, the number of bacteria decreases.

5. No; she has based her conclusion on the shape of the second graph, which does not start at Nu 0, compared to the shape of the first graph which does start at Nu 0. The graphs actually show the same increase in profit over this time period for both years.

6. No; *sample response*: While the graph shows there are more terrestrial species in danger of extinction than marine, you do not know the total numbers of each type of species, including those that are not in danger of extinction.

7. *Sample response*: On average, Brand B has more raisins per box than Brand A. But it is possible to get a Brand A box with more raisins, for example, a Brand A box could have as many as 30 raisins and a Brand B box could have as few as 22 raisins.

8. *Sample response*: Circle graphs, bar graphs, and double bar graphs. These are used for comparisons and do not show relationships between variables that could lead to extrapolation or interpolation.

4.2.1 Determining and Comparing Probabilities p. 168

1. a) $\frac{15}{36}$ or $\frac{5}{12}$ **b)** $\frac{16}{36}$ or $\frac{4}{9}$

c) $\frac{17}{36}$ **d)** $\frac{12}{36}$ or $\frac{1}{3}$

2. a) *Sample response*: $\frac{5}{30}$; very low

b) *Sample response*: $\frac{11}{30}$; a bit low

c) *Sample response*: $\frac{18}{30}$; a bit high

d) *Sample response*: $\frac{11}{30}$; a bit low

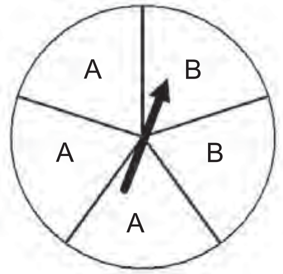
3. a) $\frac{16}{49}$ **b)** $\frac{25}{49}$ **c)** $\frac{9}{49}$ **d)** $\frac{24}{49}$

4. a) $\frac{7}{64}$ **b)** $\frac{52}{64}$ or $\frac{13}{16}$ **c)** $\frac{16}{64}$ or $\frac{1}{4}$

5. a) $\frac{9}{25}$ **b)** $\frac{10}{25}$ or $\frac{2}{5}$ **c)** $\frac{3}{25}$

6. a) $\frac{4}{25}$ **b)** $\frac{9}{25}$ **c)** $\frac{3}{25}$

7. *Sample response*:



8. *Sample response*: Outcome A might be a multiple of 3 because the probability of rolling a multiple of 3 twice is $\frac{4}{36}$.

9. A chart, tree diagram, or list makes it easier to be sure that each possible outcome is considered because it forces you to think systematically.

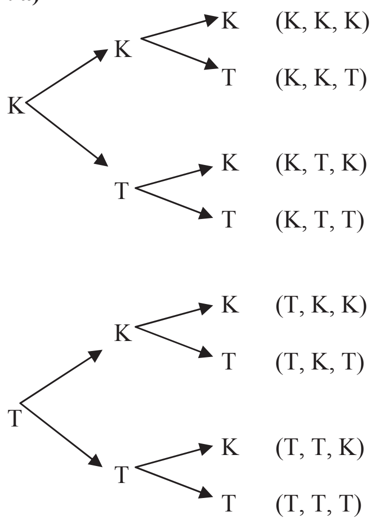
4.2.2 Calculating Probability of Two Independent Events pp. 172–173

1.

	1	2	3	4	5	6	7
K	(1,K)	(2,K)	(3,K)	(4,K)	(5,K)	(6,K)	(7,K)
T	(1,T)	(2,T)	(3,T)	(4,T)	(5,T)	(6,T)	(7,T)

a) $\frac{1}{14}$ **b)** $\frac{3}{14}$ **c)** $\frac{6}{14}$ or $\frac{3}{7}$ **d)** $\frac{2}{14}$ or $\frac{1}{7}$

2. a)



b) P(one K) = $\frac{3}{8}$ **c)** P(two Ts) = $\frac{3}{8}$

d) P(no Ts) = $\frac{1}{8}$

e) *Sample response*: What is the probability of getting more Tashi Ta-gyes than Khorlos when tossing three coins? ($\frac{4}{8}$ or $\frac{1}{2}$)

3. a) $\frac{20}{81}$ **b)** $\frac{25}{81}$

4. a) P(two 8s) = $\frac{1}{144}$

b) P(two odds) = $\frac{36}{144}$ or $\frac{1}{4}$

c) P(number > 3, number < 6) = $\frac{45}{144}$ or $\frac{15}{48}$

5. Game 1; In Game 1, the probability of not spinning a 1 on each spin is $\frac{2}{3}$ and the spins are independent. So, for Game 1, P(scoring) = $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ = 0.444; In Game 2, the probability of spinning less than 6 on each spin is $\frac{5}{8}$ and the spins are independent. So, for Game 2, P(scoring) = $\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$ = 0.391; He has a better chance with Game 1.

6. a) $\frac{1}{36}$ **b)** $\frac{1}{36}$ **c)** $\frac{9}{36}$ or $\frac{1}{4}$

d) $\frac{18}{36}$ or $\frac{1}{2}$ **e)** $\frac{3}{36}$ or $\frac{1}{12}$

7. a) $\frac{1}{2704}$ **b)** $\frac{32}{2704}$ or $\frac{2}{169}$ **c)** $\frac{169}{2704}$ or $\frac{1}{16}$

8. $\frac{2}{16}$ or $\frac{1}{8}$

9. a) *Sample response*: a spinner with six equal sectors numbered 1 to 6

b) *Sample response*: a spinner with ten equal sectors numbered 1 to 10

c) a spinner with no number 6 on it

d) *Sample response*: a spinner with two equal sectors with 6 on one and any number on the other.

10. $\frac{100}{900}$ or $\frac{1}{9}$

11. Since the events are dependent you cannot calculate by multiplying the probability of each event together as if they are independent events.

4.2.3 Randomness: Experimental Versus Theoretical Results pp. 177–178

1. a) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ or 44.4%

b) About 4 or 5 times

c) Experimental probability is $\frac{4}{10}$ or 40%, which is close to but not the same as the theoretical probability.

d) No; due to randomness, the experimental probability can vary every time she repeats the experiment.

2. C and D

C: Just because the experimental and theoretical probabilities are different does not mean they are incorrect. Experimental and theoretical probabilities often differ due to chance.

D: It is likely that the experimental and theoretical probabilities would be closer if he rolled the dice 100 times because it's a greater number of trials.

3. a) *Sample response*: My events are to draw a club and then draw a 6 from a deck of cards. The first card is replaced before the second draw.

b) *Sample response*: I will draw two cards from a deck, with replacement, 100 times.

c) *Sample response*: $\frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$ or 1.92% so I predict drawing a club and then a 6 (with replacement) two times in 100 trials.

d) *Sample response*: Everyone has different results and some are closer to the theoretical probability than others.

e) *Sample response*: Our combined results are closer to what we predicted because more trials are being combined.

4. *Sample response*: When you are comparing probabilities as fractions with different denominators, it is sometimes easier if both are either decimals or percentages.

5. The theoretical probability is a much better indicator of what will happen because it is based on two known facts: the number of ways the event can occur and the total number of possible outcomes.

4.2.4 Conducting a Simulation p. 183

1. a) any could be used to model 50% or $\frac{1}{2}$

b) spinner, and cards if only cards numbered 1 to 8 were included

c) dice, or cards if only cards numbered 1 to 6 were included

d) spinner, or cards

2. a) $\frac{1}{10}$ or 10% **b)** $\frac{1}{16}$ or 0.0625%

c) $\frac{1}{16} \neq \frac{1}{10}$

3. *Sample response*: Use a die. If you roll a 1 or a 2, it represents a win. A roll of 3, 4, 5, or 6 is a loss. Roll the die five times for each trial and do 25 trials. Count the number of trials that had three rolls of 1 or 2 out of five rolls. Create a fraction with the number of trials that had a roll of 1 or 2 three times out of 5 as the numerator and 25 as the denominator.

4. *Sample response*: Use a spinner with four equal sections numbered 1 to 4. A spin of 2, 3, or 4 represents a successful penalty kick. A spin of 1 means a miss. Spin five times for each trial and do 50 trials. Count the number of trials a 2, 3, or 4 is spun at least four times. Create a fraction with the number of trials where a 2, 3, or 4 was spun at least four times as the numerator and 50 as the denominator.

5. When you conduct a simulation, you use the theoretical probability of the probability device to model the probability, but the results are the experimental probability.

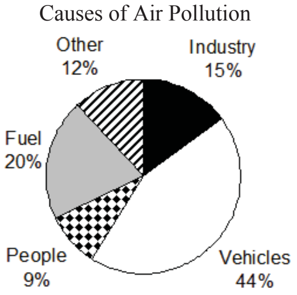
CONNECTIONS: Computer Simulations p. 185

1. Scientists conduct computer simulations to look at patterns of events and to make predictions. They are able to vary the parameters of a computer simulation to see how this will affect the outcome and thus discover how best to deal with situations that could be modified to have more favourable outcomes.

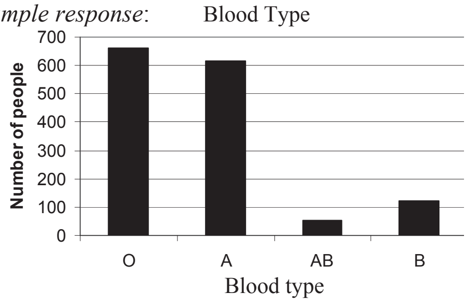
2. *Sample response*:
- weather patterns, such as temperature trends (global warming and its impact on the ecosystems of the world)
- hurricane and typhoon models, to predict the intensity and the path of such storms

UNIT 4 Revision pp. 186–188

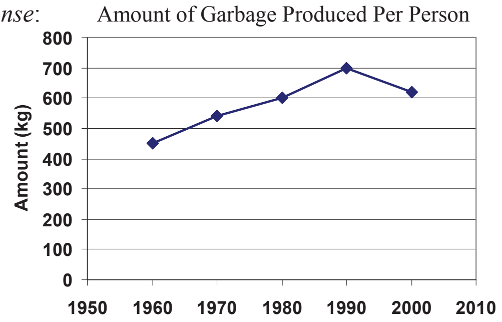
1. **a)** *Sample response*:



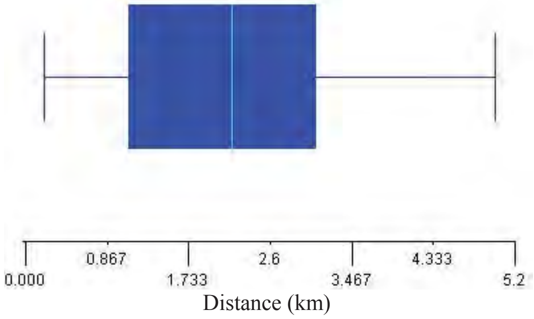
b) *Sample response*:



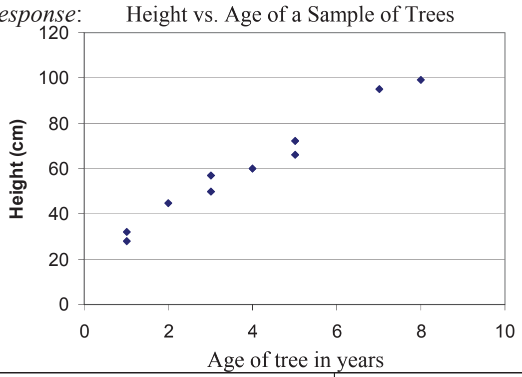
c) *Sample response*:



d) *Sample response including median in Q1 and Q3*:



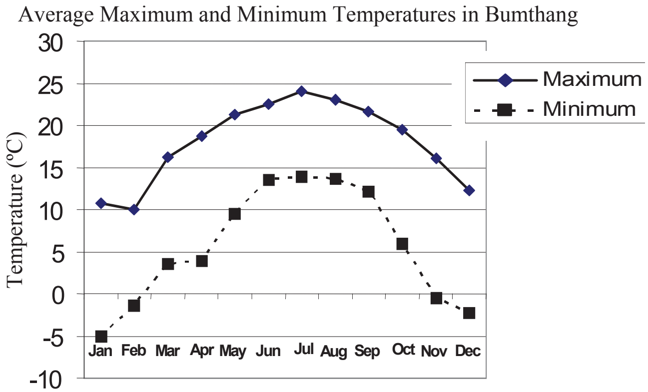
e) *Sample response*:



2. Disagree; a broken line graph is a poor choice because the data is in categories and it is not about a trend over time. A circle graph or bar graph would be better.

3. a) You are not given all countries of the world so you cannot represent the data as percentages or parts of a whole.
b) *Sample response*: A bar graph would allow you to compare the number of vehicles per thousand people for the countries listed.

4.



5. *Sample response*:
- Wrong use of scale
- Misplaced zero point
- Wrong choice of graph type
- Improper use of shading and visual effects

6. No; he has interpreted the graph incorrectly. He can conclude that there are a relatively small number of people living in forest ecosystems in North America compared to other parts of the world but the graph gives no information about the amount of forest ecosystems in these regions.

7. a) C
b) Previous rolling of a die has no effect on the next roll. Each roll is independent of all other rolls.

8. 2 times because P(7) = $\frac{1}{6}$ and

P(10) = $\frac{3}{36}$ or $\frac{1}{12}$

9. a) $\frac{1}{8}$ **b)** $\frac{1}{4}$ **c)** $\frac{1}{26}$ **d)** $\frac{1}{104}$

UNIT 4 Revision [Continued] **pp. 186–188**

10. $\frac{1}{169}$

11. P(same) = $\frac{7}{30}$

12. a) $\frac{4}{45}$ **b)** $\frac{8}{75}$ **c)** $\frac{4}{45}$ **d)** $\frac{1}{9}$

13. C; the difference becomes less because the experimental probability approaches the theoretical probability as the number of trials increases

14. Yes, the experimental probability could be anything. If it turns out to be zero it means that the event has not occurred in any of the trials conducted. If you increase the number of trials then the experimental probability will approach the theoretical probability.

15. a) *Sample response*: Toss a coin. Toss the coin to simulate a person walking into the room. If a K comes up count it as female. If a T comes up count it as male. Toss the coin four times for each trial and conduct ten trials. The experimental probability is the number of trials where K is tossed all four times divided by 10, the total number of trials.

b) $\frac{1}{16}$

16. *Sample response*: Roll a die. Roll the die to simulate playing one game. If 1, 2, or 3 comes up, Lema wins. If 4, 5, or 6 comes up, Maya wins. Roll the die four times for each trial and conduct 25 trials. Count the number of trials where one of the players wins in all four rolls in a trial. The experimental probability is the number of trials where one player wins in all four rolls divided by 25, the total number of trials.

17. a) *Sample response*: Roll a die. If a 1 comes up, it means she wins. If 2 to 6 come up, she loses. Roll the die twice for each trial and conduct 50 trials. The experimental probability is the number of trials where a 1 is rolled in both rolls divided by 50, the total number of trials.

b) P(wins 2 in a row) = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

c) The two probabilities will differ and the amount of difference will depend on the number of trials used in the simulation. The greater the number of trials, the closer will be the experimental probability to the theoretical probability.

UNIT 5 GEOMETRY pp. 189–221

Getting Started—Skills You Will Need p. 189

1. b) PR ≈ 5.55 cm, PQ ≈ 4 cm,
RQ ≈ 6.8 cm;
$\angle R \approx 35^\circ$, $\angle P \approx 90^\circ$, $\angle Q \approx 55^\circ$
c) 180º
d) $PR^2 + PQ^2 = RQ^2$
$5.55^2 + 4^2 \approx 30.8 + 16 = 46.8$
$6.8^2 = 46.24$
(The results of **part d)** are not exactly the same as **part b)** because of measurement imprecision.**)**

2. a) 6
b) 30
c) 4.52

3. a) $AB \parallel FC \parallel ED$, $AF \parallel EB \parallel CD$, $BC \parallel AD \parallel EF$
b) rotational symmetry of order 6 about O; 6 lines of reflective symmetry intersecting at O

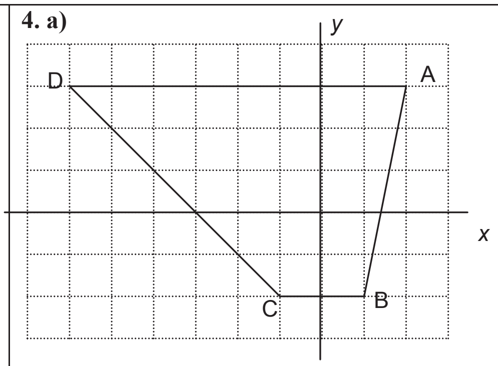
c) ∠AOB = ∠EOD because they are vertically opposite angles

∠OAB = ∠AOF because they are alternate angles formed by transversal AO intersecting parallel lines AB and OF

∠BOC = ∠BED because they are corresponding angles formed by transversal BE intersecting parallel lines CF and ED

∠COE + ∠DEO = 180º because they are interior angles formed by transversal BE intersecting parallel lines CF and ED

4. a)



b) trapezoid **c)** ∠D = 45º

5. Subtract 8

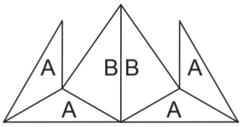
5.1.2 Congruent Triangles p. 194

1. a) SAS for ΔWXY and ΔWZY; ΔWXY ≅ ΔWZY
b) ASA for ΔSVR and ΔSTU; ΔSVR ≅ ΔSTU

2. a) ΔABC ≅ ΔKML using SSS or SAS
b) ΔABC ≅ ΔFDE using AAS
c) ΔABC ≅ ΔJGH using SAS

3. Given the hypotenuse and another side of a right triangle, you can use the Pythagorean theorem to find the third side. So, being given two sides of a right triangle is the same as being given all three sides (SSS).

4.



5. a) ASA **b)** 78 m **c)** AB ≈ 34.41 m

6. a) knowing ∠ACB = ∠ACD would allow ASA; knowing ∠ABC = ∠ADC would allow AAS; knowing AB = AD would allow SAS
b) knowing FG = HG would allow SSS; knowing ∠FEG = ∠HEG would allow SAS
c) knowing any corresponding sides are equal would allow AAS or ASA

7. You do not need to know all the measurements to know if triangles are congruent. Knowing SSS, SAS, ASA, or AAS is enough; *sample response*: For example, there is only one triangle possible if you know all three side lengths, so if two triangles have the same side lengths (SSS), they must be congruent. It works the same with SAS, ASA, and AAS.

5.1.3 Similar Triangles p. 198

1. a) ∠BAC = ∠CAD ≈ 37º
∠ABC = ∠ACD ≈ 87º
b) A corresponds with A, B corresponds with C, and C corresponds with D
c) *Sample response*: ΔCDA ~ ΔBCA

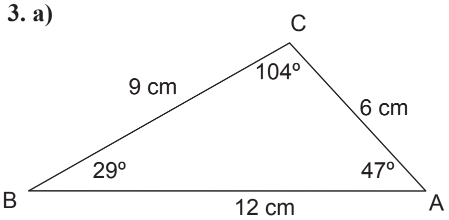
1. d) AD ≈ 6.4 cm, AC ≈ 5.3 cm, AB ≈ 4.4 cm, BC ≈ 3.2 cm, CD ≈ 3.9 cm; $\frac{AD}{AC}=\frac{AC}{AB}=\frac{CD}{BC}\approx 1.2$
(The ratios might not be exactly the same because of measurement imprecision.)

2. b and c are always true

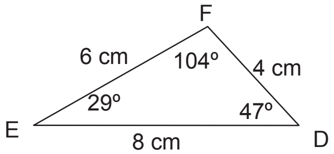
5.1.3 Similar Triangles [Continued]

p. 198

3. a)



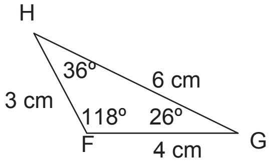
b)



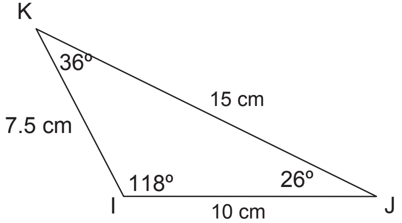
c) Angles are the same in both triangles
d) 27 cm and 18 cm
e) AAA

4. *Sample response:* using scale factor 2.5

a)



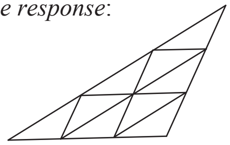
b)



4. c) Angles are the same in both triangles
d) 13 cm and 32.5 cm
e) AAA

5. a) You need only two angles because the third angle in any triangle can be found by subtracting the other two from 180º.
b) You need only one angle, the one between the pairs of corresponding sides (for SAS)

6. a) 13 similar triangles: 9 small ones, 3 middle-sized ones, and the large triangle.
b) Yes, it works for any triangle.
Sample response:



c) It cannot be done with two or three triangles, but it can be done with four.

7. a) If ∠B = ∠D or ∠A = ∠E, there would be two equal angles, or if CD = 6.75 m ($\frac{9}{6.75} = \frac{12}{9}$), there would be two pairs of corresponding sides in equal ratio with an equal angle between them

8. Congruent triangles have more requirements and are a subgroup of similar triangles.

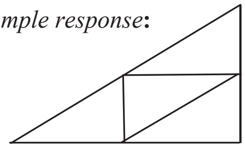
5.1.4 Solving Problems with Similarity

p. 202

1. a) $x = 3.2$ cm **b)** $h \approx 8.8$ m **c)** $y = 9$ m

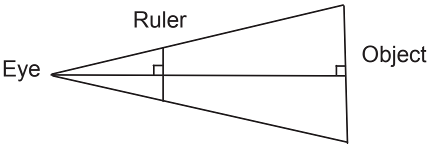
2. 4.55 m

3. a) *Sample response*:



3. b) Each side in the smaller triangle is exactly half the length of its corresponding side in the larger triangle. This means the triangles are similar.
c) This is true for all triangles.

4. the distance from your eye to the object (or person), the distance from your eye to the ruler, and the length of the ruler



5. a) The angle between the ground and the sun's rays is constant and the angle between vertical things and the ground is constant, so there will still be two equal angles in the two triangles, making them similar (AAA).
b) If the ground is hilly or uneven, triangles cannot be formed.

6. *Sample response*: A triangle with one side length of 2.3 cm is drawn on a map with the scale 1 cm represents 80 km. What is the actual distance of the side length? (184 km)

5.2.1 Translations pp. 205-206

1. a) (7, 3) **b)** (3, 0) **c)** (0, 0)

2. a) (–2, –2) **b)** (–3, 6)

3. a) [–2, 3] **b)** $(x, y) \rightarrow (x - 2, y + 3)$

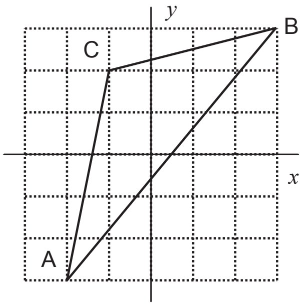
4. PQ ∥ P'Q', PR ∥ P'R', QR ∥ Q'R'

5. a) [5, – 3] **b)** [6, 1]
c) [– 3, – 6] **d)** [– 3, 5]

6. a) $(x, y) \rightarrow (x + 5, y - 3)$
b) $(x, y) \rightarrow (x + 6, y + 1)$
c) $(x, y) \rightarrow (x - 3, y - 6)$
d) $(x, y) \rightarrow (x - 3, y + 5)$

7. a) A'(–4, 0), B'(–3, 2), C'(1, 0), D'(0, –2)
b) A"(0, 1), B" (1, 3), C"(5, 1), D"(4, –1)
c) A translation of [3, 2] and then a translation of [4, 1] is the same as one translation of [7, 3].
d) [–7, –3]
e) To return to the original position, you need the opposite translation, [–7, –3].

8. a)



b) [3, –6] or $(x, y) \rightarrow (x + 3, y - 6)$
c) A'(1, –9), B'(6, –3)
d) Yes; the sides are the same lengths because a translation does not affect size
e) AB ∥ A'B', AC ∥ A'C', and BC ∥ B'C'
f) AA" ∥ BB" ∥ CC"
g) same lengths

9. a) [2, 2] or [–2, –2]
b) Because the other vertices could be on either side of the given vertices
c) [–3, 1] or [3, –1]

10. *Sample response*: For a translation, all vertices move the same distance. The top vertex of A would have to move a different distance than the other vertices.

11. All the vertices move the same distance and in the same direction.

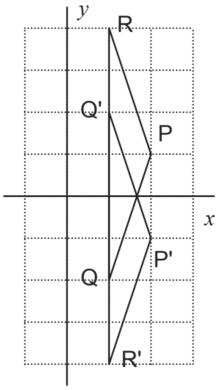
5.2.2 Reflections and Rotations pp. 209–211

1. a) X'(–1, –2), Y'(–3, 0), Z'(–1, –4)
b) X'(1, 2), Y'(3, 0), Z'(1, 4)

2. a) X'(–2, –1), Y'(0, –3), Z'(–4, –1)
b) X'(2, 1), Y'(0, 3), Z'(4, 1)
c) X'(1, –2), Y'(3, 0), Z'(1, –4)
d) X'(1, –2), Y'(3, 0), Z'(1, –4)

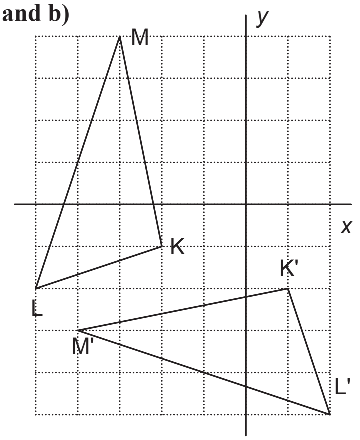
3. a) XY ‖ X'Y' only in the 180º rotations. A line segment that is rotated 90º will be perpendicular to the original.
b) The image ends up in the same place as the original shape in a 180º ccw rotation; and 180º ccw = 180º cw

4. a) and b)



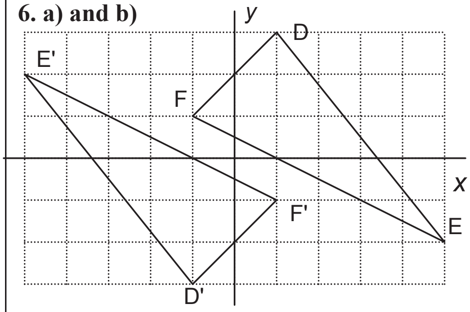
c) reflection in the *x*-axis
d) PQ is not parallel to P'Q' and PR is not parallel to P'R', but QR and Q'R' are collinear
e) reflection in the *y*-axis; corresponding sides would still not be parallel

5. a) and b)



c) rotation around the origin 90º ccw

6. a) and b)



c) 180º rotation around origin
d) rotation about the origin; the angle would be 90º ccw or 270º cw

7. a) A(–2, 1), B(–1, 3), C(3, 4), D(2, 2) and A'(1, –2), B'(3, –1), C'(4, 3), D'(2, 2)
b) $(x, y) \rightarrow (y, x)$

8. a) For a 90º cw rotation, one coordinate changes sign, but for a 180º cw rotation, both change sign. For a 90º cw rotation, the *x*- and *y*-coordinates switch positions, but for a 180º cw rotation, the coordinates stay in the same positions.
b) For a reflection in the *x*-axis, the *y*-coordinate changes sign, but for a reflection in the diagonal line, the *x*- and *y*-coordinates switch positions.

8. c) For a reflection in the y-axis, the x-coordinate changes sign, but for a reflection in the diagonal line, the x- and y-coordinates switch positions.

9. a) Both, because the lengths of the sides and the angle measures do not change, and if triangles are congruent, they are also similar.
b) The orientation is reversed. If the vertex labels are cw on the original shape, they will be ccw on the image.

10. a) Both, because the lengths of the sides and the angle measures do not change, and if triangles are congruent, they are also similar.
b) The orientation does not change. If the vertex labels are cw on the original shape, they will still be cw on the image.

5.2.3 Dilatations pp. 214–215

1. a) P'(2, –4), Q'(6, 4), R'(10, 0)
b) S'(2, 2), T'(–2, 2), U'(0, –4)

2. a) W'(2, 2), X(3, 0), Y(0, $-\frac{3}{2}$), Z(–1, $\frac{1}{2}$)
b) $\frac{1}{2}$; coordinates were all multiplied by $\frac{1}{2}$
c) Draw a line through each original vertex and its image vertex and the lines should meet at (0, 0).
d) Yes, because there are no rotations or reflections to change them.

3. a) 2
b) $\frac{1}{3}$
c) $\frac{2}{3}$

4. a) A(1, 2), B(1, –1), C(–2, 0)
b) *Sample response*: using scale factor 4: A(4, 8), B(4, –4), C(–8, 0), or using scale factor 0.5: A(0.5, 1), B(0.5, –0.5), C(–1, 0)
c) You can use any scale factor.

5. a) The angles in each pair are the same (AAA) because of the parallel lines; or, the ratios of pairs of corresponding sides are equal (SSS).
b) pair A: No; because if I draw lines through pairs of corresponding vertices, they do not all meet at the origin;
pair B: Yes; because if I draw lines through pairs of corresponding vertices, they all meet at the origin

6. One vertex was at the centre of dilatation.

7. The x- and y-coordinates must be multiplied by the same scale factor.

8. Draw lines through pairs of corresponding vertices. They all meet at the centre of the dilatation.

9. a) Always similar and only congruent if the scale factor is 1
b) Corresponding angles are the same.

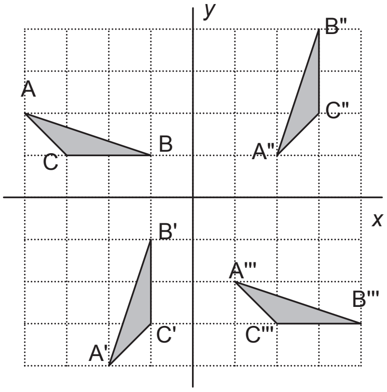
10. Dilatation images are similar because the ratio between any side length in the original shape and its corresponding side length in the image is the same for all pairs of sides. Not all similar triangles are dilatations because position is important in a dilatation. An example of similar triangles that are not a dilatation can be seen in **question 5, pair A**.

5.2.4 Combining Transformations p. 219

1. a) X'(3, –4), Y'(0, –3), Z'(4, –1)
b) $(x, y) \rightarrow (x, -y)$ is the reflection
$(x, y) \rightarrow (x + 1, y - 2)$ is the translation
c) $(x, y) \rightarrow (x + 1, -y - 2)$
d) The coordinates would be X'(3, 0), Y'(0, 1), Z'(4, 3).

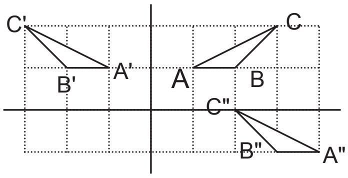
2. *Sample responses*:
a) Dilatate larger triangle using centre (0, 0) and scale factor $\frac{1}{2}$, then translate [–1, 2]
b) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ and
$(x, y) \rightarrow (x - 1, y + 2)$
c) $(x, y) \rightarrow (\frac{1}{2}x - 1, \frac{1}{2}y + 2)$

3. a) and b) *Sample response*: ΔABC has vertices A(–4, 2), B(–1, 1), and C(–3, 1).
- Rotate 90° ccw around the origin so the image (ΔA'B'C') is in the 3rd quadrant.
- Translate [4, 5] so the image (ΔA"B"C") is in the 1st quadrant
- Rotate 90° cw around the origin so the image (ΔA'''B'''C''') is in the 4th quadrant
- Translate [–5, 4] so the image (ΔA''''B''''C'''') maps onto the original shape (ΔABC) in the 2nd quadrant.



4. *Sample responses*:
a) Pair A: ΔPOQ ≅ ΔSRQ using SSS
Pair B: ΔCOD ≅ ΔOCB using AAS
b) Pair A: Rotate ΔPOQ 90° ccw around O and then translate [1, –5]. The final image is ΔSRQ.
Pair B: Rotate ΔCOD 180° cw around O and then translate up 4 units. The final image is ΔOCB.

5. *Sample responses*:
a) ΔABC maps onto ΔA"B"C" using $(x, y) \rightarrow (5 - x, y - 2)$



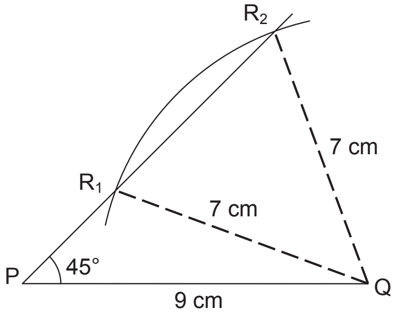
b) A reflection in the y-axis maps ΔABC onto ΔA'B'C', followed by the translation [5, –2], which maps ΔA'B'C' onto ΔA"B"C".
c) $(x, y) \rightarrow (-x, y)$ followed by $(x, y) \rightarrow (x + 5, y - 2)$

6. Yes; a reflection in the x-axis is mapped with $(x, y) \rightarrow (x, -y)$ and a reflection in the y-axis is mapped with $(x, y) \rightarrow (-x, y)$.
If you combine them, you get $(x, y) \rightarrow (-x, -y)$, which is the mapping for a 180° rotation.

7. a) Rotate 90° ccw around the origin $(x, y) \rightarrow (-y, x)$, then translate $(x, y) \rightarrow (x + 1, y + 5)$.
b) Translate $(x, y) \rightarrow (x - 1, y - 5)$, then rotate 90° cw around the origin $(x, y) \rightarrow (y, -x)$
c) Reflect in the y-axis $(x, y) \rightarrow (-x, y)$, then translate $(x, y) \rightarrow (x, y - 5)$
d) Translate $(x, y) \rightarrow (x, y + 5)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$

UNIT 5 Revision pp. 220–221

1. a) In ΔPQR_1, $\angle R_1 = 115°$, $PR_1 \approx 3.5$ cm
In ΔPQR_2, $\angle R_2 = 62°$, $PR_2 \approx 9.3$ cm



b) When establishing congruence, knowing the lengths of two sides and any angle is not enough. The angle must be contained between the two sides.

2. a) $\Delta ABD \sim \Delta CBD$ because two angles are the same (AAA). $\Delta ABD \cong \Delta CBD$ because of AAS.
b) BC = 13 m because it corresponds to AB in the congruent triangle.

3. *Sample response*: I constructed three triangles, each with sides 7 cm, 8 cm, and 5 cm and measured all the angles. The corresponding angles in each triangle were the same. I repeated this with other sets of triangles and the results were the same. So SSS also means AAA.

4. VZ = VX would show SAS; $\angle Y = \angle W$ would show ASA; and $\angle Z = \angle X$ would show AAS
5. a) P'(0, 4), Q'(2, 5), R'(1, 12)
b) $(x, y) \rightarrow (x - 3, y + 5)$

6. a) $\Delta HFE \sim \Delta HJG$ because two angles are the same (AAA)
b) 6 cm

7. a) K'(–3, –1), L'(3, 1), M'(–3, 7)
b) $(x, y) \rightarrow (x, -y)$
c) K'(1, –3), L'(–1, 3), M'(–7, –3)
d) $(x, y) \rightarrow (y, x)$
e) In both cases the orientation is reversed.

8. a) S'(1, –2), T'(4, –2), U'(4, –4)
b) S'(–1, 2), T'(–4, 2), U'(–4, 4)
c) Part a) is a rotation of 90º ccw around the origin. **Part b)** is a rotation of 90º cw around the origin
d) In both cases the orientation is the same.

9. *Sample response*:
a) triangle with vertices (0, 0), (1, 0), (0, 3)
b) triangle with vertices (0, 0), (3, 0), (0, 9)
c) Similar because of SSS

10. a) A'(0, 2), B'(3, 1), C'(1.5, –1)
b) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$
c) The orientation is the same.

11. a) reflection in y-axis
b) dilatation, centre (0, 0), scale factor 2
c) rotation of 180º around the origin
d) rotation of 90º cw around the origin
e) translation [–3, 2]

12. a) $\Delta OJI \cong \Delta FGH$ using SAS or SSS.
b) *Sample response*: Reflect ΔFGH in the y-axis and then translate [0, 1]. The final image is ΔOJI.

13. *Sample response*:
a) Dilatate with centre at the origin $(x, y) \rightarrow (2x, 2y)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$, and finally translate $(x, y) \rightarrow (x + 1, y - 8)$
b) Translate $(x, y) \rightarrow (x - 1, y + 8)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$, and finally dilatate with centre at the origin $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$
c) Rotate 90º ccw around the origin $(x, y) \rightarrow (-y, x)$, then translate $(x, y) \rightarrow (x + 5, y)$
d) Translate $(x, y) \rightarrow (x - 5, y)$, then rotate 90º cw around the origin $(x, y) \rightarrow (y, -x)$

14. a) congruent, similar, same orientation
b) congruent, similar, opposite orientation
c) congruent, similar, same orientation
d) not congruent, similar, same orientation

UNIT 6 MEASUREMENT pp. 223–264

- *In this and every lesson in this unit, there may be minor discrepancies between your responses and these answers due to rounding.*
- *The use of the ≈ symbol means a rounded approximation.*

Getting Started — Skills You Will Need pp. 223–224

1. a) cube **b)** square-based pyramid **c)** octagon-based prism

2. a) 26 cm^2 **b)** 4.16 m^2 **c)** 13.85 cm^2 or 4.41π cm^2

3. a) 125 **b)** 17.58 **c)** 3.66

4. a) 5.22 m **b)** 10 mm or 1 cm **c)** 9.80 cm **d)** 9 cm

5. a) 1897.5 mm^2 **b)** 374.12 m^2

6. a) 50.27 cm or 16π cm **b)** 4.77 m

7. a) 2400 g **b)** 0.23 L **c)** 30,000 cm^2 **d)** 5 cm^3 **e)** 4300 cm^3

6.1.1 Volume of Prisms and Cylinders pp. 229–230

1. a) 540 cm^2 **b)** 1076.63 cm^3 **c)** 348 cm^3

2. a) 1130.97 cm^3 **b)** 2.53π or 7.94 cm^3

3. 3319.06 cm^3

4. a) 75.71 cm^3 **b)** 855 cm^3

5. a) 4.74 cm **b)** 10 cm

6. a) 28.9 L or 28,900 mL
b) 107.04 cm or 1.07 m

7. 1079.92 cm^3

8. 646,592 mm^3 or 646.59 cm^3

9. 1273.24 cm^3

10. 1570.78 cm^3

11. *Sample responses*:
- You could quadruple the height because $V = Ah$ so $4 \times V = A \times 4 \times h$.
- You could quadruple the area because $4 \times V = 4 \times A \times h$.
- You could double the area and the height because $4 \times V = 2 \times A \times 2 \times h$.
- You could double each base dimension because $4V = 2l \times 2w \times h$.

6.1.3 Volume of Pyramids and Cones pp. 236–238

1. a) 75 m^3 **b)** 2.09 m^3
c) 14.34π or 45.04 m^3 **d)** 4.93 cm^3

2. a) 62.35 cm^2 **b)** 203.69 cm^3

3. 180.41 m

4. 3.6 m^3

5. a) 14.21 cm **b)** 17.84 cm

6. a) 12 cm **b)** 6 cm
c) 144π or 452.39 cm^3
d) 18π or 56.55 cm^3
e) 126π or 395.84 cm^3

7. a) A is 6 m^3, B is 4 m^3.
b) To find the volume of a square pyramid, you square the base dimension but not the height, so it makes a difference which dimension applies to the base and which applies to the height.
c) 3:2

8. a) A is 120π or 376.99 m^3;
B is 100π or 314.16 m^3.
b) To find the volume of a cone, you square the radius but not the height, so it makes a difference which dimension applies to the base and which applies to the height.
c) 12:10 or 6:5

9. 22π or 69.11 m^3

10. a) 96π or 301.59 m^3
b) 93.60π or 294.05 mm^3

10. c) 146.21 mm^3

11. *Sample response*: Two cones with diameter 6 cm. One has a height of 3 cm and the other a height of 5 cm.

12. *Sample response*: Visualize tapering the prism from bottom to top so it becomes a wedge with half the original volume. A pyramid with the same base and height is even smaller than this, so it must be less than half.

6.1.4 Volume of Spheres and Composite Shapes pp. 241–242

1. a) $1\frac{1}{3}\pi$ m^3 **b)** $10\frac{2}{3}\pi$ m^3

c) 36π m^3 **d)** $85\frac{1}{3}\pi$ m^3

2. a) and b)

Question	Radius	Volume	Factor
a)	1	$1\frac{1}{3}\pi$ m^3	× 1
b)	2	$10\frac{2}{3}\pi$ m^3	× 8
c)	3	36π m^3	× 27
d)	4	$85\frac{1}{3}\pi$ m^3	× 64

c) Each factor is the radius cubed.

d) $166\frac{2}{3}\pi$ m^3

3. The volume is multiplied by n^3.

4. about 1394.37 times larger

5. 3.33 cm.

6. a) 696.91 mL
b) 6.16 cm

7. 1970 L

8. 8.96 kg

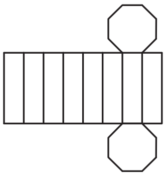
9. a) 1843.29π or 5790.87 cm^3
b) 11,059.76 cm^3
c) The cube is about 2 times larger.

6.2.1 Surface Area of Prisms pp. 246–247

1. 524 cm^2

2. a) 240 cm^2 **b)** 6090 mm^2

3. *For the area of the lateral surface*:
The eight rectangles together make one large rectangle. Use $A = l \times w$ to find the area ($20 \times 9 = 180$ cm^2).



For the area of the two regular octagon bases:
Divide into eight congruent triangles.
Use $A = \frac{1}{2}bh$ to find the area of one triangle and then multiply by 8 to find the area of all of them and then by 2 to find the area of both bases ($\frac{1}{2} \times 2.5 \times 3 \times 8 \times 2 = 60$ cm^2).
For the total SA:
Add areas together ($180 + 60 = 240$ cm^2)

6.2.1 Surface Area of Prisms [Continued] pp. 246–247

4. a) The total surface area of the large prism is 2520 m^2 and the total surface area of the small prism is 630 m^2.
b) 2:1 **c)** 4:1
d) Each face has both dimensions doubled, so it is × 2 × 2, which is the same as × 4.

5. a) 82.49 m^2 **b)** 52 m^3

6. 148.43 m^2

7. 12,696.07 cm^2

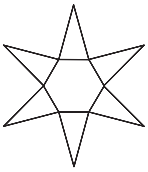
8. a) 2.94 cm
b) *Sample response*: 2 cm × 2 cm × 5.5 cm
c) 2.04 cm

9. 7.7 m^2

10. a) Find the cube root of the volume to find the edge, or side length. Square the side length to find the area of each square face. Multiply by 6 because there are 6 square faces.
b) *Sample response*: No. For example, if a prism has a base area of 20 cm^2 and a height of 5 cm, its volume is the same as a different prism with a base area of 5 cm^2 and a height of 20 cm, but the total surface areas of the two prisms would be very different.

6.2.2 Surface Area of Pyramids pp. 251–252

1. a) 438.48 cm^2 **b)** 378 cm^2 **c)** 222 cm^2



2. *For the area of the six triangles that form the lateral faces*:
Each triangle has a base of 8.7 cm (side length of the base hexagon) and a height of 9.3 cm (slant height of pyramid). Use $A = \frac{1}{2}bh$ to find each area and then multiply by 6 to find the area of all six triangles ($\frac{1}{2} \times 8.7 \times 9.3 \times 6 = 242.73$ cm^2).
For the area of the base:
Divide it into six congruent triangles, find the dimensions of one triangle (8.7 cm by 7.5 cm, the apothem), use $A = \frac{1}{2}bh$ to find its area and then multiply by 6 to find the base area ($\frac{1}{2} \times 8.7 \times 7.5 \times 6 = 195.75$ cm^2).
For the total SA:
Add the two areas to get a total surface area of 438.48 cm^2.

3. a) 5 m
b) 96 m^2

4. a) 624.67 cm^2
b) 266.39 m^2

5. 5250 cm^2

6. Two cans of wall paint and two cans of roof paint (although there will a lot of roof paint left over)

7. a) No; *sample response*: For a square pyramid with a 6 m base and height 4 m, the *SA* is 96 m^2. When the height is doubled, the *SA* is 139 m^2, which is less than 192 m^2 (double 96 m^2).
b) Yes; $V = \frac{1}{3}Ah$, so if you double the height you double the volume: $2 \times V = \frac{1}{3} \times A \times 2h$

6.2.3 Surface Area of Cylinders pp. 255–256

1. a) 9 cm **b)** 19 cm **c)** 28.27 cm^2

2.a) 87.96 m^2 **b)** 132.45 cm^2
c) 282.74 m^2

3. a) 49.74 cm^2 **b)** 31.83 cm^2

4. 3141.59 cm^2

5. 65.74 cm

6. 125.42 cm^2

7. 778.33 mm^2; 570.99 mm^3

8. 617.48 mm^2

9. The area of each base is quadrupled (× 4) if the curved surface is doubled and the height is maintained.

6.2.4 Surface Area of Cones p. 259

1. a) 942.48 cm^2 **b)** 1685.96 cm^2

2. Cone A

3. a) 137.44 cm^2 **b)** 153.94 mL

4. a) 47.12 cm^2
b) The total *SA* of each small cone is less than half the total *SA* of the original cone.

4. c) The total *SA* of each small cone is more than half the total *SA* of the original cone.

5. 78.54 m^2

6. The total *SA* of the pyramid is greater.
SA_{pyramid} = 96 cm^2; SA_{cone} = 75.40 cm^2

6.2.5 Surface Area of Spheres pp. 261–262

1. a) 452.39 cm^2 **b)** 1017.88 m^2
c) 651.44 mm^2 **d)** 0.32 m^2

2. 31.81 m^2

3. a) 111.02 m^2 **b)** 98.29 m^3

4. 1571.99 m^2

5. 510.54 m^2

6. 9.61 m^2

7. If you triple the diameter, the surface area is multiplied by 9.

UNIT 6 Revision pp. 263–264

1.a) 14.5 cm^3 **b)** 466.79 cm^3
c) 56.55 cm^3 **d)** 1017.88 m^3

2. 7116.66 g or 7.12 kg

3. a) 1280 cm^3 **b)** 328 m^3

4. a) 301.59 cm^3 **b)** 59.61 cm^3

5. 11494.04 mm^3 or 11.49 cm^3

6. 282.22 mL

7. a) 36.25 cm^2 **b)** 448.8 m^2
c) 94.25 cm^2 **d)** 565.49 m^2

8. a) 800 cm^2 **b)** 295.2 m^2

9. a) 301.59 cm^2 **b)** 94.25 cm^2

10. 2058.87 cm^3

11. *Sample response*: Volume is easier since you just do two multiplications instead of two multiplications and adding

12. a) 30 m **b)** 8.9 cm

13. Vol $\approx$ 341.64 cm^3
T.S.A $\approx$ 578.31 cm^2

UNIT 6 Revision [Continued] pp. 263–264

14. a) A: 245.04 cm^2; B: 282.74 cm^2
b) 0.87 **c)** 0.87
d) Only one thing is different when you calculate the lateral surface of each cone, $s = 13$ vs. $s = 15$. Because s in both cases is multiplied by the same value (6π), the ratio must be the same, $13 \div 15$.

15. 4412.76 cm^3; 1863.88 cm^2

16. a) 485.90 m^3 **b)** 314.16 m^2

17. a) *Sample response*: Cone on top of a square prism, hemisphere in the middle, cylinder at the bottom
b) Measure the diameter and slant height of the cone, height and side length of the prism, diameter of the hemisphere (and cylinder), and height and diameter of the cylinder.

UNIT 7 COMMERCIAL MATHEMATICS pp. 265–283

Getting Started p. 265

1. a) Nu 213 **b)** Nu 299

2. a) Nu 450 **b)** Nu 494

3. a) Nu 251 **b)** Nu 697

4. a) Nu 400 **b)** Nu 165

7.1.1 Income and Expenditures p. 269

1. *Weekly incomes were found by dividing by 52. Other answers close to these are also acceptable.*
a) month Nu 9166.67; week Nu 2115.38
b) month Nu 6666.67; week Nu 1538.46
c) month Nu 11,833.33; week Nu 2730.77
d) monthly Nu 13,000; week Nu 3000

2. a) Nu 344 **b)** Nu 960
c) Nu 1450 **d)** Nu 1237.50

3. a) *Sample response:* part iii); since the amount deposited is greatest and 4.75% interest is almost 4.95%, which is the greatest percentage.
b) i) Nu 549 **ii)** Nu 551
iii) Nu 1049.75 **iv)** Nu 977.63

4. a) Nu 2097 **b)** Nu 3789.50 **c)** Nu 3336

5. a) Nu 14,666.67 **b)** Nu 16,477.27
c) Nu 9787.23 **d)** Nu 9062.50

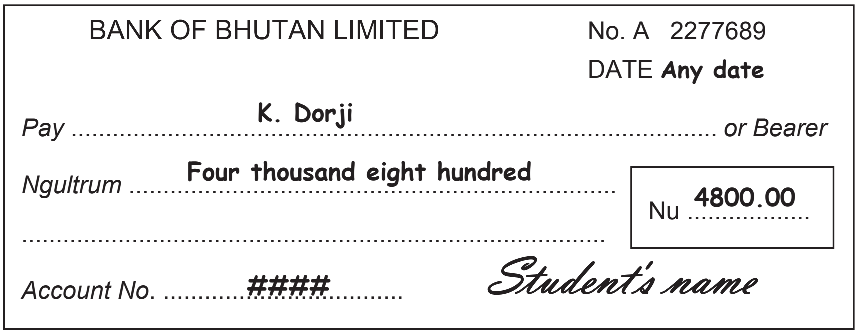
6. a) Nu 9259 **b)** Nu 6364
c) Nu 7759 **d)** Nu 7903

7. *See answer at bottom of page.*

8. Nu 57,000

9. They would keep track of their income so they know how much they can spend on rent, food, etc., and they would also keep track of expenses to make sure they do not spend more than they can afford.

7.



BANK OF BHUTAN LIMITED
No. A 2277689
DATE Any date
Pay K. Dorji or Bearer
Ngultrum Four thousand eight hundred
Nu 4800.00
Account No. ####
Student's name

7.1.2 Budgets pp. 272–273

1. a) Sonam’s monthly budget

Income source	From parents			…..			Total
Amount	900						900
Expense	For clothes	For school supplies	For recreation	For snacks	For gift for mother	Other	Total
Amount	250	100	50	300	150	50	900

b) Dechen’s father’s monthly budget

Income source	From Salary	From allowance	From interest	…..			Total
Amount	10,200	3500	200				13,900
Expense	For rent	For food	For TV/phone	For clothing, toys	For money to Dechen	Other	Total
Amount	5800	3100	700	200	500	3600	13,900

2. Sonam: Nu 50; Dechen’s father: Nu 2600

3. Rent: Nu 2850; Food: Nu 2850: Household: Nu 1425; Recreation: Nu 475; Savings: Nu 760; Other: Nu 1140

4. *Sample response:*
Ugyen’s budget

Income source	From all sources						Total
Amount	12,200						12,200
Expense	For rent	For food	For TV, phone	For clothing	For savings	Other	Total
Amount	5000	3500	1000	600	1500	600	12,200

5. a) If you do not have a lot of money, you need to make sure that you have enough for the necessities.
b) Even a person with more money could end up spending too much on unnecessary things and not have enough left for necessities.

6. a) *Sample response:* Income would be in one category. I would separate expenditures into these categories: for school, for food, for recreation, for savings, and other expenses, so I could see how I spend my money and find out if I could put more into savings.
b) *Sample response:* A budget could help me keep track of my expenses so I could explain to my parents why I need a little extra for something like recreation.

7.2.1 Reporting Income and Taxes p. 277

1. a) The first two rows in each chart are the same.
The third row in one chart says 15000 in place of 10% of the next 150,000 in the other chart, but 0.10 × 150,000 = 15,000, so these rows are also the same in each chart.
The fourth row in one chart says 52,500 in place of 10% of the 150,000 and 15% on the next 250,000 in the other chart, but
0.10 × 150,000 + 0.15 × 250,000 =
15,000 + 37,500 = 52,500, so these rows are also the same in each chart.
The last row in one chart says 1,52,000 in place of 10% of the 150,000, 15% on the next 250,000, and 20% on the next 500,000 in the other chart, but
0.10 × 150,000 + 0.15 × 250,000 + 0.20 × 500,000 = 15,000 + 37,500 + 100,000 = 1,52,500, so these rows are also the same in each chart.
b) It saves time since some of the calculations are done for you.
c) It shows the rate you are paying on each part of your income.

2. a) Nu 72,500 **b)** Nu 12,500
c) Nu 0 **d)** Nu 41,000

3. a) Nu 76 **b)** Nu 4,350
c) Nu 1250 **d)** Nu 377

4. a) He is in the second tax slab which means he only pays 10% of Nu 47,000 (247,000 – 200,000), which is a lot less than 10% of Nu 247,000.
b) He is paying 10% of less than half his total income, so the rate has to be less than half of 10%, which is less than 5%.

5. Nu 124,000

6. People who have less income use a larger percentage of their income for basic necessities like food and rent than those with a greater income, so they have less that they can afford for taxes. People who have more should share.

7. Usually, but not always. Even if they pay the same tax rate, the person who has the higher income will pay more because that percentage is applied to a greater amount. For example, if one person has an income of Nu 210,000 and another has an income of Nu 240,000, they both pay 10% but the first person pays 10% on 10,000 (which is Nu 1000) and the other person pays 10% on 40,000 (which is Nu 4000). The exception is if the income is Nu 200,000 or less.

7.2.2 Income Deductions p. 280

1. a) Nu 241,450 **b)** Nu 49,020

2. a) Nu 810 **b)** Nu 1278
c) Nu 2945 **d)** Nu 10,000

3. a) Nu 222,222.22 **b)** Nu 200,000
c) Nu 250,000 **d)** Nu 166,666.67

4. a) Nu 10,000 **b)** Nu 4770 **c)** Nu 4290

5. a) Nu 40,000 **b)** Nu 47,619.05
c) Nu 33,333.33 **d)** Nu 5,555.56

6. Nu 17,916.67

7. The dividend

CONNECTIONS: Taxation around the World p. 281

1. a) Some recent income tax rates are listed below:

Country	Low–High	Range
Australia	15–46%	31
Austria	21–50%	29
Belgium	25–50%	25
Bhutan	0–15%	15
Bulgaria	10–24%	14
Canada	15–29%	14
China	5–45%	40
Czech Rep.	12–32%	20
Denmark	38–59%	21
France	10–48.09%	38
Germany	15–42%	27
Greece	0–40%	40
Hungary	18–36%	18
India	10–30%	20
Ireland	20–42%	22
Israel	10–49%	39
Italy	23–43%	20
Japan	10–37%	27
Mexico	3–29%	26
Morocco	0–41.5%	41.5
Netherlands	0–52%	52
Norway	28–51.3%	23.3
Pakistan	7.5%–35%	27.5
Philippines	5–32%	27
Poland	19–40%	21
Portugal	10.5–40%	29.5
Russia	13%	0
South Africa	18–40%	22
Spain	15–45%	30
Taiwan	6–40%	34
Thailand	5–37%	32
Turkey	15–35%	20
U.K.	0–40%	40
U.S.	0–35%	35

The greatest range is 52 in the Netherlands and the smallest ranges are Russia at 0, followed by Canada and Bulgaria both at 14 and then Bhutan at 15.

2. *Sample response*: Most ranges are about 20 and Bhutan's is 15, so Bhutan's range is smaller than most. Bhutan is like a lot of other countries by having a 0% rate, but its high end rate is much lower than most.

Tax Ranges

0	0
1	4 4 5 8
2	0 0 0 0 1 1 2 2 3 5 6 7 7 7 8 9
3	0 0 1 2 4 5 8 9
4	0 0 0 2
5	2

Tax Rates (low end)

0	0 0 0 0 0 0 3 5 5 5 6 8
1	0 0 0 0 0 1 2 3 5 5 5 5 5 8 8 9
2	0 1 3 5 8
3	8

Tax Rates (high end)

1	3 5
2	4 9 9
3	0 2 2 5 5 5 6 7 7
4	0 0 0 0 0 0 2 2 2 3 5 5 6 8 9
5	0 0 1 2 9

UNIT 7 Revision pp. 282–283

1. a) Nu 126,000 **b)** Nu 103,480
c) Nu 117,520 **d)** Nu 105,600

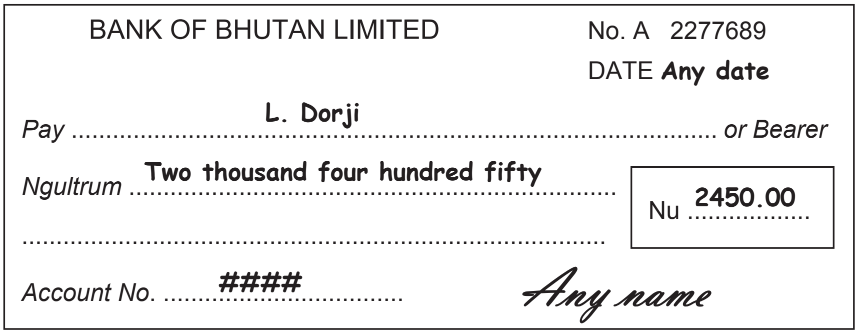
2. a) Nu 935 **b)** Nu 883.50
c) Nu 1485.75 **d)** Nu 803

3. a) Nu 6000 **b)** Nu 5850
c) Nu 10,608 **d)** Nu 2633.75

4. a) Nu 9435 **b)** Nu 10,495
c) Nu 15,989 **d)** Nu 13,989

UNIT 7 Revision [Continued] **pp. 282–283**

5. a)



BANK OF BHUTAN LIMITED

No. A 2277689

DATE **Any date**

Pay **L. Dorji** *or Bearer*

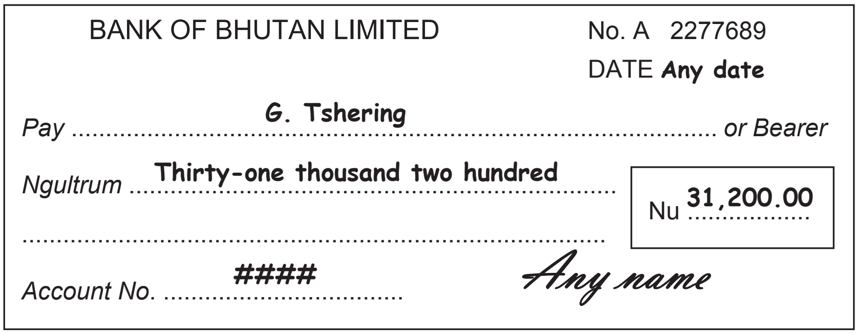
Ngultrum **Two thousand four hundred fifty**

Nu **2450.00**

Account No. **####**

Any name

b)



BANK OF BHUTAN LIMITED

No. A 2277689

DATE **Any date**

Pay **G. Tshering** *or Bearer*

Ngultrum **Thirty-one thousand two hundred**

Nu **31,200.00**

Account No. **####**

Any name

6. a) *Sample response:*

For four months

Income source	From parents					Total
Amount	3000					3000
Expense	For clothes	For school supplies	For recreation	For snacks	Other	Total
Amount	1000	250	300	800	650	3000

b) For one month

Income source	From salary						Total
Amount	12,800						12,800
Expense	For rent	For food	For household expenses	For clothing	For loan payment	Other	Total
Amount	4800	4200	800	300	1500	1200	12,800

7. Sonam: 21.7%; U. Pem: 9.4%

8. To make sure you do not spend more than you have, to see how much you spend in different categories like rent and food, and to make sure you are not overspending in one category.

9. a) Nu 2410 **b)** Nu 2602.80
c) Nu 1446 **d)** Nu 482
e) Nu 2699.20

10. a) Nu 18,929 **b)** Nu 20,357
c) Nu 17,666

11. a) Nu 13550 **b)** Nu 41,250
c) Nu 76,250 **d)** Nu 260,000

12. a) Nu 1230 **b)** Nu 39
c) Nu 6000 **d)** Nu 180.80

13. People with less money spend a greater percentage of their money on necessities. The tax rates take this into account.

14. To make sure money comes in regularly to the government to pay for services and to make sure taxpayers do not run out of money to pay their taxes when they are due.

15. a) 24.24% **b)** Nu 48,888.89

16. B

17. a) Nu 710 **b)** Nu 9150

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