Teacher's Guide to

Understanding Mathematics

Textbook for Class IX

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Department of Curriculum and Professional Development
Ministry of Education
Royal Government of Bhutan

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MINISTER

ROYAL GOVERNMENT OF BHUTAN
MINISTRY OF EDUCATION
THIMPHU : BHUTAN

FOREWORD

Provision of quality education for our children is a cornerstone policy of the Royal Government of Bhutan. Quality education in mathematics includes attention to many aspects of educating our children. One is providing opportunities and believing in our children's ability to understand and contribute to the advancement of science and technology within our culture, history and tradition. To accomplish this, we need to cater to children's mental, emotional and psychological phases of development, enabling, encouraging and supporting them in exploring, discovering and realizing their own potential. We also must promote and further our values of compassion, hard work, honesty, helpfulness, perseverance, responsibility, *thadamtsi* (for instance being grateful to what I would like to call '*Pham Kha Nga',* consisting of parents, teachers, His Majesty the King, the country and the Bhutanese people, for all the goodness received from them and the wish to reciprocate these in equal measure) and *ley-ju-drey* — the understanding and appreciation of the natural law of cause and effect. At the same time, we wish to develop positive attitudes, skills, competencies, and values to support our children as they mature and engage in the professions they will ultimately pursue in life, either by choice or necessity.

While education recognizes that certain values for our children as individuals and as citizens of the country and of the world at large, do not change, requirements in the work place advance as a result of scientific, technological, and even political advancement in the world. These include expectations for more advanced interpersonal skills and skills in communications, reasoning, problem solving, and decision-making. Therefore, the type of education we provide to our children must reflect the current trends and requirements, and be relevant and appropriate. Its quality and standard should stem out of collective wisdom, experience, research, and thoughtful deliberations.

Mathematics, without dispute, is a beautiful and profound subject, but it also has immense utility to offer in our lives. The school mathematics curriculum is being changed to reflect research from around the world that shows how to help students better understand the beauty of mathematics as well as its utility.

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The development of this textbook series for our schools, *Understanding Mathematics,* is based on and organized as per the new School Mathematics Curriculum Framework that the Ministry of Education has developed recently, taking into consideration the changing needs of our country and international trends. We are also incorporating within the textbooks appropriate teaching methodologies including assessment practices which are reflective of international best practices. The *Teacher's Guides* provided with the textbooks are a resource for teachers to support them, and will definitely go a long way in assisting our teachers in improving their efficacy, especially during the initial years of teaching the new curriculum, which demands a shift in the approach to teaching and learning of Mathematics. However, the teachers are strongly encouraged to go beyond the initial ideas presented in the Guides to access other relevant resources and, more importantly to try out their own innovations, creativity and resourcefulness based on their experiences, reflections, insights and professional discussions.

The Ministry of Education is committed to providing quality education to our children, which is relevant and adaptive to the changing times and needs as per the policy of the Royal Government of Bhutan and the wish of our beloved King.

I would like to commend and congratulate all those involved in the School Mathematics Reform Project and in the development of these textbooks.

I would like to wish our teachers and students a very enjoyable and worthwhile experience in teaching, learning and understanding mathematics with the support of these books. As the ones actually using these books over a sustained period of time in a systematic manner, we would like to strongly encourage you to scrutinize the contents of these books and send feedback and comments to the Curriculum and Professional Support Division (CAPSD) for improvement with the future editions. On the part of the students, they can and should be enthusiastic, critical, venturesome, and communicative of their views on the contents discussed in the books with their teachers and friends rather than being passive recipients of knowledge.

Trashi Delek!

Lyonpo Thinley Gyamtsho

MINISTER

Ministry of Education

January of 2007

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INTRODUCTION

HOW MATHEMATICS HAS CHANGED

Mathematics is a subject with a long history. Although newer mathematical ideas are always being created, much of what your students will be learning is mathematics that has been known for hundreds of years, if not longer.

The learning of mathematics helps a person to solve problems. While solving problems, skills related to, representation of mathematical ideas, making connections with other topics in mathematics and connections with the real world, providing reasoning and proof, and communicating mathematically would be required. The textbook is designed to promote the development of these process skills.

Worldwide, there is now a greater emphasis on the need for students to understand the mathematics they learn rather than to memorize rote procedures.

Worldwide, there is now a greater emphasis on the need for students to understand the mathematics they learn rather than to memorize rote procedures. There are many reasons for this. • In the long run, it is very unlikely that students will remember the mathematics they learn unless it is meaningful. It is much harder to memorize "nonsense" than something that relates to what they already know.

- Some approaches to mathematics have not been successful; there are many adults who are not comfortable with mathematics even though they were successful in school.

In this new program, you will find many ways to make mathematics more meaningful.

It is important to always talk about why something is true, not simply that it is true.

- We will always talk about why something is true, not simply that it is true. For example, the reason that the volume of a cube with a particular surface area is greater than the volume of a different rectangular prism with that same surface area is demonstrated and not just stated.
- Mathematics should be taught using contexts that are meaningful to the students. They can be mathematical contexts or real world contexts. The student textbook uses both Bhutanese and international real world contexts. For example, in Unit 6 (Measurement), a task with a Bhutanese context involves calculating the surface area of a cylindrical prayer wheel. A task with an international context involves estimating the volume of the Great Pyramid in Giza. These contexts will help students see and appreciate the value of mathematics.

Calculating the surface area of a prayer wheel and the volume of a pyramid

• When discussing mathematical ideas, we expect students to use the processes of problem solving, communication, reasoning, making connections (connecting mathematics to the everyday world and connecting mathematical topics to each other) and representation (representing mathematical ideas in different ways, such as manipulatives, graphs, and tables). For example, to help students work with polynomials, we will connect operations with polynomials to operations with numbers, have students use reasoning to see why different representations of polynomials are equivalent and ask them to explain their thinking while solving problems with polynomials independently.

We expect students to use the processes of problem solving, communication, reasoning, making connections, and representation.

• There is an increased emphasis on problem solving because the reason we learn mathematics is to solve problems. Once students are adults, they are not told when to factor or when to multiply; they need to know how to apply those skills to solve certain problems. Students will be given opportunities to make decisions about which concepts and skills they need in certain situations.

A significant amount of research evidence has shown that these more meaningful approaches work. Scores on international tests are higher when the emphasis is on higher level thinking and not only on the application of skills.

Scores on international tests are higher when the emphasis is on higher level thinking and not only on the application of skills.

THE DESIGN OF THE STUDENT TEXTBOOK

Each unit of the textbook has the following features:

• a *Getting Started* to review prerequisite content

• two or three chapters, which cluster the content of the unit into sections that contain related content

• regular lessons and at least one *Explore* lesson

• a *Game* (usually)

• at least one *Connections* feature

• a *Unit Revision*

Getting Started

There are two parts to the *Getting Started*. They are designed to help you know whether the students are missing critical prerequisites and to remind students of knowledge and terminology they have already learned that will be useful in the unit.

The Getting Started is designed to help you know whether the students are missing critical prerequisites and to remind students of knowledge and terminology they will need for the unit.

• The *Use What You Know* section is an activity that takes 20 to 30 minutes. Students are expected to work in pairs or in small groups. Its purpose is consistent with the rest of the text's approach, that is, to engage students in learning by working through a problem or task rather than being told what to do and then just carrying it through.

• The *Skills You Will Need* section is a more straightforward review of required prerequisite skills for the unit. This should usually take about 30 minutes.

Regular Lesson

• Each lesson might be completed in one or two hours (i.e., one or two class periods). The time is suggested in this *Teacher's Guide*, but it is ultimately at your discretion.

• Lessons are numbered #.#.#. The first number tells the unit, the second number the chapter, and the third number the lesson within the chapter. For example, Lesson 4.2.1 is Unit 4, Chapter 2, Lesson 1.

Lessons are numbered #.#.#. The first number tells the unit, the second number the chapter, and the third number the lesson within the chapter.

• Each lesson is divided into five parts:
- A *Try This* task or problem
- The exposition (the main points of the lesson)
- A question that revisits the *Try This* task, called *Revisiting the Try This* in this guide
- *Examples*
- *Practising and Applying* questions

Try This

• The *Try This* task is in a shaded box, like the one below from lesson 1.1.1.

$2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

A. Suppose you write $2^9 = 2^a \times 2^b$.
i) Find two pairs of values for *a* and *b* that would make this true.
ii) Find another pair of values.
iii) What do you notice about how the values of *a* and *b* are related in each pair?

The Try This is a brief task or problem that students complete in pairs or small groups to motivate new learning.

• The *Try This* is a brief task or problem that students complete in pairs or small groups. It serves to motivate new learning. Students can do the *Try This* without the new concepts or skills that are the focus of the lesson, but the problem is related to the new learning. It should be completed in 5 to 10 minutes. The reason to start with a *Try This* is that we believe students should do some mathematics independently before you intervene.

• Some lessons have *Try This* activities that take longer (e.g., in Unit 4, where students might be graphing). In these cases, the lesson itself is likely to be a two-period lesson.

• The answers to the *Try This* questions are not found in the back of the student book (but they are in this *Teacher's Guide*).

The Exposition

• The exposition presents the main concepts and skills of the lesson. Examples are often included to clarify the points being made.

The exposition presents the main concepts and skills of the lesson and is taught in different ways, as suggested in this guide.

• You will help the students through the exposition in different ways (as suggested in this *Teacher's Guide*). Sometimes you will present the ideas first, using the exposition as a reference. Other times students will work through the exposition independently, in pairs, or in small groups.

• Key mathematical terms are introduced and described in the exposition. When a key term first appears in a unit, it is highlighted in **bold type** to indicate that it is found in the glossary (at the back of the book).

• Students are not expected to copy the exposition into their notebooks either directly from the book or from your recitation.

Revisiting the Try This

• The *Revisiting the Try This* question follows the exposition and appears in a shaded lozenge, like this example from lesson 1.1.1.

B. Was **part A** an example of the product or quotient law? Explain.

• The *Revisiting the Try This* question links the *Try This* task or problem to the new ideas presented in the exposition. This is designed to build a stronger connection between the new learning and what students already understand.

The Revisiting the Try This question links the Try This task to the new ideas presented in the exposition.

Examples

• The *Examples* are designed to provide additional instruction by modelling how to approach some of the questions students will meet in *Practising and Applying*. Each example is a bit different from the others so that students have multiple models from which to work.

The Examples model how to approach some of the questions students will meet in Practising and Applying.

• The *Examples* show not only the formal mathematical work (in the left hand *Solution* column), but also student reasoning (in the right hand *Thinking* column). This model should help students learn to think and communicate mathematically. Photographs of students are used to further reinforce this notion.

The Examples show the formal mathematical work in Solution column and model student reasoning in the Thinking column. This model should help students learn to think and communicate mathematically.

• Many of the *Examples* present two or even three different solutions. The example below, from lesson 1.1.1, shows two possible ways to approach the task, *Solution 1* and *Solution 2*.

Example 1 Expressing Powers Using the Product Law	
List at least two possible ways of writing 3^8 as a product of powers of 3.	
Solution 1 $3^8 = (3 \times 3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^5 \times 3^3$ $3^8 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)$ $= 3^4 \times 3^4$ $3^8 = (3 \times 3) \times (3 \times 3 \times 3) \times (3 \times 3 \times 3)$ $= 3^2 \times 3^3 \times 3^3$	**Thinking**  • I knew I needed 8 threes multiplied together because that's what 3^8 means. • I grouped them in different ways to get different possibilities.
Solution 2 $3^8 = 3^{2+6} = 3^2 \times 3^6$ $3^8 = 3^{1+7} = 3^1 \times 3^7$ $3^8 = 3^{3+5} = 3^3 \times 3^5$	**Thinking**  • I used the product law: $a^r \times a^s = a^{r+s}$ and looked for two exponents that added to 8. • I could have written $3^1 \times 3^7$ as 3×3^7 because any number to the power of 1 is the number itself.

• The treatment of *Examples* varies and is discussed in the *Teacher's Guide*. Sometimes the students work through these independently, sometimes they work in pairs or small groups, and sometimes you are asked to lead them through some or all of the examples.

• A number of the questions in the *Practising and Applying* section are modelled in the *Examples* to make it more likely that students will be successful.

Practising and Applying

• Students work on the *Practising and Applying* questions independently, with a partner, or in a group, using the exposition and *Examples* as references.

• The questions usually start like the work in the *Examples* and get progressively more conceptual, with more explanations and more problem solving required later in the exercise set.

• The last question in the section always brings closure to the lesson by asking students to summarize the main learning points. This question could be done as a whole class.

Students work on the Practising and Applying questions independently, with a partner, or in a group, using the exposition and Examples as references.

Explore Lessons

• *Explore* lessons provide an opportunity for students to work in pairs or small groups to investigate some mathematics in a less directed way. Often, but not always, the content is revisited more formally in a regular lesson immediately before or after the *Explore* lesson. The *Teacher's Guide* indicates whether the *Explore* lesson is optional or core.

• There is no teacher lecture in an *Explore* lesson, so the parts of the regular lesson are not there. Instead, a problem or task is posed and students work through it by following a sequence of questions or instructions that direct their investigation.

• The answers for these lessons are not found in the back of the book, but are found in this *Teacher's Guide*.

Explore lessons provide an opportunity for students to work with a partner or in small groups to investigate some mathematics in a less directed way.

Connections

• The *Connections* is an optional feature that takes many forms. Sometimes it is a relevant and interesting historical note. Sometimes it relates the mathematical content of a unit to the content of a different unit. Other times it relates the mathematical content to a real world application.

• There are always one or more *Connections* features in a unit. The placement of a *Connections* feature is not fixed; it depends on the content knowledge required.

• The *Connections* feature always gives students something to do beyond simply reading it.

• Students usually work in pairs or small groups to complete these activities.

The Connections feature takes many forms. Sometimes it is a relevant and interesting historical note. Sometimes it relates the mathematical content of a unit to the content of a different unit. Other times it relates the mathematical content to a real world application.

Game

• There is usually one *Game* per unit, and sometimes there are two. If there is no *Game*, there is an extra *Connections* feature.

• The *Game* provides an enjoyable way to practise skills and concepts introduced in the unit.

• Its placement in the unit is based on where it makes most sense in terms of the content required to play the *Game*.

• In most *Games* students work in pairs or small groups, as indicated in the instructions.

• The required materials and rules are listed in the student book. Usually, there is a sample shown to make sure that students understand the rules.

• Most *Games* require 15 to 20 minutes, but students can often benefit from playing them more than once.

The Game provides an enjoyable way to practise skills and concepts introduced in the unit.

Unit Revision

• The *Unit Revision* provides an opportunity for review for students and for you to gather informal assessment data. *Unit Revisions* review all lesson content except the *Getting Started* feature, which is based on previous class content. There is always a mixture of skill, concept, and problem solving questions.

• The order of the questions in the *Unit Revision* generally follows the order of the lessons in the unit. Sometimes, if a question reflects more than one lesson, it is placed where questions from the later lesson would appear.

• Students can work in pairs or on their own, as you prefer.

• The *Unit Revision*, if done in one sitting, requires about one hour. If you wish, you might break it up and assign some questions earlier in the unit and some questions later in the unit.

The Unit Revision provides an opportunity for review for students and allows you to gather informal assessment data.

Glossary

• At the end of the student book, there is a glossary of key mathematical terminology introduced in the units. When new terms are introduced in the units, they are written in **bold type**. All of these terms are found in the glossary.

• The glossary also contains important mathematical terms from previous classes that students might need to refer to.

• In addition, there is a set of instructional terms commonly used in the *Practising and Applying* questions (for example, justify, explain, predict,…) along with descriptions of what those terms require the student to do.

The glossary contains key mathematical terminology introduced in the units, important terms from previous classes, and a set of instructional terms.

Answers

• Answers to most numbered questions are provided in the back of the student book. In many cases, the full solutions are not shown, only the final answers.

• There is often more than one possible answer. This is indicated by the phrase *Sample Response*.

• Full solutions to many of the questions are provided in the *Teacher's Guide*, as are the answers to the lettered questions (such as A or B) in the *Try This* and the *Explore* lessons.

The answers to most of the numbered questions are found in the back of the student book. This Teacher's Guide contains a full set of answers.

THE DESIGN OF THE TEACHER'S GUIDE

The *Teacher's Guide* is designed to complement and support the use of the student textbook.

• The sequencing of material in the guide is identical to the sequencing in the student textbook.

• The elements in the *Teacher's Guide* for each unit include:

- a *Unit Planning Chart*
- *Math Background* for the unit
- a *Rationale for Teaching Approach*
- a brief overview of *Technology in This Unit* (as required)
- support for each lesson
- a *Unit Test*
- a *Performance Task*
- an *Assessment Interview* (Units 2 and 6 only)

The Teacher's Guide is designed to complement and support the use of the student textbook.

The support for each lesson includes:

- *Curriculum Outcomes* covered in that lesson
- *Outcome relevance* (*Lesson relevance* in the case of *Explore* lessons)
- *Pacing* in terms of hours
- *Materials* required to teach the lesson
- *Prerequisites* that the lesson assumes students possess
- *Main Points to be Raised* explicitly in the lesson
- suggestions for working through the parts of the lesson
- *Suggested assessment* for the lesson
- *Common errors* to be alert for
- *Answers*, sometimes with more complete solutions than are found in the student text
- suggestions for *Supporting Students* who are struggling and/or for enrichment

Unit Planning Chart

This chart provides an overview of the unit and indicates, for each lesson, which curriculum outcomes are being covered, the pacing, the materials required, and suggestions for which questions to use for formative assessment.

The Unit Planning Chart provides an overview of the unit.

Math Background and Rationale for Teaching Approach

This explains the organization of the unit or provides some background for you that you might not have, particularly in the case of less familiar content. In addition, there is an indication of why the material is approached the way it is.

This section provides information about the critical mathematics behind the unit, an explanation of why the math is approached the way it is, and an overview of how technology is used in the unit.

Technology in This Unit

Most units include a brief overview of the role of technology in the unit. It is assumed that students have access to scientific calculators. Any reference to the use of the Internet, graphing software, or spreadsheets, for example, is considered optional.

Regular Lesson Support

Regular lesson support includes grouping and instructional strategies, alerts for common errors, suggestions for assessment, and teaching tips.

• Suggestions for grouping and instructional strategies are offered under the headings *Try This, Revisiting the Try This, The Exposition—Presenting the Main Ideas, Using the Examples,* and *Practising and Applying–Teaching Tips.*

• *Common errors* are sometimes included. If you are alert for these and apply some of the suggested remediation, it is less likely that students will leave the lessons with mathematical misunderstandings.

• A number of *Suggested assessment questions* are listed for each lesson. This is to emphasize the need to collect data about different aspects of the student's performance—sometimes the ability to apply skills, sometimes the ability to solve problems or communicate mathematical understanding, and sometimes the ability to show conceptual understanding.

• It is not necessary to assign every *Practising and Applying* question to each student, but they are all useful. You should go through the questions and decide where your emphasis will be. It is important to include a balance of skills, concepts, and problem solving. You might use the *Suggested assessment questions* as a guide for choosing questions to assign.

• You may decide to use the last *Practising and Applying* question to focus a class discussion that revisits the main ideas of the lesson and to bring closure to the lesson.

Explore Lesson Support

• As with regular lessons, for *Explore* lessons there is an indication of the curriculum outcomes being covered, the relevance of the lesson, and whether the lesson is optional or core.

• Because the style of the lesson is different, the support provided is different than for the regular lessons. There are suggestions for grouping the students for the exploration, a list of *Observe and assess* questions to guide your informal formative assessment, and *Share and reflect* ideas on how to consolidate and bring closure to the exploration.

Because the style of an Explore lesson is different than a regular lesson, the support provided in the Teacher's Guide is also different.

Unit Test

A pencil-and-paper unit test is provided for each unit. It is similar to the unit revision, but not as long. If it seems that the test might be too long (for example, if students would require more than one class period to complete it) some questions may be omitted. It is important to balance the items selected for the test to include questions involving skills, concepts, and problem solving, and to include at least one question requiring mathematical communication.

If the test seems too long, some questions may be omitted but is important to ensure a balance of questions involving skills, concepts, problem solving, and communication.

Performance Task

• The *Performance Task* is designed as a summative assessment task. Performance on the task can be combined with performance on a *Unit Test* to give a mark for a student on a particular unit.

• The task requires students to use both problem solving and communication skills to complete it. Students have to make mathematical decisions to complete the task.

• It is not appropriate to mark a performance task using percentages or numerical grades. For that reason, a rubric is provided to guide assessment. There are four levels of performance that can be used to describe each student's work on the task. A level is assigned for each different aspect of the task and, if desired, an overall profile can be assigned. For example, if a student's performance is level 2 on most of the aspects of the task, but level 3 on one aspect, an overall profile of level 2 might be assigned.

• A sample solution is provided for each task.

The Performance Task is designed as a summative assessment task that can be combined with a student's performance on the Unit Test to give an overall mark.

Unit Assessment Interviews

• Selected units (2 and 6) provide a structure for an interview that can be used with one or several students to determine their understanding of the outcomes.

• Interviews are a good way to collect information about students since they allow you to interact with the students and to follow up if necessary. Interviews are particularly appropriate for students whose performance is inconsistent or who do not perform well on written tests, but who you feel really do understand the content.

• You may use the data you collect in combination with class work or even a unit test mark.

Interviews are a good way to collect information about students, since interviews allow you to interact with the students and to follow up if necessary.

ASSESSING MATHEMATICAL PERFORMANCE

Forms of Assessment

It is important to consider both formative (continuous) and summative assessment.

Formative

• Formative assessment is observation to guide further instruction. For example, if you observe that a student does not understand an idea, you may choose to re-teach that idea to that student using a different approach.

Formative assessment is observation to guide further instruction.

• Formative assessment opportunities are provided through
  - prerequisite assessment in the *Getting Started*
  - suggestions for assessment questions in each regular lesson
  - questions that might be asked while students work on the *Try This* or during an *Explore* lesson
  - the *Unit Revision*
  - the unit *Assessment Interview* (Units 2 and 6 only)

• Formative assessment can be supplemented by
  - everyday observation of students' mathematical performance
  - formal or informal interviews to reveal students' understanding
  - journals in which students comment on their mathematical learning
  - short quizzes
  - projects
  - a portfolio of work so students can see their progress over time, for example, in problem solving or mathematical communication (see *Portfolios* below)

Summative

• Summative assessment is used to see what students have learned and is often used to determine a mark or grade.

Summative assessment is used to see what students have learned and is often used to determine a mark.

• Summative assessment opportunities are provided through
  - the *Unit Test*
  - the *Performance Task*
  - the *Assessment Interview*

• Summative assessment can be supplemented with
  - short quizzes
  - projects
  - a portfolio that is assessed with respect to progress in, for example, problem solving or communication

Portfolios

One of the advantages of using portfolios for assessment is that students can observe their own growth over time, leading to greater confidence. In each unit, items are identified in the section on math background as pertaining to the mathematical processes. Student work items related to one of the mathematical processes: problem solving, communication, reasoning, or representation, could form the basis of the portfolio.

One of the advantages of using portfolios for assessment is that students can observe their own growth over time, leading to greater confidence.

Assessment Criteria

• It is right and fair to inform students about what will be assessed and how it will be assessed. For example, students should know whether the intent of a particular assessment is to focus on application or on problem solving.

It is right and fair to inform students about what will be assessed and how it will be assessed.

• A student's mark and all assessments should reflect the curriculum outcomes for Class IX. The proportions of the mark assigned for each unit should reflect both the time spent on the unit and the importance of the unit. The modes of assessment used for a particular unit should be appropriate for the content. For example, if the unit focuses mostly on skills, the main assessment might be a paper-and-pencil test or quizzes. If the unit focuses on concepts and application, more of the student's mark should come from activities like performance tasks.

The modes of assessment used for a particular unit should be appropriate for the content.

• The focus of this curriculum is not on procedures but on a conceptual understanding of mathematics so that it can be applied to solve problems. Procedures are important too, but only in the context of solving problems. All assessment should balance procedural, conceptual, and problem solving items, although the proportions will vary in different situations.

All assessment should balance procedural, conceptual, and problem solving items, although the proportions may vary in different situations.

• Students should be informed whether a test is being marked numerically, using letter grades, or with a rubric. If a rubric is being used, then it should be shared with students before they begin the task to which it is being applied.

Determining a Mark

• In determining a student's mark, you can use the tools described above along with other information such as work on a project or poster. It is important to remember that the mark should, as closely as possible, reflect student competence with the mathematical outcomes of the course. The mark should not reflect behaviour, neatness, participation, and other non-mathematical aspects of the student's learning. These are important to assess as well, but not as part of the mathematics mark.

It is important to remember that the mark should reflect student competence with the mathematical outcomes of the course and not behaviour, neatness, participation, and so on.

• In looking at a student's mathematics performance, the most recent data might be weighted more heavily. For example, suppose a student does poorly on some of the quizzes given early in Unit 1, but later you observe that he or she has a better understanding of the material in the unit. You might choose to give the early quizzes less weight in determining a student's mark for the unit.

• At present, you are required to produce a numerical mark for a student, but that should not preclude your use of rubrics to assess some mathematical performance. One of the values of rubrics is their reliability. For example, if a student performs at level 2 on a particular task one day, he or she is very likely to perform at the same level on that task another day. On the other hand, a student who receives a test mark of 45 one day might have received a mark of 60 on a different day if one question had changed on the test or if he or she had read an item more carefully.

• You can combine numerical and rubric data using your own judgment. For example, if a student's marks on tests average 50%, but the rubric performance is higher, for example, level 3, it is fair and appropriate to use a higher average for that student's class mark.

THE CLASSROOM ENVIRONMENT

This new curriculum requires a change in the classroom environment to include more pair and group work and an increased emphasis on communication. It is only in this way that students will really become engaged in mathematical thinking instead of being spectators.

This new curriculum requires a change in the classroom environment to include more pair and group work and an increased emphasis on communication.

• In every lesson, students should be engaged in some pair or small group work (for the Try This, selected Practising and Applying questions, or during an Explore lesson).

• Students should be encouraged to communicate with each other to share responses that are different from those offered by other students or from the responses you might expect. Communication involves not only talking and writing but also listening and reading.

Pair and Group Work

• There are many reasons why students should be working in pairs or groups, including

- to ensure that students have more opportunities to communicate mathematically (instead of competing with the whole class for a turn to talk)
- to make it easier for them to take the risk of giving an answer they are not sure of (rather than being embarrassed in front of so many other people if they are incorrect)
- to see the different mathematical viewpoints of other students
- to share materials more easily

• Sometimes students can work with the students who sit near them, but other times you might want to form the groups so that students who are struggling are working together. Then you can help them while the other students move forward. Students who need enrichment can also work together so that you can provide an extra challenge for them all at once.

• For students who are not used to working in pairs or groups, you need to set down rules of behaviour that require them to attend to the task and to participate fully. You need to avoid a situation where four students are working together, but only one of them is really doing the work. You might post Rules for Group Work, as shown here.

• Once students are used to working in groups, you might sometimes be able to base assessment on group performance rather than on individual performance.

Communication

Students should be communicating regularly about their mathematical thinking. It is through communication that they clarify their own thinking as well as show you and their classmates what they do or do not understand. When they give an answer to a question, you can always be asking questions like, *How did you get that? How do you know? Why did you do that next?*

It is through communication that students clarify their own thinking as well as show you and their classmates what they do or do not understand.

• Communication is practised in small group settings, but is also appropriate when the whole class is working together.

• Students will be reluctant to communicate unless the environment is risk-free. In other words, if students believe that they will be reprimanded or made to feel badly if they say the wrong thing, they will be reluctant to communicate. Instead, show your students that good thinking grows out of clarifying muddled thinking. It is reasonable for students to have some errors in their thinking and it is your job to help shape that thinking. If a student answers incorrectly, it is your job to ask follow-up questions that will help the student clarify his or her own thinking. For example, suppose a problem requires a student to tell the slope and intercept of a line and the student gives the correct slope, but the wrong intercept. Instead of saying that the student is wrong, you could ask:

- *What does the slope tell you?*
- *What does the intercept tell you?*
- *How do you know your slope is correct?*
- *How did you figure out your intercept?*

Students will be reluctant to communicate unless the environment is risk-free.

• Many of the questions in the textbook require students to explain their thinking. The sample *Thinking* in the *Examples* is designed to provide a model for mathematical communication.

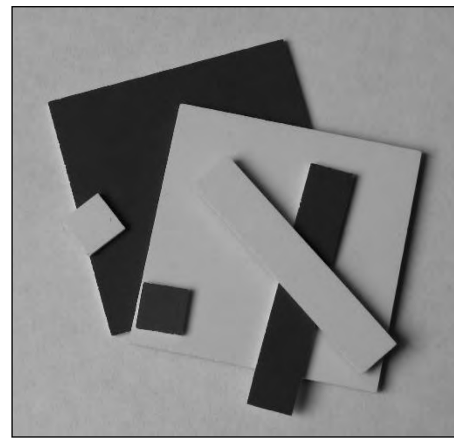
The sample Thinking in the Examples is designed to provide a model for mathematical communication.

• One of the ways students communicate mathematically is by describing how they know an answer is right. Even when a question does not ask students to check their work, you should encourage them to think about whether their answer makes sense. When they check their work, they should usually check using a different way than the way they used to find their answer so that they do not make the same reasoning error twice. This will enhance their mathematical flexibility.

MATHEMATICAL TOOLS

Manipulatives

• As the students move up the classes, there will be less use of manipulative (hands-on) materials than in earlier classes. Nevertheless, there is value in using manipulative materials even for older students who are good at mathematics. For example,
Unit 2 makes frequent use of algebra tiles. Although some students can be successful without these materials, all students benefit from their use. Students start to see not only how to perform algebraic manipulations, but why they are done the way they are.



• Manipulative materials are important in Class IX particularly, but not exclusively, in the units on polynomials, geometry, and measurement.

Appropriate Calculator Use

• In Class IX, the calculator should be used as a regular tool. Just like a pencil, a calculator is a mathematical tool. At this point in their mathematical education, students are no longer asked simply to perform routine calculations. Calculations are now part of more sophisticated mathematical tasks that are the real focus of their learning.

• Because calculators can change from year to year and because not all students have the same calculator, it is your responsibility to teach students how to use their calculators correctly. There may be some mention of general calculator skills in the text or guide, but the specifics vary from calculator to calculator.

THE STUDENT NOTEBOOK

It is valuable for students to have a well-organized, neat notebook to look back at to review the main mathematical ideas they have learned. However, it is also important for students to feel comfortable doing rough work in that notebook or doing rough work elsewhere without having to waste time copying it neatly into their notebooks. In addition to the things you tell students to include in their notebooks, they should be allowed to make some of their own decisions about what to include in their notebooks.

Students should be allowed to make some of their own decisions about what to include in their notebooks.

CLASS IX CURRICULUM

STRAND A: NUMBER

KSO Number *By the end of **Class 10** students should*
♦ *demonstrate an understanding of the real number system through appropriate application of concepts and procedures related to real numbers.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-A1 Large and Small Numbers: scientific notation to standard form and vice versa
• translate numbers from one form to another
• relate small numbers to large numbers to see the difference between the two in scientific notation form
• recognize situations where scientific notation is useful

9-A2 Square Roots: solve problems
• determine if the solution to a problem involves both values of the square root or just the principal square root

9-A3 Square Roots: approximate
• develop an awareness that square roots are often irrational
• understand that appropriate approximations in some situations are beneficial

9-A4 Integers and Real Numbers: write solution sets for equations and inequalities
• relate the language of inequality to the symbols of inequality
• graph, when given a set notation, and produce the set notation, when given a graph

9-A5 Irrational Numbers: demonstrate and understand meaning
• place irrational numbers on a number line relative to known rational numbers

9-A6 Real Numbers: interrelationships of subsets
• determine and justify if a given number is rational or irrational
• give examples of rational and irrational numbers

STRAND B: OPERATIONS

KSO Operations *By the end of **Class 10** students should*
♦ *derive, analyse, and apply computational procedures in situations involving all representations of real numbers,*
♦ *derive, analyse, and apply algebraic procedures in problem situation, and*
♦ *recognize and use the relationship between algebraic and arithmetic operations to solve problems.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-B1: Exponent Laws: integral exponents
• understand and apply the following exponent laws:

$a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^n b^n$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$; $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$

9-B2 Scientific Notation: model, solve, and create problems
• solve problems involving addition, subtraction, multiplication and division with numbers in scientific notation
• apply the laws of exponents to numbers written in scientific notation

9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation
• use estimation skills when calculating with rational numbers

9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form
• use mental computation, whenever appropriate, when solving problems

9-B5 Order of Operations: rational number computation
• apply knowledge of order of operations conventions with rational numbers

9-B6 Polynomial Expressions: interpreting
• consolidate an understanding of what polynomials are and when they are used

9-B7 Polynomial Expressions: add and subtract concretely, pictorially, and symbolically
• add and subtract polynomials concretely, pictorially, and symbolically

9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically
• multiply concretely, pictorially, and symbolically: a monomial by a monomial, a scalar by a monomial, a scalar by a polynomial, a monomial by a polynomial, and a binomial by a binomial
• divide concretely, pictorially, and symbolically: a monomial by a monomial; a polynomial by a scalar; a polynomial by a monomial, and a polynomial by a binomial

9-B9 Polynomial Factors: dimensions of a rectangle
• factor quadratic binomials and trinomials concretely

9-B10 Polynomial Expressions: evaluate
• compare the value of a polynomial prior to and after being simplified

9-B11 Income, Taxes, and Deductions: estimate and calculate
• estimate and calculate various types of income
• estimate and calculate taxes on income
• estimate and calculate income deductions

9-B12 Budgets: solve problems
• solve problems relating to personal budgets

STRAND C: PATTERNS AND RELATIONSHIPS

KSO Patterns and Relationships *By the end of **Class 10** students should*
♦ *model real-world problems using functions, equations, inequalities, and discrete structure,*
♦ *represent functional relationships in multiple ways and describe connections among those representations,*
♦ *perform operations on and between functions,*
♦ *analyse and explain the behaviors, transformations, and general properties of types of equations and relations, and*
♦ *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-C1 Patterns and Relationships: determine non-algebraic representations
• describe verbally and symbolically, patterns given in tables, charts, graphs, pictures, and /or by problem situations
• use models such as tables, graphs, and symbolic statements to assist in examining patterns and relationships
• relate the data in a table representing a linear, quadratic, or exponential relationship to its graph
• use first and second differences to determine if a table represents a linear, quadratic, or exponential relationship
• explore linear, exponential, and quadratic curves
• determine if a table represents a linear relationship by plotting the points

9-C2 Scatter Plots: characteristics of relationships
• consider whether data represented by a scatter plot are continuous or discrete and whether interpolation is meaningful
• distinguish between independent and dependent variables represented in a scatter plot

9-C3 Graphs of Linear Relations: interpret and create
• use the term slope to represent rise/run
• relate the y-intercept to the value of the y-coordinate where the graph crosses the y-axis
• relate the x-intercept to the value of the x-coordinate where the graph crosses the y-axis
• determine the slope and y-intercept by examining a table or graph
• sketch the graph of a linear relation given the slope and y-intercept
• sketch the graph of a linear relation given in standard form

9-C4 Equation of a Line: use graph to determine equation
• determine the equation of a line ($y = \mathrm{m}x + \mathrm{b}$) given the slope (m) and y-intercept (b)
• determine the equation of a linear relationship by calculating the slope and the y-intercept from the graph

9-C5 Lines of Best Fit: sketch and determine equations
• use the eyeball method to draw the line of best fit and then use the slope and y-intercept to determine the equation of the line
• understand that the line of best fit is drawn to show a relationship between two variables
• recognizes the relationship between both the dispersion around the line of best fit and the slope of the line of best fit and a description of the correlation between the variables

9-C6 Single Variable Equations: solve algebraically and graphically
• solve equations algebraically
• solve problems involving equations with coefficients that may be integers or rational numbers

9-C7 Two Linear Equations: find solutions to a problem by graphing
• solve problems by graphing pairs of linear equations

9-C8 Inequalities: solve and verify
• solve single variable linear inequalities

STRAND D: MEASUREMENT

KSO Measurement *By the end of* ***Class 10*** *students should*
♦ *measure quantities indirectly, using techniques of algebra, geometry, and trigonometry,*
♦ *determine measurements in a wide variety of problem situations, and consider accuracy and precision, and*
♦ *apply measurement formulas and procedures in a wide variety of contexts.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders
• estimate and calculate the volume of prisms and cylinders
• estimate and calculate the surface area of prisms and cylinders
[memorization of formulas is not intended at this level]

9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres
• estimate and calculate the volume of pyramids, cones, and spheres
• estimate and calculate the surface area of pyramids, cones, and spheres
• solve problems that involve finding the dimensions of a shape when the volume is given
• solve problems that involve finding the dimensions of a shape when the surface area is given
[memorization of formulas is not intended at this level]

9-D3 Volume and Surface Area: estimate and calculate for composite 3-D shapes
• estimate and calculate the volume and surface area of a variety of composite shapes

9-D4 SI Units: solve measurement problems involving conversion
• apply prior measurement skills
• solve problems involving mass and capacity units, as well as linear, area, and volume units

STRAND E: GEOMETRY

KSO Geometry *By the end of* ***Class 10*** *students should*
♦ *make and test conjectures about, and deduce properties of and relationships between, 2- and 3-dimensional shapes in multiple contexts,*
♦ *analyse and apply Euclidean transformations, including representing and applying translations,*
♦ *represent problem situations with geometric models and apply properties of shapes, and*
♦ *demonstrate an understanding of reasoning, justification, and proof.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-E1 Congruent Triangles: properties and minimum sufficient conditions
• understand, through investigation, that if two triangles are congruent through: SSS, SAS, ASA, or AAS, then the other corresponding parts of the triangle are also congruent
• interpret and use the symbol $\cong$, which is read as "is congruent to"

9-E2 Unique Triangles: minimum sufficient conditions
• examine what pieces of information are needed to guarantee a unique triangle
• understand that the following are necessary in order to produce unique triangles: three sides; two sides and a contained angle; two angles and a contained side; two angles and a non-contained side
• discover that AAA and SSA do not result in a unique triangle

9-E3 Similar Triangles: understand and apply proportions
• understand that, in similar triangles, the ratios of side lengths of one triangle are equal to the ratio of the corresponding side lengths of the second triangle
• understand that the ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides

9-E4 Similar Triangles: apply properties
• understand properties of similar triangles: that the corresponding angles are congruent (AAA) and the corresponding sides are in proportion (SSS)
• understand that two triangles are similar when two pairs of corresponding sides are in proportion and the pair of included corresponding angles are congruent (SAS)
• understand that two triangles are also similar when two angles of one triangle are congruent to two corresponding angles of another triangle (AAA)

9-E5 Triangles: relate congruency and similarity
• compare and contrast congruence and similarity as they relate to triangles

9-E6 Transformations (mapping notation): represent and interpret
• apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation
• describe the nature of a transformation based on a given mapping

9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation
• understand, through hands-on investigation, properties of each transformation:

▪ *Translations*
• line segments joining points on the original shape to their images are parallel and equal in length
• translation image of any shape is congruent
• orientation of a translation image is the same as that of the original shape
• translation images of lines or line segments are parallel or collinear to the original lines and line segments

▪ *Reflections*
• line segments joining points on the original shape to their images are perpendicular to the reflection line and have their midpoint on the reflection line
• reflection image of any shape is congruent to the original shape
• orientation of a reflection image is the opposite of the original shape
[Reflections are limited to reflections in the axes and the line $x = y$.]

▪ *Rotations*
• for a rotation of $a°$ about point X, a line segment joining a point to X and a line segment joining its image to X are equal in length and form an angle of $a°$
• rotation image of any shape is congruent
• orientation of a rotation image is the same as that of the original shape
[Rotations are limited to 90 and 180 degrees, clockwise (cw) and counterclockwise (ccw), around (0, 0).]
• for 90° rotations, horizontal line segments become vertical, vertical line segments become horizontal, and any line segment and its image are perpendicular
• for 180° rotations, line segments are parallel or collinear to their images

▪ *Dilatations*
• dilatation centre (a point) and its image form a line
• ratio of the distance between the dilatation centre and a point on the shape, to the distance between the dilatation centre and a corresponding point on the image, is the same as the scale factor
• ratio of the length of a line segment in the original shape to the length of a corresponding line segment in the image is the same as the scale factor
• dilatation image of any shape is similar to the original shape
• angle measures in the original shape are the same as corresponding angle measures in the image
[Dilatations are limited to positive scale factors around centre (0, 0).]

9-E8 Transformations (mapping notation): analyse and represent composite transformations
• analyse a transformation given in mapping notation and represent a transformation using mapping notation
• identify the transformations when given the original shape, and the image after a combination of transformations

STRAND F: DATA MANAGEMENT

KSO Data Management *By the end of* ***Class 10*** *students should*
♦ *determine, interpret and apply as appropriate a wide variety of statistical measures and distribution and*
♦ *use curve fitting to determine the relationship between, and make predictions from, sets of data and be aware of bias in the interpretation of results.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-F1 Displaying Data: draw inferences and make predictions
• interpolate and extrapolate using a data set
• draw inferences and conclusions from a number of data displays, particularly scatter plots

9-F2 Displaying Data: most appropriate methods
• determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose

9-F3 Data Analysis: evaluate arguments and interpretations
• compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness
• examine how the choice of certain graphs can lead to errors in judgment

STRAND G: PROBABILITY

KSO Probability *By the end of* ***Class 10*** *students should*
♦ *represent and solve problems involving uncertainty,*
♦ *make predictions and carry out simulations to answer real world issues of interest, and*
♦ *determine theoretical probability for dependent and independent events and apply to real life issues.*

Toward this, students in **Class 9** will be expected to master the following **SO** (Specific Outcomes):

9-G1 Theoretical Probability: independent events
• determine the number of possible outcomes for independent events using outcome charts, organized lists, and tree diagrams
• calculate the probability of two independent events, A and B, as $P(A) \times P(B)$

9-G2 Simulations and Experiments: dependent and independent events
• distinguish between theoretical and experimental probability
• conduct and design simple simulations involving both dependent and independent events
• determine experimental probabilities for simulations

UNIT 1 NUMBER AND OPERATIONS

UNIT 1 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology, and pre-assessment	1 h	None	All questions
Chapter 1 Exponents				
1.1.1 Introducing the Exponent Laws	**9-B1: Exponent Laws: integral exponents** • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$	1 h	None	Q1, 3, 5, 9, 10
GAME: Rolling Powers	Apply and practice exponent laws $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$ in a game situation	20–30 min	• Dice	N/A
1.1.2 The Power Law of Exponents	**9-B1: Exponent Laws: integral exponents** • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^nb^n$; $(a^m)^n = a^{mn}$	1 h	None	Q1, 3, 6, 9
1.1.3 Negative and Zero Exponents	**9-B1: Exponent Laws: integral exponents** • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^nb^n$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$	1 h	None	Q1, 4, 7, 9, 10
1.1.4 Fractional Exponents	**9-B1: Exponent Laws: integral exponents** • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^nb^n$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$; $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$	1 h	None	Q1, 3, 5, 7

UNIT 1 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Scientific Notation				
1.2.1 Scientific Notation with Large Numbers	**9-A1 Large and Small Numbers: scientific notation to standard form and vice versa** • translate numbers from one form to another • recognize situations where scientific notation is useful **9-B2 Scientific Notation: model, solve, and create problems** • solve problems involving addition, subtraction, multiplication and division with numbers in scientific notation • apply the laws of exponents to numbers written in scientific notation **9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	2 h	• Calculators	Q1, 4a, 5, 7, 11
CONNECTIONS: The Richter Scale	Explore a real world application of scientific notation	15 min	None	N/A
1.2.2 Scientific Notation with Small Numbers	**9-A1 Large and Small Numbers: scientific notation to standard form and vice versa** • translate numbers from one form to another • relate small numbers to large numbers to see the difference between the two in scientific notation form • recognize situations where scientific notation is useful **9-B2 Scientific Notation: model, solve, and create problems** • solve problems involving addition, subtraction, multiplication and division with numbers in scientific notation • apply the laws of exponents to numbers written in scientific notation	2 h	• Calculators	Q1, 3, 5, 7, 9

	9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems			
Chapter 3 Rational and Real Numbers				
1.3.1 Estimation with Rational Numbers	**9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	1 h	• Calculators	Q1, 4, 5, 6
1.3.2 Order of Operations	**9-B5 Order of Operations: rational number computation** • apply knowledge of order of operations conventions with rational numbers **9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	1 h	• Calculators	Q1, 2, 4, 8
1.3.3 Square Roots	**9-A2 Square Roots: solve problems** • determine if the solution to a problem involves both values of the square root or just the principal square root **9-A3 Square Roots: approximate** • develop an awareness that square roots are often irrational • understand that appropriate approximations in some situations are beneficial	1 h	• Calculators	Q1, 4, 7, 10, 11, 12

UNIT 1 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 3 Rational and Real Numbers [Continued]				
1.3.4 EXPLORE: Representing Square Roots (optional)	**9-A3 Square Roots: approximate** • develop an awareness that square roots are often irrational • understand that appropriate approximations in some situations are beneficial	1 h	• Rulers (cm) • Protractors	Observe and Assess questions
1.3.5 Representing Real Numbers	**9-A4 Integers and Real Numbers: write solution sets for equations and inequalities** • relate the language of inequality to the symbols of inequality • graph, when given a set notation, and produce the set notation, when given a graph **9-A5 Irrational Numbers: demonstrate and understand meaning** • place irrational numbers on a number line relative to known rational numbers **9-A6 Real Numbers: interrelationships of subsets** • determine and justify if a given number is rational or irrational • give examples of rational and irrational numbers	1 h	None	Q1, 3, 4, 7
UNIT 1 Revision	Review the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 1 Test	Assess the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 1 Performance Task	Assess the concepts and skills in the unit	1 h	• Paper or light cardboard for cards • Calculators	Rubric provided

Math Background

• This number unit is a way to gently move students into Class IX with some familiar and new ideas.

• The focus of the unit is on extending students' familiarity with number systems, from integers and fractions to real numbers. This is accomplished by focusing on exponential form, first with positive powers of whole numbers and then with negative powers (leading to fractions), fractional powers of whole numbers, fractions and decimals, and finally a description of real numbers as the union of rational and irrational numbers.

• The exponent laws developed in the unit, the techniques for graphing inequalities, and the work with radicals will serve students well in further work in mathematics.

• The work on scientific notation will help students understand numbers they encounter in science and other disciplines.

• As students proceed through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections. For example:

• Students use problem solving in **question 6** in **lesson 1.1.3**, where they might use an organized list to consider all possibilities in order to solve a problem, **question 9** in **lesson 1.2.1**, where they compare heart rates of people and birds, and **question 7** in **lesson 1.2.2,** where they explore human growth rates.

• They use communication frequently as they explain their thinking in answering questions, e.g., **question 10** in **lesson 1.1.1,** where they explain one of the exponent laws**, question 7** in **lesson 1.3.2**, where they describe the effects of brackets, and **question 8** in **lesson 1.3.2,** where they explain why a particular calculation is easy to perform. As well, the last question in most lessons usually requires an element of communication in bringing closure to the lesson.

• They use reasoning in answering questions such as **question 3** of **lesson 1.1.2**, where they make sense of exponential notation to order various representations of powers, **questions 4 and 7** in **lesson 1.3.3**, where they reason why different square roots are rational or irrational, and **question 9** in lesson **1.1.2**, where they reason that the representation of a product of factors with even powers must be a perfect square.

• They consider representation in **question 9** in **lesson 1.1.2,** where they see the value of representing a number in its factored form to determine its square root, **question 1** in **lesson 1.2.1,** where they represent a number in standard form in scientific notation, and when they graph inequalities in **lesson 1.3.5** on number lines.

• Students use visualization skills in the **Getting Started** activity where they visualize the pattern in the ones digits for the powers of various numbers.

• They make connections in situations like those in **question 9** in **lesson 1.1.3,** where they look at two different approaches for calculating a negative power of a fraction, **questions 9** and **12** in **lesson 1.2.1,** where they apply scientific notation to heart rates and population, and in **lesson 1.3.3,** where they work with physical and numerical representations of square roots.

Rationale for Teaching Approach

• This unit is divided into three chapters. The first chapter introduces the exponent laws along with zero, negative, and fractional exponents. The second chapter focuses on scientific notation. The third chapter examines rational, irrational, and real numbers, and explores how to graph inequalities.

• Students use patterns and reasoning initially to see how zero and negative exponents, and then fractional exponents should be defined.

• They apply the exponent laws developed in **Chapter 1** to calculations with numbers in scientific notation in **Chapter 2**.

• Rational numbers are addressed in **Chapter 3**. Irrational numbers are then described as those decimals that do not represent rational numbers. The connection between square roots, introduced in terms of fractional exponents in the first chapter, and rational and irrational numbers is part of **Chapter 3**. It is only once real numbers have been considered that inequalities are graphed.

• The **Explore** section allows students to see physical representations of some irrational numbers. The **Connections** section helps them see some real world uses of scientific notation.

• The **Game** in **Chapter 1** provides an opportunity to practice work with exponents in a pleasant way.

• Throughout the unit, it is important to encourage flexibility in computation and to accept a variety of approaches from students. Being efficient and not recording every step should be welcomed and not discouraged.

Technology in This Unit

There is substantial use of calculators in this unit, particularly for working with scientific notation and for calculating square roots. A regular scientific calculator is all that is required.

Getting Started

Curriculum Outcomes	Outcome relevance
7 Large Numbers: model **8** Square Roots: modelling and representing **8** Negative Exponents: develop meaning concretely and symbolically **8** Perfect Squares: pattern between 1 and 144 **8** Square Roots: exact square root and its decimal approximation **8** Square Roots: find using an appropriate number	Students will find the work in the unit easier after they review the meanings of positive exponents, negative powers of ten, squares, and square roots.

Pacing	Materials	Prerequisites
1 h	None	• meaning of exponent • meaning of square root, both the symbol and the concept • meaning of perfect squares

Main Points to be Raised

• Powers involve repeated multiplication by a fixed amount, namely, the base.

• Recognition of patterns and application of number sense permit greater flexibility with calculations. The ones digit can be a "check digit."

• Relatively few numbers are perfect squares and these numbers are farther apart as numbers increase.

• Perfect squares have special qualities that can be represented visually as areas of squares with sides of integer length.

• Any square root can be represented as the side length of a square with a given area.

Use What You Know—Introducing the Unit

• Students can work alone or in pairs to complete the **Use What You Know** activity. If they wish, students could sketch the circle shown in the text once and use different colours for different bases. Or, they could sketch several circles.

• If students have forgotten, take the time to re-define exponentiation as repeated multiplication.

• Number sense should be stressed as the different cyclic patterns are brought forth. For example, the powers of two are only even numbers; powers of five and six always have the same ones digit, and so on, as explained later.

Observe students as they work. You might ask:

• *How do you know when the picture of the pattern is complete?* (When I get back to a digit again, then it will repeat itself. So the picture is finished.)

• *Why do only odd numbers appear on your pattern for powers of three?* (Three is an odd number. If you are always multiplying an odd number by 3, which is also odd, the product will always be odd.)

• *Would it be easier to find the ones digit for 3^{30} or 6^{30}? Why*? (The ones digit for 6^{30} is easier to find because 6×6 ends in 6 and so any power of six ends in 6. The ones digits for all powers with a base of 3 change and I would have to find a pattern first to see what it would be when the exponent is 30.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

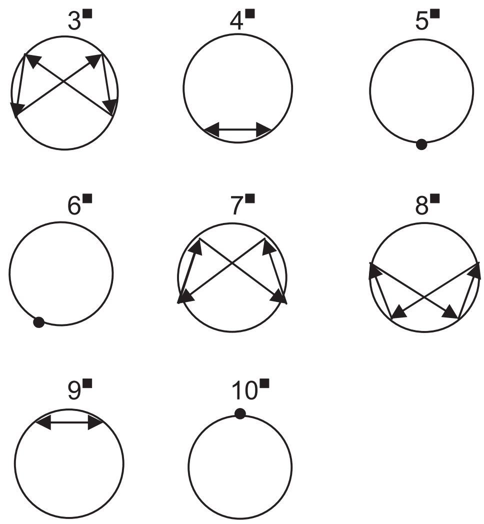
• Students can work individually.

Answers

A. i) *Sample response*: a butterfly or a symmetrical cross-cross pattern

A. ii) It would keep repeating the same zig-zag pattern because the ones digit of the powers of two repeats: 2, 4, 8, 6, over and over.

B. i), ii), iii)



B. iv) *Sample responses*:
- only one possible digit for powers of 5, 6, and 10, so there is only a dot
- only two possible digits for powers of 4 and 9, so there is only one two-way arrow
- same pattern for 3 and 7 as well as for 2 and 8 (although in reverse order each time)

v) *Sample response*:
Since 5^2 has the same ones digit as 5^1, 6^2 as 6^1, and 10^2 as 10^1, I could have predicted the patterns that are just dots.

C. I could use patterns to figure out what the 20th number in the pattern would be. If there are, for example, four numbers in the pattern, as with powers of 2, 3, 7, and 8, the 20th power would end in the last digit in the group of four. So for powers of three, the numbers 3, 9, 7, 1 repeat, so the 20th number would end in 1.

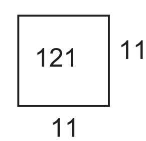
1. a) 0.001 **b)** 0.05

c) 300 **d)** 0.4

2. a) –8 **b)** –16 **c)** 9

3. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289

4. *Sample response*:



5. 12 and 13

6. *Sample response:*
I know the last digit of 1.414 × 1.414 must be 6 because 4 × 4 = 16

Supporting Students

Struggling students

• If a student is struggling with circle patterns, do not require all nine of the suggested patterns. Focus on only the patterns for 3, 5, and 8, for example.

• It may be necessary to reinforce work with perfect squares, square roots, and recognition of factors that will be needed in the unit.

• Ensure all students have basic familiarity with the powers of ten and their role in our place value system now, before further problems arise with powers and scientific notation involving powers of ten.

Enrichment

Use the circle patterns to predict the ones digit for the various bases with large exponents, such as 2007.

Chapter 1 Exponents

1.1.1 Introducing the Exponent Laws

Curriculum Outcomes	Outcome relevance
9-B1: Exponent Laws: integral exponents • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$	Applications of the exponent laws arise in various mathematical contexts that will be revisited in secondary school. Understanding the principles associated with the laws will facilitate the transition into work with radical expressions, more complicated exponents, and applications such as exponential growth.

Pacing	Materials	Prerequisites
1 h	None	• fractions as implied division • fraction simplification • powers of small whole numbers

Main Points to be Raised

• When two powers of the same base are multiplied, the product is a power of the same base. The resulting exponent is the sum of the original two exponents. (This idea can be extended to more than two powers with products.) This is called the product law.

• When two powers of the same base are divided, the quotient is a power of the same base. The resulting exponent is the difference of the original two exponents. This is called the quotient law.

Try This—Introducing the Lesson

A. Allow students to work alone or with a partner. Observe while students work. You might ask:

• *How do you know that your answer is correct?* (If I split the nine 2s in the product into two parts and multiply them together, all of the 2s are included.)

• *Do you think there are other pairs of values that would work?* (Yes. Any pair of numbers that add to 9 would work. A pair could be 1 and 8, or 2 and 7, or 3 and 6, or 4 and 5.)

• *How would you answer the question if three powers of two were required to make the product?* (I would need three numbers that add to 9. I could make them each equal to 3. Or, I could make them 2, 3, and 4.)

The Exposition—Presenting the Main Ideas

• Write the expression 2^5 on the board. Ask students what it represents. Point out which part is the base and which part is the exponent and tell them that the whole thing is called the *fifth power of 2*. Write out the product $2 \times 2 \times 2 \times 2 \times 2$. Ask them how many 2s there would be if you multiplied the expression by 2^3 (make sure they see why it is 2^8) and what the quotient would be if you then divided by 2^5 (make sure they see why it is 2^3). Ask why one might record what they suggested as $2^5 \times 2^3 = 2^8$ and $2^8 \div 2^5 = 2^3$. Help students see how these ideas can be generalized to the product and quotient laws.

• Ask students to read through the exposition to consolidate the work you have done together. Bring attention to the terms **product law** and **quotient law.** You may wish to display the laws in the classroom.

Revisiting the Try This

B. This question provides an opportunity to reinforce the language of the exponent laws. It may be handled as a class, although it would be helpful to have students begin by answering the question orally in pairs. Speaking and hearing the terminology supports the development of understanding and the retention of the concepts.

Using the Examples

Write the questions from both examples on the board. Ask students to try to solve them independently. Then have them compare their solutions with those in the text. Make sure they understand that the right-hand side (**Thinking**) describes what went through a sample student's mind as he or she solved the problem and the left-hand side (**Solution**) is what he or she would write down on paper.

Practising and Applying

Teaching points and tips

Q 1 and 2: Students might write out the expressions in long form to understand the calculations and appreciate the significance of the product and quotient laws.

Q 3: Students who have any difficulties with **parts a) and b)** could omit the subsequent parts until the power law has been introduced in **lesson 1.1.2**. Note that **question 6** is related to this question.

Q 6: Students may need help in verifying equivalence without completing a calculation. You can change the exponents to 40 and 20 rather than 4 and 2 in order to force them to think about the underlying reasons rather than calculating the quantities.

Q 10: You may wish to have students provide an answer to this question orally and/or in writing. Some students may find one method easier than the other.

Common errors

Students will often change the base when they only need to change the exponents in calculations using the exponent laws. Some will find ideas related to the power law idea, such as **questions 3 c) to e)** and **question 6** to be a bit advanced at this point. Feel free to emphasize those ideas after doing some work in **lesson 1.1.2**.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can apply the exponent rules directly
Question 3	to see if students can represent a power as products and quotients in multiple ways
Question 5	to see if students can use exponentiation in an application situation
Question 9	to see if students can explain the product law when a suitable example is given
Question 10	to see if students can explain the quotient law and provide a suitable example

Answers

A. i) *Sample response:* 3, 6 and 4, 5 **ii)** *Sample response:* 1, 8 **iii)** they add to 9	**B.** product law since you are multiplying powers with the same base and not dividing them
1. a) 6 **b)** 15 **c)** 7 **2.** A, C, and D **3. a)** *Sample response:* $5^4 \times 5^4$ **b)** *Sample response:* $5^{10} \div 5^2$ **c)** 25^4 **d)** *Sample response:* $25^2 \times 25^2$ **e)** *Sample response:* $25^{10} \div 25^6$	**4. a)** 2^6 **b)** *Sample response:* $2^4 \times 2^2$ **c)** *Sample response:* $2^2 \times 2^2 \times 2^2$ **5. a)** 2^1 **b)** 2^2 **c)** 2^7 **6.** $3^4 = 3 \times 3 \times 3 \times 3$ and $9^2 = 9 \times 9$, since $9 = 3 \times 3$, then $(3 \times 3) \times (3 \times 3) = 9 \times 9$

Answers [Continued]

7. 7^2, since $7^2 \times 7^2 = 7^4$ **8. a)** No, since $a + b = 11$ and you cannot add two even numbers and get 11. **b)** 5 pairs (each in either order): 1 and 10, 2 and 9, 3 and 8, 4 and 7, 5 and 6	**9.** Add the exponents (4, 7, and 19) to get 5^{30}, because the base is 5 in each power. **10.** When the bases are the same, you can subtract the exponent of the divisor from the exponent of the dividend; *Sample response:* $5^6 \div 5^4 = 5^{6-4} = 5^2$

Supporting Students

Struggling students

• Allow struggling students to use more examples with small exponents so that the multiplications they represent can be written out in full.

• Rephrase **questions 7 and 8** in a less symbolic way and allow students to respond more informally. For example, restate **question 7** like this:
A number is multiplied by itself and the answer is 7^4. How do you know it is not 7^1? What could it be?

Enrichment

You could encourage students to generalize the results of questions. They could also increase the powers to large numbers and find suitable alternative sets of numbers that work in various situations corresponding to the questions posed.

GAME Rolling Powers

Some variations of the game are suggested below.

• It will be easier to predict how long the game will take if students do a fixed number of rolls. For example, each player may be given eight rolls rather than being required to acquire a specific number of points. The players could record the rolls and continue with the game individually. Then their scores would be totalled and compared upon completion. If time permits, they could take turns and observe one another.

• Another option is to roll three dice together and allow the player to select which die represents the base. Suppose that the dice land on 2, 3, and 6. The player may choose 2 as the base. This is worth 2 points because the exponent is greater than 8 and the power can be expressed as a power of 2. If the player selects a base of 3, only 1 point is scored for a power with a whole number square root. If 6 is selected as the base, 0 points are awarded.

• The number of dice could be increased to four in any of the versions of the game. The exponents would tend to become greater and you may decide to raise the exponent required to earn one point to some higher amount such as 11. The rules could also be changed to award points for other results. The game could be played again later on as a form of revision.

1.1.2 The Power Law of Exponents

Curriculum Outcomes	Outcome relevance
9-B1: Exponent Laws: integral exponents • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^n b^n$; $(a^m)^n = a^{mn}$	Students can use the power law as an efficient extension of the product and quotient laws. It is particularly valuable for converting and comparing numbers that are powers of a common base but appear initially to have different bases.

Pacing	Materials	Prerequisites
1 h	None	• terminology: base, power, and exponent • product and quotient exponent laws

Main Points to be Raised

• The power law can be thought of as a way of describing a repeated product. It is therefore a special case of the product law. For example:
$(5^4)^3 = 5^4 \times 5^4 \times 5^4 = 5^{4+4+4} = 5^{12}$
The power law allows you to directly represent the value as $5^{4 \times 3} = 5^{12}$.

• The power of a product law and the power of a quotient law can be used to simplify rational number calculations.

• The three laws introduced in this lesson can also be used in reverse to suggest alternative representations of powers.

Try This—Introducing the Lesson

A. Allow students to work alone or in pairs. Observe while students work. You might ask:

• *What is similar in each of the expressions?* (The same power is multiplied by itself.)

• *How is the product law related to the answers you got in* ***part A****?* (The product law is being used. It is just that the exponents stay the same as well as the base. If the exponents were different, I would add them. Since they are the same, I can multiply them by the number of powers in the expression.)

The Exposition—Presenting the Main Ideas

• Write out $(5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5)$ on the board. Ask students how to express each of the three factors as a power (5^4). Then ask them to represent the final product as a power (5^{12}). Finally, ask why the expression can be represented as either $5^4 \times 5^4 \times 5^4 = 5^{12}$ or as $(5^4)^3 = 5^{12}$. Ask the students why the 12 is represented as the sum 4 + 4 + 4 in the first situation and as the product 3 × 4 in the second situation.

• Next, write $(2 \times 3)^5$ on the board. Ask why it could be written as $(2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3)$. Then rewrite the expression grouping the twos and threes together, as $(2 \times 2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3 \times 3)$. Ask students to write the altered expression as the product of two powers ($2^5 \times 3^5$). Ask how they could have predicted that the factors could be separated, with the power applied to each.

• Finally, write the expression $(15 \div 3)^4$. Ask students to talk in pairs to consider why it could be written either as $15^4 \div 3^4$ or as 5^4. Have them offer suggestions for discussion.

• Lead students through the exposition. Display the power law, the power of a product law, and the power of a quotient law in the classroom. Make sure students understand why $\frac{2}{3}$ is a representation of 2 ÷ 3.

Revisiting the Try This

B. Most students will view this as an example of the power law. Some students might cite the product law, which is also correct.

Using the Examples

• Write the exercises from **Example 1** on the board. Encourage students to try them with their books closed. Encourage them to express their comparisons both orally and in written form. Students can check their work against the thinking and the solution in the text for **Example 1**.

• Ask students to read through **Example 2** independently. Answer any questions they might have about the calculations in the example. Make sure that they realize that they could write out an expression like the one in **part a)** in long form to see that $n = 5$.

Practising and Applying

Teaching points and tips

Q 1: Make sure that the students can do this question before proceeding with the other questions.

Q 3: Some students may wish to compare the quantities without using a calculator. Others may need a calculator. Students who depend entirely on the calculator for comparison could be encouraged to use exponents that are 10 times those shown.

Q 4: Some students might use $n = 1$ in both cases. Ask them to find another value of n (and hence, m) for which the statements are true.

Q 9: See if students connect the explanation here to their work with **question 8**.

Q 10: This question is well-suited to general review and discussion with the class.

Common errors

Students will often

• add exponents, as they did with the product law, rather than multiplying them as intended with the power law. It is helpful to write out the expression in full to see the difference between $(2^5)^4$ and $2^5 \times 2^4$, for example.

• mistake quantities such as -2^3 as $(-2)^3$ instead of the $-(2^3)$ it represents.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can directly apply the power law in a simple situation
Question 3	to see if students can represent different quantities with a single base for comparison
Question 6	to see if students can apply each of the laws they learned about in the lesson
Question 9	to see if students can recognize conditions for using exponent laws to simplify determination of square roots

Answers

A. i) 2^{20} **ii)** 3^{12} **iii)** 4^{10} **B.** You can add the exponents because of the product law. However, since both the bases and the exponents remain the same, you can keep the base and get the exponent by multiplying it by the number of times it appears: $2^{5+5+5+5}$ is $2^{4 \times 5}$	**C.** The power law could have been used because $2^5 \times 2^5 \times 2^5 \times 2^5 = (2^5)^4 = 2^{20}$.
1. a) 5^{15} **b)** 13^{36} **c)** 9^{25} **d)** 2^{21} **2. a)** $b = 6$ **b)** $b = 12$ **c)** $b = 9$ **3.** -2^{30}, $(-2)^8$, $(2^7)^2$, 8^5, $(2^4)^4$, $(4^3)^3$ **4. a)** *Sample response:* $m = 3$ and $n = 1$ **b)** *Sample response:* $m = 3$ and $n = 2$ **5.** $n = 2$	**6. a)** *Sample response:* $(8^2)^5$ **b)** *Sample response:* $(2 \times 4)^{10}$ **c)** *Sample response:* $(16 \div 2)^{10}$ **7.** 10^6 **8. a)** 10^2; $5^2 \times 2^2$ **b)** 12^2; *Sample response:* $4^2 \times 3^2$ **c)** 40^2; *Sample response:* $8^2 \times 5^2$

9. *n* must be even since you have to be able to divide it by 2 using the power law to create a perfect square; *Sample response:* $2^6 \times 3^6 \times 4^6 = (2 \times 3 \times 4)^6 = (24)^6 = (24^3)^2$, making $2^6 \times 3^6 \times 4^6$ a perfect square because $24^3 \times 24^3 = 2^6 \times 3^6 \times 4^6$	**10.** *Sample response:* - power law: $5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{4 \times 3} = 5^{12}$ - product law: $5^8 \times 2^8 = 10^8$ - quotient law: $20^8 \div 2^8 = 10^8$

Supporting Students

Struggling students

• Use smaller numbers and write out expressions in full, if necessary. Emphasis should be placed on the concepts rather than on doing many questions. Reinforce the connection between the product law and the power law.

• Questions that involve solving for *n* can be postponed. Also, comparisons of multiple quantities could be reduced to two quantities at the beginning. Multiple quantities can be included later.

Enrichment

Ask students to complete calculations involving powers of different, but related, bases. For example, they could try to simplify expressions like $3^4 \times 9^3$ and compare them to other expressions like $(3^2)^5 \div 9^2$.

1.1.3 Negative and Zero Exponents

Curriculum Outcomes	Outcome relevance
9-B1: Exponent Laws: integral exponents • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^n b^n$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$	Introducing students to zero and negative integer exponents will allow them to extend their proficiency with integer calculations involving powers to calculations involving rational numbers.

Pacing	Materials	Prerequisites
1 h	None	• positive integer powers

Main Points to be Raised

• $a^0 = 1$ as long as $a \neq 0$. The condition is required because there would otherwise be ambiguity about what to call 0^0. Since every other $a^0 = 1$, a case could be made to define 0^0 as 1. This inconsistency is one reason for not giving 0^0 a value. Another is that a^0 could be thought of as, for example, $a^5 \div a^5$, but it is not possible to divide by 0 so 0^0 is undefined.

• The quotient law produces negative exponents when the exponent of a base in the denominator is greater than the exponent of the same base in the numerator.

• A negative exponent can be interpreted as a reciprocal of the same quantity with a positive exponent. For example:

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 \qquad 2^{-5} = (\frac{1}{2})^5$$

• The idea of expressing a whole number, n, as a fraction of the form $\frac{n}{1}$ can be helpful in work with negative exponents.

Try This—Introducing the Lesson

A. Allow students to work in pairs. Observe while students work. You might ask:

• *How do you know that Karma will probably not win?* (I know that both 2^9 and 9^2 are quite large and not very close to 1.)

• *Why do you think Tshewang gets a point?* (He gets a point because $1^7 = 1$. Any power of 1 would get points.)

The Exposition—Presenting the Main Ideas

• Begin by writing this expression on the board: $\frac{7^{10}}{7^{10}}$. Ask students how the quotient could be expressed using a power ($7^{10-10} = 7^0$). Ask why the value also has to be 1. (a number is divided by itself.) Discuss why bases other than 7 could have been used to lead to the same result (a power divided by itself is the same as a number raised to the 0 power, but is also a name for 1). Ask why 0 is not be a good base to use (division by 0 is not allowed).

• Then display the expression: $\frac{7^{10}}{7^{12}}$. Again, ask how the value could be represented as a single power, using the quotient law, or why it could also be expressed as $\frac{1}{7^2}$. Using similar examples with the same base, but a greater power in the denominator than the numerator, help students see that the result will always be the reciprocal of a positive power of the base.

• Ask students to write a fraction to represent, for example, 3^{-1}. If necessary, present $\frac{3^5}{3^6}$ as a hint. Make sure that they can see that 3 raised to any negative power is a positive power of 3^{-1}, for example, $3^{-4} = (3^{-1})^4$.

• Display the table of decreasing powers of 2 from the main idea section. Help students see the pattern in the table.

• Suggest to students that they can use the main idea section as a reference as they work through the examples and the exercises.

Revisiting the Try This

B. For most students, the only new idea is that Dodo's roll of 0 produced a point. It would be worth asking if all rolls of 0 would produce points. (A roll of 0 would produce a point provided that both rolls are not 0.)

Using the Examples

Write the questions in the example on the board. Encourage students to try them on their own and then to check their results by reading the example in the text.

Practising and Applying

Teaching points and tips

Q 3: Some students might find this problem to be too difficult, as it involves changing a base. This problem could be omitted or postponed.

Q 4: Students should be aware that multiple answers are possible. You may encourage students to compare answers and/or to seek answers other than those they initially wrote down.

Q 6: You may decide to do an example of this question with a smaller set of numbers, if necessary.

Q 7: Place the emphasis on reasoning so that the results are not simply calculated and compared.

Q 9: This question is designed to show that another way of calculating a negative power of a fraction is to calculate the positive power of its reciprocal.

Common errors

• Students will often confuse $a^0 = 1$ with $0^a = 1$. They might benefit by having to write out the full power, for example, $0^4 = 0 \times 0 \times 0 \times 0$, to help see why they are incorrect.

• A second common error is to apply laws to bases that are unequal. Again, it might be helpful to have the student write out the full expression, for example, $2^3 \times 3^4 \neq 5^7$, since $2^3 \times 3^4 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$ and there are no 5s involved.

Students might find it confusing that negative exponents do not always produce negative results and that negative exponents can make quantities greater, for example, $(\frac{2}{5})^{-1} = \frac{5}{2}$.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can do straightforward calculations involving non-positive integer exponents
Question 4	to see if students can provide examples of suitable powers to meet certain conditions
Question 7	to see if students can use reasoning to apply positive and negative exponents
Question 9	to ensure that students can describe an alternative way of calculating a negative power of a fraction
Question 10	to see if students can communicate their understanding of the core concept of the lesson concerning an exponent of 0

Answers

A. Karma: 2^9 or 9^2 Dodo: 3^0 or 0^3 Tshewang: 7^1 or 1^7 **B.** *Sample response:* I think Karma and Dodo will not get points this round.	**C.** *Sample response:* Yes. I think that only Karma will not get a point this round. Because $3^0 = 1$ and $1^7 = 1$, Dodo and Tshewang will also get a point.
1. a) 1 **b)** $\frac{1}{25}$ **c)** $\frac{1}{25}$ **d)** 1 **2. a)** $\frac{1}{9}$ and $\frac{1}{9}$ **b)** 9 and 9 **c)** Since $9 = 3^2$, whatever power 9 is raised to should be doubled if 3 is raised to that power.	**3.** 9 **4. a)** *Sample response:* $a = 3$, $b = -3$ **b)** *Sample response:* $a = 3$, $b = -2$ **c)** *Sample response:* $a = 2$, $b = 49$ **d)** *Sample response:* $a = 0$, $b = 0$

Answers [Continued]

5. a) $\frac{1}{2} = 2^{-1}$ and $\frac{1}{3} = 3^{-1}$ **b)** $\frac{1}{6}$

c) The values are equal since $\frac{1}{2} \times \frac{1}{3} = 2^{-1} \times 3^{-1} = (2 \times 3)^{-1} = 6^{-1}$, using the power of a product law.

6. Five possibilities: 0, 1, 2, $\frac{1}{2}$, and $\frac{1}{4}$

7. a) 3^{-5} since it is a positive fraction and $(-5)^3$ is a negative number

b) $(-5)^4$ since it is the same as 5^4, which is a positive whole number but 4^{-5} is a fraction less than 1

c) $(-9)^2$ since it is the same as 9^2 which is 81, but -9^2 is -81

d) equal since they both represent the opposite of 9^3

8. a) if n is odd, the value is negative; if n is even, the value is positive

b) If the greater value is odd, the result is negative and therefore less; *sample response*: $(-6)^3 < (-6)^2$

9. $\left(\frac{3}{10}\right)^{-2} = \frac{1}{\left(\frac{3}{10}\right)^2} = 1 \div \frac{9}{100} = 1 \times \frac{100}{9} = \frac{100}{9}$ and $\frac{100}{9} = \left(\frac{10}{3}\right)^2$

10. *Sample response*:
A number divided by itself is equal to 1 and $a^b \div a^b = a^0$ so $a^0 = 1$. For example, $4^3 \div 4^3 = 1$, and $4^3 \div 4^3$ is also $4^{3-3} = 4^0$ so $4^0 = 1$

Supporting Students

Struggling students

Assist struggling students with the changing bases. For example, have them write the powers of 2: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, …. Then show how $4^5 = (2^2)^5 = 2^{10}$ or $16^3 = (2^4)^3 = 2^{12}$ or, going the other way, $2^{12} = (2^2)^6 = 4^6$. Give them additional practice with this idea using other simple bases instead of 2, such as 3 and 9. Help them recognize it is not new to use equivalent names for the same number. They have already done this with fractions, for example.

Enrichment

- Revisit **question 4**, asking students to find all possible values that work in each part.
- Encourage students to create a problem like **question 6** for which there are a different number of resulting values.

1.1.4 Fractional Exponents

Curriculum Outcomes	Outcome relevance
9-B1: Exponent Laws: integral exponents • understand and apply the following exponent laws: $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(\frac{a}{b})^n = \frac{a^n}{b^n}$; $(ab)^n = a^n b^n$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$; $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$	Although it is not essential that students become highly proficient with fractional exponents, it is important for them to have a sense of what they represent. Connecting fractional exponents to the notions of cube root and square root will also make simplification of these roots make more sense.

Pacing	Materials	Prerequisites
1 h	None	• integer exponents • meaning of square root • meaning of cube root • perfect squares and cubes

Main Points to be Raised

• *Squaring* and *cubing* are names given to the act of raising to the power of 2 and 3 respectively. The inverse processes, finding a square root or cube root, refer to finding numbers that when squared or cubed produce the number whose root is sought.

• This inverse process provides an entry point for fractional exponents. That is, if you want to represent a number as the square or cube of another number, you have to represent the roots as $n^{\frac{1}{2}}$ or $n^{\frac{1}{3}}$ since $n^{\frac{1}{2}} \times n^{\frac{1}{2}} = n^1 = n$ and $n^{\frac{1}{3}} \times n^{\frac{1}{3}} \times n^{\frac{1}{3}} = n^1 = n$.

• Any fractional exponent can be thought of as a positive power of a root. This is explained using the power law. For example, $n^{\frac{5}{2}}$ is $\left(n^{\frac{1}{2}}\right)^5$, the fifth power of the square root of n.

• The values of fractional exponents are consistent with what one might expect. Since $1 < \frac{3}{2} < 2$, a number raised to the power of $\frac{3}{2}$ will have a value between the number itself (a power of 1) and its square (a power of 2). That is, $n < n^{\frac{3}{2}} < n^2$.

• The exponent laws students have already learned also pertain to fractional exponents.

Try This—Introducing the Lesson

A. Allow students to work alone or in pairs. Observe while students work. You might ask:

• *What is the relationship between the number in brackets and the result in each case in* ***part i)****?* (You double the exponent of the number in brackets to get the result.)

• *How do you find the square root each time in* ***part ii)****? Are any of them more difficult than others?* (The square root has the same base with half the value of the exponent. But I am not sure what to do with the last one because if I took half, I would get a fraction in the exponent.)

The Exposition—Presenting the Main Ideas

• Remind students what perfect squares are. Check to see if they remember what square roots are by asking what the principal square root of 100 is and why.

• Write the equation [ ] × [ ] = 100 on the board. Ask what number belongs in the [ ] to make this true ([ ] =10). Now ask why it seems reasonable that $100^{\frac{1}{2}} \times 100^{\frac{1}{2}} = 100^1 = 100$ should also be true (the product law).

The Exposition—Presenting the Main Ideas [Continued]

Use that example to help students see why $n^{\frac{1}{2}}$ should be a representation of the square root of n. • Ask students to read the exposition. Then ask why it makes sense that the cube root of a number is represented by $n^{\frac{1}{3}}$. Finally, ask them why it makes sense that $a^{\frac{1}{c}}$ is the number that must be multiplied by itself c times to result in a (we call it the cth root of a) and why $a^{\frac{b}{c}} = (a^{\frac{1}{c}})^b$.

Revisiting the Try This

B. Make sure that students who struggled with the square root of 6^1 now realize that it must be $6^{\frac{1}{2}}$.

Using the Examples

Write all four questions from the example on the board. Encourage students to try them on their own before reading the solutions in the text.

Practising and Applying

Teaching points and tips

Q 1: It would be helpful to check answers to this question before students proceed to other questions.

Q 5: Students might compare their answers to others' to see that there is often more than one possible solution to a question.

Common errors

• Some students will change their answers to fractional form, assuming a fractional exponent should result in a fractional answer. For example, they might write $6^{\frac{1}{2}}$ as $\frac{6}{2}$ or 3. Remind them that fractional exponents describe roots (which can be whole numbers or decimals).

• Some students understand the concepts but struggle with the symbolism. While they many not know what to do if asked to find the value of $\sqrt{25}$ or $25^{\frac{1}{2}}$, they may be successful if you simply ask, "What is the square root of 25?" Approach questions in a less symbolic way with these students.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can do basic calculations with fractional exponents
Question 3	to see if students can apply the power law when fractions are involved
Question 5	to see if students can generalize concepts involving fractional exponents
Question 7	to see if students can communicate an understanding of how factoring can be used to determine square roots

Answers

A. i) $(9^4)^2 = 9^{4\times 2}$; $(4^2)^2 = 4^{2\times 2}$; $(7^3)^2 = 7^{3\times 2}$ **ii)** 9^4; 5^5; 17^4; $6^{\frac{1}{2}}$; I took half of the exponent	**B.** You halve the exponent of the expression to find its square root.
1. a) 12 **b)** 5 **c)** $\frac{1}{8}$ **d)** 64 **2. a)** 6 **b)** 36 **c)** *Sample response:* $\frac{2}{4} = \frac{1}{2}$ and the $\frac{1}{2}$ power means the square root	**3. a)** 63^{24} **b)** 30^{31} **c)** 118^{13} **4.** Using the power law, $(49^{\frac{1}{2}})^3 = 49^{\frac{3}{2}}$ and $(49^3)^{\frac{1}{2}} = 49^{\frac{3}{2}}$; since $7 = 49^{\frac{1}{2}}$, then $7^3 = (49^{\frac{1}{2}})^3 = 49^{\frac{3}{2}}$

5. a) *Sample response:* 3, 6, 9, 12, 15 **b)** *Sample response:* $(5^3)^{\frac{1}{3}} = 5^1 = 5$ and $(5^6)^{\frac{1}{3}} = 5^2 = 25$ **c)** No; when you multiply $100 \times \frac{1}{3}$ using the power law, you do not get a whole number exponent, since 100 is not a multiple of 3.	**6. a)** 4 **b)** 0 **c)** 3 **7.** The base stays the same and you take half the value of the exponent to get the exponent of the square root; *sample response*: the square root of 5^8 is $(5^8)^{\frac{1}{2}}$, which equals 5^4.

Supporting Students

Struggling students

Continue to use fractional exponents with perfect squares or cubes until students get a better sense of how fractional exponents operate. In this way, they can actually calculate the numbers with which they are working.

Enrichment

Ask students to create examples of integers raised to fractional exponents that will be easy to calculate and have them explain why they are easy to calculate.

Chapter 2 Scientific Notation

1.2.1 Scientific Notation with Large Numbers

Curriculum Outcomes	Outcome relevance
9-A1 Large and Small Numbers: scientific notation to standard form and vice versa • translate numbers from one form to another • recognize situations where scientific notation is useful **9-B2 Scientific Notation: model, solve, and create problems** • solve problems involving addition, subtraction, multiplication and division with numbers in scientific notation • apply the laws of exponents to numbers written in scientific notation **9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	Scientific notation is an important convention that students will encounter in science courses. It is also a new tool for estimating quantities.

Pacing	Materials	Prerequisites
2 h	• Calculators	• place value notation • multiplication by powers of ten • exponent laws

Main Points to be Raised

• Scientific notation is a convention where a number is expressed as the product of a multiplier and a power of ten. The multiplier must be at least 1 and less than 10. Every number has a unique representation in scientific notation.

• Numbers may be compared using scientific notation. The number with the greatest power of ten is greatest; if the powers are equal, the multipliers must be compared.

• Numbers greater than or equal to 10 will have positive exponents associated with the power of ten. (Note: The next section considers the use of negative exponents.)

• A number with a multiplier of 1 is a power of ten. For example, one million is 1×10^6.

• The exponent laws for products, quotients, and powers can be applied to calculations with scientific notation because powers of ten are always involved.

Try This—Introducing the Lesson

A. Allow students to work in pairs to compare answers. Observe while students work. You might ask:

• *Why did you select the first expression as greatest?* (This one has the greatest exponent, 5.)

• *Would you change your choice?* (Maybe, since the 6 is so small compared to 6000.)

The Exposition—Presenting the Main Ideas

• Record two very large numbers such as 3,200,456,135 and 453,217,789 on the board but do not line them up. Ask students what strategies they have for comparing these values. After listening to their approaches, explain that you will be showing them another way, called scientific notation.

• Provide some elementary examples of writing numbers in scientific notation, beginning with single-digit multiples of powers of ten: 200 as 2×10^2, 8000 as 8×10^3, and 4,000,000 as 4×10^6.

• Then introduce a double digit multiple of a power of ten: 16,000 first as 16×10^3 and then as 1.6×10^4. Explain that in scientific notation the multiplier of the power of ten must always be between 1 and 10.

• Model how to rewrite a whole number in scientific notation, using the ideas in the text, for example, $1627 = 1.627 \times 10^3$ since 10^3 is 1 thousand and 1,627 is between 1 and 2 thousand. Point out how the digit (1) that was in the thousands place moves over 3 places when it moves to the ones place in the multiplier.

• Lead students through the exposition, placing emphasis on these principles and ensuring that students understand why they apply:
- If two numbers in scientific notation are compared, the one with the higher power of ten is greater. If the powers of ten are equal, the multipliers must be compared.
- If the leftmost digit in a number is in the 10^n place, then the number can be written in scientific notation in the form of $b \times 10^n$, where $1 \leq b < 10$.
- The multiplier must be greater than or equal to 1 and less than 10, or the power of ten used to describe the number will be misleading. For example, writing 10,000 as 1×10^5 gives a better sense of its size than writing it as 10×10^4 or 100×10^3.
- As well, if the multiplier is at least 1 and less than 10, it is easier to compare the magnitude of numbers. For example, if a multiplier is multiplied by 10^3, then we know that the number is within the range of numbers from 1000 to 9999.999…, for example, $3.7 \times 10^3 = 3700$ is between 1000 and 9999.999…. The chart here shows the magnitude range for the first five positive powers of ten. Notice, for example, that if you multiplied 10^3 by a number less than 1, the number would be below the 1000 to 9999.999… range ($0.37 \times 10^3 = 370$). If you multiplied 10^3 by 10 or a number greater than 10, the number would be above the 1000 to 9999.999… range ($37 \times 10^3 = 37{,}000$).

Number	Magnitude range
$m \times 10^1$	10 to 99.999…
$m \times 10^2$	100 to 999.999…
$m \times 10^3$	1000 to 9999.999…
$m \times 10^4$	10,000 to 99,999.999…
$m \times 10^5$	100,000 to 999 999.999…

- To multiply or divide numbers in scientific notation, the exponent laws can be applied. Sometimes there must be a multiplier "adjustment." For example, $5 \times 10^2 \times 3 \times 10^3 = 15 \times 10^5$ must be rewritten as 1.5×10^6.
- To add or subtract numbers in scientific notation, the numbers need to be written using the same power of ten. For example, to add $3 \times 10^2 + 4 \times 10^3$, the second number must be rewritten as 40×10^2 so the sum of 42×10^2 can be calculated. The number then must be changed back into scientific notation.

• Show students how numbers are entered or how they appear in scientific notation on their calculators. If it is scientific notation mode, you should see SCI in the display somewhere.

• To enter numbers in scientific notation form on most calculators, enter the multiplier, press [EXP] and then the exponent if it is positive or press [(–)] and then the exponent if it is negative and then press [=].

Revisiting the Try This

B. Students should find all of the expressions to be equivalent when expressed in scientific notation.
The first expression from **part A ii)** is already in scientific notation. Students should now see that the others are equivalent.

Using the Examples

• List the four parts in **example 1** on the board. Encourage students to try them and then to check their work against the thinking in the text.

• Put students in pairs to work through **examples 2 and 3**. Ask one student in the pair to focus on **example 2** and the other to work on **example 3**. When they have finished, they should each explain what they have read to the other student. Ensure students know that 1 billion is as 1000 million.

Practising and Applying

Teaching points and tips

Q 1: Make sure that the students can do this question before proceeding with the other questions.

Q 4: If students seem to be struggling with **part a)**, you may suggest they write 42,357,200 in scientific notation for **part b)**.

Q 5: Remind students that when two numbers are expressed in scientific notation, it is the power of ten that is most important. The number multiplied by that power is only important for comparison if the powers of ten are the same.

Q 7: Students might attempt to do this question without using scientific notation. Encourage the use of scientific notation, particularly as the answer is to be an estimate.

Q 10: Students who correctly answer both parts of this question likely have a strong grasp of the material.

Q 11: This is a core idea and needs to be understood to make the use of scientific notation meaningful.

Q 12: You might point out that this value is about 6 billion or 6×10^9.

Common errors

• Many students forget to ensure that the multiplier of the power of ten is at least 1 and less than 10. You might display a reminder.
• Students may assume their answers in problems involving estimation are incorrect when their answers are different than those of their peers. Remind students that estimation will produce different, but not necessarily incorrect, results.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can express quantities in scientific notation
Question 4 a)	to see if students can apply scientific notation in a numerical situation
Question 5	to see if students can use scientific notation to compare numbers
Question 7	to see if students can apply scientific notation in a computational situation
Question 11	to see if students can communicate about scientific notation

Answers

A. i) *Sample response*: the third one looks greatest **ii)** 6×10^9; 60×10^8 or 6×10^9; 6000×10^6 or 600×10^7 or 60×10^8 or $6 \times 10^{9.}$ They are all actually $6 \times 10^{9,}$ though they may appear different**.**	**B.** 6×10^9; 6×10^9; 6×10^9. They are all the same.
1. a) 2×10^6 **b)** 4.357893389×10^9 **c)** 3.6×10^{13} **d)** 2.04783895×10^7 **2.** 1.6×10^{30} **3.** 1.4×10^9 **4. a)** *Sample response:* $423{,}572 \times 10^2$; $4{,}235{,}720 \times 10^1$; $42{,}357.2 \times 10^3$ **b)** *Sample response:* No, since the multiplier was never between 1 and 10. **5. a)** $(2 \times 10^6) \times (7 \times 10^{13}) > 10^{20}$ because $(2 \times 10^6) \times (7 \times 10^{13}) = 14 \times 10^{19} = 1.4 \times 10^{20}$; $10^{20} = 1 \times 10^{20}$ and $1.4 > 1$ **b)** $(2 \times 10^4)^3 > 250$ billion because 250 billion = $250 \times 10^9 = 2.5 \times 10^2 \times 10^9 = 2.5 \times 10^{11}$; $(2 \times 10^4)^3 = 2^3 \times 10^{12}$ and $10^{12} > 10^{11}$ **c)** $375 \times 10^6 > 3.5893 \times 10^8$ because $375 \times 10^6 = 3.75 \times 10^8$ and $3.75 > 3.5893$ **6. a)** 3750×10^{36} **b)** 3.750×10^{39}	**7.** *Sample response*: $46.6 \times 10^9 \times 46.5 \approx 50 \times 50 \times 10^9 = 2500 \times 10^9 = 2.5 \times 10^{12}$ **8.** *Sample response for a 15-year-old*: $15 \times 365 \times 24 \times 60 \times 12 \approx 94{,}608{,}000$ or 9.4608×10^7 **9.** The parrot's; *sample response*: $45 \times 365 \times 24 \times 60 \times 550 > 75 \times 365 \times 24 \times 60 \times 70$ because $45 \times 550 > 75 \times 70$ **10. a)** $N = 2{,}000$ **b)** No; $500 \times 7.5 \times 10^5 = 5 \times 10^2 \times 7.5 \times 10^5 = 3.75 \times 10^8$, which is much greater than 3.4×10^7 **11.** The greatest value for the first number is $9.9\ldots \times 10^9$, which is almost 10^{10}, and the least value for the second number is $1.0 \times 10^{10} = 10^{10}$, so the first value is always less. **12.** *Sample response*: $6{,}536{,}211{,}569 \approx 6{,}000{,}000{,}000 = 6 \times 10^9$ and the multiplier for 6,536,211,569 must be greater than or equal to 1 and less than 10, so 6,536,211,569 must be 6.536211569×10^9.

Supporting Students

Struggling students

Some students might need to review what the powers of ten represent. Others might need to review multiplication and division by powers of ten.

CONNECTIONS: The Richter Scale

1. The increase in magnitude from 5.7 to 8.1, which is an increase of 2.4 ($8.1 - 5.7 = 2.4$), means the 1897 earthquake was $10^{2.4} \approx 251.19$ times stronger.

1.2.2 Scientific Notation with Small Numbers

Curriculum Outcomes	Outcome relevance
9-A1 Large and Small Numbers: scientific notation to standard form and vice versa • translate numbers from one form to another • relate small numbers to large numbers to see the difference between the two in scientific notation form • recognize situations where scientific notation is useful **9-B2 Scientific Notation: model, solve, and create problems** • solve problems involving addition, subtraction, multiplication and division with numbers in scientific notation • apply the laws of exponents to numbers written in scientific notation **9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	Students will now be able to apply scientific notation to explore situations involving very small numbers as well as very large numbers.

Pacing	Materials	Prerequisites
1 h	• Calculators	• multiplication and division by powers of ten • scientific notation with large numbers • decimal calculations • decimal representations for simple fractions (**question 4**)

Main Points to be Raised

• Numbers less than 1 cannot be represented in scientific notation using positive powers of ten because the multiplier must be at least 1. Negative exponents are needed to represent these numbers.

• The same principles of scientific notation that apply to describing numbers greater than 1 apply to describing numbers less than 1.

• Numbers in scientific notation, including those less than 1, can be displayed on a calculator.

Try This—Introducing the Lesson

A. Allow students to work in pairs. Observe while students work. You might ask:

• *Why does our place value system need negative exponents of ten?* (Sometimes you have to describe numbers that are less than 1. I know that negative exponents represent fractions, so 10^{-1} represents $\frac{1}{10}$ and that is a decimal value.)

• *What does the exponent tell you?* (the position of the nonzero digit the farthest to the left)

The Exposition—Presenting the Main Ideas

• Copy the place value chart from the exposition onto the board. Ask students to use it to explain why $0.6 = 6 \times 10^{-1}$ but $0.006 = 6 \times 10^{-3}$.

• Next, deal with a decimal like 0.43. Ask students how they would read 0.43 (43 hundredths) and why that would suggest they could write it as 43×10^{-2}. Talk about why that is not in scientific notation (the multiplier is at least 1 and less than 10) and why it must be rewritten as $4.3 \times 10^{1} \times 10^{-2} = 4.3 \times 10^{-1}$. Have students notice how the digit 4 in 0.43 moved one place to the left to write it in scientific notation.

• Ask students to read through the exposition and ask any questions they might have.

Revisiting the Try This

B. Most students will not have thought about scientific notation as they solved **part A.** This is an opportunity to make that connection.

Using the Examples

• Display the calculations for **example 1** on the board and encourage students to complete them individually.

• Ask students to have their calculators out as they read through and follow **example 2**.

• Suggest that students use **example 3** as a reference as they work through the lesson.

Practising and Applying

Teaching points and tips

Q 1: Students might want to use a negative exponent in **part d)**. Remind them that positive exponents are also needed.

Q 3: Some students might use calculators to answer **part b)**. Encourage them to explain the answer without using a calculator.

Q 4: Students must already be aware of decimal representations of common fractions such as $\frac{1}{3}$ to complete this question.

Q 6: Students should develop the skill to blend written calculations and the use of a calculator, as suggested here.

Q 7: This is a suitable problem for group work.

Q 8: Students will need to do a number of calculations to successfully complete this problem.

Q 9: For some students, you may want to restate the question in a less symbolic way.

Common errors

Students will often mix up exponents when working with scientific notation in application problems. Make an effort to get students to check the reasonableness of results. For example, an answer of 0.12 may seem small, but in **question 7 b)**, it would mean growing about a kilometre in height every 8 days!

Suggested assessment questions from Practising and Applying

Question 1	to see if students can express quantities less than 1 in scientific notation
Question 3	to see if students can work with calculator displays of scientific notation
Question 5	to see if students can calculate with scientific notation involving positive and negative exponents
Question 7	to see if students can apply scientific notation in an application problem involving measurement
Question 9	to see if students can reason about calculations involving scientific notation

Answers

A. i) $1 \times 10^{-4} = 1 \times \frac{1}{10^4} = 1 \times \frac{1}{10{,}000}$ or 0.0001 **ii)** $3 \times 10^{-4} = 3 \times \frac{1}{10^4} = 3 \times \frac{1}{10{,}000} = \frac{3}{10{,}000}$ or 0.0003	**B. i)** 1×10^{-4} and 3×10^{-4} **ii)** In part A, 0.0001 and 0.0003 were each written as a multiplier of a power of ten. That is scientific notation when the multiplier is at least 1 and less than 10
1. a) 7×10^{-5} **b)** 1.34893×10^{-3} **c)** 8.8×10^{-14} **d)** 3.561587×10^{5} **2. a)** $4.5(0000)^{-01}$ **b)** $7.394(00)^{-02}$ **3. a)** 10,584,100 **b)** the ones digit should be 4 and not 0 because the product of the ones digit of the factors is 4 because $4 \times 6 = 24$ **4.a) i)** $\frac{1}{3}$ **ii)** $\frac{1}{2}$	**4. b)** The first one is not exact since the 3s keep repeating, but the second likely is since it ends with a lot of zeros. **5.** 2.1×10^{-4} **6.** 9.2567×10^{10} **7. a)** 2.64×10^{-7} km/day **b)** 9.636 cm/year **8.** 350 min (moves about 0.16 m/h) **9. a)** yes, if $m \times n \geq 10$ **b)** yes, if $1 \leq m \times n < 10$ **c)** no, since the least possible values are $1 \times 10^p \times 1 \times 10^q$ and the exponent is $p + q$

Supporting Students

Struggling students

You may encourage struggling students to use their calculators more frequently.

Enrichment

More work with applied problems could be encouraged, including estimates of hypothetical quantities such as the number of kilometres of roadway in Bhutan or the total number of cups of tea consumed in a year.

Chapter 3 Rational and Real Numbers

1.3.1 Estimation with Rational Numbers

Curriculum Outcomes	Outcome relevance
9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	Students need to expand their familiarity with numbers beyond integers. It makes sense to begin with rational numbers.

Pacing	Materials	Prerequisites
1 h	• Calculators	• integer operations • fraction operations

Main Points to be Raised

• Rational numbers are numbers that can be expressed as the quotient of two integers. These numbers include integers, fractions (e.g., $\frac{3}{5}$) and their opposites (e.g., $-\frac{3}{5}$), and decimals that terminate (e.g., 0.23) or repeat (e.g., 0.1414141414...) and their opposites (e.g., –0.23 or –0.1414141414…).

• Every rational number has an opposite.

• Calculations with rational numbers parallel those with integers and fractions.

• It is often appropriate to estimate computations rather than to perform exact calculations.

Try This—Introducing the Lesson

A. Allow students to compare answers with partners. This exercise should be quick, allowing the partners to possibly generate additional pairs as well as those they have each listed.

Observe while students work. You might ask:

• *How do you know that the quotient of your fraction is about 1.5?* (I divided the numerator by the denominator to check.)

• *How did you know which numbers to try?* (I know that 1.5 is 3 ÷ 2, so I thought about the fraction $\frac{3}{2}$ and I multiplied the numerator and the denominator by the same amount.)

The Exposition—Presenting the Main Ideas

• Write several calculations on the board involving integers and fractions and give students a few minutes (but not enough time to do the calculations) to write down estimates for each of the calculations. The students can then pair up or group themselves to discuss their methods for estimating.

• Introduce the ≈ sign as a replacement for an equal sign to indicate that a calculation has been estimated and not calculated.

• Have students suggest situations where an estimate would be more appropriate than a more exact calculation.

• Have students read through the exposition on their own. Then ask them to demonstrate their understanding by responding to these questions:

- *Describe three situations where an estimate would be appropriate.*
- *Why is there always more than one possible correct estimate?*

Revisiting the Try This

B. Students should be required to explain their rationale to other students in their small group. This can instead be handled collectively in a whole class discussion. If you do it as a whole class, you may ask each student to write down a pair of fractions that would produce a particular quotient, such as 1.25.

Using the Examples

• Display the three questions for **example 1** on the board. Ask students to estimate, not calculate, the results and check their thinking with the thinking in the text example. Take the opportunity to observe students' fluency with rational number calculations.

• Ask students to read through **example 2** to see whether the reasons for estimating and the process of estimating make sense to them.

Practising and Applying

Teaching points and tips

Q 1: In **part d)** students might find it helpful to know that 2^{10} is approximately 1000.

Q 3: Some students might focus on the number of workers, whereas others might focus on the number of person-days of labour required. It is probable that the two approaches will lead to different estimates.

Q 4: Students should use a calculator in **part b)** after doing the estimation without a calculator in **part a)**. The two parts of the question effectively contrast the good use of (mental) estimation with the exactness of a calculator.

Q 6: This question should be discussed in pairs, if possible.

Common errors

Students will often make estimates that appear to be reasonable but may not be close to the exact result. While students may be discouraged, remind them that estimation is a skill that needs to be developed. Errors are often due to the effect of compound calculations (as in the example of calculating the number of seconds in one week). Sometimes the magnitude of the answer may make good estimates seem bad because the difference ends up being, for example, millions (e.g., if a fraction of a billion).

Suggested assessment questions from Practising and Applying

Question 1	to see if students have the basic computational skills to succeed in subsequent work in the unit
Question 4	to see if students can organize the presentation of a compound calculation both as an estimate and an exact calculation
Question 5	to see if students can estimate in a contextual situation
Question 6	to see if students can estimate in more than one way

Answers

A. *Sample response:* $\frac{8}{9} \div \frac{3}{5}$; $\frac{9}{10} \div \frac{2}{3}$; and $\frac{1}{2} \div \frac{3}{10}$	**B.** *Sample response:* For two of the pairs I used fractions that were close to 1 for the dividend and close to $\frac{2}{3}$ for the divisor because $1 \div \frac{2}{3} = 1\frac{1}{2}$ or 1.5. For the last pair of fractions I used $\frac{1}{2}$ for the dividend and close to $\frac{1}{3}$ for the divisor because if $1 \div \frac{2}{3} = 1\frac{1}{2}$ or 1.5, then $\frac{1}{2} \div \frac{1}{3}$ is also $1\frac{1}{2}$ or 1.5.

1. a) *Sample response:* about 200 **b)** *Sample response:* about 7 **c)** *Sample response:* about 180 **d)** *Sample response:* about 2 million	**2.** *Sample response:* about 200 **3. a)** *Sample response:* about 6 days; high because I estimated it would take twice as long with half as many people, but really there are more than half as many people.

3. b) *Sample response:* They might not have $\frac{1}{2}$ to estimate the fraction of people. **4. a)** *Sample response:* 60 × 60 × 20 × 10 = 720,000 **b)** 604,800; *sample response*: high since I used 10 instead of 7 (days /week) even though I used 20 instead of 24 (h/day). **5.** *Sample response:* about 1750 **6. a)** *Sample response:* $-7 \times \left(\frac{2}{3}\right)^4 \approx -7 \times \left(\frac{1}{2}\right)^2 \approx -2$ or $-7 \times \left(\frac{2}{3}\right)^4 \approx -7 \times (0.4)^2 \approx -1.2$	**6. b)** *Sample response:* $7.06 \div 0.3 \approx 7 \div \frac{1}{3} = 21$ or $7.06 \div 0.3 \approx 7 \times 3.3 \approx 21 + 2 = 23$ **c)** *Sample response:* $0.25 \times 465 \approx \frac{1}{4} \times 480 = 120$ or $0.25 \times 465 \approx \frac{1}{4} \times 464 = 116$ **d)** *Sample response:* 1078 × 512 ≈ 1000 × 500 = 500,000 or 1078 × 512 ≈ 1100 × 500 = 550,000 **7.** *Sample response:* comparing the distance from Venus to Earth with the distance from Saturn to Earth

Supporting Students

Struggling students

Some students will need to revisit integer, decimal, or fraction calculations. Take these students aside and work with them on areas where they have difficulty.

Enrichment

Ask students to create calculations involving three or four numbers and several operations with a particular resulting estimate, such as –2.5 or 800.

1.3.2 Order of Operations

Curriculum Outcomes	Outcome relevance
9-B5 Order of Operations: rational number computation • apply knowledge of order of operations conventions with rational numbers **9-B3 Reasonableness of Results: square roots, rational numbers, scientific notation** • use estimation skills when calculating with rational numbers **9-B4 Add, Subtract, Multiply, and Divide: rational numbers in fractional and decimal form** • use mental computation, whenever appropriate, when solving problems	As students progress through the grades, they will be required to calculate with a broader range of numbers. They need to know that the order of operations with which they are already familiar applies in these new situations.

Pacing	Materials	Prerequisites
1 h	• Calculators	• order of operation rules for fractions and integers • fraction and integer operations

Main Points to be Raised

• The same order of operations applicable to fractions and integers applies to calculations with any rational numbers.

• Order of operation rules ensure that everyone calculating a mathematical expression obtains the same result..

Try This—Introducing the Lesson

A. Allow students to work alone or with a partner.

Observe while students work. You might ask:

• *Why did you use a subtraction sign there?* (If I had used a + sign, the answer would have been 5 and not 3.)

• *How do you start looking for a solution?* (I decided to multiply the two negative values and figure out how to arrange the others to subtract 5.)

The Exposition—Presenting the Main Ideas

• On the board, write a calculation involving brackets, several operations, and exponents, for example, $25 \div (3 + 1 \div 0.5) \times 2^3 - 4$. Ask students to evaluate the results (36 for the given one). If different answers are suggested, find out the sequence that the students used to come up with their answers.

• Have students look at the list of operations at the bottom of the page.

• Suggest that students use the provided calculations for reference as they continue to work.

Revisiting the Try This

B. Students have already met the order of operations rules and might have been using them in adding or removing brackets for **part A i)**, but some students might benefit from this more explicit reference. They would see, for example, that putting brackets around $3 \times (-2)$ in $3 \times (-2) + 5$ would have no effect since the multiplication would have to be completed first anyway.

Using the Examples

• Display the calculations in both examples on the board and ask students to complete them individually or in pairs, then check their responses against those in the text. Identify problems and errors with the examples so that the basic operational work is cleared up prior to proceeding with the **Practising and Applying** questions.

• Point out the value of brackets for organisation, even when results are not affected.

• Address the use of square brackets in notation.

Practising and Applying

Teaching points and tips

Q 1: Students do not need to complete the calculations; they only need to consider the order. This question could be done as a whole class.

Q 2: Some students might know the order of operations but get incorrect results due to errors in calculation. Calculation errors need to be revisited, as the numbers used in the question lend themselves to work without a calculator.

Q 4: This may be the most difficult question for some students. You may encourage them to proceed with others and come back to it later if they prefer.

Q 5: Encourage students to compare answers with others.

Q 8: Students should be able to mentally calculate the value of this expression as 10 and explain why the brackets are so helpful.

Q 9: This application demonstrates the importance of the order of operations. The difference between written and oral presentation of mathematical calculations becomes evident. Encourage students to translate Yeshey's method of calculation into a mathematical expression.

Common errors

Students will often make mistakes that have nothing to do with the order of operations. Any errors involving operations with decimals, fractions, integers, and exponents need to be minimized so that students can confidently proceed with further work in mathematics as the year progresses.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can perform calculations in the right sequence
Question 2	to see if students can carry out calculations with integers, exponents, decimals, and fractions
Question 4	to see if students can insert brackets to produce specified results
Question 8	to see if students can recognize the efficacy that may come from using brackets

Answers

A. i) *Sample responses:* $[(-2) \times (-4)] \div (5 - 3) - 1 = 3$ or $1 - [3 \times (-4) - 5 \times (-2)]$ or $(1 + (-2)) \times 3 \div [-(5 + (-4))]$ **ii)** *Sample responses:* $[(-2) \times (-4)] \div 5 - 3 - 1 = -2\frac{2}{5}$ or $1 - 3 \times (-4) - 5 \times (-2)] = 23$ or $(1 + (-2)) \times 3 \div -5 + (-4) = -3\frac{2}{5}$	**B.** *Sample response:* When I removed the brackets from 5 – 3 in $[(-2) \times (-4)] \div (5 - 3) - 1$, I got an answer of $-2\frac{2}{5}$ instead of 3: $[(-2) \times (-4)] \div 5 - 3 - 1 = -2\frac{2}{5}$. This was because the order of operations made me divide the result of $[(-2) \times (-4)]$, which is 8, by 5 instead of by 2 because the brackets were no longer there to make me subtract 5 – 3 first.
1. a) 4.3×5.7 **b)** $5.8 \div 3.6$ **c)** $-6.7 \div 3.2$ **d)** $(2.3)^3$ **2. a)** 32.5 **b)** $8\frac{1}{9}$ **c)** –3.8 **d)** 0.128 **3.** A, C, and D **4. a)** *Sample response:* no brackets needed	**4. b)** *Sample response:* $\left(\frac{2}{3}\right)^{-1} - \left(1 - \frac{2}{3}\right) \times 3 = \frac{1}{2}$ **c)** *Sample response:* $\frac{1}{4} \times \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{4} \div \frac{1}{4} = 1$ **d)** *Sample response:* $\frac{2^3}{5} - 2 \times 3 \div (4 + 1) = \frac{2}{5}$ **5. a)** *Sample response:* $1 \times \left(\frac{2}{3} + 3\right) \times \left(4 + \frac{2}{5}\right) = \frac{242}{15}$ **b)** *Sample response:* no brackets needed

Answers [Continued]

6. No, adding is commutative so the order in which you add the numbers does not affect the answer; *sample response*: $[-2 + 3] + (-4) + 5 + (-6)$ $= -2 + 3 + [(-4) + 5 + (-6)]$ $= -4$ **7.** Yes; *Sample response:* $4 - (\frac{1}{2} - 3) + 6 \neq (4 - \frac{1}{2}) - 3 + 6$ **8.** The expression in the brackets does not even have to be calculated since $a^0 = 1$ and the expression $\neq 0$ since it is the sum of positive numbers.	**9.** Yes; total area should be $2.5^2 + 3.7^2 = 19.94$ not $(2.5 + 3.7)^2 = 38.44$ m^2 **10.** *Sample response:* Some expressions change value if the operations are performed in a different order, so without rules people could get different answers for the same question; *sample response*: $20 \div 4 \times 5 = 25$ but if you multiply first, it would be 1.

Supporting Students

Struggling students

- Provide more practice opportunities if students struggle with particular types of calculations.
- Struggling students may omit **questions 4 and 5**, which are more difficult.

Enrichment

The topic lends itself to challenges such as inserting brackets to get particular values, as in **question 4**. Students can readily create such challenges for one another by creating and calculating expressions involving brackets, then removing the brackets and asking others to determine where they are needed to obtain specific values.

1.3.3 Square Roots

Curriculum Outcomes	Outcome relevance
9-A2 Square Roots: solve problems • determine if the solution to a problem involves both values of the square root or just the principal square root **9-A3 Square Roots: approximate** • develop an awareness that square roots are often irrational • understand that appropriate approximations in some situations are beneficial	Students will need to be familiar with square roots to solve applied problems in Class IX and to work with quadratics in Class X.

Pacing	Materials	Prerequisites
1 h	• Calculators	• formula for the area of a square • vocabulary and meaning of the term square root • exponent laws • whole number factoring • repeating decimals as fractions (**question 4**)

Main Points to be Raised

• A positive number has two square roots, a positive one and a negative one. The positive one is called the principal square root.

• $\sqrt{n}$ means the principal square root of n.

• The principal square root of a number represents the side length of a square with an area equal to the indicated number.

• A negative number does not have a square root because if you square any number, positive or negative, you never end up with a negative value.

• The square root of n is less than n if n is greater than 1, but it is more than n if n is a fraction between 0 and 1. If n is negative, it does not have a square root.

• An irrational number is a number that is not the quotient of integers. Many square roots, but not all of them, are irrational.

• Writing whole numbers as products of powers with even exponents makes it easier to determine their square roots.

Try This—Introducing the Lesson

A. Allow students to work individually or in pairs. Observe while students work. You might ask:

• *How do you know the length has to be greater than 70 m?* (If it was 70 m × 70 m, the area would be 4900 m^2 and it is more than that.)

• *How do you get that answer?* (I could not find a number that would work. I know that 80 × 80 = 6400. So my guess is that the length is 78 m.)

The Exposition—Presenting the Main Ideas

• Ask students to name some perfect squares. Take some of them and ask for the square root. For example, suppose that 144 is used. Then $\sqrt{144}$ is 12. Why? (12 × 12 = 144) Draw a picture of a square to reinforce the idea of the principal square root as the length of a side of a square of a given area. Ask students why –12 is also a square root of 144. Then ask why it would not be used as the length of the side of the square (a length cannot be negative).

• Ask students how they know that the square root of $\frac{4}{9}$ must be $\frac{2}{3}$. Point out that although both whole numbers and fractions can have square roots, there is a difference. Notice that the square root of a whole number other than 0 or 1 is less than the number, e.g., 12 is much less than 144, but the square root of a fraction between 0 and 1 is greater than the number, for example, $\frac{2}{3} > \frac{4}{9}$. Help students see that this makes sense since the product of whole numbers greater than 1 increases the values being multiplied, but multiplying by a proper fraction reduces the value of the other factor.

The Exposition—Presenting the Main Ideas [Continued]

• Write the product 144 × 9 × 100 on the board. Ask why it is easy to see that the square root is 12 × 3 × 10. Point out that factoring a number sometimes makes the calculation of its square root simpler.

• Point out that the square roots of most numbers, for example, 2, are not rational. Therefore, it is not possible to calculate their exact values. Estimation is required. Record the word **irrational** on the board to emphasize that these are the numbers that are not rational.

• Have students read through the exposition. Make sure they are comfortable with the material by asking, once they have finished reading, how they might calculate the square root of 1.44.

Revisiting the Try This

B. The idea of a square root as a numerical value should be emphasized. At the same time, you should reinforce the physical interpretation of the square root of 6200 as the length of a side of a square having an area of 6200 square units.

Using the Examples

• Ask students to read through the two examples on their own. The experience with **Try This** should have guided students to recognize $\sqrt{3500}$ as being the required length in **example 1**.

• Discourage the use of a calculator in **example 1**, but encourage it in **example 2**. The contrasting experiences will help students develop an appreciation for proper use of a calculator.

Practising and Applying

Teaching points and tips

Q 1: Students might be encouraged to mentally estimate the values without doing any checking. Place the emphasis on the placement of the roots between two successive integers for **parts a), b), and c)**, or between successive multiples of ten for **part d)**.

Q 2: Some students might wish to use a calculator. Remind them that estimates should not be based on values from a calculator.

Q 4: Students might not be aware that 0.4444... is $\frac{4}{9}$.

If necessary, review some common decimal representations of fractions or intentionally insert such examples into the presentation of relevant topics.

Q 6: Students can use their calculators to explore the relationship and then use reasoning to see why it makes sense.

Q 7: You may discuss this question as a class and ask for other numbers that could have been used in place of 8.

Q 11: Factoring is useful in working with radicals and square roots. Students will need to be competent in factoring to work effectively in algebra and other secondary level topics.

Q 12: The connection between scientific notation and square roots should be reinforced. Note how the use of even powers of ten facilitates the estimation process.

Q 13: This question lends itself to discussion as some students will use scientific notation and others will consider different methods.

Suggested assessment questions from Practising and Applying

Question 1	to see if students have a working knowledge of perfect squares to facilitate estimation
Question 4	to see if students recognize situations where changing the form of a number may simplify calculation of its square root
Question 7	to see if students can explain why a number is not rational
Question 10	to see if students can explain the relationship between the size of a number and the size of its square root
Question 11	to see if students can use factoring as a tool for calculating square roots
Question 12	to see if students can apply the principles of scientific notation to calculating square roots

Answers

A. between 78 m and 79 m

B. You square the side length of a square to get the area, so the side length is the square root of the area.

1. a) 6 **b)** 10

c) 17 **d)** 80

2. B since 30 × 30 = 900

3. a) 82 **b)** 820

4. 0.4444… = $\frac{4}{9}$ and the square root is $\frac{2}{3}$, which is a rational number

5. a) 70 × 70 = 4900, which is close to 4823

b) less **c)** 69.4 m

6. a) i) equal; *Sample response:* both are about 5.3
ii) equal; *Sample response:* both are about 17.3
iii) If you break a number into factors, you can multiply the square roots of the factors to calculate the square root of the number.

b) *Use the product of powers law*:
$(20\times4)^{\frac{1}{2}} = 20^{\frac{1}{2}} \times 4^{\frac{1}{2}}$
By calculating:
$\sqrt{20\times4} = 8.94427191$;
$\sqrt{20} \times \sqrt{4} = 4.472135955 \times 2 = 8.94427191$

7. Since $8 = 4 \times 2$, $\sqrt{8} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2}$ and $\sqrt{2}$ is not rational, so $2 \times \sqrt{2}$ cannot be rational.

8. All numbers have a positive square root and a negative square root, in this case, 29.4 and –29.4.

9. a) i) 4.5 s **ii)** 14.2 s **iii)** 100.6 s

b) number of seconds cannot be negative

10. If you square any whole number greater than 1, the result is a greater whole number. So the square root of any whole number must be less than the number.

11. a) 9 × 7 × 6 × 9 × 7 × 6 = 142,884

b) $9216 = 9 \times 1024 = 9 \times 2^{10}$, so $\sqrt{9216} = 3 \times 2^5 = 96$

12. a) 4.6216×10^4, so about 200

b) 62.6147×10^4, so about 800

13. *Sample response*:
39,417 ≈ 40,000 = 4 × 10,000, so
$\sqrt{39{,}417} \approx \sqrt{4} \times \sqrt{10{,}000} = 2 \times 100 = 200$

Supporting Students

Struggling students

• Allow struggling students to use estimates that are less precise. This is easier when they recognize some of the familiar perfect squares, for example, 400 as 20^2, 8100 as 90^2, 10,000 as 100^2, and so on. It is also appropriate to allow them to use a calculator to help them become familiar with square root values.

• Help students understand a perfect square from both a geometrical and algebraic perspective. Drawings and examples of factoring should be encouraged.

Enrichment

• Challenge students with some more abstract questions, e.g.

- *How do I know that* $\sqrt{29{,}300{,}568}$ *is irrational without factoring?* (No perfect squares end in 8.)

- *How is it possible that* $\sqrt{29{,}300{,}569}$ *is rational?* (Perfect squares can end in 9. In fact, $\sqrt{29{,}300{,}569} = 5413$.)

1.3.4 EXPLORE: Representing Square Roots

Curriculum Outcomes	Lesson relevance
9-A3 Square Roots: approximate • develop an awareness that square roots are often irrational • understand that appropriate approximations in some situations are beneficial	This optional exploration of the Archimedes spiral provides another physical model to make the abstract concept of the square root more meaningful.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Protractors or set square and compass	• Pythagorean theorem

Exploration

• Ask each student to draw and extend the diagram (called the Archimedes spiral) as indicated in **part A** and **part B**. Students can complete the measurements for **part C** using their own diagrams. Before proceeding further, they can pair up to compare measurements.

• Students should make sure they have similar measurements before proceeding to **part D**. When the partners agree on a set of measures, they can work on **parts D to G** together.

Observe while students work. You might ask:

• *How do your measures compare?* (They are not exactly the same but they were all close except for one. We checked that measure and I found an error with my measurement.)

• *How did you notice that pattern?* (I just added 1 each time.)

• *If I change the number from 40 to 49, what would you answer?* (the 48th side) *Is there anything else you would know about that length? Why?* (It would be 7 because 7 × 7 = 49.)

Observe and Assess

As students are working, notice:
• Do they draw carefully, thus reducing the likelihood of measurement error?
• How do partners compare and reconcile any differences in measurement?
• Do they generalize in **part G** to state that any time the hypotenuse length is the square root of a perfect square the length will be a rational number? Do they extend the pattern to include $\sqrt{16}$, $\sqrt{25}$, … ?

Share and Reflect

After students have had sufficient time to answer **parts A, B, and C**, they should pair up or form small groups to complete the exploration. Subsequently groups can come forward to discuss their observations.

Answers

C. 1.4 cm, 1.7 cm, 2.0 cm, 2.2 cm, 2.4 cm, 2.6 cm, 2.8 cm, 3.0 cm, 3.2 cm, 3.3 cm, 3.5 cm

D. $\sqrt{2}$ cm, $\sqrt{3}$ cm, $\sqrt{4}$ cm, $\sqrt{5}$ cm, $\sqrt{6}$ cm, $\sqrt{7}$ cm, $\sqrt{8}$ cm, $\sqrt{9}$ cm, $\sqrt{10}$ cm, $\sqrt{11}$ cm, $\sqrt{12}$ cm

E. i) As you go to the next triangle, the number you are taking the square root of is 1 greater; it seems like the centimetre measures increase by about 0.2 each time

ii) *Sample response:*
The pattern of the number being the next square root will continue because each time you are squaring the square root of a number and adding 1 to it and then taking the square root again. But the pattern of increasing by about 0.2 doesn't continue.

F. the hypotenuse of the 39th triangle

G. *Sample response:*
The hypotenuse lengths of the 3rd and 8th triangles are rational since they would be $\sqrt{4}$ and $\sqrt{9}$ cm and these are integer lengths.

Supporting Students

Struggling students

• If some students have problems with the diagram and are missing the exploration, you may have them join another group and encourage them to draw the diagram again later.

• You might provide a previously photocopied copy of the diagram for these students to use.

Enrichment

Ask students:

When would the perimeter of the spiral be rational? (when the longest, or exterior hypotenuse is rational)

1.3.5 Representing Real Numbers

Curriculum Outcomes	Outcome relevance
9-A4 Integers and Real Numbers: write solution sets for equations and inequalities • relate the language of inequality to the symbols of inequality • graph, when given a set notation, and produce the set notation, when given a graph **9-A5 Irrational Numbers: demonstrate and understand meaning** • place irrational numbers on a number line relative to known rational numbers **9-A6 Real Numbers: interrelationships of subsets** • determine and justify if a given number is rational or irrational • give examples of rational and irrational numbers	The classes of numbers with which students will extend as they move forward in mathematics. Students need to see how these different kinds of numbers are alike, but also how they are different.

Pacing	Materials	Prerequisites
1 h	None	• inequality symbols • meaning of square roots • meaning of π

Main Points to be Raised

• The definition of rational numbers as quotients of integers should be reinforced. Since repeating decimals can be represented as such quotients, they are rational. Terminating decimals are rational.
The remaining decimals are irrational.

• Roots, e.g., square roots, are only rational in special cases.

• Combining rational and irrational numbers by adding, subtracting, multiplying, or dividing sometimes produces rational results and other times produces irrational results.

• Real numbers are defined as the combination (union) of rational and irrational numbers.

• Inequalities can be graphed on a number line.
All numbers to be included in the graph are represented by solid points or a line segment or a ray (going indefinitely toward one end of the number line). If a particular point is to be excluded but those near it included, an open circle is used at that point.

• The possible solutions of an inequality are affected by the *domain* (permissible values), which may be integers, real numbers, or some other restricted set of values.

Try This—Introducing the Lesson

A. Encourage students to create their own lists of numbers and compare them with others.

Observe while students work. You might ask:

• *Is it possible to have no integers in your list?* (Yes, I could use a decimal and just change it slightly by putting another digit on the end.)

• *How do you know that the number is in the proper range?* (Since 2 × 2 = 4, I can use values very close to 2 and still be less than the square root of 5.)

• *Is there any number that could be on all three lists?* (No, it is impossible for a number listed in **part i)** to be great enough to be listed in **part ii)**.)

The Exposition—Presenting the Main Ideas

• Review the definition of a rational number. List some rational numbers, including repeating decimals. Make a second list of irrational numbers. Students will have to take your word that these values are irrational, since the proofs are generally beyond the students' level of mathematical sophistication. Include on the list numbers such as $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$, $4\sqrt{2}$, $\sqrt{3}$, π, $\frac{\pi}{2}$, and 3π. Mention that when these numbers are expressed on a calculator, the display always shows an estimate, never an exact value, since the digits go on forever in an unpredictable way.

• When it appears that the distinction is clear, suggest operations involving some of the numbers shown and ask whether the result is rational or irrational.

• Then ask students to list some irrational numbers and rational numbers in a certain range, e.g., between 3 and 4. Examples of rational numbers include 3.1, 3.2, $3\frac{1}{3}$, and so on. Examples of irrationals include π, $\pi + 0.1$ and $2 + \sqrt{2}$.

• Model how to graph the inequality $2 < x < 3$ by including all the real numbers between 2 and 3 on the number line and using open circles at 2 and at 3 to indicate their exclusion..

• Invite students to read through the exposition to see how inequalities are graphed if one or both end points are included and if only integers are to be graphed.

Revisiting the Try This

B. Have students compare and check the irrational numbers listed. Ask why they selected the inequality that they did. All three graphs can be shared with the whole class to minimize problems with the subsequent exercises.

Using the Examples

• Have students read through **example 1** independently. Ask them to create two other values for a partner to check to determine rationality or irrationality. Circulate as they share their numbers to check their work.
• Students can also read through **example 2** on their own. They should get their work checked before proceeding to the questions. **example 2** could be assigned to groups, with each student in the group completing at least one of the graphs and others checking their work.

Practising and Applying

Teaching points and tips

Q 1: Students might not recognise C as repeating because it appears to behave differently than A, which repeats from the start. Others might assume anything involving π is irrational, such as F. Review such misconceptions.

Q 2: Remind students of the formula for the area of a circle, if needed.

Q 5: This question helps remind students of the role of the domain in graphing an inequality.

Q 9: Students who can answer this question have a strong understanding of numbers. Have all students try the question so that you can later discuss both methods outlined in the answers.

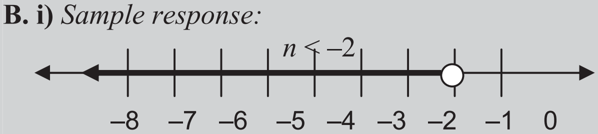
Common errors

Students will often assume a repeating decimal is irrational since it goes on forever. Remind students that any repeating decimal can be expressed as a fraction and hence is rational.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can classify numbers as being rational or irrational
Question 3	to see if students can identify the results of operations as being rational or irrational
Question 4	to see if students can list rational and irrational numbers in a specified range
Question 7	to see if students can graph inequalities and sets of given values

Answers

A. i) *Sample response:* $-2.4, -2\frac{2}{3}, -2.6, -3.8, -4\frac{7}{8}$ **ii)** *Sample response:* $6.9, 7, 7\frac{1}{2}, 8, 8\frac{1}{4}$ **iii)** *Sample response:* $1.5, 1\frac{4}{5}, 2.0, 2.1, 2.2$	**B. i)** *Sample response:*  **ii)** for **i)** a possible rational value for n is –3; a possible irrational value for n is -10π for **ii)** a possible rational value for n is 10; a possible irrational value for n is 10π for **iii)** a possible rational value for n is 1.7; a possible irrational value for n is $\sqrt{3}$

Answers [Continued]

1. A, B, C, and F:
A. repeating decimals are rational
B. terminating decimals are rational
C. repeating decimals are rational
F. $3\pi^0 = 3$, which is rational

2. estimates: $\frac{88}{7}$, 3.14×4, $3.14 \times \sqrt{16}$; each uses an estimate for π

3. A, B, D, and F:
A. adding $0.8 + 0.4 = 1.2$, which is a terminating decimal, which makes it rational
B. subtracting two fractions gives a fraction and fractions are rational
D. dividing a number (other than 0) by itself gives 1, which is a rational number
F. adding a terminating and repeating decimal gives a repeating decimal, which is rational

4. a) *Sample response:* 4.5

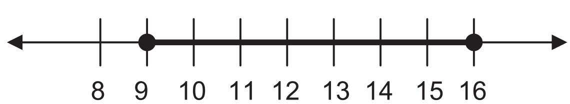
b) *Sample response:* $\sqrt{17}$

c) *Sample response:* $\frac{5}{2}$

d) *Sample response:* $\pi - 5.7$

5. a) 9 is least and 16 is greatest

b)



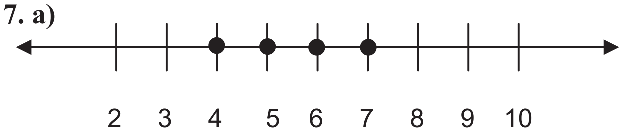
c) You would put solid circles only at each integer from 9 to 16 inclusive

6. a) F; the answer is 0, which is rational

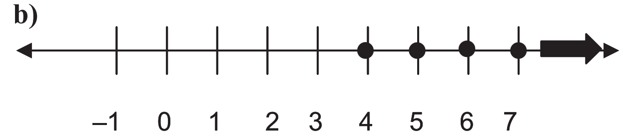
b) T; adding a rational to an irrational usually results in an irrational

c) T; the answer is 1, which is rational

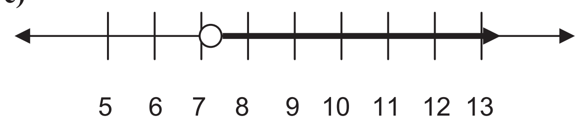
7. a)



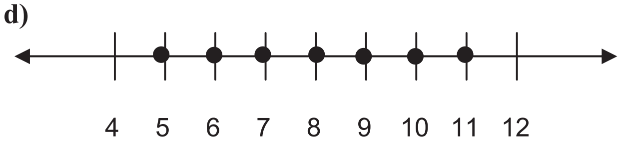
b)



c)



d)



8. a) $9 \le n \le 6\sqrt{3}$, n is a real number

b) $\sqrt{40} \le n < 13$, n is a real number

9. *Sample response:*
$-\frac{7}{9} = -\frac{14}{18}$ and $-\frac{8}{9} = -\frac{16}{18}$, so
$-\frac{15}{18}$ is between $-\frac{14}{18}$ $(-\frac{7}{9})$ and $-\frac{16}{18}$ $(-\frac{8}{9})$

Supporting Students

Struggling Students

Focus on graphing inequalities involving real numbers with end points that are rational numbers before asking them to work with end points that are irrational.

Enrichment

Try other examples like **question 9** until these students can confidently find a real number between any two others, regardless of how close they may be. A game can be played in which the first person begins by naming a real number and the second names another real number. The next person must name a real number that falls between the last two real numbers that have been named. This can be continued indefinitely.

UNIT 1 Revision

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1 – 4	Lessons 1.1.1, 1.1.2, 1.1.3, and 1.1.4
5	Lessons 1.1.4, 1.2.1, and 1.3.3
6 – 8	Lessons 1.2.1 and 1.2.2
9, 10	Lesson 1.3.1
11	Lesson 1.3.2
12 – 16	Lesson 1.3.3
17 – 19	Lesson 1.3.5

Revision Tips

Q 5: Encourage students to make the work easier by using the power law and the idea of the reciprocal in **part a)** and by converting to an expression with an even power of ten in **part c).**

Q 10: Ensure students can represent the multi-step calculation in written form. Note that the estimation may stop when it is clear that the claim is invalid. (You may want to ask students to do a second question: Estimate and then calculate how many years one million seconds is.)

Q 11: Focus on the order of operations rather than on the actual values. Ask students to calculate by hand.

Q 15: See if students can factor 1764 into a form where the square root becomes apparent.

Q 18: Students need to articulate clearly written justifications of their responses.

Q 19: Observe whether students properly graph the inequalities.

Answers

1. a) 8^{29} **b)** 5^{19} **c)** $2^3 \times 3^8$

2. a) 10,000 **b)** 100,000

3. a) 4 **b)** 16 **c)** 0

4. $-\frac{2}{3}$

5. a) $\frac{1}{16}$ **b)** 196 **c)** 600,000

6. Since *a* and *b* are between 1 and 10, N is greater because its power of ten has a greater exponent.

7. a) 2,340,000
b) 2.34000 ^{06} = 2.34×10^6 = 2,340,000;
sample response: this means you move the digit 2 six places to the left so 2.34×10^6 becomes 2,340,000

8. a) 3.518×10^{-7}
b) Yes; $0.0003518 = 3.518 \times 10^{-4} > 4 \times 10^{-7}$ because $10^{-4} > 10^{-7}$ and 3.518 and 4 are both ≥ 1 and < 10

9. *Sample response:* $1 \times -7 = -7$; $\frac{4}{5}$ of -7.5 is -6

10. No, a 15-year-old is about 9,000,000 min old
($15 \times 365 \times 24 \times 60 \approx 15 \times 400 \times 25 \times 60 = 9{,}000{,}000$)

11. a) 158 **b)** 16.3

12. a) two solutions, a positive and negative square root
b) no solutions because there is no square root for a negative number

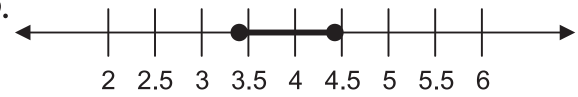
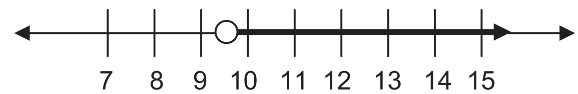
13. The positive square root since you cannot have a negative thickness.

14. a) *Sample response:* about 29
b) *Sample response:* about 0.9
c) *Sample response:* about 2500
d) *Sample response:* about 48,000

15. $1764 = 9 \times 196 = 3 \times 3 \times 14 \times 14$, so
$\sqrt{3 \times 3 \times 14 \times 14} = 3 \times 14 = 42$

16. It is between about 0.6 and 0.7 (or between –0.6 and –0.7).

Answers [Continued]

17. C and E: - C is a repeating decimal, which is rational - E is $\frac{3}{7}$, which is rational **18. a)** True; *Sample response:* $\sqrt{2} \times \sqrt{2} = 2$, which is rational **b)** False; *Sample response:* 0.020406081012… is non–terminating and never repeats	**19.** **a)**  *Sample response:* rationals: 3.5, 3.6, 3.7 irrationals: $\sqrt{2} + 2.1$, $\sqrt{2} + 2.2$, $\sqrt{2} + 2.3$ **b)**  *Sample response:* rationals: 13, 205, 327 irrationals: 4π, 5π, 6π

UNIT 1 Number and Operations Test

1. Solve for n.

a) $15^0 \times (15^4)^3 = 15^n$

b) $(15^n)^7 \times 15^n = 15^{16}$

c) $3^6 \times 3^8 \div 3^n = 3^9$

d) $2^n = \left(\frac{1}{4}\right)^{-3}$

2. Which is greater in each pair? Justify your choice.

a) $(6 \times 10^9) \times (9 \times 10^3)$ or 10^{14}

b) $16^{\frac{3}{4}}$ or $36^{\frac{1}{2}}$

c) $\left(\frac{2}{3}\right)^3$ or $\left(\frac{2}{3}\right)^5$

3. Explain why $a^0 = 1$ $(a \neq 0)$. Provide an example using the quotient law of exponents to support your explanation.

4. Express each in scientific notation.

a) 5193.45

b) 0.03584

c) 73 billion

d) $(1.4 \times 10^{20}) \times (2 \times 10^{-7})$

5. A classmate reports that the answer shown on a calculator was 1.44000^{10}.

a) What number does this represent?

b) Explain to your classmate how you would find the square root of the value displayed without using a calculator.

6. Estimate the number of minutes each year that you spend in school classes. Show your work.

7. Estimate.

a) $\sqrt{907}$

b) 19.61×0.243

c) $\sqrt{5.023 \times 10^7}$

8. Evaluate:

a) $25 - (7 + 18 \div 6)^2 + 20 \times (9.2 - 4.1)$

b) $\sqrt{55 + 6^2 - 5 \times 2}$

9. a) Give an example of an irrational number that becomes a rational number when it is multiplied by $\sqrt{2}$.

b) Give an example of a rational number between $\sqrt{2}$ and $\sqrt{3}$.

c) Which of the following numbers are rational?

i) $2 + \sqrt{2}$

ii) 1.8181818...

iii) $6\pi \div 3$

iv) $\sqrt{\frac{9}{49}}$

10. a) Graph the following inequality:

$\pi < n \leq \sqrt{23}$, n is a real number

b) List two rational numbers and two irrational numbers that satisfy the inequality in **part a)**.

UNIT 1 Test

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1	Lessons 1.1.1, 1.1.2, and 1.1.3
2	Lessons 1.1.2, 1.1.4, and 1.2.1
3	Lesson 1.1.3
4, 5	Lessons 1.2.1 and 1.2.2
6	Lesson 1.3.1
7	Lesson 1.3.2
8	Lesson 1.3.3
9, 10	Lesson 1.3.5

Select questions to assign according to the time available.

Answers

1. a) 12 **b)** 2 **c)** 5 **d)** 6

2. a) 10^{14} is greater since
$(6 \times 10^9) \times (9 \times 10^3) = 54 \times 10^{12} = 5.4 \times 10^{13}$, and
5.4×10^{13} is less than 10^{14}.

b) $16^{\frac{3}{4}}$ is greater since

$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$ and $36^{\frac{1}{2}} = 6$

c) $(\frac{2}{3})^3$ is greater because each time you multiply by $\frac{2}{3}$ the value gets smaller

3. *Sample response*:
A number divided by itself is equal to 1, so $4^3 \div 4^3 = 1$, but $4^3 \div 4^3$ is also $4^{3-3} = 4^0$ so $4^0 = 1$. This will be true not just for 4 but for any number that is not equal to 0.

4. a) 5.19345×10^3

b) 3.584×10^{-2}

c) 7.3×10^{10}

d) 2.8×10^{13}

5. a) 1.44×10^{10} or 14,400,000,000

b) *Sample response*:
Since the exponent of 10 is even, I can take half of 10 to get 5 as the exponent of the square root. I must also take the square root of 1.44, which is 1.2. The square root of the displayed value is 1.2×10^5 or 120,000.

6. *Sample response*:
I am in school a bit more than half the days of the year, or about 200 days. Each day in school I spend about 5 h in classes. That is 1000 h. Each hour has 60 min. $1000 \times 60 = 60{,}000$ min, so I spend about 60,000 min a year in classes at school.

7. *Sample responses*:
a) 30

b) 4

c) 7000

8. a) 27

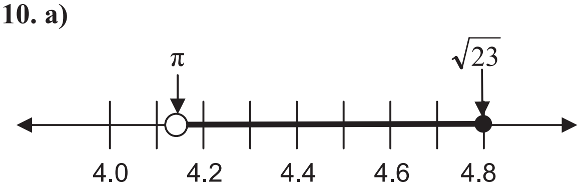
b) 9

9. *Sample responses*:
a) $\sqrt{2}$

b) 1.5

c) ii) 1.8181818... and iv) $\sqrt{\frac{9}{49}}$

10. a)



b) *Sample response:*
rational numbers: 4 and 4.1
irrational numbers: $\pi + 0.01$ and $\sqrt{21}$

UNIT 1 Performance Task—Number Match Game

To play this game you first need to create ten pairs of matching cards.
One number in each pair is one of the numbers below and the other is a calculation that results in that number (see the instructions below for making the other cards).

Number Cards

$\pi - 2$	$6\sqrt{29}$	$-3\frac{2}{5}$	4.1×10^4	3.24×10^{-3}
10π	$8(\sqrt{2} + \sqrt{3})$	-8.9	$5^{\frac{2}{3}}$	42.7×10^{-1}

Instructions for creating the matching cards

A. at least two cards must include negative exponents

B. at least two cards must include the exponent 0

C. at least two cards must use a fractional exponent

D. at least two cards must include three operations (three of +, –, ×, and ÷)

E. at least two cards must involve a square root

F. at least two cards must include both rational and irrational numbers,

G. at least two cards must require the use of an exponent law

Note that some cards will meet several of the conditions above

How to play the game

• Play with a partner. Shuffle all 20 cards. Place them face down on a table and make four rows of five cards.

• Take turns. On your turn, turn two cards over. If they match, keep the cards and take another turn. If they do not match, turn them face down again. A player may take no more than two turns in a row.

• Once all the cards have been removed, the player with the most cards wins.

UNIT 1 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-A1 Large and Small Numbers: scientific notation to standard form and vice versa **9-A3** Square Roots: approximate **9-A6** Real Numbers: interrelationships of subsets **9-B1** Exponent Laws: integral exponents **9-B2** Scientific Notation: model, solve, and create problems **9-B3** Reasonableness of Results: square roots, rational numbers, scientific notation **9-B4** Add, Subtract, Multiply and Divide: rational numbers in fractional and decimal form **9-B5** Order of Operations: rational number computation	1 h	• Paper or light cardboard for cards • Calculators

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used, if you wish, as enrichment material for some students. You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution

$8 + \pi - 2\sqrt{29}^{0} \div 2 - 9$ (matches $\pi - 2$ and B, D, E, F, G)	$(30^2 - 60 + 1)^{\frac{1}{4}} \times (5 + 1)$ (matches $6\sqrt{29}$ and C, D)	$-\left(\frac{\sqrt{16}}{17} + \frac{11 - 5 \times 2}{17}\right)^{-1}$ (matches $-3\frac{2}{5}$ and A, D, E, G)	$2.05 \times 10^5 \div 10^3 \times 2 \times 10^2$ (matches 4.1×10^4 and G)	$\sqrt{0.09} \times \sqrt{1.1664} \times (10^{-1})^2$ (matches 3.24×10^{-3} and E, F, G)
$8 + 2^2 \times \sqrt{6.25}\,\pi - 8$ (matches 10π and C, D, E, F)	$2^{\frac{7}{2}} + (15 - \sqrt{49}) \times 9^{\frac{1}{4}}$ (matches $8(\sqrt{2} + \sqrt{3})$ and C, D, E, F)	$-3 \times (5 - 2) + \left(\sqrt{100}\right)^{-1}$ (matches -8.9 and A, D, G)	$(50 \div 2)^{\frac{2}{3}} \times \left(\sqrt{16} + 1\right)^{\frac{5}{3}}$ (matches $5^{\frac{2}{3}}$ and C, D, E, F, G)	$7 \times 10^0 \times \sqrt{40 - 5^2 \times 10^{-1} - 0.29} \times 10^4 \div 10^5$ (matches 42.7×10^{-1} and A, B, E, F, G)

UNIT 1 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Correctness of calculations	Almost completely correct calculations to match the ten cards and meet the specifications	Mostly correct calculations to match the ten cards but not always meeting the specifications	Many correct calculations to match the ten cards but with many errors and often not meeting the specifications	Errors in most calculations and most do not meet the specifications
Creativity of calculations	Use of a wide variety of strategies to create calculations	Use of a number of strategies to create calculations	Repetitive use of a few strategies to create calculations	No obvious strategies used to create calculations
Proper use of square roots	Consistently correct use of square roots involving both numbers greater than 1 and those less than 1	Usually correct use of square roots involving numbers greater than 1 and occasionally correct use of square roots less than 1	Correct use of square roots involving numbers greater than 1	Correct use of only some square roots involving numbers greater than 1
Proper use of exponents	Consistently correct use of integer and fractional exponents	Frequently correct use of integer and fractional exponents	Correct use of integer exponents only	Correct use of positive integer exponents only

UNIT 2 POLYNOMIALS

UNIT 2 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	Algebra tiles: x, $-x$, 1, and -1	All questions
Chapter 1 Introducing Polynomials				
2.1.1 Interpreting Polynomials	**9-B6 Polynomial Expressions: interpreting** • consolidate an understanding of what polynomials are and when they are used	1 h	None	Q1, 2, 3, 4, 7, 10, and 11
2.1.2 Adding and Subtracting Polynomials	**9-B7 Polynomial Expressions: add and subtract concretely, pictorially, and symbolically** • add and subtract polynomials concretely, pictorially, and symbolically	2 h	Algebra tiles	Q1, 2, 3, 4, 5, and 9
Chapter 2 Multiplying Polynomials				
2.2.1 Multiplying a Polynomial by a Monomial	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • multiply concretely, pictorially, and symbolically: a monomial by a monomial, a scalar by a monomial, a scalar by a polynomial, a monomial by a polynomial	1 h	Algebra tiles	Q2, 6, and 7
2.2.2 Multiplying a Binomial by a Binomial	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • multiply concretely, pictorially, and symbolically a binomial by a binomial **9-B10 Polynomial Expressions: evaluate** • compare the value of a polynomial prior to and after being simplified	2 h	Algebra tiles	Q4, 5, 7, and 9
2.2.3 Multiplying Polynomials Symbolically	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • multiply symbolically: a monomial by a monomial, a scalar by a monomial, a scalar by a polynomial, a monomial by a polynomial, and a binomial by a binomial y **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	2 h	None	Q1, 6, 9, and 10
GAME: Polyprod	Practise multiplying polynomials symbolically in a game situation	20–30 min	Dice	N/A

UNIT 2 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 3 Dividing Polynomials				
2.3.1 Dividing a Polynomial by a Monomial	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • divide concretely, pictorially, and symbolically: monomial by a monomial; polynomial by a scalar; polynomial by a monomial **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	1 h	Algebra tiles	Q2, 4, and 5
2.3.2 EXPLORE: Dividing a Polynomial by a Binomial (Optional)	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • divide concretely a polynomial by a binomial	1 h	Algebra tiles	Observe and assess questions
2.3.3 Dividing a Polynomial by a Binomial	**9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically** • divide concretely, pictorially, and symbolically a polynomial by a binomial **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	2 h	Algebra tiles	Q2, 4, and 6
2.3.4 EXPLORE: Creating Rectangles to Factor	**9-B9 Polynomial Factors: dimensions of a rectangle** • factor quadratic binomials and trinomials concretely	1 h	Algebra tiles	Observe and assess questions
CONNECTIONS: Using Number Patterns to Factor	Use patterns to factor polynomials	1 h	None	N/A
UNIT 2 Revision	Review the concepts and skills in the unit	1 h	Algebra tiles	All questions
UNIT 2 Test	Assess the concepts and skills in the unit	1 h	Algebra tiles	All questions
UNIT 2 Performance Task	Assess the concepts and skills in the unit	1 h	Algebra tiles	Rubric provided
UNIT 2 Assessment Interview	Assess the concepts and skills in the unit	30 min	Algebra tiles	All questions
UNIT 2 Blackline Masters	Algebra Tiles			

Math Background

• Work with polynomials is designed primarily to prepare students for algebraic work in higher grades. Although some real-life applications are included in this unit, they are fairly minimal—mostly the use of formulas for measurement.

• The concept of collecting like terms, an idea students have already encountered in Class VIII, is extended to addition and subtraction of polynomials. Both addition and subtraction are modelled first with algebra tiles and then abstracted to a more symbolic approach. Several models for subtraction are provided. The first is a take-away model, which is useful when the terms to be subtracted are already present in the minuend. The second model uses the zero principal and is used when the terms to be subtracted are not present in the minuend. The third builds on the relationship between adding and subtracting, i.e., that subtracting is a matter of finding a missing addend. Here students begin with the subtrahend and add enough to make the minuend. Subtracting by adding the opposite is mentioned, but this is not stressed since it is more a rule than a meaningful approach.

• Scalar multiplication is introduced prior to the multiplication of two binomials. Work with polynomial multiplication beyond two binomials is left to higher classes. Again, work with algebra tiles precedes any symbolic work, and the symbolic work is limited Students are shown the strategy of using a picture that looks like an algebra tile model to help them recognize the parts that must be multiplied in order to properly apply the distributive property when they multiply two factors.

• Students learn how to divide a polynomial by simple monomials, again first using algebra tiles and then symbolically. Expectations for symbolic division are quite limited, but they are addressed. Division is used as an introduction to the factoring of quadratic trinomials. Factoring is addressed in a very exploratory way using both algebra tiles and patterns based on numerical relationships.

• As students work through this unit, they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- They use problem solving in **question 6** of **lesson 2.2.3**, where they solve a fairly symbolic problem relating to polynomial multiplication, as well as in **question 9** of **lesson 2.2.3**, where the application is more practical.

- They use communication frequently as they explain their thinking in answering **questions B and E** of **lesson 2.3.2** and **question 10** of **lesson 2.2.3**. The last question in most lessons involves an element of communication in bringing closure to the lesson.

- They make connections in all of the **Try This** problems, where the new learning is connected to a problem they can already solve. Specific connections are also made for them. For example, **question 11** of **lesson 2.1.1** helps students connect the concept of a polynomial to what they already know about formulas and the **Connections** feature connects factoring polynomials to factoring numbers.

- They use reasoning in answering questions such as **question 11** of **lesson 2.2.2**, where they look at an alternative way to represent a product of polynomials and reason through it.

- They use representation and visualization regularly as they visualize or create algebra tile models for symbolic expressions. Examples include **questions 1 and 4** of **lesson 2.1.2**, where students are asked to relate the symbolic form of a polynomial sum or difference to its algebra tile representation.

Rationale for Teaching Approach

• The unit is divided into three chapters. **Chapter 1** allows students to recall and extend work with polynomials from Class VIII, including polynomial addition and subtraction. **Chapter 2** is designed to introduce polynomial multiplication, and **Chapter 3** introduces polynomial division and early explorations with factoring of polynomials.

• Polynomial multiplication and division are built primarily on an area model so that the operations are meaningful. Subsequent work with polynomials in Class X will be more symbolic, so the work in Class IX, particularly in the areas of multiplication, division and factoring, is very concrete and visual. There is some symbolic work with multiplication and division and even more with addition and subtraction.

• There are two **Explore** lessons, **lesson 2.3.2** explores division using algebra tiles, and **lesson 2.3.4** explores factoring using algebra tiles. The **Connections** feature allows students to link factoring polynomials and factoring numbers. The **Game** allows students to practice polynomial multiplication in a pleasant way.

• Algebra tiles are used extensively and can be made using cardboard. A template (blackline master) is provided at the end of Unit 2 in this teacher's guide.

Getting Started

Curriculum Outcomes	Outcome relevance
8 Add and Subtract Simple Algebraic Terms: solve problems **8** Polynomial Expressions: add and subtract concretely and pictorially **8** Multiply Polynomials by a Scalar: concretely, pictorially, and symbolically **8** SI Units: solve measurement problems **6** Parallelograms: relate the bases, height, and area and a parallelogram **6** Area of a Triangle: relate to area of a parallelogram	Students will need to review some calculation skills from previous grades, particularly the use of the zero principle, to help them work with positive and negative expressions. They are also reminded of what they know about representing variables with models (simple algebra tiles).

Pacing	Materials	Prerequisites
1 h	Algebra tiles: x, $-x$, 1, and -1	• addition, subtraction, and multiplication of integers

Main Points to be Raised

• To represent sums and differences properly, you need to collect like terms before the sum or difference is considered to be in final form.

• In a subtraction situation, the zero principle allows you to add pairs of "opposite" tiles with no value so that tiles can be taken away or subtracted. This is an important strategy to deal with situations when the minuend does not contain the tiles that need to be taken away.

Use What You Know—Introducing the Unit

• You may need to show students how to use algebra tiles or remind them if they have forgotten what algebra tiles are and how to use them.

• Point out that opposite tiles are shaded differently. The positive tiles are white and the negative are darker. Assigning the colours in this way is simply a matter of convention; the positive tiles could just as easily be dark and the negative tiles could be white. Show students that although the x-tile is 1 unit wide, just like the 1-tile, it is a different length, which is called x. The $2x$ is modelled using two x-tiles, $-3x$ with three $-x$-tiles, and $x + 2$ with one x-tile and two 1-tiles.

• Remind students of the zero principle (**part D**): any value and its opposite are 0. Pairs of opposite tiles can be added or removed without affecting the value of the quantity being modelled. For example, all three models below show $2x$.

$2x$ $2x$ $2x$

Observe students as they work. You might ask:

• *How do you know that you need to get rid of four tiles?* (If you start with 9 tiles and 5 tiles, you would have 14 tiles. But you are supposed to only have 10 tiles and that is 4 fewer.)

• *How do you know that one of the expressions is likely to have terms that are opposites or are very close to being opposites?* (You start with 18 tiles and end up with only 2, so most of them have to disappear. This can only happen if they are opposites.)

• *How do you know that you will have to use the zero principle to add pairs of opposite tiles so you can be left with seven tiles?* (If you could take the 7 tiles from the 10, there would only be 3 left and not 7. You will have to add extra tiles without changing the value, and that can only happen if you use the zero principle.)

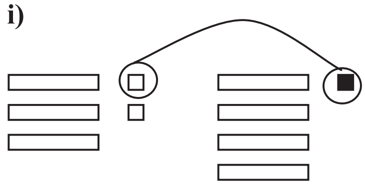
Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually.

Answers

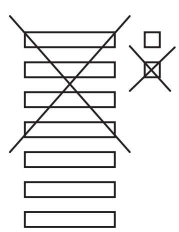
A. i)



$3x + 2$ + $4x - 1$
$= 7x + 1$

ii) You need 3 tiles to represent $3x$ and 2 tiles to represent +2, so there are 5 tiles for $3x + 2$.
You need 4 tiles to represent $4x$ and 1 tile to represent –1, so there are 5 tiles for $4x - 1$.
When you add them, +1 and –1 balance out to 0, so you take them away and you are left with only 8 tiles.

B. i)



$(7x + 2) - (4x + 1) = 3x + 1$

B. ii) You need 7 tiles to represent $7x$ and 2 tiles to represent +2, so there are 9 tiles for $7x + 2$. Take away $4x$ and 1, and only 4 tiles are left.

C. You need 3 tiles to represent $3x$ and 1 tile to represent –1, so there are 4 tiles altogether.
You need 3 tiles to represent $-3x$ and 1 tile to represent 1 so there are 4 tiles altogether.
Combine $3x$ and $-3x$ to get 0, so take those 6 tiles away.
Combine –1 and +1 to get 0, so take those 2 tiles away—0 tiles are left.

D. i) The $-x$ and $+x$-tiles and the –1 and +1 tiles that you add do not change the value of the expression because their value is 0 but they allow you to subtract three x-tiles and two 1-tiles.
ii) 2 tiles: one $-x$-tile and one –1-tile

E. *Sample responses*:
i) $(4x - 5) + (3x + 2) = 7x - 3$
ii) $(-7x + 3) + (6x - 2) = -x + 1$
iii) $(6x - 4) - (5x + 2) = x - 6$
iv) $(4x - 2) - (x - 5) = 3x + 3$

F. *Sample response*: $(4x - 2) + (-3x - 1) = x - 3$
Start with an expression represented by 6 tiles and subtract an expression represented by 4 tiles. The difference is an expression with 4 tiles.

1. a) –7	**b)** –7
c) +6	**d)** +4
2. a) $6x + 12$	**b)** $20x - 8$
c) $-6 + 3x$	**d)** $-8x + 20$

3. a) 1.89 square units
b) 4.41 square units
c) 1.62 square units
d) 3.14 or π square units

4. a) 6 units
b) 6.28 or 2π units
c) 9 units

5. a) –1
b) 19
c) –4.5
d) –2.4

Supporting Students

Struggling students

• If students are struggling with the use of the algebra tiles, you may have to show more examples like those in **parts A and B** before students can proceed.

• Some students might need to be reminded of the area formulas for triangles, parallelograms, and/or circles in order to complete **questions 3 and 4**.

Enrichment

Students might make up additional problems beyond those in **part F**. Make these problems available for subsequent problem solving experiences for other students.

2.1.1 Interpreting Polynomials

Curriculum Outcomes	Outcome relevance
9-B6 Polynomial Expressions: interpreting • consolidate an understanding of what polynomials are and when they are used	Students need to become reacquainted with what polynomials are and when they are used. Students will make the connection between polynomials and measurement formulas they already know.

Pacing	Materials	Prerequisites
1 h	None	• perimeter and circumference formulas • area formulas for triangles, rectangles, parallelograms, and circles • meaning of exponents • addition and subtraction of positive and negative integers

Main Points to be Raised

• A polynomial is an algebraic expression. In a polynomial, all of the powers to which the variables are raised must be whole numbers. More than one variable can be involved.

• Polynomials are often classified by how many terms are involved or by their degree.

• Terms in the polynomials are described by referring to the variable involved along with its coefficient or the constant, if no variable is involved.

• Before polynomials can be classified according to the number of terms, like terms must be collected.

• Polynomials are evaluated by substituting each occurrence of a variable with the value for that variable.

• Polynomials can be used to represent general situations. For example, when we say $4s$ represents the perimeter of a square with side length s, we are using a polynomial to represent a general situation.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. These questions provide an opportunity to use a polynomial that is familiar, namely $\frac{1}{2}bh$.

Observe while students work. You might ask:

• *What do you need to know in order to use the formula?* (the value of the base and the value of the height)

• *What substitutions can you use in the area formula?* ($\frac{1}{2}b$ for the h)

• *Can you think of another situation where you can calculate an area when you know just one measurement?* (the area of a circle if you know the radius — $A = \pi r^2$; the area of a square if you know the side length — $A = s^2$)

The Exposition—Presenting the Main Ideas

• You may wish to record a number of polynomials and a few non-polynomials on the board. The polynomials should be of different degrees, with a variety of coefficients (not all whole numbers). In some of the polynomials, include like terms that have not been combined. For example, you could write:

$3x^2$ $\quad$ $4.4x$ $\quad$ $3.2t + 9m$ $\quad$ $x + y + z + w$ $\quad$ $4^{4x} - 1$ $\quad$ $\sqrt{3x}$ $\quad$ $\frac{3x}{x^2}$ $\quad$ $5m + 3mn - 2m$

Use the polynomials and non-polynomials to discuss:

- what makes something a polynomial;
- how to determine the degree;
- how to name the polynomial according to the number of terms;
- what a variable is; what a coefficient is; what a constant is;
- how to evaluate a polynomial; and how to simplify by collecting like terms.

• Once this has been completed, encourage students to read the exposition on their own and allow them to ask questions about anything that is still unclear.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between what was done in **parts A and B** to the new, more formal polynomial language now that the main ideas in the Exposition have been presented. You might handle **part C** as a whole class.

Using the Examples

• Students can read through the examples, which provide example of how polynomials are used (**example 1**) and how to classify polynomials (**example 2**).

• Although **example 1** asks for polynomials and the solutions are monomials, students can be encouraged to find their own polynomials. For example, $3n + 2$ could describe the number of hours needed to drive n round trips with a provision for a one-hour lunch break during each one-way trip. Ask students why it is a good idea to choose letter variables that reflect what they represent, for example, r for number of round trips.

Practising and Applying

Teaching points and tips

Q 5: Students might need support in creating their own polynomials. Refer them back to **example 2** and to the part of the exposition that talks about the constant term.

Q 9: Refer students back to **example 2** for help with this question.

Q 11: Make sure students understand that most of the measurement formulas with which they are familiar are polynomials using one or more variables.

Q 12: Some students may need support, particularly with **part c)**. Remind them of how to write a percentage as a decimal and help them to see why you would multiply the test mark by 0.7 and the project mark by 0.3, and then add the parts.

Suggested assessment questions from Practising and Applying

Questions 1 to 4	to see if students can apply polynomial vocabulary
Question 7	to make sure students understand what like terms are and how to combine them
Question 10	to see if students can apply polynomials to describe situations
Question 11	to see if students recognize measurement formulas as polynomials

Answers

A. i) $A = \frac{1}{2}bh$ **ii)** $A = \frac{1}{4}b^2$

B. Usually there would be two, but since the height can be written in terms of the base, you need only one.

C. i) monomial of degree 2 **ii)** $A = h^2$

1. a) binomial **b)** trinomial
c) monomial **d)** binomial
e) binomial **f)** monomial

2. a) 1 **b)** –2
c) 2.3 **d)** $\frac{3}{4}$

3. B, C, and D

4. a) 3 **b)** 4
c) 2 **d)** 1

Answers [Continued]

5. *Sample response*: $x^2 - 3xy - 3$; $2y + (-3)$ **6. a)** 184 **b)** 4 **c)** 93 **d)** -6 **7. a)** $3t$ and $7t$; $10t - 3t^2$ **b)** $-2n$ and $-7n$; $3 + 8m - 9n$ **c)** $3p$ and $-17p$ as well as $4q$ and $-2q$; $-14p + 2q$ **d)** There are no like terms. **8.** *Sample responses*: **a)** $3x^3$ **b)** $2m + 4$ **c)** $p^3 - 5p^2 + p$ **d)** $2m + 1$ **e)** $-\frac{1}{2}(a^2 + b^2)$ when $a = 2$ and $b = -2$ **9.** *Sample response*: $2m - 3$, $2m^2 - 3$, $-3 + 2m$, $-3 + 2m^2$, $-3m - 2$, $-3m^2 - 2$, $2 - 3m$, $2 - 3m^2$, $2 - m$, $2 - m^2$, $-3 - m$, $-3 - m^2$	**10.** $6l + 4s$ **11.** Any two of $2r$ (diameter), πr^2 (area), and $2\pi r$ (circumference) **12.** *Sample responses*: **a)** $80b + 85c$ to describe the cost of b kg of beef and c kg of chicken; or, $85c - 80b$ to describe the difference in cost between b kg of beef and c kg of chicken **b)** $2.25n$ to describe the time in hours for n trips from Thimphu to Punakha; or, $154r$ to describe the distance in kilometres for r round trips between Thimphu and Punakha **c)** $0.7e + 0.3p$ to calculate the final mark **d)** e^3 to describe the volume; or, $6e^2$ to describe the total surface area **13.** *Sample response*: Same: binomials, variable s, worth 8 if $s = 1$ Different: one is degree 1 and one is degree 2

Supporting Students

Struggling students

Some students might have difficulty creating polynomials. You might support those students by allowing them to spend more time describing (with words) the polynomials given to them.

Enrichment

Students could prepare conditions like in **question 9** to exchange with others. They would be required to find several answers in each case. The number of terms, degree, and other components of polynomials could be integrated into these challenges.

2.1.2 Adding and Subtracting Polynomials

Curriculum Outcomes	Outcome relevance
9-B7 Polynomial Expressions: add and subtract concretely, pictorially, and symbolically • add and subtract polynomials concretely, pictorially, and symbolically	Because of the connection between adding and subtracting polynomials and simplification by combining like terms, this outcome supports students' later work with multiplication, division, and factoring of polynomials.

Pacing	Materials	Prerequisites
2 h	Algebra tiles	• addition and subtraction of integers

Main Points to be Raised

• The tiles representing y and x are different lengths to help distinguish them. As a result, the square tiles for x^2 and y^2 are also different sizes.

• The rectangular tiles representing xy and $-xy$ have x as one dimension and y as the other. This is consistent with the notion that the tile represents an area, which is the product of the two dimensions of the rectangle.

• The tiles representing the negative values of x^2, y^2, xy, x, and 1 have slightly different shadings in the student book as an additional cue (besides the dimensions) to help students distinguish them.

• Adding polynomials amounts to combining like terms.

• You can subtract polynomials using a take-away model, using the zero principle, finding the missing addend model, or by adding the opposite.

• The zero principle (that a tile and its opposite can be removed or added without affecting the overall value) is essential for simplifying polynomials. The zero principle also makes it possible to subtract in certain situations.

Try This—Introducing the Lesson

A. Students can solve this individually or with a partner.

Observe while students work. You might ask:

• *How might you use multiplication to help you complete the addition?* (When you have to add the same value several times, like 60 or 45, you can just multiply instead.)

• *What products would you use?* (6 × 60 and 4 × 45)

• *How does using multiplication simplify the task?* (It takes a lot less time to multiply 60 by 6 than to add 60 six times.)

Have students share how they approached the problem. Some may not have multiplied, for example, 120 + 120 + 120 + 90 + 90 = 540 and 540 ÷ 10 = 54.

The Exposition—Presenting the Main Ideas

• Begin by making sure students recall different ways to subtract integers.

For example, present the question 3 – 5 = ♦.

Ask students how they solve this. They might figure out how to move from 5 to 3 on the number line (move 2 to the left so –2) or they might use the zero principle by adding 2 negative counters and 2 positive counters so that 5 negative counters can be taken away.

Contrast this with –4 – (–2) by thinking of 4 as 4 negative ones. When 2 negative ones are removed, there are 2 negative ones left, so the result must be –2. The result can be found by thinking of subtraction as take-away.

• Show how the same ideas can be used with polynomials by modelling, in a parallel way, $3x - 5x$ and $-4x - (-2x)$.

• Ask students to read through the exposition.

• Make sure they understand by asking them to model and solve a simple addition such as $(7x^2 + 5x) + (2x + 4)$ and a simple subtraction such as $(3x + 8) - (2x + 9)$ using algebra tiles.

Revisiting the Try This

B. Students are encouraged to see how gathering the 60s and the 45s in **part A** is very similar to adding like terms with polynomials. This should help them build a connection that will support further learning with polynomials.

Using the Examples

- Write the two problems used in **examples 1 and 2** on the board. Ask students to try to solve them with their books closed. Once they have completed their own solutions, students can compare their solutions with those shown in the worked examples. Make sure they understand that the crossed-out tiles represent tiles being taken away.
- Students can read **example 3** on their own.
- Provide time for students to ask questions if anything in the examples is unclear to them.

Practising and Applying

Teaching points and tips

Q 3: Some students might forget the meaning of the various tile sizes. You might want to remind them that the y-tiles are shorter than the corresponding x-tiles.

Q 6: Encourage students to decide in advance if they will use the take-away model or the missing addend model to calculate. Ask them to explain their choice.

Q 10: This closure question will allow students to synthesize what they have learned by putting it into their own words.

Common errors

Students will often make mistakes subtracting and adding negative integers. Suggest they check subtractions by performing the corresponding addition.

Suggested assessment questions from Practising and Applying

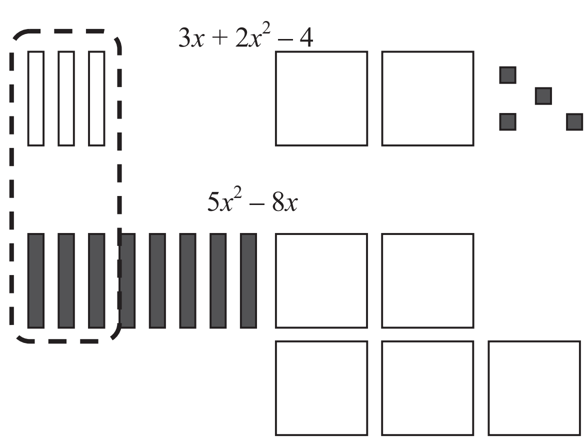
Questions 1 and 4	to see if students can perform the required operations with tiles
Questions 2 and 5	to see if students can add or subtract polynomials symbolically
Question 3	to see if students recognize an operation from the model
Question 9	to see if students can apply their knowledge of polynomials to a problem situation

Answers

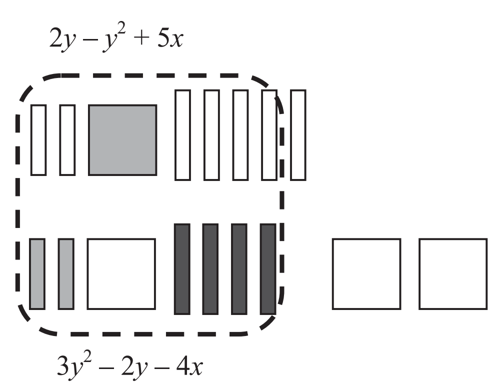
A. *Sample response*: $(6 \times 60 + 4 \times 45) \div 10 = 54$

B. *Sample response*: I combined all the 60s and all the 45s. It is just like combining all the x-terms or y-terms.

1. a) $-5x + 7x^2 - 4$



1. b) $2y^2 + x$

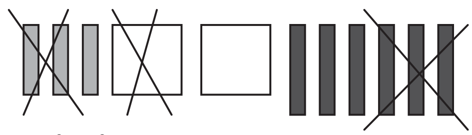


2. a) $-4x - x^2 - 4$
b) $4m^2 - 2m^3 + 7.5m + 2$
c) $7y^3 - y^2 - 11x + y$

3. $(y^2 - y + 2x + 1) + (-y^2 - y - x - x^2 - 2)$;
the sum is $x - 2y - 1 - x^2$

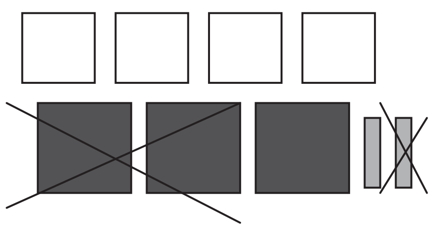
4. a) $-y + y^2 - 3x$

$(-3y + 2y^2 - 6x) - (-2y + y^2 - 3x)$



b) $4y^2 - x^2 - y$

$(4y^2 - 3x^2 - 2y) - (-2x^2 - y)$



5. a) $-6y - 2y^2 - 7x$ (get rid of 3y by adding –3y and add another $-3y$; get rid of $2y^2$; add another $-7x$)
b) $x^2 - 4y + 2y^2$ (add $2y^2$; get rid of the y by adding a $-y$ and add another $-3y$; get rid of $-x^2$ by adding a x^2)

6. a) $-2m + m^3 - 8t$
b) $-m^2 - 2m - 24$

7. a) $11x + 4y - 2y^2$
b) $-5k - 5h - 6y^2$
c) $4m^2 - 2m + 7r - 8$

8. $(-2y^2 - 3x - 2) - (-2y^2 - x) = -x - 2$

9. a) $16x + 12$
b) $4x + 15$

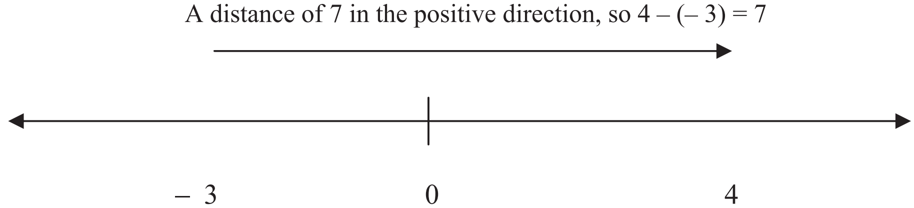
10. *Sample response*:
When you subtract, you sometimes have to decide how to subtract depending on what you're subtracting; with addition, you always just combine like terms.

Supporting Students

Struggling students

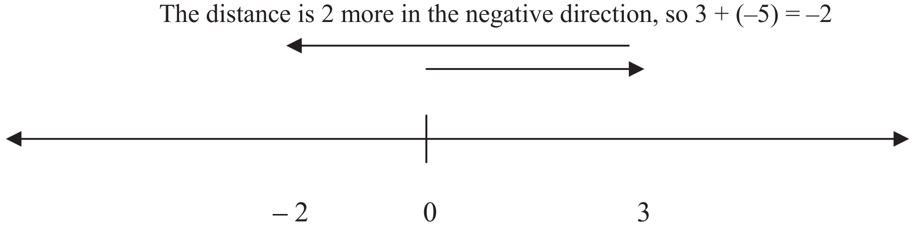
Some students who have trouble subtracting integers might benefit from using a number line model. For example, to model a subtraction such as 4 – (– 3), they note the distance and direction from –3 to 4.

A distance of 7 in the positive direction, so 4 – (– 3) = 7



To model an addition such as 3 + (–5), they can put arrows together.

The distance is 2 more in the negative direction, so 3 + (–5) = –2



Chapter 2 Multiplying Polynomials

2.2.1 Multiplying a Polynomial by a Monomial

Curriculum Outcomes	Outcome relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • multiply concretely, pictorially, and symbolically: a monomial by a monomial, a scalar by a monomial, a scalar by a polynomial, a monomial by a polynomial	Students need to be able to multiply a polynomial by a monomial before they can multiply binomials, which will be used in Class X to work with quadratics.

Pacing	Materials	Prerequisites
1 h	Algebra tiles	• multiplication of integers

Main Points to be Raised

• The various meanings of multiplication with whole numbers are still pertinent with polynomials, particularly the notion that the area of a rectangle is the product of its dimensions.

• Sometimes it is helpful to use "guide tiles" to show how wide or how high a rectangle should be (as in **example 1**).

• It can be difficult for some students to model multiplications of positives by negatives with algebra tiles. It is important to remember that, for example, $-3(2x + 4)$ could be represented by a rectangle with three rows, each showing –2x and –4, or by three separate groups with –2x and –4 in each group. To model $-x(2x + 3)$ requires a rectangle with one dimension of x and the other of –2x and –3. This is done by using two $-x^2$-tiles and three $-x$-tiles. There are no exercises requiring the multiplication of two negatives with algebra tiles because, in this instance, the concrete representation would not be helpful.

Try This—Introducing the Lesson

A. and B. Students can work alone or with a partner. To prepare students for working on **part A**, you can check to see if students know what an exchange rate is.

Observe while students work. You might ask:

• *Why might someone represent the amount as* $2300n + 1200b$ *+?* (There are 2300 ngultrums and 1200 baht. If you used the same variable, you would not be able to tell which number went with which type of money.)

• *Why is it more convenient to represent the amount with one variable?* (If you know the exchange rate you only have to make one substitution instead of two and that is quicker.)

• *Why do you have to double both the baht and the ngultrums to represent twice as much money?* (Because you have double everything the traveller has and she has both baht and ngultrums.)

The Exposition—Presenting the Main Ideas

• Ask students to read through the exposition on their own.

• Discuss with them how area models make sense for multiplying a linear (degree 1) polynomial both by another linear polynomial and by a whole number.

• Point out that to model a multiplication involving a polynomial of degree higher than 1, such as $3x(x^2 + 1)$, it is not possible to use an area model since there is no tile that can show x^2 as a linear dimension of a rectangle as x^2 is already an area. As well, $3x \times x^2 = 3x^3$ and that represent a cubic or volume situation.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between what was done in **parts A and B** within a real life context to the now more formal multiplication of a polynomial by a monomial. You might handle **part C** as a whole class.

Using the Examples

Allow students to complete both examples and then compare their answers to what is shown in the text.

Practising and Applying

Teaching points and tips

Q 1: This question requires students to use both the equal groups and area representations of multiplication.

Q 2: Some students might need help in seeing that to model **part a)** they need to use all negative tiles.

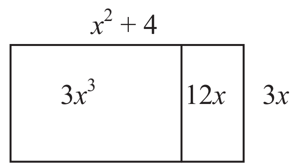
Q 5: This question provides another opportunity to apply polynomial multiplication to a contextual situation.

Q 6: Although technically this is a division question, students can solve it strictly using multiplication strategies.

Q 9: This question provides students an opportunity to synthesize what they have learned in the lesson.

Common errors

Students often try to represent a product such as $3x(x^2 + 4)$ as a rectangle with algebra tiles when it is not possible to do so. Encourage them to see that the rectangle model with algebra tiles only applies when the dimensions are linear (when the polynomials are of degree 1). A pictorial representation of an area can be shown, however, even with the product $3x(x^2 + 4)$, though the representation is not an algebra tile model.



Suggested assessment questions from Practising and Applying

Question 2	to see if students can visualize and model a product using algebra tiles
Question 6	to see if students can solve a multiplication problem by working backwards
Question 7	to see if students are able to perform monomial multiplication and simplify

Answers

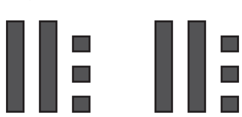
A. x represents the exchange rate; how many ngultrums it costs to buy 1 baht.

B. $4600 + 2400x$

C. $2 \times (2300 + 1200x)$

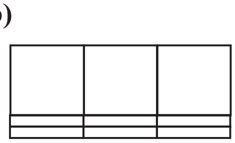
1. a) $2(-2x^2 + 1)$ **b)** $4(3x)$ **c)** $4(5x - 3)$

2. a)



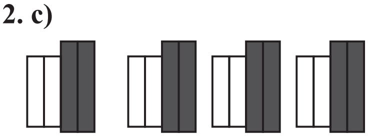
$2(-2x - 3) = -4x - 6$

b)



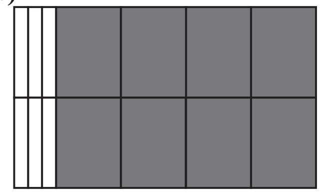
$3y(y + 2) = 3y^2 + 6y$

2. c)



$4(2y - 2x) = 8y - 8x$

d)



$2x(3 - 4y) = 6x - 8xy$

Answers [Continued]

3. a) $-3x^2 - 2xy$

b) $24 - 6t + 18t^2$

c) $2jk^2 - 6jk$

d) $15m + 5m^3 - 10r$

4. a) $6x$

b) $8x - 2$

5. a) $6000x + 6050y$

b) $6000x + (6050 \times 2x) = 18{,}100x$

c) $6000x + (6050 \times 0.5x) = 9025x$

6. *Sample responses*:
a) $3(2t^2 - t + 5)$

b) $2(4xy - 5y^2 + 3y)$

c) $2(7x - 8x^2)$

d) $4(2 - 4x + 3y)$

7. a) $50 - 2c$

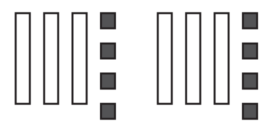
b) $10x^2 - 14xy - 6y$

c) $15y - 20x - 4y^2$

d) $2xy - 3x^2y + 4xy^2$

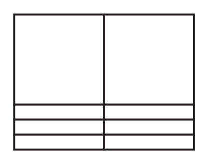
8. A monomial × a monomial = monomial: $3x(2x) = 6x^2$, but a monomial × a binomial = binomial, since it results in a sum of two unlike terms: $3x(2 + x) = 6x + 3x^2$.

9. *Sample responses*:
a)



$2(3x - 4) = 6x - 8$

b)



$2x(x + 3) = 2x^2 + 6x$

Supporting Students

Struggling students

Many students struggle with setting up the tiles when one of the factors is negative. If students find it easier to solve these questions symbolically, the tiles need not be used.

Enrichment

Students can create polynomial multiplications and then give the product to a partner to see if they can figure out the factors.

2.2.2 Multiplying a Binomial by a Binomial

Curriculum Outcomes	Outcome relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • multiply concretely, pictorially, and symbolically a binomial by a binomial **9-B10 Polynomial Expressions: evaluate** • compare the value of a polynomial prior to and after being simplified	Concrete work with multiplication of binomials using algebra tiles will help students better understand how the distributive property is used to multiply polynomials.

Pacing	Materials	Prerequisites
2 h	Algebra tiles	• multiplication of integers • area formulas for parallelogram, rectangle, trapezoid, and circle

Main Points to be Raised

• Two linear (degree 1) binomials can be multiplied by using their expressions as dimensions of a rectangle and calculating its area.

• When you multiply two linear binomials involving one variable, such as x, the rectangle that is formed has four sections: an x^2 section, a constant section, and two sections of x-tiles (one vertical and one horizontal). These sections correspond to the four partial products when the binomials are multiplied.

• When negatives are multiplied by positives, there will always be either 2 or 4 dark-tiled sections in the rectangle.

• Two linear binomials involving different variables can be modelled in a similar way, as a rectangle with four parts.

Try This—Introducing the Lesson

A. Students should solve the problem in pairs or in small groups.

Observe while students work. You might ask:

• *Why did Meto multiply numbers that are two apart?* (because if one is one more than a number and the other is one less, they have to be two apart)

• *What two numbers could Meto have multiplied to get* 48*?* (6 and 8)

• *What would Drapka say if Meto said* 120*?* (11 since it is between 10 and 12)

The Exposition—Presenting the Main Ideas

• Students should work through the exposition on their own or with a partner by using their own tiles to follow along.

• You can check understanding by asking how $(2x - 1)(3x + 2)$ would be modelled with tiles.

Revisiting the Try This

B. and C. These questions allow an opportunity to make a formal connection between what was done in **part A** and the new learning in the exposition. Many students will appreciate the opportunity to actually try out the trick.

Using the Examples

• The example provides another opportunity for students to see how guide tiles can be used to help make sure their rectangles are the correct width and height. Before students read through this example, you could read aloud the problem in the example and ask them what variables describe the two dimensions.

• Next, allow students to read through the example on their own. If you wish, you might ask them to try to set up the problem first and then check their work by reading through the example.

Practising and Applying

Teaching points and tips

Q 1: This question is designed to help students understand how negative tiles are used (and not used) in multiplication.

Q 4: Observe whether students immediately recognize that all coefficients and constants are positive for **part a)** but not for **parts b) or c)**.

Q 6: Encourage students to read the descriptions carefully to represent the dimensions of the rectangles. Once they create the binomials, they should check them against the descriptions.

Q 7: Some students might need to be reminded of the area formulas required for this question.

Q 10: Some students will notice that the sum of the absolute values of the coefficient and constant could be 5 in one binomial and 3 in the other to satisfy the conditions. It does not matter if the values are positive or negative.

Q 11: This question may be difficult for some students, particularly since they cannot actually cut the algebra tiles. They might want to make a square out of paper and follow the directions to better understand what Pema is doing.

Suggested assessment questions from Practising and Applying

Question 4	to see if students can go from the model to a symbolic representation
Question 5	to see if students can correctly model a polynomial multiplication
Question 7	to see if students can use polynomial multiplication in an application situation
Question 9	to see if students can work backwards (in preparation for dividing and factoring)

Answers

A. You add 1 and then take the square root.

B. i) $x + 1$ and $x - 1$
ii) $(x + 1)(x - 1)$; if you multiply, you get $x^2 - 1$, so add 1 to get x^2 and then the principal square root is x.

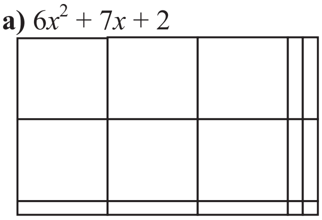
1. A, since $(2x - 1)(x + 1) = 2x^2 + 2x - 1x - 1$

2. The width is $2x + 3$ and the height is $x - 2$. The area is the product of the two dimensions.

3. $(2x - 4)(3x + 1)$;
$(2x - 4)(3x + 1)$ uses 24 tiles:
six x^2-tiles, two x-tiles, twelve $-x$-tiles, and four -1-tiles.
$(2x + 1)(3x - 4)$ uses 21 tiles:
six x^2-tiles, three x-tiles, eight $-x$-tiles, and four -1-tiles.

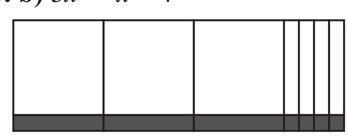
4. a) $(2x + 1)$ and $(y + 1)$
b) $(x + 1)$ and $(x - 2)$
c) y and $(3y - 1)$

5. a) $6x^2 + 7x + 2$



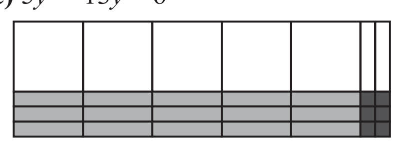
$(2x + 1)(3x + 2)$

5. b) $3x^2 + x - 4$



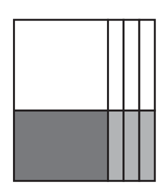
$(3x + 4)(x - 1)$

c) $5y^2 - 13y - 6$



$(y - 3)(5y + 2)$

d) $x^2 - xy + 3x - 3y$

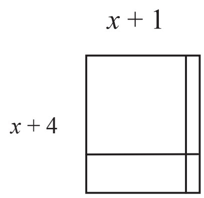


$(x + 3)(x - y)$

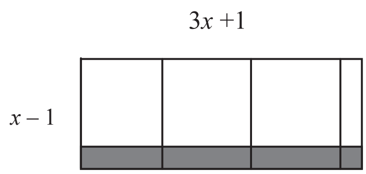
5. e) $6 - 3x + 2y - xy$

$(2 - x)(3 + y)$

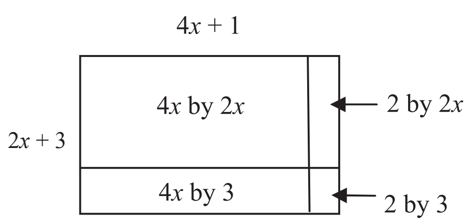
6. a) $x^2 + 5x + 4$



b) $3x^2 - 2x - 1$

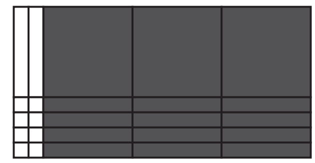


c) $8x^2 + 16x + 6$



7. a) $3x^2 + 7x + 2$ **b)** $4x^2 + 2x - 12$
c) $x^2 + 8x + 12$ **d)** $\pi(4x^2 + 24x + 36)$

8. a) $8 - 10x - 3x^2$



$(2 - 3x)(4 + x)$

b) 11
c) 11
d) 1
e) If you multiply 11 × 1, you get 11, just as $(2 - 3x)(4 + x)$ is equal to 11 for $x = -3$.

9. $6x(x + 2)$ or $3x(2x + 4)$ or $2x(3x + 6)$ or $x(6x + 12)$

10. *Sample response*: $(2x + 1)(x - 4)$ or $(x - 2)(2x + 3)$ or $(3x + 2)(2x + 1)$ or $(2x + 1)(x + 4)$ or $(x + 2)(4x + 1)$

11. The original x^2 is made up of a rectangle $(x + 1)(x - 1)$ and 1 more, so $(x + 1)(x - 1) = x^2 - 1$.

12. Both times you make a rectangle and calculate the area. When you multiply a binomial by a monomial there are only two kinds of tiles, but there could be 3 or 4 kinds of tiles when you multiply two binomials.

Common errors

Sometimes students forget to take away the guide tiles they have used to help them make sure that their rectangles are the right size, and they count them in the product. It is essential to remind students to remove them as soon as the area has been modelled.

Supporting Students

Struggling students

You might use only positive coefficients and constants for students who struggle with how to handle the negative tiles. The negative tiles can be introduced when their confidence improves.

Enrichment

Challenge students to find all possible polynomial multiplications that require a specific number of tiles (as in **question 10**).

2.2.3 Multiplying Polynomials Symbolically

Curriculum Outcomes	Outcome relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • multiply symbolically: a monomial by a monomial, a scalar by a monomial, a scalar by a polynomial, a monomial by a polynomial, and a binomial by a binomial y **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	The ability to multiply two binomials will help prepare students for work with quadratics in Class X.

Pacing	Materials	Prerequisites
2 h	None	• multiplication of integers • area formulas for trapezoid, circle, square, and triangle

Main Points to be Raised

• When two linear binomials are multiplied, there are four partial products, with each term of the first binomial being multiplied by each term of the second binomial. Sometimes the terms can be combined so the result is a binomial, but it is usually a trinomial or a polynomial with four terms if the binomials use different variables.

• If the binomials are of the form (ax + b)(cx + d), the coefficient of x^2 will be ac, the coefficient of x will be (ad + bc) and the constant will be bd.

• Regardless of the variables and coefficients contained in the binomials you multiply, you can always multiply the first terms, the outer terms, the inner terms, and the last terms and combine them.

• Though the multiplication of two binomials is shown as **F**irst terms, **O**uter terms, **I**nner terms, and then **L**ast terms (FOIL), the order does not matter as long as each term in the first binomial is multiplied by each term in the second binomial (or vice versa). Products in the student book and in the answers are shown in a variety of forms.

Try This—Introducing the Lesson

A. Students should solve the problem in pairs or in small groups using algebra tiles. This problem helps them see that patterns are sometimes useful to help develop algebraic ideas.

Observe while students work. You might ask:

• *Why are there always two x^2-tiles?* (One dimension has a 2x part and the other dimension has an x part, so that is how we get two x^2-tiles.)

• *What do you notice about the how the constant term in the product relates to the original constant terms?* (It is the product of the two constants in the binomials.)

• *Why are there more x-tiles than x^2-tiles in the product?* (The x-tiles result from multiplying two sets of terms and adding them together. Since the values are all positive, there are likely to be more than two.)

The Exposition—Presenting the Main Ideas

Students should be able to read through the exposition on their own. They can confirm what they are reading by using algebra tiles if they wish.

Revisiting the Try This

B. This question allows students an opportunity to revisit their prediction from **part A** by multiplying the polynomials directly.

Using the Examples

Before asking students to read the example, you might write the product (3x – 15)(2x + 10) on the board. Have students multiply and then combine like terms. Show them how the x terms are removed as a consequence of the zero principle since the two products involving x were opposite. Point out that they started with two binomials, multiplied, and ended up with a binomial. Contrast this with the product of (3x – 15)(2x + 4), which is a trinomial. Next, ask students to read through the example on their own.

Practising and Applying

Teaching points and tips

Q 2: Notice whether some students realize they only need to check the inner and outer terms and not perform the entire multiplication.

Q 4: This question and similar ones in other lessons allow students to begin to see that when two polynomials are multiplied to create another one, the polynomials can be evaluated separately and the values multiplied or the polynomials can be multiplied first and the product evaluated. The result is the same in both cases. This will help prepare them for the **Connections** section near the end of the unit.

Q 6: Students will need to set up and solve the equation 3▲ – 2 = 10 to solve the problem.

Q 7b: You may need to confirm that the piece of the figure on the right is a half-circle and so the formula for the area of the circle, if halved, can be used.

Q 7c: This area can be found by subtracting from the area of the rectangle with dimensions of $2y + 5$ and $4y - 3$.

Q 9: Students may need help seeing that these connections can be made:

- Since $(x + 2)^2 = x^2 + 4x + 4$, $42^2 = 400 + 160 + 4$.
- Since $(x - 1)^2 = x^2 - 2x + 1$, $79^2 = 6400 - 160 + 1$.
- Since $(x + y)(x - y) = x^2 - y^2$, $53^2 - 47^2 = (53 + 47)(53 - 47) = 600$.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can multiply binomials
Question 6	to see if students can solve a problem using polynomial multiplication
Question 9	to see if students can apply polynomial multiplication in a new context
Question 10	to see if students can communicate their understanding of polynomial multiplication

Answers

A ii) The product is $2x^2 + 61x + 30$. I did not have enough algebra tiles, so I used a pattern. I noticed that the coefficient of x^2 was always 2. I noticed that the constant kept going up by 1 and this would be the 30th constant. I noticed that the middle coefficient kept going up by 2. Since it started at 3, the 29th increase, which is the 30th coefficient, would have to be 61.

B. Yes, since if you multiply each term in the first binomial by each term in the second one, you get $2x^2 + x + 60x + 30$, which simplifies to $2x^2 + 61x + 30$. This is what I predicted.

1. a) $15x^2 + 26x + 8$
b) $24y^2 + 22y - 10$
c) $-6xy - 24y - 8x - 32$
d) $6y^2 + 34y + 48$

2. a) 3
b) 2
c) 4
d) 2

3. a) $5y^2 + 24y + 29$
b) $-18x - 8xy + 3y + 18$

4. a) $35 + 13s - 12s^2$
b) $-112 = 35 + 13(-3) - 12(-3)^2$
c) 16
d) –7
e) If you multiply the answers to **4c) and d)**, you get the same answer as **4b)**: 16 × (–7) =–112.

5. $8n^2 + 20n + 12$

6. The product of (3x – 1)(2x + ▲) is 6x² + ▲3x – 2x – ▲. The coefficient of x^2 is 6. The coefficient of x must be 10 (4 more than the coefficient of x^2). The coefficients of x are ▲3 and –2; ▲3 + (–2) = 10. The value of ▲ is 4.

7. a) Area = $\frac{1}{2}bh = \frac{1}{2}(2x - 1)(2x + 8) = 2x^2 + 7x - 4$
b) Area of a square = side² = $(4x + 2)^2$
Area of a semi-circle = $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2x + 1)^2$;
Total area = $16x^2 + 16x + 4 + \frac{1}{2}\pi(4x^2 + 4x + 1) =$
$(16 + 2\pi)x^2 + (16 + 2\pi)x + 4 + \frac{\pi}{2}$
c) Area of small rectangle = $(2y - 1)(2y + 5) = 4y^2 + 8y - 5$
Area of large rectangle = $(4y - 3)(2y + 5) = 8y^2 + 14y - 15$
Area of large rectangle – Area of small rectangle = 2 × Area of both triangles = $4y^2 + 6y - 10$
Area of both triangles = $2y^2 + 3y - 5$
Total area of shape = Area of small rectangle + Area of both triangles = $6y^2 + 11y - 10$

Answers [Continued]

8. $4ax$ **9. a)** Calculate $40^2 + 160 + 4 = 1764$ **b)** Calculate $80^2 - 160 + 1 = 6241$ **c)** Calculate $(50 + 3)^2 - (50 - 3)^2 = 600$	**10.** To calculate 32×43, multiply each part of 32 (30 and 2) by each part of 43 (40 and 3): $(30 \times 40) + (30 \times 3) + (2 \times 40) + (2 \times 3) = 1376$ For $(3x + 2)(4x + 3)$, you multiply each part of $3x + 2$ by each part of $4x + 3$: $(3x)(4x) + 3x(3) + 2(4x) + (2)(3) = 12x^2 + 17x + 6$

Common errors

Many students choose to multiply only first terms and only last terms, as in $(2x - 3)(4x + 7) = 8x^2 - 21$, which is incorrect. These students might benefit from continued use of algebra tiles. Be sure to remind them that they are always trying to make a rectangle with the length and width dimensions representing the binomials to be multiplied.

Supporting Students

Struggling students

Students who are struggling might use diagrams to support their thinking. For example, for $(5x + 2)(3x + 4)$:

$5x$	$+ 2$	
$15x^2$	$6x$	$3x$ +
$20x$	8	4

Enrichment

Some students will be ready to multiply a binomial by a trinomial , for example, $(x + 1)(x^2 + 2x + 1)$.

GAME: Polyprod

This game provides practice with binomial multiplication. The game may be faster-paced and more enjoyable if students multiply symbolically, but many will still benefit from using algebra tiles. A variation of the game would allow students to choose the order in which they place the values rolled in the binomial expressions. This may allow them to improve their score.

Some variations:

- Players can place their rolled numbers anywhere but, once placed, they cannot move them.
- Players can place their rolled numbers anywhere.

Chapter 3 Dividing Polynomials

2.3.1 Dividing a Polynomial by a Monomial

Curriculum Outcomes	Outcome relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • divide concretely, pictorially, and symbolically: monomial by a monomial; polynomial by a scalar; polynomial by a monomial **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	Students who can divide by a monomial will be able to simplify a broader range of algebraic and rational expressions.

Pacing	Materials	Prerequisites
1 h	Algebra tiles	• multiplication of binomials

Main Points to be Raised

• The distributive property applies when dividing polynomials by monomials — each term must be divided by the monomial and the resulting quotients added.

• Division is the reverse of multiplication. Because the area of a rectangle is the product of its two dimensions, dividing the area by either of the dimensions yields the other dimension.

• Division by a constant can be thought of as equal sharing. For example, dividing a polynomial by 2 involves sharing tiles equally and describing the amount in each of the two equal groups.

• The area model using algebra tiles only applies when a linear or quadratic (degree 2) polynomial is being divided by a constant or linear monomial.

• Guide tiles (as shown in **example 1**) continue to be useful to help students create rectangles of the correct size.

Try This—Introducing the Lesson

A. Students should solve the problem individually or in pairs.

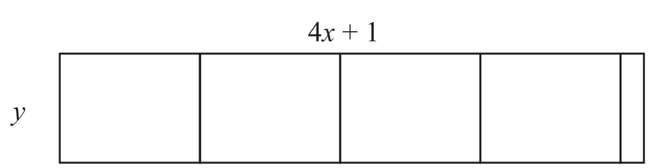
Observe while students work. You might ask:

• *How much does the first group of items cost? The second group?* ($250x$ and $60y$)

• *Why should you divide to find each share?* (Dividing is about sharing.)

The Exposition—Presenting the Main Ideas

• Write the equation $(4xy + y) = y \times (?)$ on the board. Ask students to find the missing factor. You might encourage them to use algebra tiles. Talk about how the missing factor must be $4x + 1$ and how this multiplication is represented by a rectangle with width $4x + 1$, height y, and area $(4xy + y)$.



Area = $4xy + y$

Discuss how if $y(4x + 1) = 4xy + y$, then $(4xy + y) \div y = 4x + 1$. You could deal with the second part of the exposition as a whole class, using the polynomials $(3x + x^2) \div x$.

• In addition to the last point about $\frac{2x}{2y} + \frac{6}{2y}$ not being a polynomial because $2y$ is not a factor of either $2x$ or 6, you could point out that the expression could also be written as $y^{-1}(x + 3)$ which is not a polynomial because the exponent is –1.

Revisiting the Try This

B. This question allows an opportunity for students to connect sharing to dividing by two.

Using the Examples

• Have students read through **example 1** on their own.
• Ensure that students are comfortable with rewriting a division statement as a fraction before asking them to read through **example 2.** Also make sure they recognize that when the numerator and denominator have a common factor, the fraction can be simplified. For example, since there is a common factor of 3 in the numerator and denominator of $\frac{9y}{6}$, the fraction can be simplified to $\frac{3y}{2}$.

Practising and Applying

Teaching points and tips

Q 1: Encourage students to use guide tiles so they do not change the length of the known dimension.

Q 2: Students can solve this with tiles or symbolically.

Q 4b: Observe whether students realize that one of the terms must have both a negative coefficient and a negative constant since all the tiles are negative and the other binomial must consist of only positive coefficients.

Q 5: To solve this problem, students can multiply $3y - 2$ by any monomial or binomial.

Common errors

When working with a binomial, students sometimes forget to divide the second term by the denominator after having divided the first term. Remind students to check by multiplying their quotient by the divisor to make sure the result is the dividend.

Suggested assessment questions from Practising and Applying

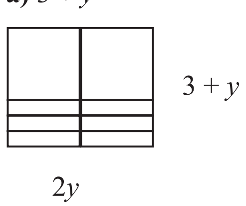
Question 2	to see if students can divide by a monomial
Question 4	to see if students can interpret a multiplication model as a division
Question 5	to see if students can work backwards, relating multiplication and division of polynomials

Answers

A. i) $250x + 60y$ **ii)** $125x + 30y$

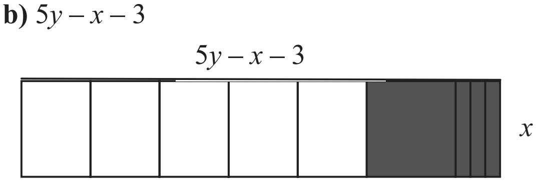
B. $(250x + 60y) \div 2$

1. a) $3 + y$



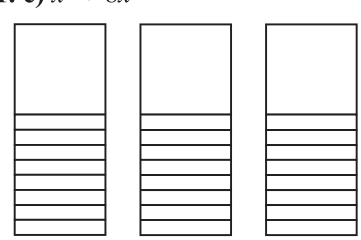
$(6y + 2y^2) \div 2y$

b) $5y - x - 3$



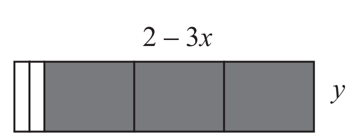
$(5xy - 3x - x^2) \div x$

1. c) $x^2 + 8x$



$(3x^2 + 24x) \div 3$

d) $2 - 3x$



$(2y - 3xy) \div y$

2.a) $3m^2 - mn$ **b)** $2s^2 + 1 - 5t$ **c)** $4m^2 - 2m$ **3. a)** $2k + 3m$; 5; –20; –4; if you divide –20 by –4, you get 5 **b)** $2k - 2$; –6; 36; –6; if you divide 36 by –6, the quotient is –6 **4. a)** $(4y^2 + 6) \div 2 = 2y^2 + 3$ **b)** $(-4xy - 6y) \div 2y = -2x - 3$ **c)** $(2x^2 - 3x) \div x = 2x - 3$	**5.** *Sample response*: $(6y - 4) \div 2$; $(3xy - 2x) \div x$; $(9y - 6) \div 3$; $(3x^2y - 2x^2) \div x^2$ **6.** To divide, think of a missing factor in the related multiplication; *sample response*: $(2xy + y^2) \div y \rightarrow$ $y \times ? = 2xy + y^2$

Supporting Students

Struggling students

Some students might benefit from continuing to work on multiplication questions with monomials and binomials, each time rewriting the question as a division, for example:

$5x(2x + 7) = 10x^2 + 35x$ becomes $10x^2 + 35x \div 5x = 2x + 7$.

2.3.2 EXPLORE: Dividing a Polynomial by a Binomial

Curriculum Outcomes	Lesson relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • divide concretely a polynomial by a binomial	This optional exploration allows students to divide polynomials concretely will support them as they work more symbolically.

Pacing	Materials	Prerequisites
1 h	Algebra tiles	• multiplication with algebra tiles

Main Points to be Raised

• As in the previous lesson, students divide polynomials by setting up the dividend as the area and the divisor as one of the dimensions. The quotient is the other dimension.

• Students begin to notice that although the product of polynomials is always a polynomial, the same is not true of quotients.

• There is a first opportunity to examine how one might "estimate" a quotient by comparing the highest power terms in the dividend and the quotient. This will be further explored in the next lesson.

Exploration

• Ask students to work on **parts A to E** with a partner or in a small group.

• Make sure they know that they can check their divisions for **parts A through C** by multiplying. They might want to divide up the work for **part A** among three students and then "teach" each other the question for which they were responsible, while discussing **part B**. The work in **part C** can then be shared.

Observe while students work. You might ask:

• *If you used guide tiles, what tiles would you use and how would you use them?* (For **part C**, I would take two x-tiles and one –1-tile and put them in a line.)

• *Could you divide* $x^2 + 4x + 4$ *by* $x + 2$*?* (Yes. The quotient is $x + 2$.)

Observe and Assess

As students are working, notice:

• Do they immediately recognize where to position tiles or do they need to try many possibilities?

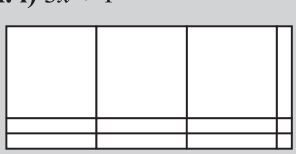
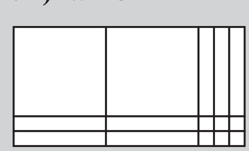
• Do they choose to draw the tiles or to use the concrete representations and dismantle them?

• Can they provide many examples of quotients that are not polynomials, or only one or two?

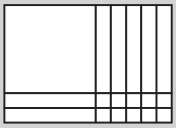
Share and Reflect

After students have had enough time to answer **parts B and D**, encourage different groups to come forward and describe how they answered those questions.

Answers

A. i) $3x + 1$  $x + 2$ $(3x^2 + 7x + 2) \div (x + 2)$	**A. ii)** $2x + 3$  $x + 2$ $(2x^2 + 7x + 6) \div (x + 2)$

A. iii) $x + 5$

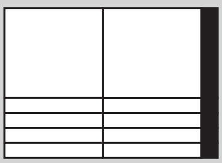


$x + 2$

$(x^2 + 7x + 10) \div (x + 2)$

B. i) They were all the same height: $x + 2$.
ii) Some were vertical to show the +5 in $x + 5$ and some were horizontal to show the +2 in $x + 2$.

C. i) $x + 4$

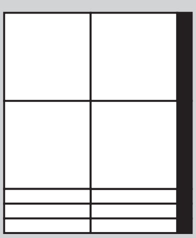


$x + 4$

$2x - 1$

$(2x^2 + 7x - 4) \div (2x - 1)$

ii) $2x + 3$

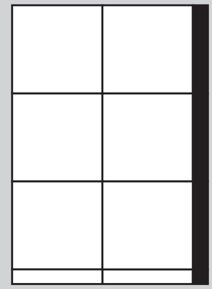


$2x + 3$

$2x - 1$

$(4x^2 + 4x - 3) \div (2x - 1)$

C. iii) $3x + 1$



$3x + 1$

$2x - 1$

$(6x^2 - x - 1) \div (2x - 1)$

D. i) They were all $2x - 1$ wide, so the top row had 2 white squares and a black strip.
ii) Because of the –1 in $2x - 1$, the things under –1 have to be negative.

E. If the first term of the dividend is the x^2-term and the first term of the divisor is the x-term, you can figure out the number of rows of x^2-terms there will be by dividing. This gives you part of the quotient.

2.3.3 Dividing a Polynomial by a Binomial

Curriculum Outcomes	Outcome relevance
9-B8 Polynomial Products and Quotients: concretely, pictorially, and symbolically • divide concretely, pictorially, and symbolically a polynomial by a binomial **9-B10 Polynomial Expressions: evaluate** • examine the value of polynomials prior to and after being simplified	Division of a trinomial by a linear binomial will be a good introduction to factoring, which will be useful later in this chapter and in Class X work with quadratics.

Pacing	Materials	Prerequisites
2 h	Algebra tiles	• multiplication of polynomials with and without algebra tiles

Main Points to be Raised

• Sometimes you need to use the zero principle to introduce extra pairs of opposite tiles in order to calculate a quotient using algebra tiles.

• To estimate a quotient symbolically, it makes sense to divide the highest power term in the dividend by the highest power term in the divisor.

• The degree of the quotient, if it is a polynomial, is the difference in degree between the dividend and the divisor.

Try This—Introducing the Lesson

A. Students can work alone or with a partner.

Observe while students work. You might ask:

• *What multiplication goes with that division?* ($(5x + ▲)(ax + b) = 12x^2 + x – 6$)

• *How would it help to write this as a multiplication?* (If I multiply, I see that 5a = 12, which is not possible with integers.)

• *Why did it matter that the question talked about integer coefficients?* (I could have tried a = $\frac{12}{5}$ and it might have worked.)

The Exposition—Presenting the Main Ideas

• Write on the board the question $(6x^2 + 7x – 3) \div (3x – 1)$, discussed in the exposition. Help students see why the zero principle must be used to solve the problem. Then have students read the first part of the exposition to consolidate their understanding.

• Then ask students to divide these polynomials: $3x^2 \div x$, $3x^3 \div x$, and $3xy \div x$. Each time, talk about how the answer makes sense. They can then read the second part of the exposition.

• Once they have finished reading the second part, make sure they understand by asking them how you might estimate 1640 ÷ 8 by dividing the 1600, but not the 40, by 8. Ask how this same idea applies to polynomials.

Revisiting the Try This

B. This question allows students to use an estimate of the quotient (as developed in the exposition) to help them find the actual quotient of $12x^2 \div 5x$.

Using the Examples

Encourage students to read through both examples to see how both algebra tiles and symbolic manipulations can be used to calculate quotients of polynomials.

Practising and Applying

Teaching points and tips

Q 2: Students should recognize that, as with numbers, polynomial division can be represented in fractional form.

Q 4: Observe whether students realize that all they need to do is select any polynomial for the quotient and then multiply by $2x + 1$ to create the dividend.

Q 6: Some students may include polynomial quotients such as $(3x^2 + 2x) \div 2x = \frac{3}{2}x + 1$ because they assume that the coefficients cannot be fractions. Remind students that a polynomial coefficient can be an integer, a fraction, a decimal, or an irrational number (such as $\sqrt{2}$ or π).

Q 7: This question assumes a polynomial quotient. A non-polynomial quotient is possible, for example, $x^4 \div (x^2 + 3)$.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can divide polynomials
Question 4	to see if students can recognize and use the relationship between multiplication and division
Question 6	to see if students can communicate effectively about why polynomial division does not always result in a polynomial

Answers

A. If you multiply $5x$ + ▲ by something like ax + b, the coefficient of x^2 will be a multiple of 5. 12 is not a multiple of 5.

B. I would have tried to divide $12x^2$ by $5x$ to realize I would get more than $2x$, but not $3x$.

1. a) $2x + 3$ **b)** $3x - 4$ **c)** $2x + 5$ **d)** $4x + 9$ **2. a)** $2x^2 - 1$ **b)** $4x - 2$ **c)** $x^3 + 1$ **d)** $4 - 3x^2$ **3. a)** –1, –1, and 1 **b)** –24, 4, and –6 **c)** 0, –2, and 0 **d)** 6, 6, and 1 Each time, I noticed that when I divided the first two values, the last value was the quotient.	**4.** *Sample response*: $4x^2 - 1$ and $2x - 1$; $2x^2 - 5x - 3$ and $x - 3$; $2x^3 + 3x^2 + x$ and $x^2 + x$ **5. a)** $x + 1$ **b)** $4x - 12$ **6.** *Sample response*: $(10x^2 + 3y) \div y$ is not a polynomial since $\frac{10x^2}{y}$ is not a polynomial. **7.** Two; when you divide polynomials with exponents, you subtract the exponents using the exponent quotient law; *sample response*: $4x^4 \div 2x^2 = 2x^2$

Supporting Students

Struggling students

Some students might benefit from continuing to work on multiplication questions, each time rewriting the question as a division, for example, $(2x + 3)(4x + 5) = 8x^2 + 22x + 15$ becomes $(8x^2 + 22x + 15) \div (2x + 3) = 4x + 5$ or $(8x^2 + 22x + 15) \div (4x + 5) = 2x + 3$.

Enrichment

Some students may be ready to work more symbolically with polynomial divisions involving fractional coefficients such as $(x^2 - \frac{3}{2}x - 1) \div (2x + 1)$.

2.3.4 EXPLORE: Creating Rectangles to Factor

Curriculum Outcomes	Lesson relevance
9-B9 Polynomial Factors: dimensions of a rectangle • factor quadratic binomials and trinomials concretely	Before students consider factoring formally in Class X, they need some informal concrete exploration. This core lesson allows for this.

Pacing	Materials	Prerequisites
1 h	Algebra tiles	• multiplication with algebra tiles

Main Points to be Raised

• To factor a second degree polynomial (a quadratic), you can represent it as the area of a rectangle. The linear dimensions of the rectangle are the factors.

• Sometimes it is necessary to the use the zero principle to find the factors.

• Students may observe that they can factor the coefficient of x^2 to determine the coefficients of x for the two linear factors and that they can then factor the constant to determine the constants in the two linear factors.

Exploration

• Ask students to complete **parts A and B** with a partner or in a small group. They should use algebra tiles.

Observe while students work. You might ask:

• *Why did you try those factors for* ***part A v)****?* (The x^2 coefficient was 3 so I needed $3x$ in one of the factors and x in the other. I also noticed that $4 = 2 \times 2$ or 4×1 so I tried both and moved them around until they fit.)

• *How did you know that one of the constants would be negative for* ***part A ii)****?* (The constant in the polynomial being factored is negative.)

• *How did you decide which of the two values in the factor pair to make negative for* ***part A ii)****?* (I noticed that the coefficient of x was negative, so I knew that I should make the 3 negative and the 1 positive.)

Observe and Assess

As students are working, notice:

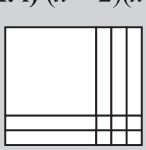
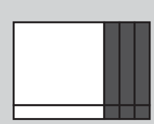
• Do they struggle more with negative constants than with positive constants?

• Do they recognize that if one pair of factors does not work, they can try a different pair?

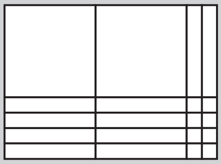
Share and Reflect

Ask students which polynomials they found easiest to factor and why they found them easiest. Have them describe the process they went through to calculate each pair of factors. On the board, record students' responses to **part B.** Show how the coefficients in the factors relate to the coefficients of the polynomial, how the constants in the factors relate to the constant in the polynomial, and how the coefficients and constants in the factors relate to the coefficient of x in the polynomial.

Answers

A. i) $(x + 2)(x + 3)$	**A. ii)** $(x - 3)(x + 1)$
 $x + 2$ $x + 3$ $x^2 + 5x + 6$	 $x + 1$ $x - 3$ $x^2 - 2x - 3$

A. iii) $(2x + 2)(x + 4)$

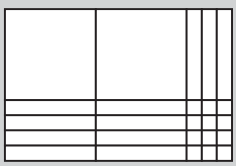


$x + 4$

$2x + 2$

Or, $(2x + 8)(x + 1)$, or $2(x + 1)(x + 4)$
$2x^2 + 10x + 8$

iv) $(2x + 3)(x + 4)$

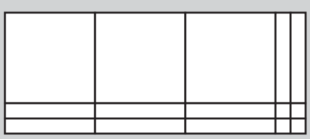


$x + 4$

$2x + 3$

$2x^2 + 11x + 12$

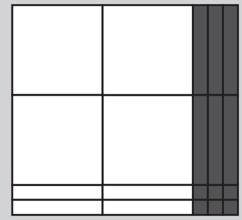
v) $(3x + 2)$ $(x + 2)$



$x + 2$

$3x + 2$
$3x^2 + 8x + 4$

A. vi) $(2x - 3)(2x + 2)$



$2x + 2$

$2x - 3$

Or, $2(x + 1)(2x - 3)$, or $(4x - 6)(x + 1)$
$4x^2 - 2x - 6$

B. i) The product of the coefficients of the x-terms in the factors is the coefficient of the x^2-term in the polynomial being factored.

ii) The product of the constant terms in the factors is the constant term in the polynomial being factored.

iii) The coefficient of x in the polynomial being factored is the sum of two products. The first is the product of the coefficient of x in the first factor and the constant in the second factor. The second is the product of the constant in the first factor and the coefficient of x in the second factor.

Supporting Students

Enrichment

- Some students could try to factor polynomials where the coefficient of x^2 is negative.
- Students can try to find all the possible ways to factor **part A i) and vi**.

CONNECTIONS: Using Number Patterns to Factor

This feature allows students to see how number patterns can be used to help with factoring. It also builds the connection between factoring numbers and factoring polynomials.

There is often more than one possible pair of factors that can be used to describe the numbers in the column on the right. Students should be encouraged to try different combinations until a pattern emerges.

Answers

1.

x	$x^2 + 5x + 6$	Factors
1	12	3, 4
2	20	4, 5
3	30	5, 6
4	42	6, 7

The factors are $(x + 2)$ and $(x + 3)$.

2.

x	$4x^2 + 4x + 1$	Factors
1	9	3, 3
2	25	5, 5
3	49	7, 7
4	81	9, 9

The factors are $(2x + 1)$ and $(2x + 1)$.

3.

x	$x^2 – 1$	Factors
1	0	2, 0
2	3	3, 1
3	8	4, 2
4	15	5, 3

The factors are $(x + 1)$ and $(x – 1)$.

4.

x	$x^2 – x – 2$	Factors
1	–2	2, –1
2	0	3, 0
3	4	4, 1
4	10	5, 2
5	18	6, 3

The factors are $(x + 1)$ and $(x – 2)$.

5.

x	$9x^2 – 6x + 1$	Factors
1	4	2, 2
2	25	5, 5
3	64	8, 8
4	121	11, 11

The factors are $(3x – 1)$ and $(3x – 1)$.

UNIT 2 Revision

Pacing	Materials
1 h	Algebra tiles

Question(s)	Related Lesson
1	Lesson 2.1.1
2, 3, 4	Lesson 2.1.2
5, 6, 7	Lesson 2.2.1
8, 9, 10, 11	Lesson 2.2.2
12, 13	Lesson 2.2.3
14	Lesson 2.3.1
15	Lesson 2.3.2
16	Lesson 2.3.3
17	Lesson 2.3.4

Revision Tips

Q 3: Students should feel free to solve this with or without algebra tiles.

Q 7: It may be necessary to put up a visual to remind students which tile is which.

Q 11: Observe whether students realize that they might multiply a polynomial represented by 4 tiles by one represented by 5 tiles (or 10 × 2 or 1 × 20).

Q 17: Students might use algebra tiles or they might use a numerical connection. For example, this table suggests that the factors of $x^2 - 9$ are $(x - 3)$ and $(x + 3)$.

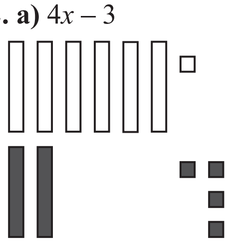
x	$x^2 - 9$
1	$-8 = -2 \times 4$
2	$-5 = -1 \times 5$
3	$0 = 0 \times 6$
4	$7 = 1 \times 7$

Answers

1. a) degree 1; binomial; like terms are $3x$ and $4x$ as well as $-2y$ and $6y$; *Sample response*: the perimeter of a triangle with dimensions $6y$, $4x$, and $3x - 2y$

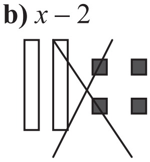
b) degree 2; monomial; no like terms; *Sample response*: the area of a rectangle with dimensions of 16 and x^2

2. a) $4x - 3$



$(-2x - 4) + (6x + 1)$

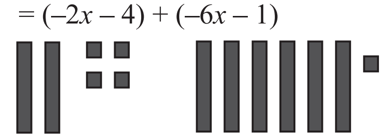
b) $x - 2$



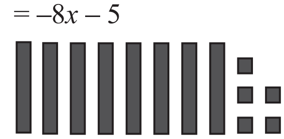
$(2x - 4) - (x - 2)$

2. c) $-8x - 5$

$(-2x - 4) - (6x + 1) = (-2x - 4) + (-6x - 1)$



$= -8x - 5$



3. a) $-2x^2 + x + 8 - y$
b) $x^2 - 4x + 5$
c) $4x - 7x^2 + 8x^3 + y^2$

4. Because a polynomial is not in its simplest form to model until you have combined like terms.

Answers [Continued]

5. a) $6y - 8y^2$
b) $8 - 12x$
c) $-6xy - 24x$
d) $-15x + 6x^2$

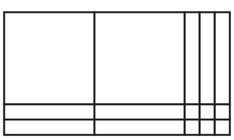
6. a) 6, –9, and –54
b) 4, 8, and 32
c) 12, 7, and 84
d) 9, 6, and 54
I noticed that multiplying the evaluated factors is equal to multiplying the factors and then evaluating the expression.

7. a) $y(x - 2y)$
b) $x(-3x - 2)$
c) $2(x^2 + 2xy - x)$

8. a) $12x^2 - 6x$
b) $9\pi y^2$

9. a) $(x + 5)(x + 1) = x^2 + 6x + 5$
b) $(2x + 3)(x - 2) = 2x^2 - x - 6$

10. a) $2x^2 + 7x + 6$

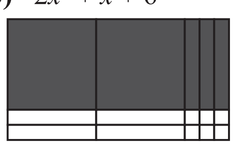


$x + 2$

$2x + 3$

$(2x + 3)(x + 2)$

b) $-2x^2 + x + 6$

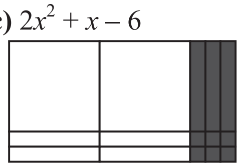


$-x + 2$

$2x + 3$

$(2x + 3)(-x + 2)$

c) $2x^2 + x - 6$



$x + 2$

$2x - 3$

$(2x - 3)(x + 2)$

10. d) $-2y^2 - 7y - 6$

$-y - 2$

$2y + 3$

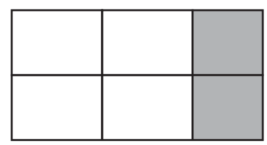
$(2y + 3)(-y - 2)$

11. $(2x - 3)(3x + 1)$ or $(x + 3)(4x + 1)$

12. The factors are degree 1 × degree 2 because of the exponent product law, 1 + 2 = 3.

13. a) $(70 + 1)^2 = 70^2 + (2 \times 70) + 1 =$
$4900 + 140 + 1 = 5041$
b) $(71^2 - 69^2) = (71 + 69)(71 - 69) = 140 \times 2 = 280$

14. a) $2x - y$



$2y$

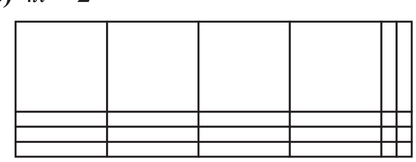
$2x - y$

$(4xy - 2y^2) \div 2y$

b) $3y^2 - 2xy$

$(6y^2 - 4xy) \div 2$

c) $4x + 2$



$x + 3$

$4x + 2$

$(4x^2 + 14x + 6) \div (x + 3)$

15. $(4x^2 + 2x - 2) \div (2x - 1)$ or $(4x^2 + 2x - 2) \div (2x + 2)$

16. a) $x^2 + 3x$ **b)** $-3x + 1$

17. a) $(3x - 1)(3x + 2) = 9x^2 + 3x - 2$
b) $(x + 3)(x - 3) = x^2 - 9$

UNIT 2 Polynomials Test

1. Write a polynomial to fit each description:

a) degree 3 binomial

b) degree 2 trinomial

2. Simplify. Then represent each polynomial with algebra tiles.

a) $3x - 2x + 3xy + 4y$

b) $2x^2 + 4y^2 - 3y - 5x^2$

3. Simplify.

a) $(4y^2 + 2y + x^2 + 8) + (-3x^2 - 2x + 4)$

b) $(4y^2 + 2y + x^2 + 8) - (-3x^2 - 2x + 4)$

c) $(8y - 4x - 3) - (2y - x - 2)$

4. Simplify.

a) $8(4 - 2y)$

b) $3x(-2y - 4)$

c) $(2x + 5)(4x - 5)$

5. Simplify.

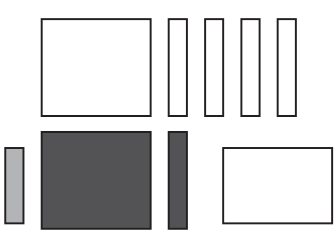
a) $(4x^2y - 14x^2 + 8xy) \div 2x$

b) $(6x^2 + x - 1) \div (2x + 1)$

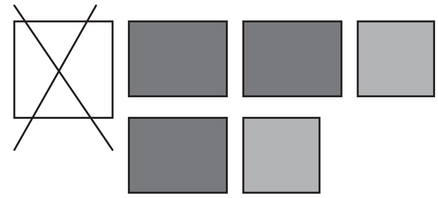
c) $(10y^2 + 11y - 6) \div (2y + 3)$

6. Identify the two polynomials that were used in the calculation in each algebra tile model. Tell which operation each model represents as well as the result of the operation.

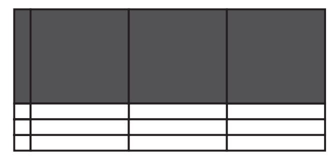
a)



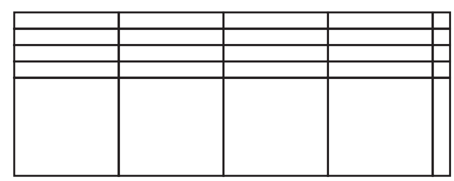
6. b)



c)



d)



7. Use algebra tiles to factor $x^2 + 4x - 5$.

8. Show how each could occur:

a) The sum of two trinomials is a binomial.

b) The difference of two binomials is a trinomial.

c) The product of two binomials is a binomial.

9. The area of a triangle is $6x^2 - 9x$. What might be the dimensions of the base and height?

10. Explain why the difference of two polynomials has to be a polynomial, but the quotient of two polynomials does not have to be a polynomial.

UNIT 2 Test

Pacing	Materials
1 h	Algebra tiles

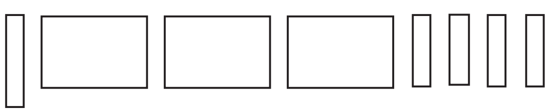
Question	Related Lesson(s)
1	Lesson 2.1.1
2	Lessons 2.1.1 and 2.1.2
3	Lesson 2.1.2
4	Lessons 2.2.1 and 2.2.3
5	Lessons 2.3.1 and 2.3.3
6	Lessons 2.1.2, 2.2.2, and 2.3.2
7	Lesson 2.3.4
8	Lessons 2.1.2, 2.2.2, and 2.2.3
9	Lesson 2.2.1
10	Lessons 2.3.3 and 2.1.2

Select questions to assign according to the time available.

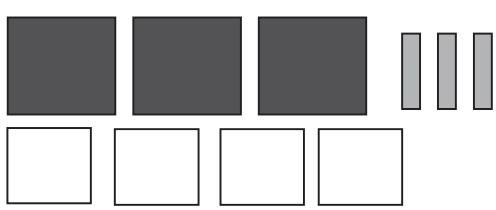
Answers

1. *Sample responses*:
a) $x^3 - x^2$
b) $2x^2 + 5x + 3$

2. a) $x + 3xy + 4y$



b) $-3x^2 + 4y^2 - 3y$



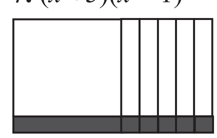
3. a) $4y^2 + 2y - 2x^2 - 2x + 12$
b) $4y^2 + 2y + 4x^2 + 2x + 4$
c) $6y - 3x - 1$

4. a) $32 - 16y$
b) $-6xy - 12x$
c) $8x^2 + 10x - 25$

5. a) $2xy - 7x + 4y$
b) $3x - 1$
c) $5y - 2$

6. a) $(x^2 + 4x) + (-y - x^2 - x + xy) = -y + 3x + xy$
b) $(x^2 - 3xy - 2y^2) - x^2 = 3xy - 2y^2$
c) *Sample response*: $(-3x - 1)(x - 3) = 3x^2 + 8x + 3$
d) *Sample response*: $(4x + 1)(x + 4) = 4x^2 + 17x + 4$

7. $(x + 5)(x - 1)$



8. *Sample responses*:
a) $(x^2 - x + 5) + (2x^2 + x + 1) = 3x^2 + 6$
b) $(x + 4) - (y - 2) = x - y + 6$
c) $(x + 1)(x - 1) = x^2 - 1$

9. $3x$ and $2x - 3$

10. When you find the difference, all you do is change positive signs to negative signs or negatives to positives, but the powers do not change. When you subtract the second polynomial, you are still adding or taking away polynomial terms, so what you have left is a polynomial. But when you divide polynomials, the result could contain a non-polynomial term, for example, $\frac{3}{n}$ is not a polynomial.

UNIT 2 Performance Task – Modelling Polynomials

Part 1
For each question, use at least five positive and five negative tiles. Each time, sketch your model and write an algebraic equation to describe your calculation.

1. Model and complete an addition.

2. Model and complete a subtraction.

3. Model and complete a multiplication by $3x + 2$.

4. Model and complete a division by y.

5. Model and complete a factoring.

Part 2
For each question, the answer must be $2x^2 + x - 1$. Each time, sketch your model and write an algebraic equation to describe your calculation.

1. Add two polynomials that are modelled with 12 tiles altogether.

2. Subtract two polynomials that are modelled with more than 10 tiles altogether.

3. Multiply two polynomials.

UNIT 2 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-B7 Polynomial Expressions: add and subtract concretely, pictorially, and symbolically **9-B8** Polynomial Products and Quotients: concretely, pictorially, and symbolically	1 h	Algebra tiles

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used, if you wish, as enrichment material for some students. You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

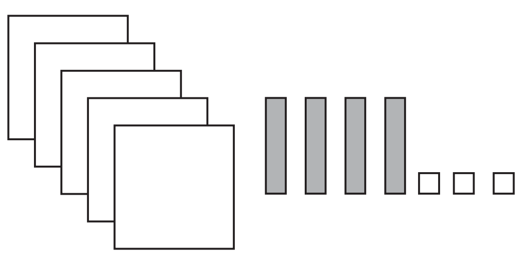
Sample solution

Part 1

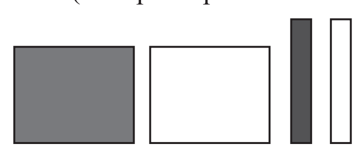
1. $(6x - 4) + (x^2 - 5) = x^2 + 6x - 9$

$6x - 4$ $\quad + \quad$ $x^2 - 5$ $\quad = \quad$ $x^2 + 6x - 9$

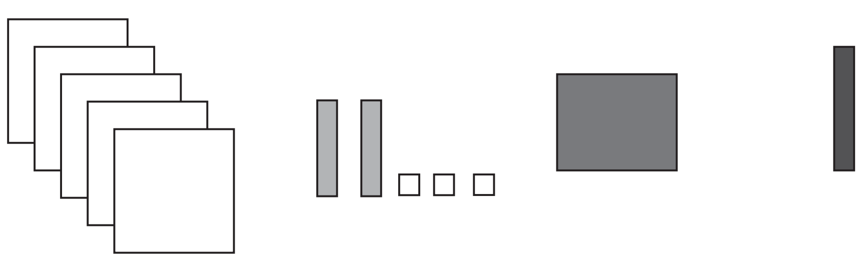
2. $(5x^2 - 4y + 3) - (xy - 2y + x) = 5x^2 - x - 2y - xy + 3$



Add (zero principle



Subtract $xy - 2y + x$



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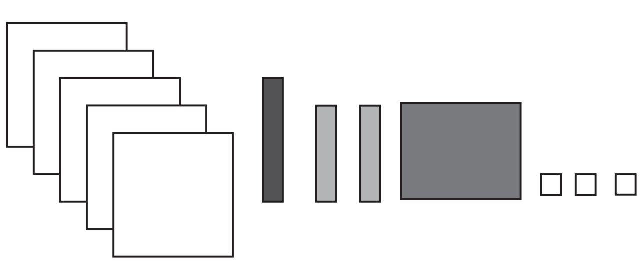
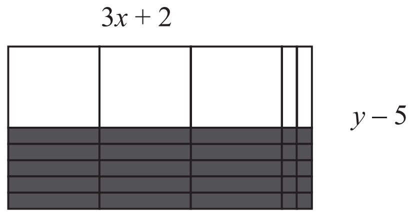
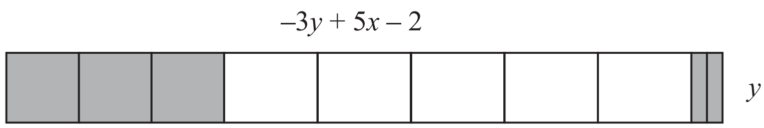


$5x^2 - x - 2y - xy + 3$

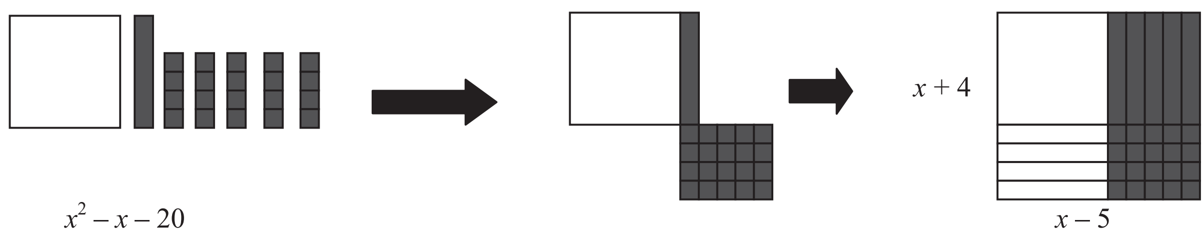
3. $(3x + 2)(y - 5) = 3xy - 15x + 2y - 10$



4. $(-3y^2 + 5xy - 2y) \div y = -3y + 5x - 2$

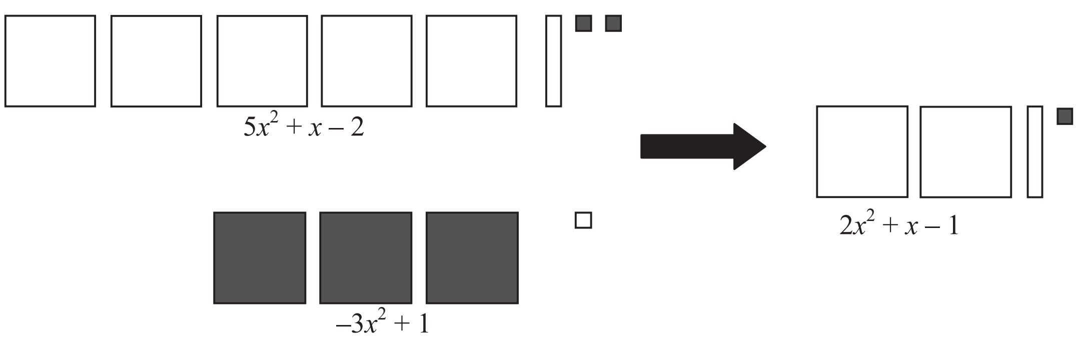


5. $x^2 - x - 20 = (x - 5)(x + 4)$

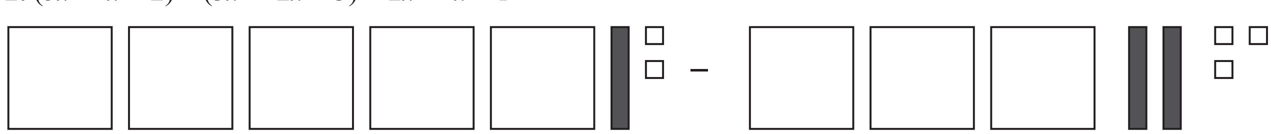


Part 2

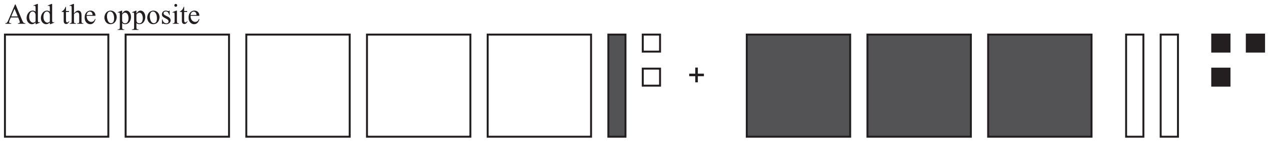
1. $(5x^2 + x - 2) + (-3x^2 + 1) = 2x^2 + x - 1$



2. $(5x^2 - x + 2) - (3x^2 - 2x + 3) = 2x^2 + x - 1$



Add the opposite



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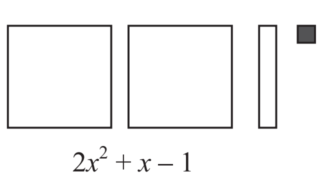
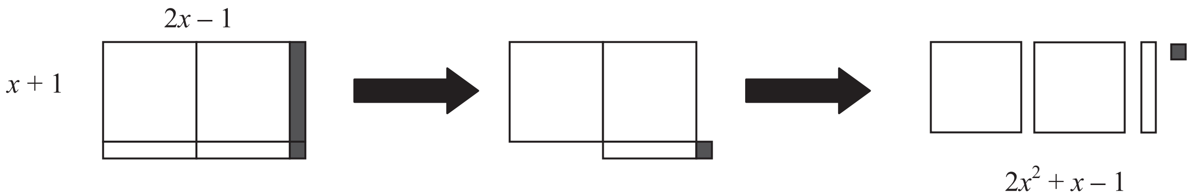


Sample solution [Continued]

3. $(2x - 1)(x + 1) = 2x^2 + x - 1$



UNIT 2 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Ability to add and subtract	Correctly sets up all additions and subtractions; correctly calculates	Correctly sets up most additions and subtractions; correctly calculates	Correctly sets up many additions and subtractions; correctly calculates most of the time	Frequently does not correctly set up additions or subtractions; makes incorrect calculations often
Ability to multiply, divide, and factor	Correctly sets up all multiplications, divisions, and factoring situations; correctly calculates	Correctly sets up most multiplications, divisions, and factoring situations; correctly calculates	Correctly sets up most multiplications and at least one division or factoring situation; correctly calculates most of the time	Does not correctly set up multiplication situations or at least one factoring or division situation
Problem solving	Works backwards effectively to determine polynomials yielding a given sum, difference, or product	Works backwards to determine polynomials yielding at least two of a given sum, difference, or product	Works backwards to determine polynomials yielding at least one of a given sum, difference, or product	Does not work backwards to determine polynomials yielding a given sum, difference, or product

Unit 2 Assessment Interview

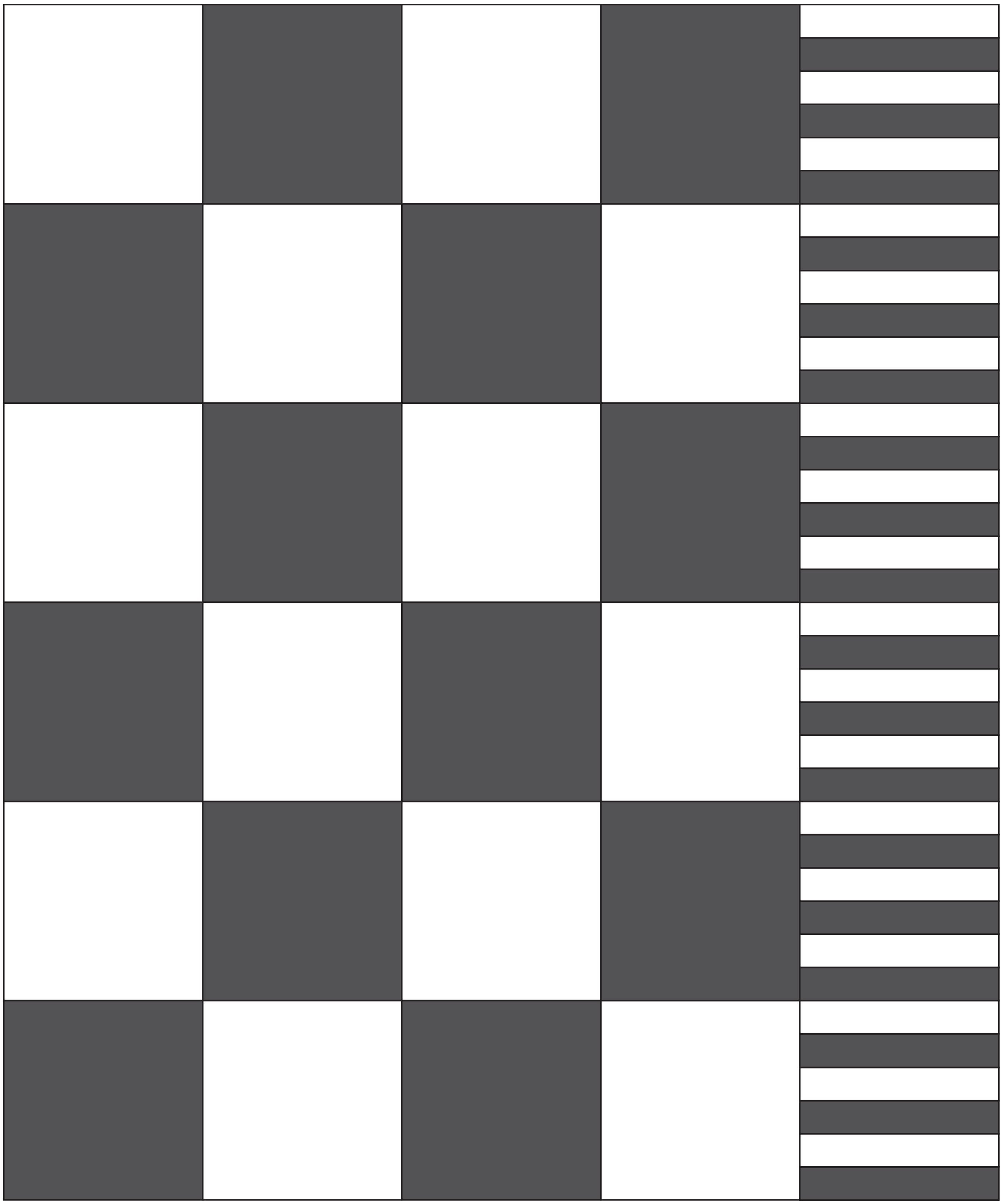
You may want to take the opportunity to interview selected students to assess their understanding of the work of this unit. The results can be used as formative assessment or, if you wish, as a piece of summative assessment data. As the student works, ask him or her to explain their thinking.

Have available a set of algebra tiles. Ask the student to show you each of these:

- a representation for $x^2 + 2x - 3$
- a representation for $2x - 3xy$
- how to add $3x + 4xy - y$ and $-2x + 2xy + 5y$
- how to subtract $3x^2 + 2x + 8$ from $2y^2 - 5x + 9$
- how to multiply $2x - 5$ and $3x + 4$

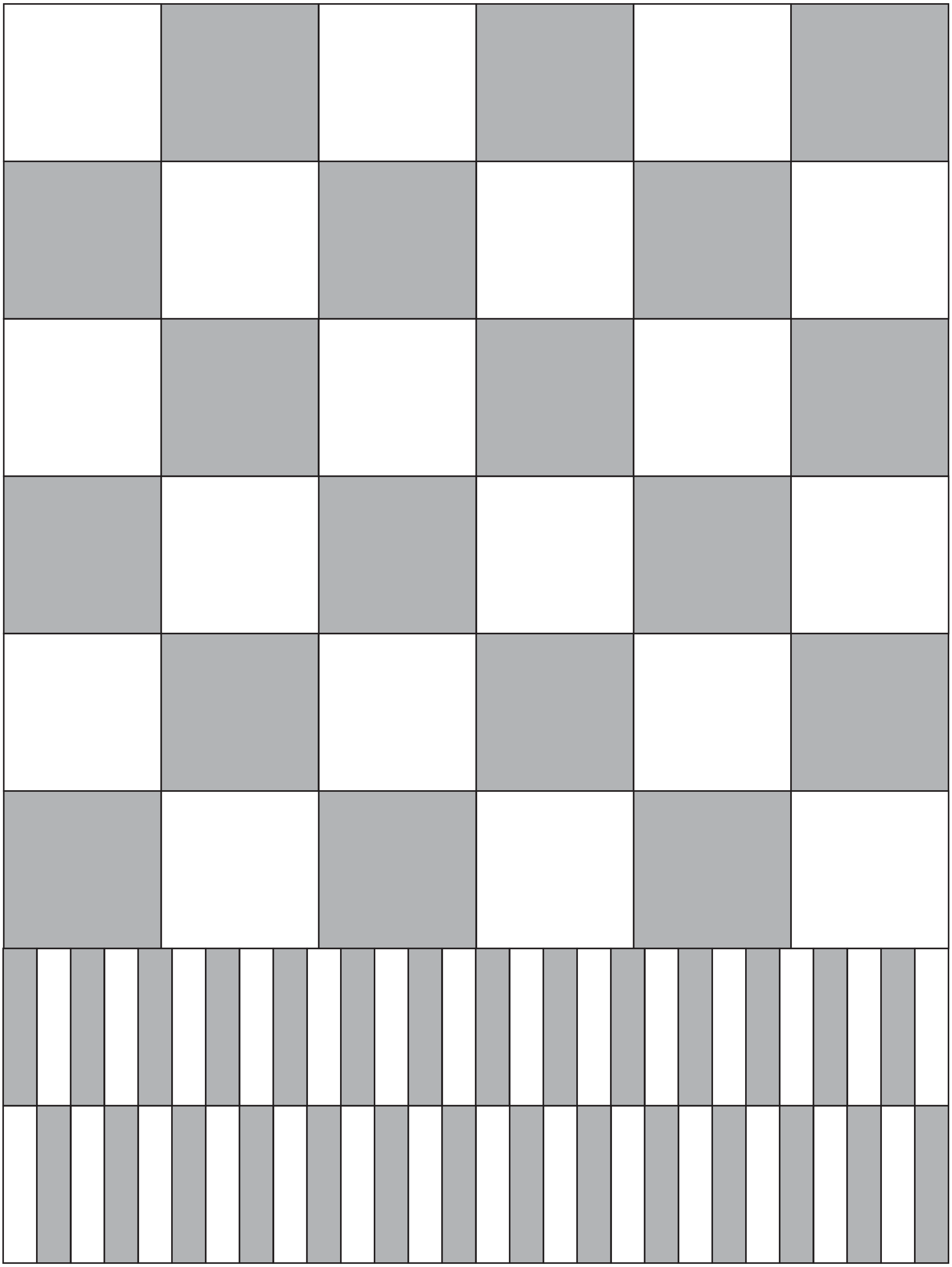
UNIT 2 Blackline Master 1

Algebra Tiles (x^2 and $-x^2$; x and $-x$)



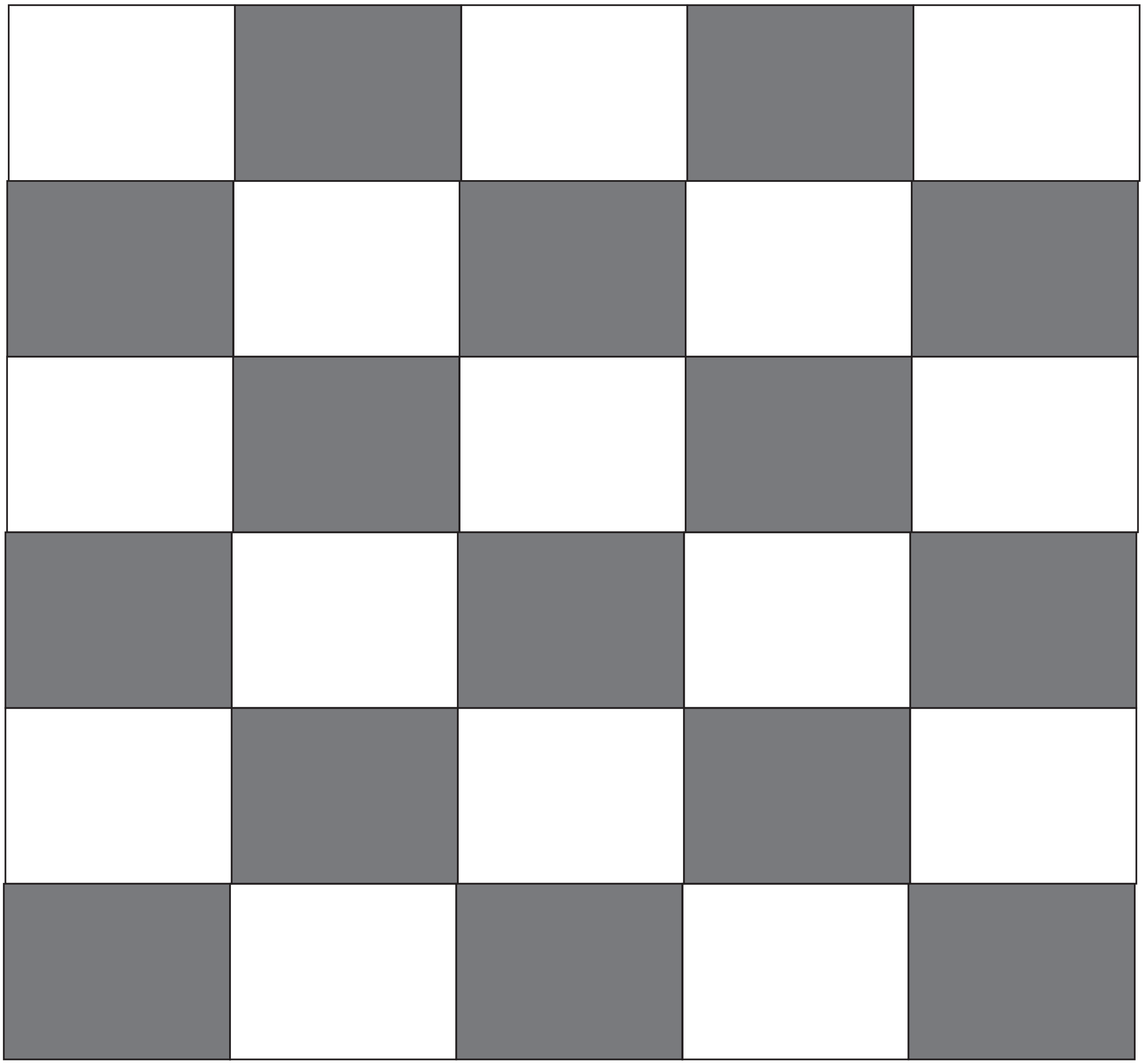
UNIT 2 Blackline Master 2

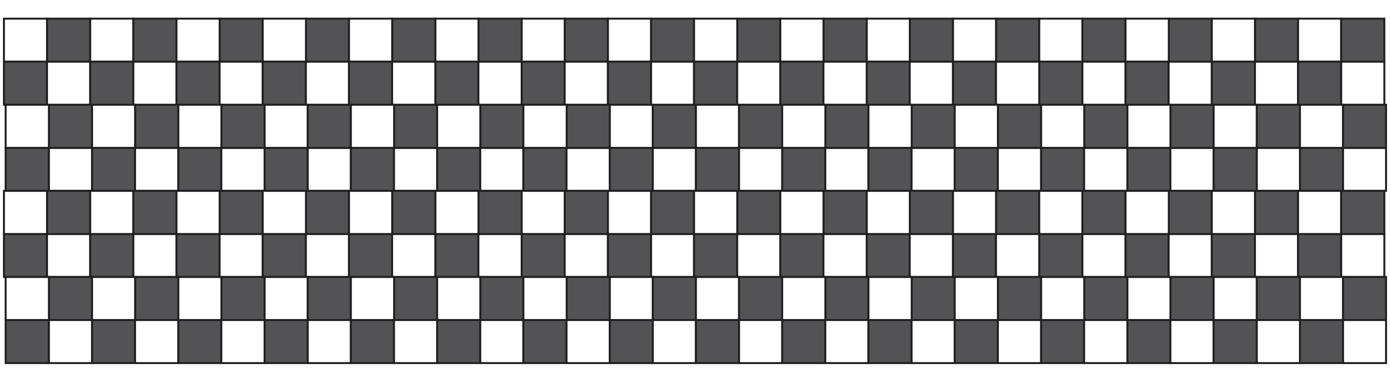
Algebra Tiles (y^2 and $-y^2$; y and $-y$)



UNIT 2 Blackline Master 3

Algebra Tiles (x*y* and –x*y*; 1 and –1)





UNIT 3 LINEAR RELATIONS AND EQUATIONS

UNIT 3 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Wooden or interlocking cubes • Grid paper • Rulers	All questions
Chapter 1 Linear and Non-Linear Relation Graphs				
3.1.1 Patterns and Relations in Tables	**9-C1 Patterns and Relationships: determine non-algebraic representations** • describe verbally and symbolically, patterns given in tables, charts, pictures, and /or by problem situations • use models such as tables, and symbolic statements to assist in examining patterns and relationships. • relate the data in a table representing a linear, quadratic, or exponential relationship to its graph • use first and second differences to determine if a table represents a linear, quadratic, or exponential relationship	2 h	• Rulers	Q1, 3, 4 8, and 9
3.1.2 Scatter Plots of Discrete and Continuous Data	**9-C1 Patterns and Relationships: determine non-algebraic representations** • describe verbally and symbolically, patterns given in graphs, pictures, and /or by problem situations • use models such as graphs to assist in examining patterns and relationships. **9-C2 Scatter Plots: characteristics of relationships** • consider whether data represented by a scatter plot are continuous or discrete and whether interpolation is meaningful • distinguish between independent and dependent variables in a scatter plot **9-F1 Displaying Data: draw inferences and make predictions** • interpolate and extrapolate using a data set	2 h	• Grid paper • Rulers	Q1, 3, 4, and 6
3.1.3 EXPLORE: Graphs of Linear and Non-Linear Relations (Optional)	**9-C1 Patterns and Relationships: determine non-algebraic representations** • explore linear, exponential and quadratic curves • describe verbally and symbolically, patterns given in graphs • use models such as graphs, and symbolic statements to assist in examining patterns and relationships.	1 h	• Grid paper • Rulers	Observe and Assess questions

UNIT 3 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 1 Linear and Non-Linear Relation Graphs [Cont'd]				
CONNECTIONS: Half-Life	Explore the application of exponential relationships in a real world context	20–30 min	• Scientific calculators	N/A
3.1.4 Graphs of Linear and Non-Linear Relations	**9-C1 Patterns and Relationships: determine non-algebraic representations** • explore linear, exponential and quadratic curves • describe verbally and symbolically, patterns given in graphs, pictures, and/or by problem situations • use models such as graphs to assist in examining patterns and relationships • relate the data in a table representing a linear, quadratic, or exponential relationship to its graph • determine if a table represents a linear relationship by plotting the points	2 h	• Grid paper • Rulers	Q1, 2, and 5
Chapter 2 Equation of a Line				
3.2.1 The Meaning of Slope and *Y*-Intercept	**9-C3 Graphs of Linear Relations: interpret and create** • use the term slope to represent rise/run • relate the *y*-intercept to the value of the *y*-coordinate where the graph crosses the *y* axis • determine the slope and *y*-intercept by examining a table or graph	2 h	• Grid paper • Rulers	Q1, 2, 4, 5, and 8
3.2.2 EXPLORE: The Equation of a Line (Optional)	**9-C3 Graphs of Linear Relations: interpret and create** • use the term slope to represent rise/run • relate the *y*-intercept to the to the value of the *y*-coordinate where the graph crosses the *y*-axis • determine the slope and *y*-intercept by examining a graph • sketch the graph of a linear relation given the slope and *y*-intercept	1 h	• Grid paper • Rulers	Observe and Assess questions
3.2.3 Slope and Y-Intercept Form	**9-C3 Graphs of Linear Relations: interpret and create** • use the term slope to represent rise/run • relate the *y*-intercept to the to the value of the *y*-coordinate where the graph crosses the *y*-axis • determine the slope and *y*-intercept by examining a table or graph • sketch the graph of a linear relation given the slope and *y*-intercept	1 h	• Grid paper • Rulers	Q1, 2, 4, 6, and 8

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
	9-C4 Equation of a Line: use graph to determine equation • determine the equation of a line (y = mx + b) given the slope (m) and y-intercept (b) • determine the equation of a linear relationship by calculating the slope and the y-intercept from the graph	1 h	• Grid paper • Rulers	Q1, 2, 4, 6, and 8
3.2.4 The Line of Best Fit	**9-C5 Lines of Best Fit: sketch and determine equations** • use the eyeball method to draw the line of best fit and then use the slope and y-intercept to determine the equation of the line • understand that the line of best fit is drawn to show a relationship between two variables • recognizes the relationship between both the dispersion around the line of best fit and the slope of the line of best fit and a description of the correlation between the variables **9-F1 Displaying Data: draw inferences and make predictions** • draw inferences and conclusions from a number of data displays, particularly scatter plots • interpolate and extrapolate using a date set	2 h	• Grid paper • Rulers	Q2, 4, and 6
3.2.5 Standard Form	**9-C3 Graphs of Linear Relations: interpret and create** • use the term slope to represent rise/run • relate the y-intercept to the to the value of the y-coordinate where the graph crosses the y-axis • relate the x-intercept to the to the value of the x-coordinate where the graph crosses the x-axis • determine the slope and y-intercept by examining a table or graph • sketch the graph of a linear relation given in standard form	2 h	• Grid paper • Rulers	Q1, 2, 4, 9, and 10
Chapter 3 Linear Equations and Inequalities				
3.3.1 Solving Linear Equations Algebraically	**9-C6 Single Variable Equations: solve algebraically and graphically** • solve equations algebraically • solve problems involving equations with coefficients that may be integers or rational numbers	2 h	• Algebra tiles (optional)	Q1, 2, 3, 6, 7, and 9
GAME: Equation Concentration	Solve equations algebraically in a game situation	1 h	• Paper to make cards	N/A
3.3.2 Solving Linear Inequalities	**9-C8 Inequalities: solve and verify** • solve single variable linear inequalities	1 h	None	Q1, 2, 4, and 7

UNIT 3 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 3 Linear Equations and Inequalities [Cont'd]				
3.3.3 Solving Linear Equations Graphically	**9-C6 Single Variable Equations: solve algebraically and graphically** • solve equations graphically • solve problems involving equations with coefficients that may be integers or rational numbers	1 h	• Grid paper • Rulers	Q2, 3, 4, and 5
3.3.4 Solving a System of Linear Equations	**9-C7 Two Linear Equations: find solutions to a problem by graphing** • solve problems by graphing pairs of linear equations	2 h	• Grid paper • Rulers	Q1, 2, and 3
UNIT 3 Revision	Review the concepts and skills in the unit	2 h	• Grid paper • Rulers	All questions
UNIT 3 Test	Assess the concepts and skills in the unit	1 h	• Grid paper • Rulers	All questions
UNIT 3 Performance Task	Assess the concepts and skills in the unit	1 h	• Grid paper • Rulers	Rubric provided

Math Background

• Much of this unit focuses on linear relationships. The work of the unit formalizes the idea of slope and relates it to the algebraic representation of linear relationships. As students work through this unit they see that the slope is the value of m in the slope-intercept form of the equation of a line ($y = mx + b$) and that the value of the y-intercept is b. Students sketch graphs of linear equations in both slope-intercept and standard form.

• Once students are familiar with relating the equation of a line to its graph, they use lines of best fit applied to scatter plots to predict values not actually recorded. They informally discuss the quality of the fit, the correlation, and the confidence they can reasonably have in the predicted values.

• Students solve single linear equations, systems of two linear equations, and linear inequalities using graphs as well as algebraic strategies.

• This unit also exposes students to non-linear relationships presented as tables of data, graphs, and algebraic expressions. They use each of the three forms of presentation to decide whether a relationship is linear, quadratic, exponential, or none of these.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections. For example:

- They use problem solving in **question 4** in **lesson 3.1.1**, where they have to create patterns to meet conditions. Other examples of problem solving are found in **question 6** in **lesson 3.3.1**, where they create an equation based on a question about a pattern and in **question 7** in **lesson 3.3.2,** where they create an inequality to model and then solve a problem.

--They use communication frequently to explain their thinking, for example, in **question D** of **lesson 3.1.3**, where they summarize an exploration, and in **question 8** in **lesson 3.3.2**, where they compare solving an inequality with solving an equation.

- They use reasoning in answering questions such as **question 4c** in **lesson 3.1.2**, where they decide when it is not appropriate to extrapolate and in **question 7** in **lesson 3.2.3**, where they talk about why it is easier to create a graph starting from the y-intercept.

- They use representation in many situations, for example, in **lesson 3.1.1** they consider different ways to represent relations. They use also visualization, for example, in **lesson 3.2.2** they explore the impact of the slope and y-intercept on the look of a line and in **question 2** in **lesson 3.1.4** they look at a graph to see whether it is quadratic, exponential, or linear.

- Students see connections, for example, in **part C** of the **Try This** in **lesson 3.2.1**, where they connect rate of change in a problem situation to slope of a line, or in the **Connection**s feature about half-life they connect what they are learning about relations to a real world situation.

Rationale for Teaching Approach

• This unit is divided into three chapters. All three emphasize the connection between numerical, graphical, and algebraic representations of relations.

• **Chapter 1** introduces the use of tables of values, scatter plots, lines or curves of best fit, and algebraic expressions to describe and differentiate among linear, quadratic, and exponential relations. This chapter provides students with a lot of experience working with relations numerically (in the form of tables of values) and graphically (scatter plots and lines or curves of best fit) before moving on to the more formal development of the relationship between the graph and the equation of a linear relation in **Chapter 2**.

• **Chapter 2** focuses on the different representations of the equation of a line and develops the roles of the slope and the y-intercept. Students should be able to determine the equation of a line from its graph or sketch the graph from its equation. Significant applications include determining the equation of a line of best fit and sketching graphs given the equation in either slope-intercept or standard form.

• **Chapter 3** develops the use of inverse operations to solve linear equations and introduces the solution of linear inequalities and the use of graphs to solve systems of linear equations.

• The **Explore** features allow open explorations of basic linear, quadratic, and exponential curves in **Chapter 1** and the effect of slope and y-intercept changes in **Chapter 2**. Students construct their own understanding before the ideas are formally presented. This provides a stronger base for understanding.

• The **Connections** feature focuses on half-life to provide students with a better understanding of a non-linear relationship, in this case an exponential one, in a real world application.

Technology in This Unit

The work on lines of best fit and the exploration of the role of the parameters m and b in $y = mx + b$ can be enhanced through the use of scientific graphing calculators or suitable statistical or graphing software if it is available. No technology is actually required in the unit.

Getting Started

Curriculum Outcomes	Outcome relevance
8 Patterns and Relationships: representation in a variety of formats **8** Single Variable Equations: solve algebraically **8** Linear and Non-linear Graphs: interpret **8** Linear and Non-linear Graphs: how changing one quantity affects the other **8** Slope: link visual characteristics with numerical values **8** Area of circles: develop formula **7** Large Numbers: model **7** Multiplication and Division of Integers and Decimals: mental calculation	Students will experience more success in this unit if they review what they already know about the relationships between graphs and tables and solving simple equations. Several of the other outcomes listed here represent contexts in which these skills are applied.

Pacing	Materials	Prerequisites
1 h	• Wooden or interlocking cubes • Grid paper (see page 232 in this *Guide*) • Rulers	• tables of values • scatter plots • slope • powers (**question 1**) • multiplication of negatives (**question 1**) • linear equations (**question 3**) • circumference and area of a circle (**question 4**)

Main Points to be Raised

• The values for the train lengths and the associated numbers of painted faces from the table of values can be used to describe ordered pairs that can be graphed in a scatter plot.

• Visualization skills are needed to figure out how many faces are painted. Students need to see that, for trains longer than 2 cubes, the end cubes are painted on 5 faces, but the inner cubes on only 4.

• The plotted points show the relationship between the train lengths and number of painted faces—the number of painted faces goes up by 4 for each increase of 1 in the train length. If a line were drawn through the points, the slope of the line would be $\frac{4}{1}$, or 1.

Use What You Know—Introducing the Unit

• This activity will be more meaningful if students use actual cubes to help them visualize the number of painted surfaces.

• Encourage students to work in small groups and discuss their results as they progress through the activity.

• Observe students as they work. You might ask:
- *How many painted faces did you start with in a train of length 1?* (6; cubes have 6 faces)
- *How many painted faces are added when you add a new cube to the train?* (4 painted faces are added)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

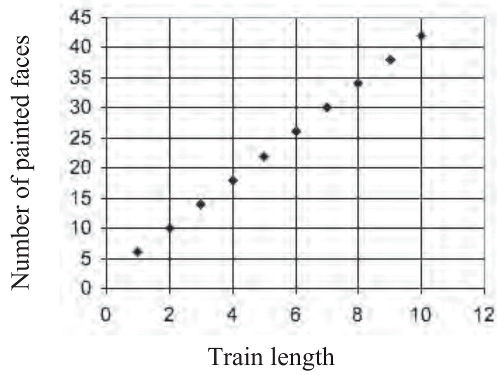
• Students can work individually, but it will be helpful if each student has a partner with whom he or she can discuss answers or difficulties.

Answers

A. and **B.**

Train length	Painted faces
1	6
2	10
3	14
4	18
5	22
6	26
7	30
8	34
9	38
10	42

C.



D. The number of painted faces increases by 4 for every 1 cube added to the train.

E. The number of painted faces is 2 more than 4 times the train length.

F. 102

1. a) 9 **b)** –8

c) 25 **d)** 1

2. a)

x	y
1	**–3**
2.5	0
3	**1**
4.5	**4**

b) 2 **c)** –2

3. a) a = –1 **b)** $x = -\frac{2}{3}$

4. $A = \pi\,(10)^2 \approx 314$ cm^2; $C = 2\pi(10) \approx 63$ cm

Supporting Students

Enrichment

Students might create an equation to represent the situation and then use it to determine the number of painted faces for a given train length and vice versa ($p = 4l + 2$, where p is number of painted faces and l is the train length).

Chapter 1 Linear and Non-Linear Relation Graphs

3.1.1 Patterns and Relations in Tables

Curriculum Outcomes	Outcome relevance
9-C1 Patterns and Relationships: determine non-algebraic representations • describe verbally and symbolically, patterns given in tables, charts, pictures, and /or by problem situations • use models such as tables, and symbolic statements to assist in examining patterns and relationships. • relate the data in a table representing a linear, quadratic, or exponential relationship to its graph • use first and second differences to determine if a table represents a linear, quadratic, or exponential relationship	Students begin to see how they can predict the type of relationship between two variables by looking at a pattern of growth.

Pacing	Materials	Prerequisites
1 h	• Rulers	• substitution into algebraic expressions • circumference and area of a circle (**question 5**)

Main Points to be Raised

• A *relation* is a property that connects the values of one variable to the values of another.

• A relation may be described in many ways:
- a table of values
- ordered pairs
- a graph
- an algebraic expression
- a verbal description

• The algebraic expression of a relation tells you its nature.
- ax + b is linear since x is raised to the power of 1
- ax^2 + bx + c is quadratic since the highest degree term is x^2
- ab^x + c is exponential since x is in the exponent

• Difference tables can help you determine the nature of a relation.
- For a linear relation, first differences are constant and second differences are zero.
- For a quadratic relation, second differences are equal, but not zero.
- For an exponential relation, the ratio of consecutive first differences is constant.

Try This—Introducing the Lesson

A. and B. Observe while students work. You might ask:

• *How could you predict the number of diamonds needed for Figures 4 and 5 in each pattern without actually drawing each picture?* (In pattern 1, each figure has as many diamonds as the figure number; Figure 4 has 4 diamonds and Figure 5 has 5 diamonds. In pattern 2, the number of diamonds is the square of the figure number; Figure 4 has $4^2 = 16$ diamonds and Figure 5 has $5^2 = 25$ diamonds. In pattern 3, you double the number of diamonds from the previous figure. Figure 4 has 8 diamonds and Figure 5 has 16 diamonds.)

• *How would an algebraic expression of the pattern rule make it easier to predict Figure 10 in each pattern?* (With an algebraic expression, you would not have to draw all of the figures from 1 to 9 to find the number of diamonds in Figure 10. You can simply substitute the figure number into an algebraic expression.)

The Exposition—Presenting the Main Ideas

• Ask one of the students to record one of his or her tables of values from **part A** in **Try This** on the board. Write the values from the first few rows in ordered pairs, e.g., (1, 1), (2, 2), (3, 4), (4, 8),… from pattern 3. Ask students why this representation of the relation is just as appropriate as the table of values for describing the relationship. Discuss how this new representation in ordered pairs makes it easier to represent the same relation on a scatter plot. Introduce the algebraic expression that describes the relation whose table of values was shown, e.g., $y = 2^{x-1}$. Discuss how this algebraic expression is really just an abbreviated way of writing a verbal description of the pattern rule. Emphasize that all five representations (table of values, ordered pairs, graph, algebraic description, and verbal description) of a relation are valid.

• Take the table of values used above and create a difference table as shown on the second page of the exposition. Make sure to show both first differences and second differences and to name them.
Explain that these differences sometimes make it easier to describe a pattern verbally as well as algebraically.

For example, for the table from pattern 3, students will see that the first differences keep doubling. Tell them that when first differences keep doubling, this is an example of an *exponential* relation. Write the term "exponential" on the board. Tell them that sometimes, instead of doubling, it might be tripling or halving or multiplication by some other factor, but that whenever multiplication (other than by 0 or 1) describes the pattern in the first difference column, the relation is exponential. Some students may notice that the original numbers also show a pattern of multiplication, as do the second differences that are the same as the first differences, just moved down one row.

Contrast this with a table of values that you create for a linear function like $y = 2x$, where the first differences are equal and the second differences are zero, or a quadratic function like $y = x^2$ where the second differences are constant but not zero. Write the terms *linear* and *quadratic* on the board to describe these types of relations. Make sure that students know that there are also other kinds of relations, but that they will not be studied at this time.

• Work through the exposition with the students to reinforce what you have shown them above. Emphasize the algebraic descriptions of linear, quadratic and exponential expressions. For quadratic relations, focus on the exponent of 2. For exponential relations, focus on the fact that the variable, e.g., x, is in the exponent. If students ask, assure them that if the exponent were, for example, $2x$ or $-x + 2$, this would still be called an exponential relation.

Revisiting the Try This

C. Students can work with a partner.
This question allows an opportunity to make a formal connection between the patterns in **part A** and the notion of relationships that they will eventually represent algebraically and graphically. These representations can then be used to find the number of diamonds in any figure from the figure number.

Using the Examples

Have students read through the two examples independently. Assure students that although some may use the difference tables to determine the nature of the relations and others may use the algebraic expression of the relation, both approaches are valued equally.

Practising and Applying

Teaching points and tips

Q 2: Some students may realize that the use of the term “square” to represent the difference in the figures forces the relationship to be quadratic. Others may have to construct the difference table.

Q 3: Encourage students to look for patterns in the tables of values before calculating differences.
For example, **part a)** is exponential because each term is 3 times the previous term and **part b)** is linear since each term increases by 2 from the previous term.

Q 4: There is only one possible linear or exponential pattern. There are many possible quadratic patterns.

Q 6: This pattern is challenging to visualize.
Students will find it easier to determine the number of balls on each level if they can actually build the pyramid using balls, marbles, or small round stones.

Common Errors

Arithmetic errors are likely to occur. Suggest that students work in pairs and double check computations as they do them.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can create tables of values from a pattern and use first and second differences to identify the nature of a relation
Question 4	to see if students can apply the characteristics of linear and exponential relations to continue a pattern
Question 8	to see if students can apply their knowledge of linear, quadratic, and exponential relations in a real-world problem context
Question 9	to see if students can calculate first and second differences and use them to identify the nature of a relation

Answers

A.

Figure	Pattern 1	Pattern 2	Pattern 3
1	1	1	1
2	2	4	2
3	3	9	4
4	**4**	**16**	**8**
5	**5**	**25**	**16**

B.

Pattern	Diamonds in Figure 10	Reason
1	10	Same as figure number
2	100	Square of figure number
3	512	Power of 2; $2^{(\text{figure number} - 1)}$

C. i) Linear; first differences are equal

ii) Quadratic; second differences are constant (2) and not zero

iii) Exponential: ratios of first differences to term values are equal

1.

Figure	**a)** Perimeter	**b)** White triangles	**c)** Small triangles
1	3	1	1
2	6	3	4
3	9	6	9
4	12	10	16

d)
part a) is linear since the first differences are all 3
part b) is quadratic since the first differences are different, but the second differences are all 1
part c) is quadratic since the first differences are different, but the second differences are all 2

2. Quadratic; second differences are 2.

3. a) E **b)** L **c)** N **d)** N

4. a)

b)

5.

Radius	**a)** Circum.	**b)** Area
0	0.00	0.00
1	6.28	3.14
2	12.57	12.57
3	18.85	28.27
4	25.13	50.27
c)	linear	quadratic

c) Circumference: linear since the formula has *r* only to the first power and first differences are constant (6.28)
Area: quadratic since the formula has *r* raised to the second power and second differences are constant and not zero (6.28)

6. a) 10 **b)** 20 (10 + 6 + 3 + 1)

c) 21 **d)** Quadratic

7. Linear; first differences are constant

8. Quadratic; second differences are all 160

9. a) *Sample response*:

x	Linear	Quad.	Exp.
1	6	3	1
2	8	6	3
3	10	11	9
4	12	18	27

b)
Linear; first differences are equal and second differences are zero
Quadratic; second differences are equal but not zero
Exponential; each term is the product of the previous term and a constant factor

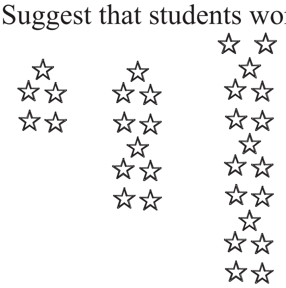
Supporting Students

Struggling students

Completing tables of values and difference tables is computationally intensive. Allow struggling students to use calculators to carry out the arithmetic. Encourage them to check for patterns in each column of differences.
If a particular term does not follow the pattern they have found, they have probably made a computational error.

Enrichment

Suggest that students work with a partner to try to come up with quadratic patterns for **question 4**. For example:



3.1.2 Scatter Plots of Discrete and Continuous Data

Curriculum Outcomes	Outcome relevance
9-C1 Patterns and Relationships: determine non-algebraic representations • describe verbally and symbolically, patterns given in graphs, pictures, and /or by problem situations • use models such as graphs to assist in examining patterns and relationships. **9-C2 Scatter Plots: characteristics of relationships** • consider whether data represented by a scatter plot are continuous or discrete and whether interpolation is meaningful • distinguish between independent and dependent variables in a scatter plot **9-F1 Displaying Data: draw inferences and make predictions** • interpolate and extrapolate using a data set	• Students begin to see how a scatter plot can reveal information about a relation. They learn to pay attention to which variable is independent and which is dependent and to consider whether data is discrete or continuous. This will serve them well in future mathematical work. • Students begin to see the strengths and limitations of interpolating and extrapolating using a graph to make predictions. This prepares them for future experiences with mathematical modelling of natural and social phenomena.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• labelling a graph • scatter plots

Main Points to be Raised

• You can use a scatter plot to show a relationship between variables.

• The independent variable is the variable whose value you can choose. Its values usually appear in the left column of a table of values and are the x-coordinates of the plotted points.

• The value of the dependent variable depends on the value of the independent variable. Its values usually appear in the right column of a table of values and are the y-coordinates of the plotted points.

• Sometimes the plotted points are joined by a line or a smooth curve. This makes it easier to see to see whether there is a relationship between the variables.

• A *solid* graph shows that any value between the plotted values could have been used as a value of the independent variable. In this case, the data is said to be continuous.

• If no values or a limited number of values exist between the plotted values, a *dashed* graph is usually used and the data is said to be discrete.

• When you predict or estimate between known coordinates you are *interpolating.* Predicting beyond known coordinates is *extrapolating*. It is important to judge whether it is appropriate to interpolate or extrapolate.

Try This—Introducing the Lesson

A. and B. Encourage students to work with a partner. Observe while students work. You might ask:

• *As you add bottles of water to the cart, why does the mass of the water increase in "jumps"? How is this shown on the graph?* (When filling the cart with bottles, the mass cannot increase by fractions of a bottle so the mass increases only in jumps of 1 kg. This is represented on the graph by breaks that increase by 1 kg on the dependent axis.)

• *As you add water to the tank, why does the mass of the water increase at a steady rate with no "jumps"? How is this shown on the graph?* (Since the water is poured continuously, there are no discrete amounts that must all be added at once. That is why the graph is a solid line with no breaks.)

The Exposition—Presenting the Main Ideas

• Have students read through the exposition independently. You may have to explain the term *elapsed time,* which describes time that passes between two events of interest. When students have finished reading, point out that in the situation described in this section, the experimenter controls the times at which the remaining volume of water is measured, so time is the independent variable. Point out that in the **Try This** situations, Dawa is able to control the volume of water added to the cart or the tank. The mass of the water depends on the volume of water Dawa has added, so the volume of water (or number of bottles) is the independent variable.

• Discuss why the example involving evaporation of water is more like the **Try This** situation of filling the tank than like adding water bottles to the cart. Explain that in the evaporation and tank-filling cases it is possible to determine a data point on the graph that corresponds to a decimal or fractional value of litres of water. It is not possible to determine a data point that corresponds to a non-whole number of bottles of water.

• Display the terms *independent variable, dependent variable, continuous, discrete, interpolate,* and *extrapolate* on the board, on a poster, or on cards on the wall. Make sure students can explain the meaning of each of these terms.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between the tables of values of discrete and continuous data in a real-life context in **part A** and the graphs that represent these data sets.

Using the Examples

• Lead students through **examples 1 and 2**, emphasizing how they are alike and how they are different.
• Allow time for students to read through **example 3** independently. When they have finished, make sure they understand why interpolation was not possible (since the data was discrete).

Practising and Applying

Teaching points and tips

Q 2: You might ask students to use a ruler on the graph to interpolate the area for a rectangle with height 3.5 cm and extrapolate the area for a rectangle with height 10 cm.

Q 5: This pattern is difficult to visualize. Students will find it easier to determine the number of balls on each level if they can build the pyramid using golf balls, marbles, or small round stones. If that is not practical, suggest they sketch each layer after counting the number of balls on one side.

Q 6: Although students will not realize this, the data in this question previews later work in trigonometry. Students might notice that the relationship appears linear up to 20º. Talk about why it is important to consider as many points as possible before deciding on the type of relationship.

Q 7: Some students might be curious about how Manju determined the number of segments.
The formula for n points is $n(n - 1) \div 2$.

Suggested assessment questions from Practising and Applying

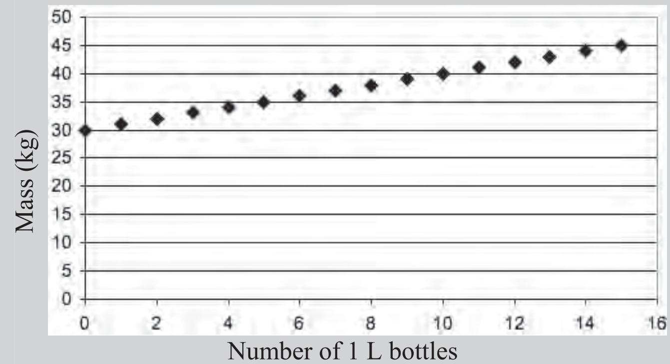
Question 1	to see if students can create a scatter plot from a table of values and determine whether the data should be treated as discrete or continuous
Question 3	to see if students can extrapolate data using a scatter plot of discrete data
Question 4	to see if students can interpolate data using a scatter plot of continuous data
Question 6	to see if students can distinguish between linear and almost-linear data

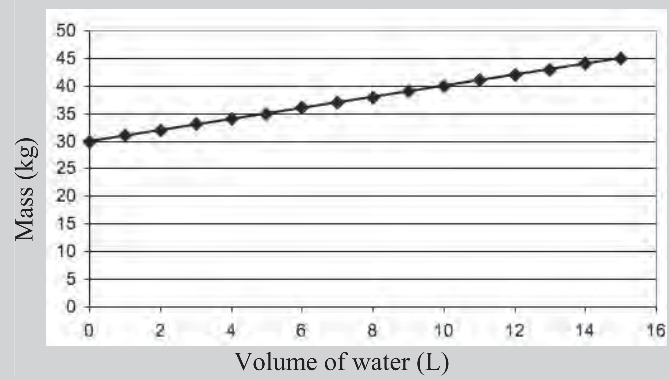
Answers

A. NOTE: the tables for **parts i) and ii)** contain the same values.

i) Number of bottles or **ii)** Litres of water	Mass (kg) of **i)** Cart of bottles or **ii)** Water in barrel
0	30
1	31
2	32
3	33
4	34
5	35
6	36
7	37
8	38
9	39
10	40

B.

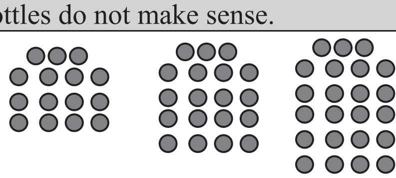




C. i) Water bottle data is discrete; water barrel data is continuous.

ii) You can extrapolate with both graphs; you can only interpolate with the water barrel graph because fractions of bottles do not make sense.

1. a)



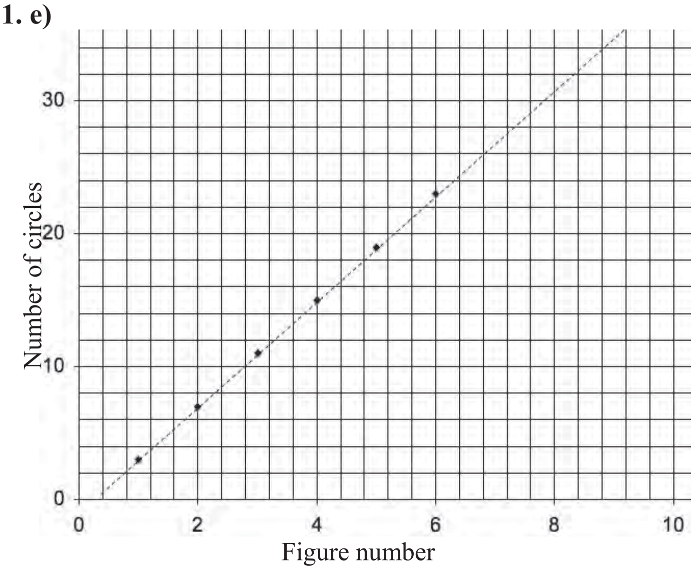
b)

Figure number	Number of circles
1	3
2	7
3	11
4	**15**
5	**19**
6	**23**

c) Discrete; does not make sense to have a fractional figure number

d) Dashed line

1. e)



f) 39

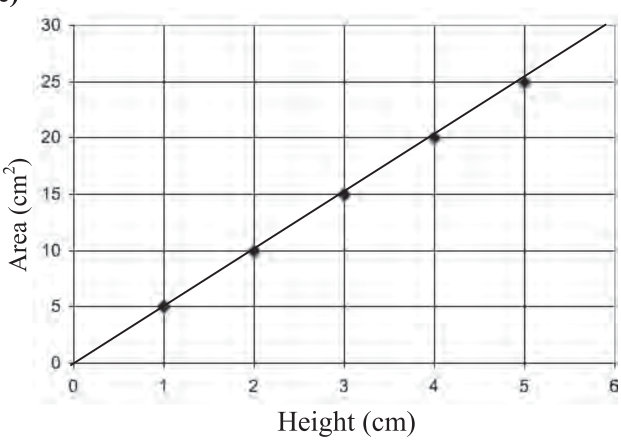
2. *Sample response*: **a)** and **b)**

Height (cm)	Area (cm^2)
1	5
1.5	7.5
2	10
3	15
4.75	23.75
5	25

c) Continuous; rectangles with a width of 5 cm can be any height

d) Solid line

e)



3. a) Day number is discrete

b) 15 trees on 7th day; 21 trees on 10th day

4. a) Time is continuous.

b) *Sample response*: How long did it take to drain 24,000 L? (120 min) How much water remained after 222 min? (35,000 L)

c) The pool is empty at 400 min and the volume of water in the pool cannot be negative.

5. a)

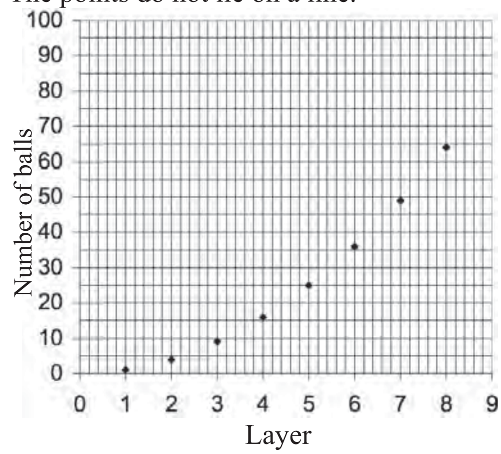
Layer number	Number of balls
1	1
2	4
3	9
4	**16**
5	**25**
6	**36**
7	**49**
8	**64**

5. b) Discrete; no fraction layer numbers

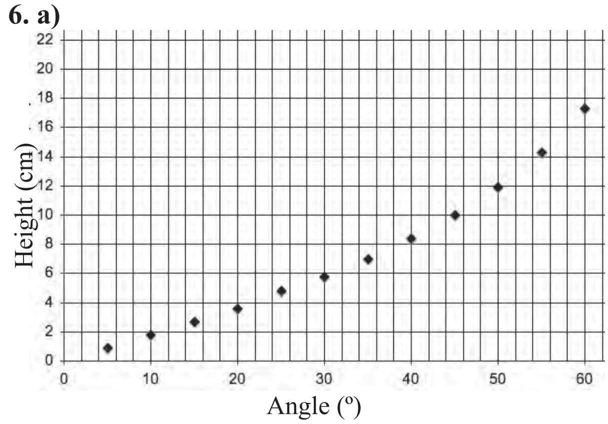
c) Dashed; data is discrete

d) *Sample response*:

The points do not lie on a line.



6. a)



b) *Sample response*:

A smooth curve; it looked like a line for a while, but then it started curving upward

c) solid; an angle measure between those plotted is possible

7. Yes, the data points follow a curved pattern, so she was right to use a curve;

No, the data is discrete so she should not have used a solid curve

Supporting Students

Struggling students

- Some students will not see patterns immediately. Encourage them to use more concrete or pictorial models if it helps them. For example, they might use markers or counters to build the pattern in **question 1**.
- Students might benefit from conferring with a partner about whether data is discrete or continuous.

Enrichment

- Students could repeat **question 5** but using the cumulative number of balls instead of the number of balls per layer.
- Some students might be ready to determine the algebraic formula for **question 7** on their own.

3.1.3 EXPLORE: Graphs of Linear and Non-Linear Relations

Curriculum Outcomes	Lesson relevance
9-C1 Patterns and Relationships: determine non-algebraic representations • explore linear, exponential and quadratic curves • describe verbally and symbolically, patterns given in graphs • use models such as graphs, and symbolic statements to assist in examining patterns and relationships.	• This optional lesson provides students with experience that will help them make more sense of subsequent formal work with different kinds of functions. • Students see how varying the parameters in linear, quadratic, and exponential relations may change the position or size of the graph, but not the characteristic shape of the graph.

Pacing	Materials	Prerequisites
1 day	• Grid paper (see page 232 in this *Guide*) • Rulers	• creating tables of values • evaluating expressions • scatter plots

Main Points to be Raised

• The graphs of linear, quadratic, and exponential relations are different from one another.

• The graph of each type of relation has its own unique characteristic shape.

Exploration

You might divide the class into groups of two to four students and have each group work on one of **parts A**, **B or C**. Ask each group to examine several different relations of their assigned type.

Observe while students work. You might ask:

• *If you were to graph different linear (or quadratic or exponential) relations, how would the graphs compare?* (For linear relations, they might have different slopes and be in different positions on the coordinate grid.)

• *What effect did the 3 have on the graph in part iii) of each? How could you tell?* (It made the graph steeper in each case.)

Observe and Assess

As students are working, notice:

• Are they able to apply order of operations rules and evaluate the expressions correctly?

• Are they able to plot the points and draw the scatter plots with reasonable accuracy?

• Are they able to recognize the characteristic shape of the graph of each type of relation?

• Do they use what they learn in one situation to predict what might happen in the next one?

Share and Reflect

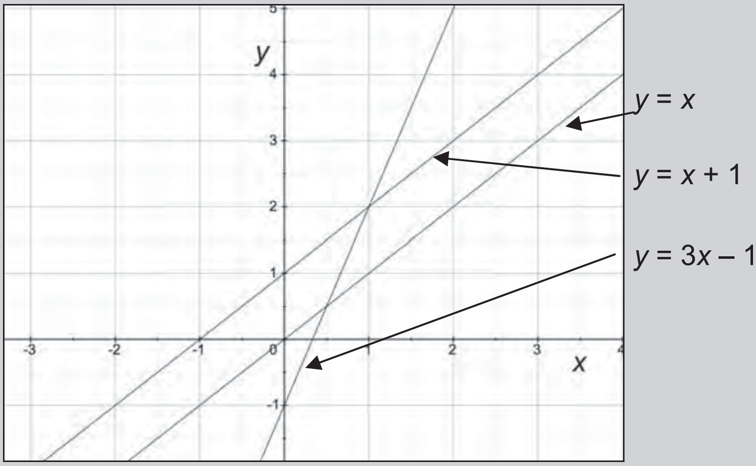
• If you had students work in groups, ask all the students who worked on the same part to get together and report what they observed about the graphs. They could record their observations on a large sheet of poster paper or on the board for the entire class to see.

• If you have graphing software available, you could ask students to print the graphs and display them in the classroom. Students could then discuss how the various graphs differed or were alike.

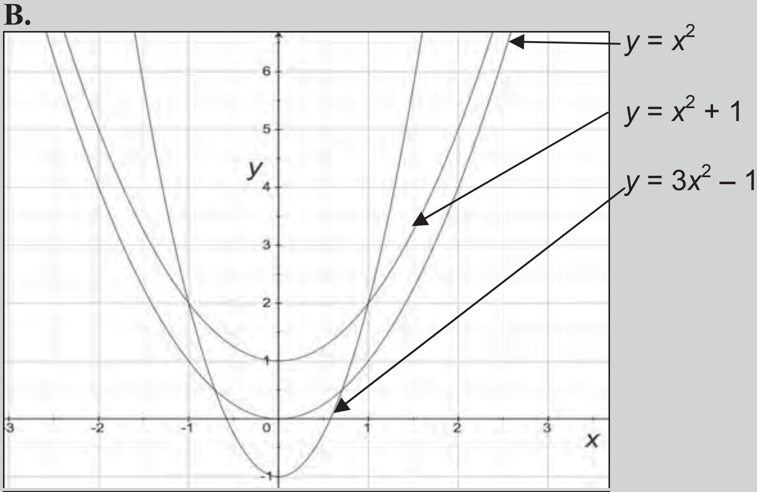
• You could encourage students to predict what other variations of each type of relation would look like to emphasize that the shape of the graph does not change even if the size and position do.

Answers

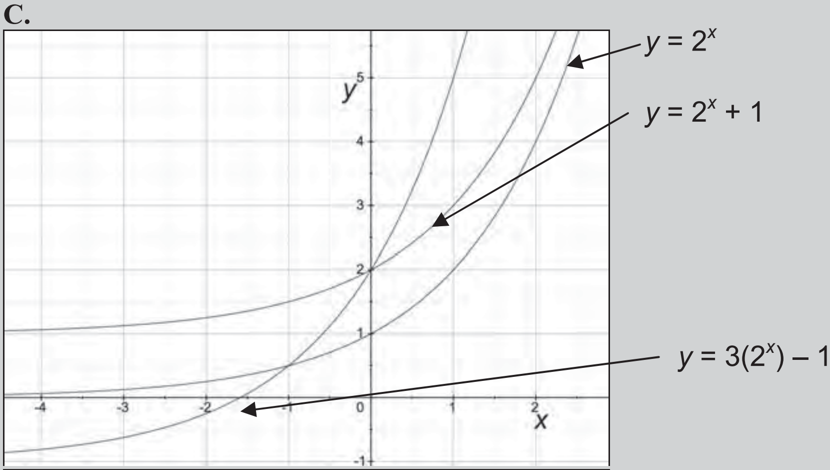
A.



B.



C.



D.

Linear graphs look like straight lines.

Quadratic graphs are symmetric curves with a U shape.

Exponential curves are steep (almost vertical) on one side and shallow (almost horizontal) on the other side.

Supporting Students

Struggling students

Assign students to work with students who you know will be able to assist them with the task.

Enrichment

Interested students might explore the graphs of types of relations other than those included in the investigation to see if similar results occur. For example, they might graph curves like $y = x^3 + k$ or $y = \frac{1}{x} + k$.

CONNECTIONS: Half-Life

This is an example of an exponential relation in which the dependent variable (mass) decreases as the independent variable (time) increases.

Note: It may be more obvious to students that this is an exponential relation if the powers of the fractions are written with the fraction as the base, e.g., $\frac{1}{2^2} = \left(\frac{1}{2}\right)^2$.

Answers

1. 128 g; 64 g; 32 g **2. a)** $\frac{1}{2}$; $\frac{1}{4}$; $\frac{1}{8}$ **b)** $\frac{1}{2^1}$; $\frac{1}{2^2}$; $\frac{1}{2^3}$ **c)** $\frac{1}{2^{10}}$ or $\frac{1}{1024}$; $\frac{1}{2^1}$ after 1 day; $\frac{1}{2^2}$ after 2 days; $\frac{1}{2^3}$ after 3 days; so $\frac{1}{2^{10}}$ or $\frac{1}{1024}$ after 10 days	**3.** Exponential; the values of successive terms are found by multiplying by a constant ($\frac{1}{2}$)

3.1.4 Graphs of Linear and Non-Linear Relations

Curriculum Outcomes	Outcome relevance
9-C1 Patterns and Relationships: determine non-algebraic representations • explore linear, exponential and quadratic curves • describe verbally and symbolically, patterns given in graphs, pictures, and/or by problem situations • use models such as graphs to assist in examining patterns and relationships • relate the data in a table representing a linear, quadratic, or exponential relationship to its graph • determine if a table represents a linear relationship by plotting the points	Students discover unique characteristics of the graphs of linear, quadratic, and exponential relations.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• scatter plots • circumference of a circle (**question 4**) • surface area and volume of a cube (**question 5**)

Main Points to be Raised

• The graph of a linear relation is a straight line.

- In a linear relation, the values of the *y*-coordinate increase by a constant value, the rise, as the values of the *x*-coordinate increase by a constant value, the run. The ratio of these values, $\frac{\text{rise}}{\text{run}}$, is the slope of the line.

- You can choose the spacing between adjacent points to determine the run. Then you use the appropriate *y*-values for the points you chose to determine the rise.

- If the line slopes up and to the right, the slope is positive. If the line slopes down and to the right, the slope is negative.

• The graph of an exponential relation is non-linear.

- The graph of the exponential relation shown becomes nearly parallel to the *x*-axis on one side and then curves upward and becomes nearly parallel to the *y*-axis on the other side.

- Sometimes, if it makes no sense to use certain values for the *x*-coordinate, you see only one of the sides becoming nearly parallel to an axis.

• The graph of a quadratic relation is non-linear.

- The graph of a quadratic relation is a U-shaped curve called a parabola.

- If the parabola opens upward, it has a minimum point; if it opens downward, it has a maximum point.

- The maximum or minimum point is called the vertex of the parabola.

- The parabolas shown both have a vertical line of symmetry.

- Sometimes, if it makes no sense to use certain values for the *x*-coordinate, you do not see the whole U shape, but only part of it.

• There are other types of relations that have graphs that are neither straight lines, U-shaped, nor curved upward. The term non-linear can be used to describe these graphs and relations.

Try This—Introducing the Lesson

A. and B. To reduce the time required to complete the **Try This**, you might assign one third of the class to investigate each choice. Suggest that students graph the values each choice provides up to week 12 as they work on **part B**.

Observe while students work. You might ask:

• *What type of relation does Choice 1 (or Choice 2 or Choice 3) seem to be?* (Choice 1 is a linear relation; Choice 2 is quadratic; Choice 3 is exponential.)

• *How did you keep track of the amounts? Did you use a table? What did it look like?* (For choice 1, to save space, I made a chart and wrote the amount for each week. Since he got paid every day, I had to multiply by 7.)

The Exposition—Presenting the Main Ideas

• Ask students to recall what the slope of a line is and how it is calculated. If necessary, review the concept of $\frac{\text{rise}}{\text{run}}$ and make sure students realize that any *run* can be used as long as the appropriate *rise* is also used. • Lead students through the main ideas in the Exposition. You may find it useful to have them refer to the graphs they drew in **lesson 3.1.3**. • Emphasize any new terminology and have students help you put together a list of the characteristics of the graphs.

Revisiting the Try This

C. Suggest that students extend their tables of values and graphs beyond 12 weeks to verify that eventually *Choice 3* returns a greater amount.

Using the Examples

• On the board, display the equations for the three graphs in **example 1**. Ask students to draw rough sketches and see how they differ. Then have them read through the example in the text.
• Draw students' attention to **example 2.** Help them see where the data in the table came from. Let them read through the example independently. Ask them why the data in **example 1** is continuous while the data in **example 2** is discrete.

Practising and Applying

Teaching points and tips

Q 3: Observe whether students predict the graph before they actually create it. Ask why the graph in the text does not go above the *x*-axis.

Q 5: You may have to remind some students of the formulas for the surface area and volume of a cube.

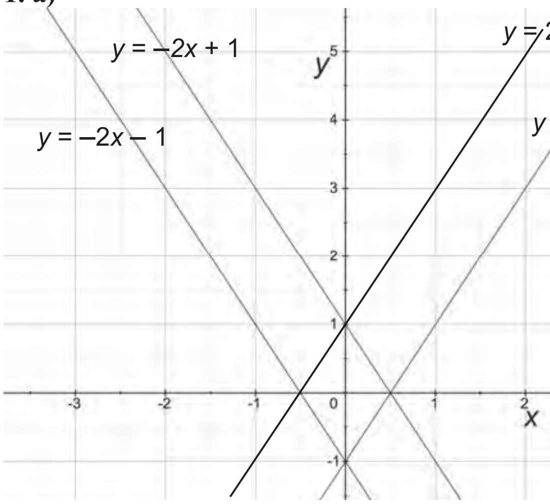
Suggested assessment questions from Practising and Applying

Question 1	to see if students recognize that the graph of a linear relation is a straight line and to see if they can determine the slope of that line from the graph
Question 2	to see if students can associate the shape of the graph of a linear, quadratic, or exponential relation with the type of the relation it represents
Question 5	to see if students can make a connection between measurement formulas with which they are familiar and new learning about types of relations

Answers

A. Choice 1: 84 Choice 2: 10 Choice 3: 2 **B.** Choice 1: 252 Choice 2: 78 Choice 3: 14	**C. i)**: The differences between the choices get smaller as more weeks pass. Choice 3 was growing the fastest and by 1 year would definitely pass the other choices. **ii)** Choice 1: linear; Choice 2: quadratic; Choice 3: exponential

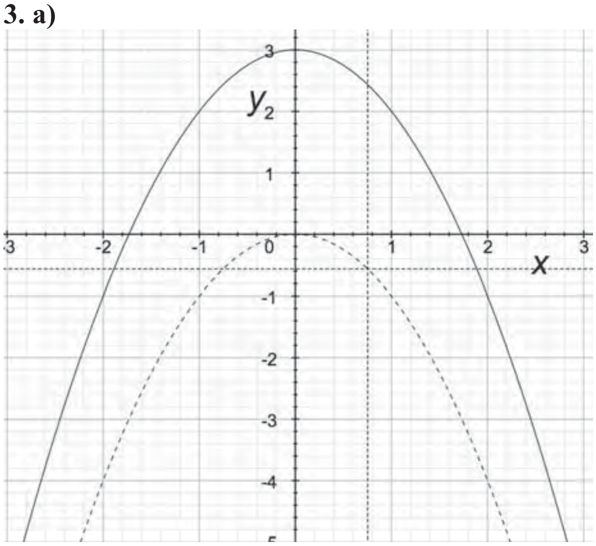
1. a)



b) They are all straight-line graphs.
c) $y = -2x - 1$ and $y = -2x + 1$ both have a slope of -2; $y = 2x + 1$ and $y = 2x - 1$ both have a slope of 2

2. a) A; it's U-shaped with a vertical line of symmetry
b) C; it's a curve that is almost horizontal on the left and almost vertical on the right

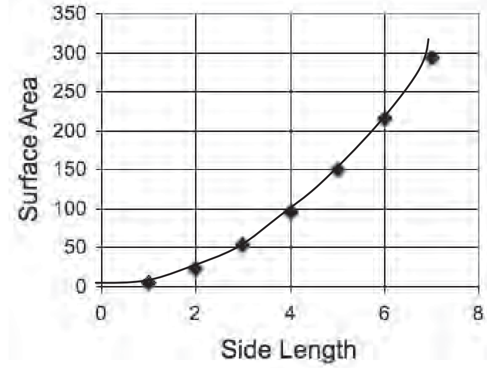
3. a)



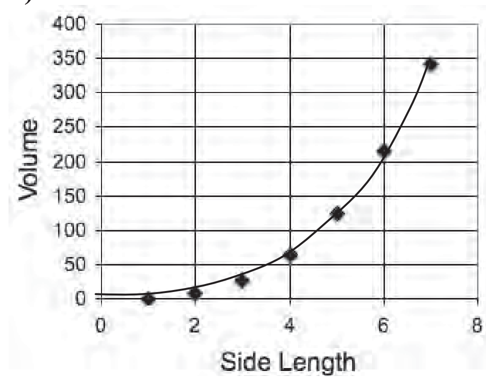
b) *Sample response*:
Both are U-shaped with a vertical line of symmetry, both open downward, and each has a maximum point
c) *Sample response*:
One goes through the origin, but not the other.

4. A line; circumference is $C = 2\pi r$ and r is raised to the first power; also, circumference is a measure of length and length is linear

5. a)



b)

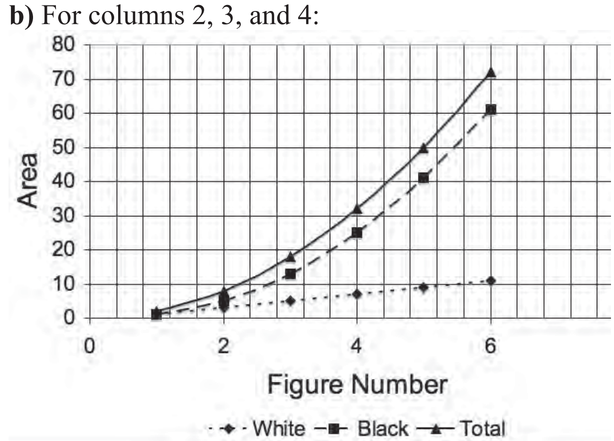


c) Neither graph has points that fall along a straight line.

6. a)

Figure number	White area	Black area	Area of big square	Part of big square that is black
1	1	1	2	0.500
2	3	5	8	0.6250
3	5	13	18	0.7222
4	7	25	32	0.7813
5	9	41	50	0.82
6	11	61	72	0.8472

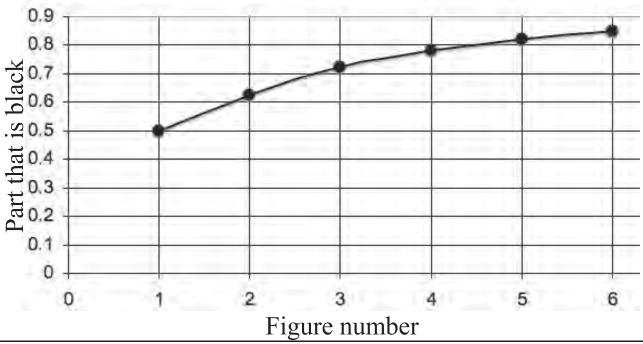
b) For columns 2, 3, and 4:



[Continued]

Answers [Continued]

6. b) [Cont'd] For last column



6. c)

White area	Linear; because it's a straight line
Black area	Quadratic; because it looks like half a parabola
Big square	Quadratic; because it looks like half a parabola
Part of big square that is black	None; because it's not a straight line parabola, or exponential curve

Supporting Students

Struggling Students

The focus in Class IX should be on linear relationships. If students are struggling with non-linear relationships, allow them to focus on work with linear relations.

Enrichment

• Interested students can extend the pattern in **example 2** and determine that there appears to be a limiting value to the area.
• You could ask interested students to determine whether there is a limiting value to the decimal that represents the portion of the big square that is black in **question 6**.

3.2.1 The Meaning of Slope and *Y*-Intercept

Curriculum Outcomes	Outcome relevance
9-C3 Graphs of Linear Relations: interpret and create • use the term slope to represent rise/run • relate the *y*-intercept to the value of the *y*-coordinate where the graph crosses the *y* axis • determine the slope and *y*-intercept by examining a table or graph	• As students begin to interpret the slope of the graph of a linear relation as a rate of change of the dependent variable in terms of the independent variable. They can apply this skill to solve many real-world problems. • As students understand the role of the *y*-intercept, they will be able to use graphs more easily to solve problems involving linear data.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• scatter plots

Main Points to be Raised

• The slope of the line tells the rate at which the *y*-variable changes in terms of the *x*-variable:
- If the slope is positive, the *y*-variable increases as the *x*-variable increases.
- If the slope is negative, the *y*-variable decreases as the *x*-variable increases.
- If the slope is zero, the *y*-variable remains constant as the *x*-variable increases. The result is a horizontal line.

• The *y*-intercept of the line is the *y*-value of the coordinates where the line meets or crosses the *y*-axis.

Try This—Introducing the Lesson

A. and B. Have students work alone or in pairs. Observe while students work. You might ask:

• *How far will he climb every 5 min?* (Since he climbs 5 m every min, he will climb 25 m in 5 min.)

• *How will this be shown in the table of values? on the graph?* (The values for the altitude in the table of values will go up by 25 for every 5-min interval. The graph will be a straight line of slope 5.)

The Exposition—Presenting the Main Ideas

• Have students read through the exposition on their own. They will already have related slope to direction and steepness of the line. You could point out that the units in which the slope is expressed (m/min) connect the value of the slope to the rate of change of the dependent variable in terms of the independent variable.

• Ask students to describe situations that might be represented by each line on the graph in the exposition.

• You could refer students to the **Try This** and **examples** in **lesson 3.1.2**. Ask what the slope and *y*-intercept are for each graph and how they should be interpreted in each situation.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between the rate of change of altitude in **part A** and the slope of a linear graph. Explain why 25 m/5 min, 150 m/30 min, and 5m/min all represent the same climbing rate and, therefore, the same slope.

Using the Examples

• Pairs of students can be assigned to read through the examples independently. One student can read **example 1** and the other can read **example 2**. They then share with each other what they learned.

• Make sure all students understand that it is because the rate of change is linear— a change of *a* units of one variable corresponds to a change of *b* units of another— that the graphs are straight lines.

Practising and Applying

Teaching points and tips

Q 1: Some students will be able to explain that the y-intercept is the value when $t = 0$, but will not necessarily relate it to the initial water temperature

Q 2: This question is one of several that encourages students to relate a rate of change in a real-world situation to the slope of the graph that describes it. It is important to help students see that both the slope and the y-intercept have meaning in relation to the context.

Q 4: Some students have more difficulty recognizing a linear relationship when the numbers decrease. Ask why the slope of the line has to be negative. Some students, but not all, will be able to determine the slope from the table, recognizing that a decrease of 450 km/h in 5 min reflects a slope of –90. km/min

Q 8: Some students will realize that the 4 cm representing the perimeter of the square would have to be the y-intercept since it remains constant.

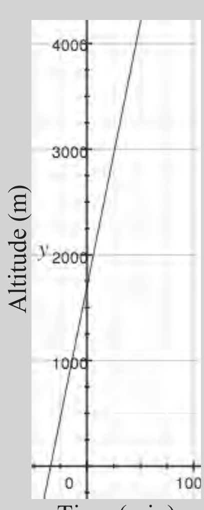
Common Errors

Some students may misread coordinates from a graph and incorrectly compute rise and run as a result. Other students may reverse rise and run in the computation of the slope. In both cases, encourage students to very careful in reading the graph and to use an equation like $\frac{\text{rise}}{\text{run}} = \square$ to report their results.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can interpret slope and y-intercept from a graph of a real-world situation
Questions 2 and 8	to see if students can determine slope and y-intercept from a verbal description of a real-world situation
Question 4	to see if students can relate the slope and y-intercept determined from a graph to the rate of change and initial value described in a verbal description of a real-world situation modelled by a linear relation
Question 5	to see if students can relate the slope and y-intercept of a graph to a pattern described by a linear relation

Answers

A. i) A constant climbing rate means that the graph has a constant slope, so it is a straight line graph **ii)** 	**B.** 1700 m; 1800 m **C. i)** It is the slope of the line. **ii)** The y-intercept is 1700 m. It represents the starting altitude of the climb.

1. a) Cooling rate (in °C/min)

b) Temperature is falling

c) Starting temperature

d) Graph would be parallel and cross the y-axis at a higher point

2. a) The cost changes at a constant rate.

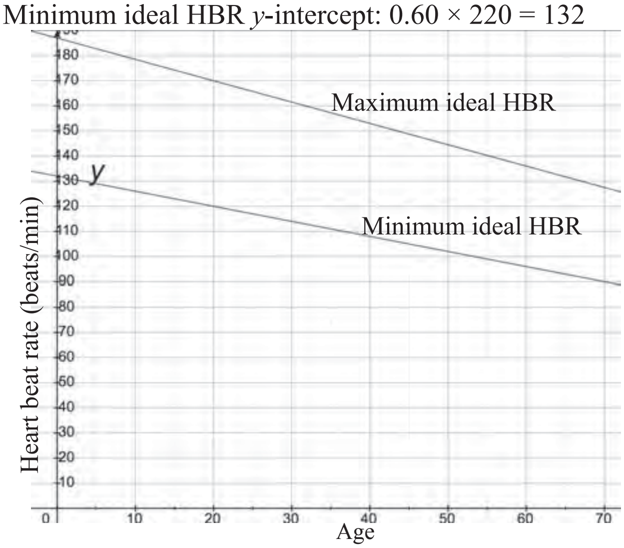
b) Cost per minute above base rate

c) Base amount

3. a) Maximum ideal HBR formula is is $t = 0.85(220 - a)$, which is a linear expression.

b) The y-intercept for the maximum ideal heart beat rate is at age 0, so its value is 85% of 220: $0.85 \times 220 = 187$.

c) and d) Minimum ideal HBR formula is $t = 0.60(220 - a)$, which is a linear expression. Minimum ideal HBR y-intercept: $0.60 \times 220 = 132$



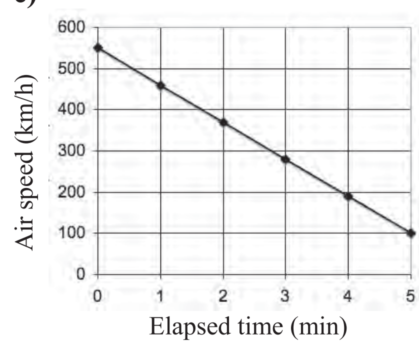
e) different slopes and y-intercepts; both are straight lines.

4. a)

Elapsed time (min)	Airspeed (km/h)
0	550
1	**460**
2	**370**
3	**280**
4	**190**
5	100

4. b) A constant rate of change means a straight line graph.

c)



d) slope = –90; y-intercept = 550

e) y-intercept is starting airspeed, or 550 km/h; slope is change in speed per minute, so dividing rise by run, e.g., $-450 \div 5 = -90$

5. a) A straight line, same y-intercept, shallower slope

b) Slope would be steeper because rise stays the same but run is reduced

6. a) Each time you add a square, the number of sticks increases by a constant amount, 3.

b) 3; for each increase of 1 in the figure number (run), there is an increase of 3 in the number of squares (rise)

c) 1; Figure 0 would have 1 stick

7. a) 2π

b) $r = 0$; the circumference of a circle with radius 0 cm is 0 cm

8. a) Perimeter is a measure of length and length is linear

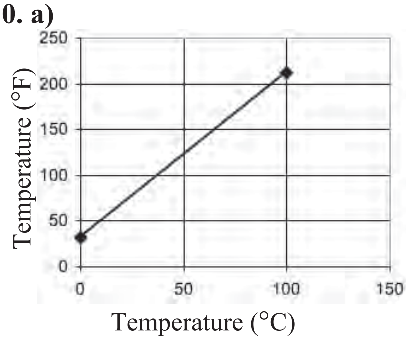
b) 2π

c) 4

Answers [Continued]

9. If you don't, you get the opposite value for slope; *sample response*: for (1, 7) and (2, 9), if you subtract the values of one point from the other, you get a slope of 2; if you subtract the x-value of the first point from the x-value of the second point but the y-value of the second point from the y-value of the first point, you get a slope of –2

10. a)



10. b) Locate 30°C on the horizontal axis and then find the corresponding point on vertical axis.

c) °F equivalent of 0°C

d) $\frac{9}{5}$ or 1.8

e) For every increase of 9°F, there is an increase of 1°C.

Supporting Students

Struggling students

Struggling students will need more support to see how the rate of change relates to the slope. Keep emphasizing that relationship in simple situations, for example, converting numbers of hours to minutes, weeks to days, or years to weeks.

Enrichment

• You could ask interested students to try to write the formula for converting a temperature given in °C to °F as an extension to **question 9**.

• A shortcut for converting °C to °F is to double the Celsius temperature and add 30. You could ask interested students to compare the graph of this relation to the one in **question 9 a)**. They could investigate whether there is a range of temperatures for which this is a reasonably good approximation of the actual Fahrenheit temperature.

3.2.2 EXPLORE: The Equation of a Line

Curriculum Outcomes	Lesson relevance
9-C3 Graphs of Linear Relations: interpret and create • use the term slope to represent rise/run • relate the *y*-intercept to the to the value of the *y*-coordinate where the graph crosses the *y*-axis • determine the slope and *y*-intercept by examining a graph • sketch the graph of a linear relation given the slope and *y*-intercept	In this optional lesson, students explore the impact of changing the values of m and b on the graph of a linear relation in $y = \mathrm{m}x + \mathrm{b}$ form. This will help them visualize graphs from equations.

Pacing	Materials	Prerequisites
1 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• scatter plots

Main Points to be Raised

• Changing the value of b changes the *y*-intercept of the line:
- If b > 0, the *y*-intercept is > 0.
- If b < 0, the *y*-intercept is < 0.
- If b = 0, the *y*-intercept is 0.

• Graphs of linear relations that have the same value of m, but different values of b, have the same slope.

• Changing the value of m changes the slope of the line:
- If m > 0, the line slopes up and to the right.
- If m < 0, the line slopes down and to the right.
- If m = 0, the line is horizontal.

• Graphs of linear relations that have the same value of b, but different values of m, have the same *y*-intercept.

Exploration

You might divide the class into several smaller groups. Some group members would focus on the effect of changing m in **part A** while the others would explore the effect of changing b in **part B**. They would then work on **part C** together.

Have students display their graphs around the room so other students can see the effects of varying the parameters. Make sure that each graph clearly shows the equation for each line. It would be helpful if you displayed the **part A** graphs in one area of the room and the **part B** graphs in another area.

Observe while students work. You might ask:

• *How did you determine the equation for each graph?* (I could tell the *y*-intercept easily (b) and figuring out the slope (m) wasn't hard and then I just used those values in the equation $y = \mathrm{m}x + \mathrm{b}$)

Observe and Assess

As students are working, notice:

• Are they contributing to the discussion in a meaningful way?

• Are they plotting the graphs using a table of values or are they using the slope and *y*-intercept?

• In **part C**, are they using a guess-and-check approach or are they relating the slope and *y*-intercept of each graph to the values of m and b?

Share and Reflect

• If you used the grouping strategy suggested, have the students come together and explain what they learned from each exploration before starting to work on **part C**.

• If you displayed the graphs for each part, you could ask students to look at the graphs in each section and discuss any patterns or relationships they see.

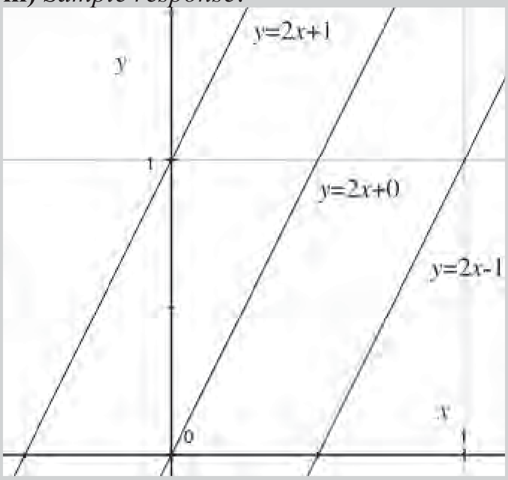
• You could use graphing software, if it is available, to prepare several graphs for each of **parts A** and **B** and display these for the class.

Answers

A. i) *Sample response*: m = 2

ii) *Sample response*: $y = 2x + 1$; $y = 2x + 0$; $y = 2x - 1$

iii) *Sample response*:



iv) All three graphs have the same slope.

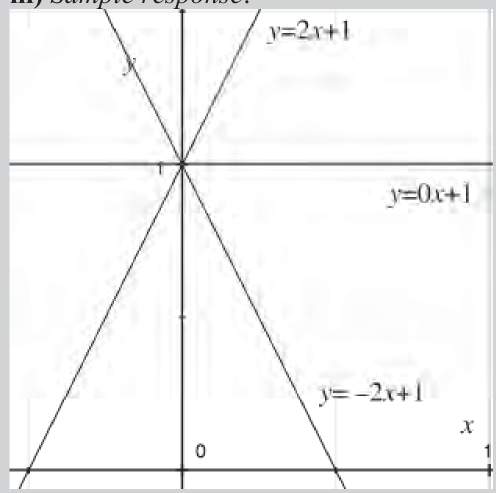
v)

b	y-intercept
Positive	Above the *x*-axis
Negative	Below the *x*-axis
0	Through the origin

B. i) *Sample response*: b = 1

ii) *Sample response*: $y = 2x + 1$; $y = 0x + 1$; $y = -2x + 1$

iii) *Sample response*:



iv) All three graphs have a *y*-intercept of 1.

v)

m	Slope
Positive	Slopes up to the right
Negative	Slopes down to the right
0	Horizontal

.

C.
Graph 1: $y = x + 6$
Graph 2: $y = -2x + 3$
Graph 3: $y = -2x - 3$
Graph 4: $y = -x + 6$

Supporting Students

Struggling students

• You could assist students who struggle by suggesting they investigate relations in which each parameter is a whole number rather than using negative integers.

• You could suggest that students calculate the first two or three equally values for a table of values and then use the growing pattern to complete the rest of the task.

• You could prepare tables of values on which the *x* values of 1, 2, 3, 4, and 5 are pre-printed and all the students have to do is fill in the corresponding values for *y*.

3.2.3 Slope and *Y*-Intercept Form

Curriculum Outcomes	Outcome relevance
9-C3 Graphs of Linear Relations: interpret and create • use the term slope to represent rise/run • relate the *y*-intercept to the to the value of the *y*-coordinate where the graph crosses the *y*-axis • determine the slope and *y*-intercept by examining a table or graph • sketch the graph of a linear relation given the slope and *y*-intercept **9-C4 Equation of a Line: use graph to determine equation** • determine the equation of a line ($y = mx + b$) given the slope (m) and *y*-intercept (b) • determine the equation of a linear relationship by calculating the slope and the *y*-intercept from the graph	By linking the equation of a line to its graph and to the values of its slope and *y*-intercept, students can develop a solid understanding of linear relations. Linear relations are one of the most important relations we use to describe many real situations.

Pacing	Materials	Prerequisites
1 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• meaning of slope

Main Points to be Raised

• When an equation for a line is in the form $y = mx + b$, it is called the slope and *y*-intercept form of the equation.

- The value of m is the slope of the line.
- The value of b is the *y*-intercept.

• If you know the slope of a line and its *y*-intercept, you can write an equation for the graph.

• You can calculate the slope and read the *y*-intercept directly from the graph.

Try This—Introducing the Lesson

A. Suggest that students work in pairs. Observe while they work. You might ask:

• *When 0 L of oil is in the drum, its mass is 25 kg. What does that tell you about the formula for the total mass of the drum and the oil in it?* (This tells you that in the graph of total mass of the drum against volume of oil in the drum, the *y*-intercept is 25 kg.)

• *What would the combined mass be if you added 10 L of oil to the drum? 20 L? 30 L?* (34.2 kg; 43.4 kg; 52.6 kg)

• *How does the volume of oil in the drum affect the combined mass?* (The combined mass increases at a constant rate in terms of the volume of oil.)

The Exposition—Presenting the Main Ideas

• Students should have guessed the relationship between m and the slope and between b and the *y*-intercept if they completed the exploration in **lesson 3.2.2**. Have them read the exposition and ask them if it confirms what they had already guessed.

• Emphasize that the value of b in $y = mx + b$ provides the *y*-coordinate of the point that is the *y*-intercept.

Revisiting the Try This

B. Students have an opportunity to relate their formula from **part A** to an equation and a graph. You could ask students to graph the relationship to confirm their responses to **B ii)**.

Using the Examples

- Present the problems displayed in **example 2** and **example 3** on the board. Ask students to try the problems before reading the examples.
- Have students then check their thinking by reading the examples in the text as well as **example 1.**
- For **example 2** and **example 3**, discuss why it is easier to start the sketch by plotting the *y*-intercept and then using the slope to determine a second point.

Practising and Applying

Teaching points and tips

Q 1: Make sure students realize that they can use any two points on the graph to determine the slope.

Q 3: Some students may need to sketch a couple of graphs to realize that more vertical means a greater slope.

Q 5: Some students may struggle with the notion of an H-intercept. Explain that sometimes rather than using *x* and *y*, some people like to use variable names that link to the variables being described by the graph.

Q 8: Some students might show the steps visually and others might use words. Both methods are appropriate.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can write the equation of a line given its graph
Question 2	to see if students can sketch the graph of a line given its slope and *y*-intercept
Question 4	to see if students can sketch the graph of a line given its equation in slope-intercept form
Question 6	to see if students recognize that the distance between heights of corresponding points on parallel lines is determined by the distance between their *y*-intercepts
Question 8	to see if students can communicate their understanding of the relationship between the equation of and the graph of a line

Answers

A. $k = 0.92l + 25$, where *k* represents the total mass in kg and *l* the volume of oil in L.

B. i) The starting value for the mass is the mass of the drum. Then I add the product of the mass for 1 L multiplied by the number of L, and that is 0.92 times *v*.

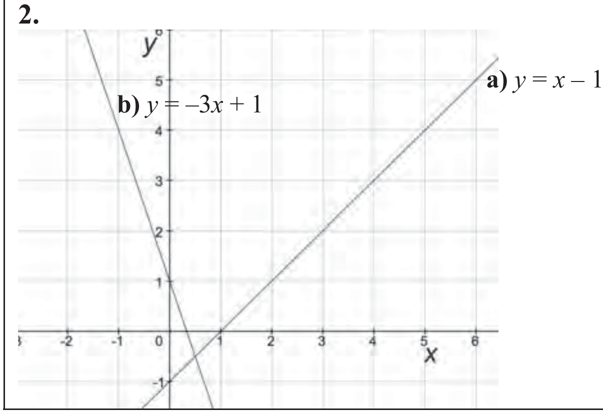
ii) The mass of 1 L of oil, 0.92, is the slope of the graph and the mass of the drum, 25, is the *y*-intercept; because the formula, $t = 0.92v + 25$, is in the $y = mx + b$ form; m is 0.92 and b = 25.

1. a) m = 2, b = 0; $y = 2x$

b) $m = \frac{-4}{3}$, $b = -2$; $y = \frac{4}{3}x - 2$

c) $m = \frac{4}{3}$, $b = 2$; $y = \frac{-4}{3}x + 2$

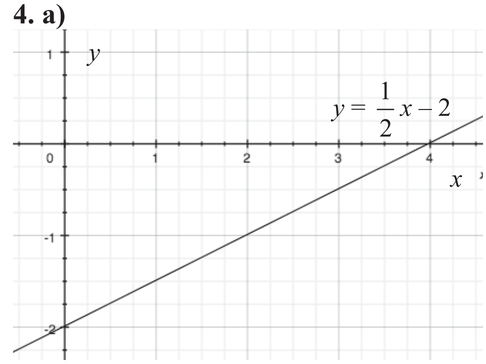
2.



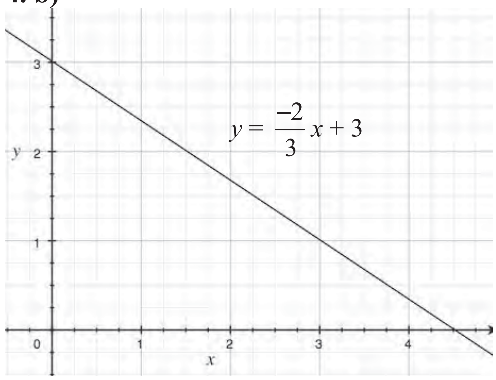
3. a) C; B

b) A; D

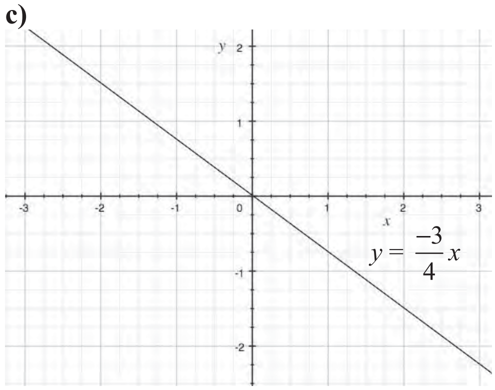
4. a)



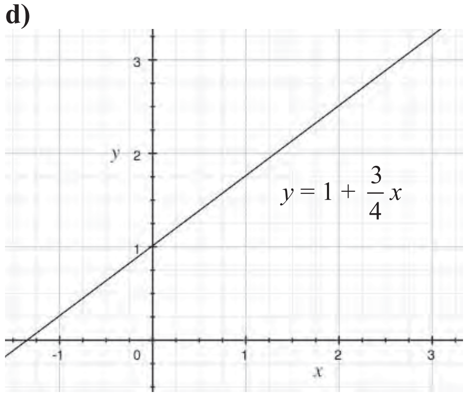
4. b)



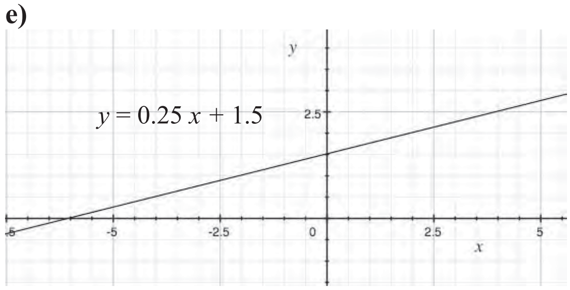
c)

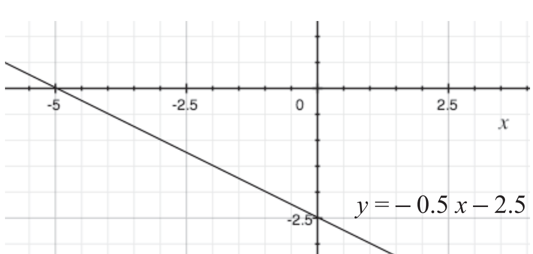


d)

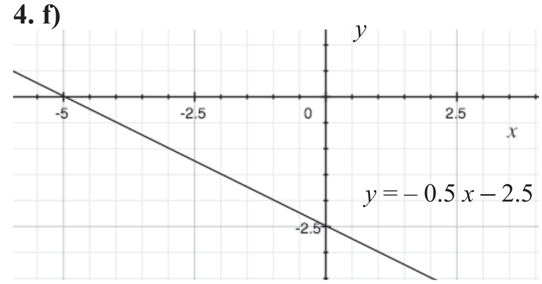


e)



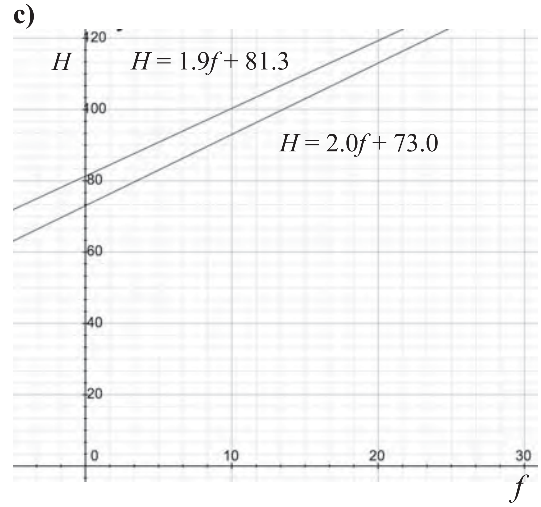


4. f)



5. a) 81.3; 73 **b)** 1.9; 2

c)

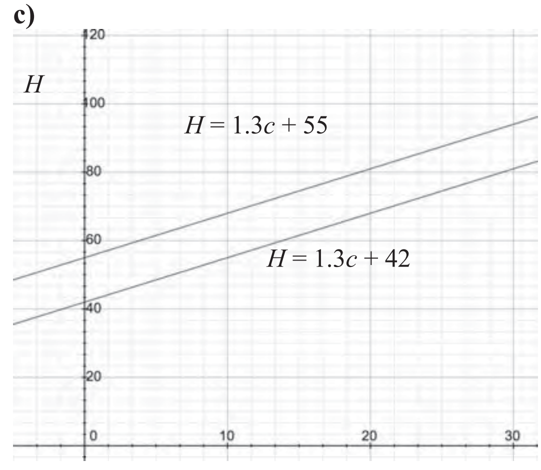


d) Nobody would have a femur of length 0 cm.

6. a) Because both equations are in the form y = mx + b, the slope (m) is 1.3 for each

b) 13; because both equations are in the form y = mx + b, 13 is the difference between the y-intercepts (b)

c)



Answers [Continued]

7. If you begin with the y-intercept, you can use the slope to locate a second point and then use a ruler to connect them. If you start with the slope, you don't know where to place the ruler.	**8.** Plot the y-intercept, move right a distance equal to the run and up (positive slope) or down (negative slope) a distance equal to the rise and plot the point, join the two points with a straight line

Supporting Students

Struggling students

Some students might need to work more with slopes that are integers rather than fractions or negative rationals. Alter some of the values in the exercises to allow for this if necessary.

Enrichment

Students can investigate the following questions:
- *If two lines have the same y-intercept, is it possible for the lines to be parallel?*
- *If two lines have the same slopes, is it possible for the lines not to be parallel?*

3.2.4 The Line of Best Fit

Curriculum Outcomes	Outcome relevance
9-C5 Lines of Best Fit: sketch and determine equations • use the eyeball method to draw the line of best fit and then use the slope and *y*-intercept to determine the equation of the line • understand that the line of best fit is drawn to show a relationship between two variables • recognizes the relationship between both the dispersion around the line of best fit and the slope of the line of best fit and a description of the correlation between the variables **9-F1 Displaying Data: draw inferences and make predictions** • draw inferences and conclusions from a number of data displays, particularly scatter plots • interpolate and extrapolate using a date set	Students learn to create a line of best fit as a mathematical model for predicting unknown information. This approach to modelling is one of the primary applications of mathematics.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• scatter plots

Main Points to be Raised

• The line of best fit is the straight line that best describes the relationship between two variables in a scatter plot of the data. It is only appropriate to use a line of best fit when the data appears to be almost linear.

• You can use the equation of the line of best fit to make predictions about unknown information.

• The linear correlation between two variables describes the strength of the linear relationship between the variables. It also suggests how reasonable it is to use the line of best fit to predict the value of one variable if you know the value of the other variable. If the correlation is weak, the prediction is not very reliable. If the correlation is strong, the prediction is reasonably reliable.

• Lines of best fit make sense both when one variable increases as the other decreases (negative correlation) and when both increase or decrease together (positive correlation).

• Sometimes a smooth curve provides a better fit to the data than a line. (Students see that this is the case in this lesson, but they are not asked to create curves of best fit.)

Try This—Introducing the Lesson

A. Assign the task to individual students or pairs. Observe while students work. You might ask:

• *Do you think there is a strong connection between how well a student did on the test and the number of hours the student studied?* (Yes. Most of my teachers say that this is true and so do my parents.)

• *Why is it hard to tell this just from looking at the data?* (From the table, it looks like most students get better marks on tests when they study longer, but the student with the best test mark studied fewer hours than some of the other students.)

• *Why can you not predict the mark just by knowing how many hours the student studied?* (The students who studied 1 hour all got different marks.)

The Exposition—Presenting the Main Ideas

• Explain that, in a real-world situation, the pattern the data points make on a scatter plot may suggest that one variable appears to grow or shrink in tandem with the other. The relationship is not as exact as when you plot points described by the equation of a line. The best you can do in these situations is to draw a line or curve that describes the relationship as closely as possible knowing that some points will be off the line or curve drawn.

• Have the students read the exposition. Make sure students are comfortable with the term *correlation.*
You might wish to mention that there are mathematical formulas to compute correlation values that measure the quality of the fit of a line to data, but that in Class IX we will limit the coverage to descriptions of strong, weak, or no correlation, and positive versus negative correlation..

Revisiting the Try This

B. This question allows students to make a formal connection between the informal analysis of data in **part A** and the line of best fit method presented in the text. In groups, the students can discuss how the scatter plot helps you see whether or not there is a strong connection between hours of study and test scores. Ask:
If you study longer than another student, would the graph predict you would get a higher test score than that student?

Using the Examples

• Ask students to read **example 1** independently and indicate whether they agree with the conclusion in **part b)** that the correlation seems strong. Ask if they would be confident using the predicted value in **part a)**.

• Work through **example 2** with the students. You might wish to point out that the student in **example 2** used the procedures of **lesson 3.2.3** to determine the equation of the line of best fit.

• Students with Internet access could find a tool that automatically creates a line of best fit when they input the data. You could display this for the students and discuss how the equation compares with the one developed by the student in the example.

Practising and Applying

Teaching points and tips

Q 1: Point out that the students would have to extend the graph to meet the vertical axis if they wished to use the method in **example 2** to determine the equation of the line of best fit.

Q 2: Using **Year number** instead of **Actual year** makes the computation of the slope and y-intercept simpler. Some students find this confusing.

Q 5: Some students will choose graph A since every point is on the graph. But help them see that graph A would not help them predict what will happen for a subsequent value of the x-variable.

Common errors

• Incorrect placement of the line of best fit may result in values of the slope and y-intercept that are substantially different from those in the answers. Remind students that some latitude is acceptable and to be expected with lines of best fit, but that the deviations should not be too great.

• Some students confuse negative correlations, like Graphs B and D in **question 4,** with weak correlations. Remind students that if they feel fairly sure they can predict the y-value when they know the x-value, the correlation is strong.

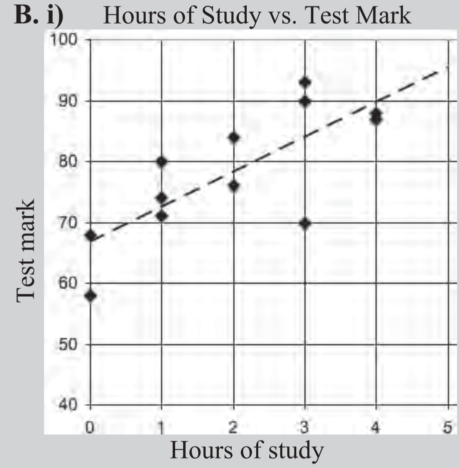
Suggested assessment questions from Practising and Applying

Question 2	to see if students can draw a line of best fit for tabulated data, determine its equation, and use the equation to predict values
Question 4	to see if students can determine the nature of the correlation between variables given a scatter plot
Question 6	to see if a student can describe the process of using a line of best fit

Answers

A. *Sample response*:
About 80; 2 h has both 76 and 84 for marks, which average to 80, and 3 h has 70, 90 and 93 as marks, which average to about 84, so 2.5 h would be 82, which is the average of 80 and 82.

B. i) Hours of Study vs. Test Mark



predicted mark: 82
ii) The prediction is about the same; using the line of best fit was easier because you only had to consider one value, and with the numbers in the chart you had to look at several values

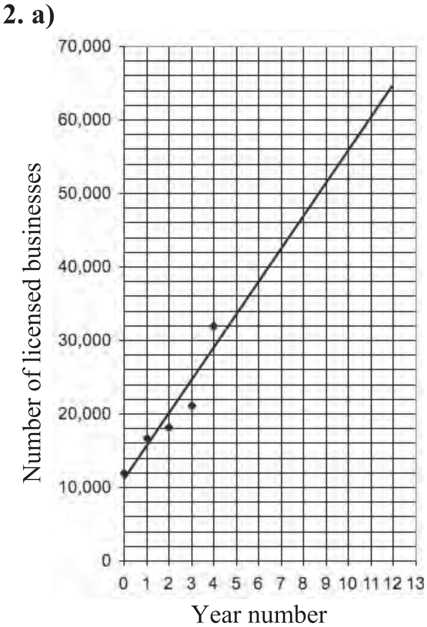
1. a) Strong positive correlation

b) About 10.4%; Since a line was used instead of a curve, the population for future years may be underestimated.

c) About 14%

d) about 7.8%; about 8.7%

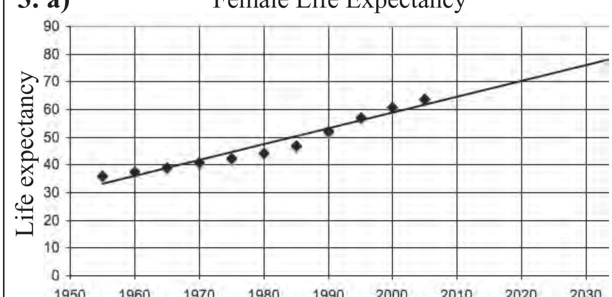
2. a)



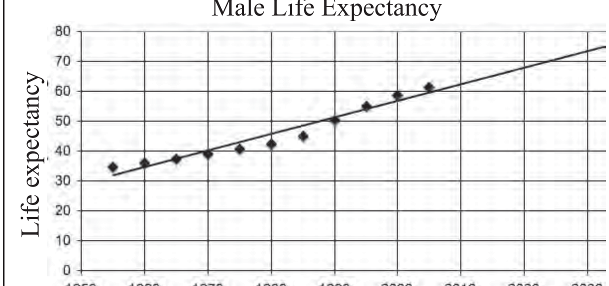
b) *Sample response*:
$y = 4700x + 11{,}000$; using the points (6, 38,000) and (3, 24,000), rise is 38,000 – 24,000 = 14,000 and run is 3; the slope is 14,000 ÷ 3 = 4666.666…, which rounds to 4700; y-intercept is a bit lower than 11,896.

c) *Sample response*: 2010 would be year 12;
$y = 4700(12) + 11{,}000 = 67{,}400$
d) *Sample response*: I would prefer more data points to be more confident because it could have been non-linear
d) *Sample response*:
I would prefer more data points, but there does seem to be a fairly strong correlation. I think that my prediction is reasonable.

3. a) Female Life Expectancy



Male Life Expectancy



b)

Year	Female	Male
2010	65	62
2030	76	74

[Continued]

Answers [Continued]

3. [Cont'd] **c)**

Year	Female	Male
1992	54	52
2002	60	58

4. a) I **b)** IV **c)** II **d)** III

5. Kinley's graph is most useful; Unlike Dorji, Kinley uses a single line of best fit to show the general correlation of all the data. The single line makes it possible to extrapolate or interpolate. Unlike Yamuna, Kinley uses a line that closely matches the distribution of the data points.

6. a) when data points fall along a line or close to a line

b) Use a ruler to draw a line that passes through as many data points as possible and make sure there are an equal number of points above as below the line.

c) Use the line to interpolate or extrapolate coordinates for points not plotted

Supporting Students

Struggling students

Suggest that students work in pairs or threes to double-check scatter plots and lines of best fit.

Enrichment

• Encourage students to investigate how slight variations in the line of bet fit affect their predictions when they interpolate and extrapolate.

• Provide data that is closer to quadratic. For example, using the data below, ask students to draw a curve of good fit.

x	1	2	3	4	5	6	7	8	9
y	+16	–8	8	29	54	88	125	170	225

3.2.5 Standard Form

Curriculum Outcomes	Outcome relevance
9-C3 Graphs of Linear Relations: interpret and create • use the term slope to represent rise/run • relate the y-intercept to the to the value of the y-coordinate where the graph crosses the y-axis • relate the x-intercept to the to the value of the x-coordinate where the graph crosses the x-axis • determine the slope and y-intercept by examining a table or graph • sketch the graph of a linear relation given in standard form	Graphing a line is easier using the x-intercept and y-intercept. Using the standard form of the equation facilitates this approach.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• substituting of values for variables • solving one-step equations of the form ax = b

Main Points to be Raised

• One form for writing the equation of a line is the slope and y-intercept form. Another form is called standard form. When an equation for a line is in the form Ax + By = C, it is said to be in standard form.

• The equation can be solved and rewritten in slope and y-intercept form: $y = \frac{-A}{B}x + \frac{C}{B}$, where $\frac{-A}{B}$ is the slope and $\frac{C}{B}$ is the y-intercept.

• The x- and y-intercepts can be used to draw the graph of an equation.

• You can easily determine both the x- and y-intercepts of the graph from standard form:

- The y-intercept is the value of the y-coordinate where the line meets or crosses the y-axis. The coordinates of the y-intercept look like (0, y). You can substitute $x = 0$ into the equation and solve it to determine the y-intercept.

- The x-intercept is the value of the x-coordinate where the line meets or crosses the x-axis. The coordinates of the x-intercept look like (x, 0). You can substitute $y = 0$ into the equation and solve it to determine the x-intercept.

Try This—Introducing the Lesson

A. and B. Encourage students to work in pairs. Observe while they work. You might ask:

• *How can you be sure you used all possible combinations of Nu 20 and Nu 50 notes?* (I know that if I use 4 Nu 50 notes, then that is Nu 200. I will try to use 0, 1, 2, 3 and 4 Nu 50 notes to make all combinations.)

• *Does it matter if you treat the number of Nu 20 notes as the dependent or the independent variable? How does your choice affect the graph?* (The number of either the Nu 20 or the Nu 50 notes can be used as the independent variable and the same combinations would be found, but the line is less steep if the number of Nu 20 notes is the independent variable.)

The Exposition—Presenting the Main Ideas

• Draw the graph of $y = 2x + 6$ on the board. Ask students to name the points where the line crosses the x- and y-axes. Introduce the value of the x-coordinate point where it crosses the x-axis as the x-intercept. Show that this can be determined by setting $y = 0$ in the equation. Repeat for the y-intercept. Then show students how the same equation can be written as $2x - y = -6$ by subtracting both y and 6 from both sides of the equation. Have students notice that in this form they can set $y = 0$, to find the x-intercept, and they can set $x = 0$, to find the y-intercept.

• Ask students to read through the exposition.

Revisiting the Try This

C. This question allows students to see that a line is completely determined if the x- and y-intercepts are known.

Using the Examples

• Work through **example 1** with the students so that you can focus on the process for first determining the intercepts using the equation and then using the intercepts to draw the graph.

• Assign pairs of students to read **example 2** and **example 3.** Have each student in the pair read one example, and ask them to explain what they read to one another.

Practising and Applying

Teaching points and tips

Q 2: Encourage students to test the equation with a few values to make sure it is correct.

Q 3: Some students will have difficulty with the decimal coefficients. Show them how to rewrite the equation as $42b + 96s = 400{,}000$. Some students will forget to multiply the 400, so watch for this.

Q 5: Some students will be more comfortable rewriting the equation without calculating the *x*-intercept.

Q 7: This question foreshadows later work with solving a system of linear equations.

Q 9 and 10: Be flexible in the format of students' presentations. They might be visual or verbal.

Common errors

• Students will often have difficulty going from standard form to slope and *y*-intercept form if the coefficient of *y* is not 1. For example, they might assume that the *y*-intercept of $2x + 3y = 6$ is 6. It might be useful to first allow them to become comfortable with equations with a coefficient of 1.

• Many students assume the *x*-intercept is the place where $x = 0$ and the *y*-intercept is the place where $y = 0$. Keep pointing out the location of the intercepts on the graph to help them see that this is not the case.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the intercepts and the slope-intercept form of the equation of a line given in standard form
Question 2	to see if students can use a linear equation in standard form to understand a real-world situation, create a graph using the intercepts, and use the graph to solve a problem
Question 4	to see if students can use standard form to describe a "combination" situation
Questions 9 and 10	to see if students can explain how to determine the intercepts of a linear relation presented in standard form and use them to sketch the graph and write the relation in slope-intercept form

Answers

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A.

Number of Nu 20 notes	Number of Nu 50 notes
0	4
5	2
10	0

B. A straight dashed line

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C. i) *x*-intercept is 10 and *y*-intercept is 20

ii) If *x* represents the number of Nu 20 notes, the *y*-intercept tells the number of Nu 20 notes needed to make Nu 200 using no Nu 50 notes; the *y*-intercept represents the number of Nu 50 notes needed to make Nu 200 using no Nu 20 notes.

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1. a) (0, 3) **b)** (2, 0)

c) $m = \frac{-3}{2}$ **d)** $y = \frac{-3}{2}x + 3$

2. a) *c* is number of correct answers and *i* is number of incorrect answers; so add 4*c* points for correct answers and subtract *i* points for incorrect answers, or $4c - i$ points; final score is 60 points, so $4c - i = 60$

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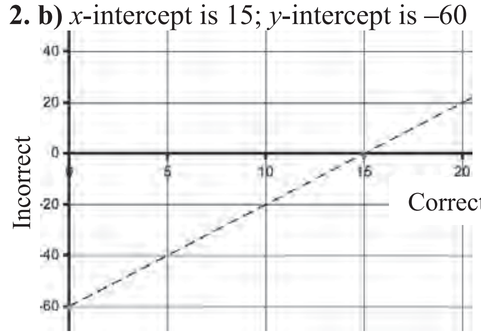
2. b) *x*-intercept is 15; *y*-intercept is –60



c) *x*-intercept means 15 correct answers with 0 errors; *y*-intercept means –60 errors and 0 correct answers; a *y*-intercept of –60 is not possible because you cannot have a negative number of answers

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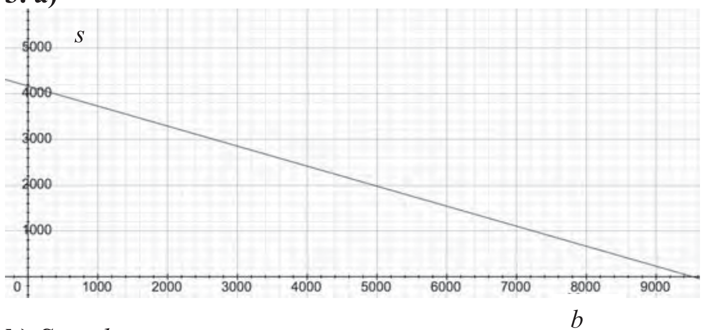
2. d) 40; if $c = i$, then $4i - i = 60$ and therefore $i = c = 20$. The total number of questions on the test is $i + c = 40$.

3. a)



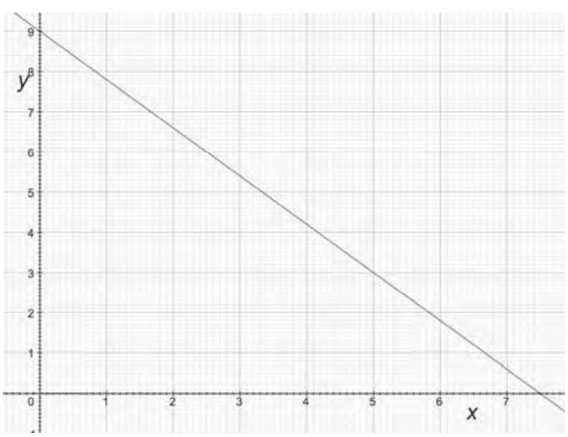
b) *Sample response*:

At 4.2% (b)	At 9.6% (s)
0	4200
9450	0
5000	2000
2600	3000

c) Cannot invest a negative amount

4. a) $600x + 500y = 4500$, x is the number of hours at Nu 600 and y is the number at Nu 500

b)



4. c) *Sample response*:

at Nu 600	at Nu 500
0	9
7.5	0
5	3

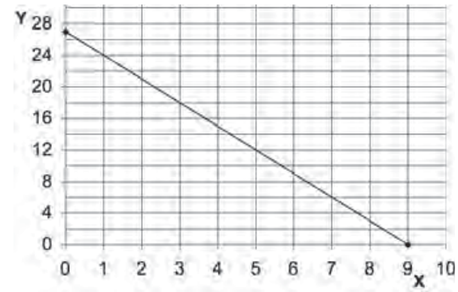
d) Cannot work a negative number of hours

5. a) $y = -\frac{2}{3}x + 4$ **b)** $y = \frac{4}{5}x - 4$

5. c) $y = -\frac{5}{2}x + \frac{5}{2}$ **d)** $y = \frac{10}{3}x - 5$

6. a) $3x + y = 27$

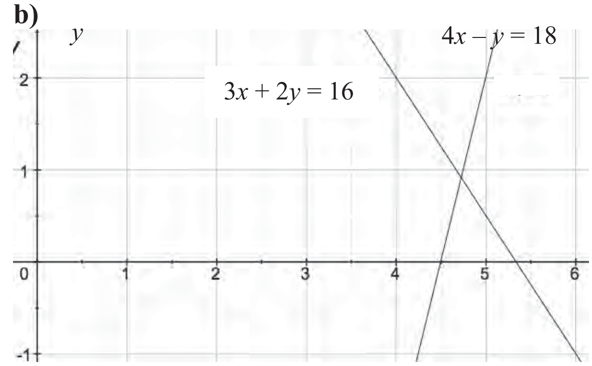
b)



c) *Sample response*: (9, 0), (5, 12), and (0, 27).

7. a) $3x + 2y = 16$ and $4x - y = 18$

b)



c) (4, 2)

8. a) $y = \frac{3}{2}x + 2$ and $\frac{3}{2}x - y = -2$

b) *Sample response*: (0, 2), (2, 5), (–2, –1), (4, 8)

c) and d) Same graph as $\frac{3}{2}x - y = -2$;

$\frac{3}{2}x - y = -2$ and $3x - 2y = -4$ are equivalent equations

9. Substitute 0 for y and solve equation to determine x-intercept, plot that value; substitute 0 for x and solve equation to determine y-intercept, plot that value; join points with a straight line

10. Determine x- and y-intercepts, calculate slope using the coordinates of intercepts, use slope and y-intercept to write equation in $y = mx + b$ form. Or, rearrange the equation algebraically.

Supporting Students

Enrichment

Interested students could be asked to make up a problem (and solve it) based on a guessing game similar to those described in **question 6** and **question 7.**

3.3.1 Solving Linear Equations Algebraically

Curriculum Outcomes	Outcome relevance
9-C6 Single Variable Equations: solve algebraically and graphically • solve equations algebraically • solve problems involving equations with coefficients that may be integers or rational numbers	Skills for solving equations in more complex situations are built on a solid understanding of how to solve linear equations.

Pacing	Materials	Prerequisites
2 h	• Algebra tiles (optional)	• modelling expressions in one variable using algebra tiles (optional) • rational number operations

Main Points to be Raised

• An equation is a mathematical statement in which the value on the left side of the equal sign is the same as the value on the right side of the equal sign.

• The solution to an equation is the value of one (or more) variable(s) that makes the equation true.

• One way to think about an equation is to use the metaphor of a pan balance. The two sides of the equation must be in balance; if you substitute a value on one side of the equation, you end up with the same number as if you substitute that same value on the other side of the equation.

• A useful strategy for solving equations is to perform the same operation on both sides of the equation. It could be adding, subtracting, multiplying, dividing, or some combination of these.
- The operations performed are usually operations that undo the operations in the equation one step at a time. Each time such an operation is performed, the new equation is equivalent to the original one, with exactly the same solutions.

• You can represent the steps required to solve an equation concretely using algebra tiles or symbolically using algebraic expressions.

Try This—Introducing the Lesson

A. and B. If possible, provide students with algebra tiles. If it is not possible, they can visualize what the tiles would show. Students should work in pairs. Observe as they work. You might ask:

• *How might you use guess-and-check to solve the equation? Would this work if the solution were not an integer?* (If I try $x = 1, 2, 3\ldots$, I find that $x = 2$ satisfies the equation. If the solution were not an integer, there would be too many choices.)

• *How do you know your solution is correct?* (I substituted my solution, $x = 2$, into both sides of the equation and found that they were equal: $2(2) - 2 = -2 + 4 = 2$.)

The Exposition—Presenting the Main Ideas

• Write the equation $2n - 3 = 1$ on the board. Ask students to discuss in small groups how they would solve the equation. Ask a few students for their strategies. Make sure that a number of different strategies arise. They could include: guess-and-check (try a number, e.g., 4, test it ($8 - 3 \neq 1$) and then adjust the guess (in this case to make it lower) until the equation is solved); performing inverse operations on both sides of the equation; reasoning it through (e.g. if something – 3 = 1, the something must be 4); or using algebra tiles.

• Work through the main ideas in the Exposition with the students. Where possible, relate what is in the text to strategies students had already proposed. Make sure students understand these terms: *inverse operations*, *isolate the variable term, isolate the variable,* and *equivalent equation*.

• Students might model the sample solution in the exposition using algebra tiles and record the symbolic step beside each drawing of an action with the algebra tiles.

Revisiting the Try This

C. You may wish to demonstrate solutions to **C i) and ii)**. Students can draw tiles for **part ii)** if they do not have them to work with.

Using the Examples

- Post the problems from each example on the board. Ask students, alone or in pairs, to try the problems and then compare their thinking with the **Thinking** in the text. The process demonstrated in **example 1** is similar to prior work in **lesson 3.2.5**.
- Help students see how **solutions 1 and 2** differ — in solution 1, the fractions are eliminated right away but in solution 2, the student works with the fractions until the end.

Practising and Applying

Teaching points and tips

Q 1: Remind students that the black tiles represent negative values.

Q 3: Provide algebra tiles, if available.

Q 4: If students have difficulty with **part a),** help them by having them write expressions for each part of the situation and then put them together. For example, they could describe depositing the same amount for 5 weeks as $5a$.

Q 6: Students should state the pattern before trying to write equations.

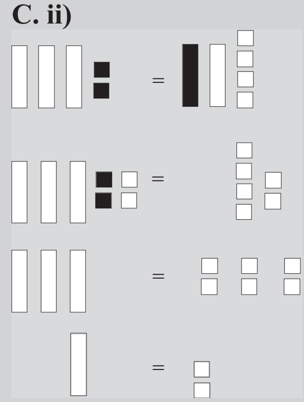
Common errors

Students may have difficulty with fraction operations. You might assign struggling students only equations with integer coefficients until they feel more comfortable with the operations.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can interpret an algebra tile model of an equation
Question 2	to see if students can model and solve an equation represented using a balance model
Question 3	to see if students can solve an equation of the form ax + b = c
Question 6	to see if students can solve a problem modelled by a linear equation
Question 7	to see if students can solve an equation of the form ax + b = cx + d
Question 9	to see if students can solve an equation with integer or fraction coefficients

Answers

A. $x = 2$ **B.** Each white square tile represents 1. Each black square tile represents –1. Each white rectangle tile represents x. Each black rectangle tile represents $-x$. **C. i)** You could add x to each side: $3x - 2 = 4$. Then add 2 to each side: $3x = 6$. Divide each side by 3: $x = 2$.	**C. ii)** 
1. a) $3x + 2 = 5$ **b)** $x + 1 = 2x$ **c)** $-2x = -x - 3$ **d)** $-2x = x - 3$ **2.** Subtract 4 from each side, then divide each side by 6. **3. a)** $x = 4$ **b)** $a = 5$ **c)** $y = -3$ **d)** $x = 2$ **e)** $x = 6.9$ **f)** $x = 40$	**4. a)** $5w - 60 = 180$ **b)** $w = 48$ **5. a)** $250 + 50p = 1030$ **b)** $p = 15.6$ **c)** 15 **6. a)** $c = 3f + 2$ **b)** $50 = 3f + 2$; $f = 16$ **c)** $100 = 3f + 2$; $f = \frac{98}{3} = 32\frac{2}{3}$, but a fractional figure number is not possible.

Answers [Continued]

7. a) $x = 3$ **b)** $x = –2$

c) $x = –1$ **d)** $x = –9$

e) $x = – 2$ **f)** $x = –8$

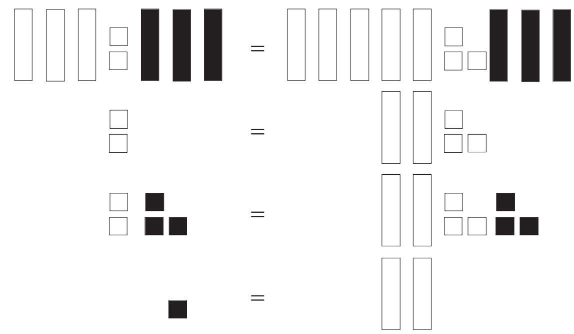
8. a) $3c = 100 – 40$ **b)** $c = 20$

9. a) $\frac{4}{5}$ **b)** –10

c) 6 **d)** 15

10. –40°F = –40°C

11. a)



b)

$$\begin{aligned} 3x + 2 &= 5x + 3 \\ -3x + 3x + 2 &= -3x + 5x + 3 \\ 2 &= 2x + 3 \\ 2 - 3 &= 2x + 3 - 3 \\ -1 &= 2x \\ x &= -\frac{1}{2} \end{aligned}$$

Supporting Students

Struggling students

- Allow students to use algebra tiles to represent equations with integer coefficients in **questions 1 to 3.**
- Students for whom arithmetic may be an obstacle should be allowed to use calculators when solving equations.

GAME: Equation Concentration

This game provides a pleasant way to review equation solving. Students can make simpler or more difficult equations depending on what is most comfortable for them. A sample set of six pairs of cards is shown here:

$2x + 5 = 11$	$4 – x = 9$	$3x + 10 = 70$	$8 + 3x = 4x$	$1.5x = 0$	$6x – 10 = 8x – 2$
3	–5	20	8	0	–4

3.3.2 Solving Linear Inequalities

Curriculum Outcomes	Outcome relevance
9-C8 Inequalities: solve and verify • solve single variable linear inequalities	Many mathematical relations that students will encounter are inequalities rather than equalities. Linear inequalities are the first step toward understanding these.

Pacing	Materials	Prerequisites
1 h	None	• solving equations with inverse operations

Main Points to be Raised

• An inequality is a mathematical statement in which the value on the left side is compared with the value on the right side using an inequality symbol. The symbol can be $<$, $>$, $\leq$, or $\geq$.

• The solution to an inequality is the set of values for the variable that makes the inequality true.

• You solve an inequality using the same steps you would use to solve the related equation.

• It is usually easier to interpret the solution if the coefficient of the variable term is positive. If the student multiplies by a negative, he or she needs to recognize that this reverses the sign of the inequality and therefore the inequality sign must be reversed in the solution. This will have to be demonstrated.

Try This—Introducing the Lesson

A. Assign the question to individuals or to pairs of students. Observe while students work. You might ask:

• *How might you change the wording of* **part A i)** *so that you could answer it by solving an equation?* (Which figure number has 50 counters?)

• *If I asked which figure numbers requires more than 50 counters, why would there be a lot of answers?* (Because once one figure needs 50 counters, every figure beyond that needs at least that many and more.)

The Exposition—Presenting the Main Ideas

• Post the inequality $2n + 5 < 10$ on the board. Ask students to give you some values of n that make this true (e.g., 0, 1, 2) and some values that make it false (e.g., 10, 11, 12, …). Help them notice that the numbers that do not make it true seem greater than the ones that do make it true. Ask where the “cross-over” point is, i.e., the place where it switches from being true to being untrue (2.5). Ask how they found it and, if they do not suggest it, point out that the equation $2n + 5 = 10$ could have been solved. Ask whether 2.5 is a solution to the inequality (it is not).

• Use another equation involving a $\geq$ symbol, e.g., $31 - 2n \geq 11$. Again, ask for some values that make the inequality true and some that make it false, and ask for the cross-over point. Once the cross-over point (10) is discovered, ask whether or not it is a solution to the inequality. This time it is. Contrast this with the previous example.

• Ask students to read through the exposition.

• You might show the solution of the equation $4 = 3n - 2$ alongside the solution of $4 < 3n - 2$ to demonstrate the similarity between solving an inequality and solving the related equation.

Revisiting the Try This

B. This question allows students to make the formal connection that counting numbers less than 17 as described in **part A** is equivalent to counting numbers that are less than or equal to 16. You might show all of the counting numbers that solve this problem on a number line to exhibit this fact.

Using the Examples

- Allow students time to read the examples independently.
- You might ask the class why $\leq$ was used in **example 1** and not $<$.
- For **example 2**, you might demonstrate the effect of isolating the variable on the left to yield $-a > -4$ and discuss why concluding that $a > 4$ would be incorrect.

Practising and Applying

Teaching points and tips

Q 1a: Encourage students to start by visualizing how they would solve the related equation.

Q 1c: When the variable remains on the right side of the inequality, remind students that they can read it from right to left to make a more direct statement about the allowed values. For example, $1 < x$ may be read as $x > 1$.

Q 3: Make sure students think about whether to use the $<$ symbol or the $\leq$ symbol to model the problem.

Q 4f: Remind students to select steps to isolate the variable so that it has a positive coefficient.

Q 6: Many students will need to write out the numbers to see the pattern. Starting with 1 cut, terms would be 3, 5, 7, 9,…. If students start with 0 cuts, the inequality needs to take that into account.

Common errors

Students may isolate the variable on a side that leaves it with a negative coefficient and then multiply by –1 without adjusting the inequality. Once students solve the inequality, make sure they check their solution by testing a value in the range indicated by their solution.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can solve a two-step inequality
Questions 2 and 7	to see if students can use a two-step inequality to model and solve a problem situation
Question 4	to see if students can solve inequalities in which the variable appears on both sides

Answers

A. i) Figure 16 **ii)** figures 1 through 16 **B. i)** $n \leq 16$ means any (whole) number less than and including 16, or 1 to 16.	**ii)** $n < 17$ means any (whole) number less than 17, which is the same as saying any figure from 1 to 16.
1. a) $x < 3$ **b)** $x \leq 3$ **c)** $2 > x$ **d)** $-2 \geq x$ **e)** $x < 1$ **f)** $2 \geq x$ **2. a)** $400 + 50t < 1000$ **b)** $t < 12$, so less than 12 min **3. a)** $100{,}000 - 150t < 40{,}000$ **b)** $t > 400$, so more than 400 min **4. a)** $a < 2$ **b)** $b < 7$ **c)** $x \leq 2$ **d)** $x < -2$ **e)** $x > 1$ **f)** $x < 2$	**5. a)** Figures 14 and greater **b)** $4n - 3 > 50$ **c)** $n > 13\frac{1}{4}$, which means $n > 13$ or $n \geq 14$ because fractional values of n are not possible **6. a)** $p = 2c + 1$ where p represents the number of pieces and c the number of cuts. **b)** $20 > 2c + 1$ **c)** $c < 9.5$, which means $c \leq 9$ or $c < 10$. **7.** 69 L **8.** You can use inverse operations to solve both equations and inequalities. Linear equations have a single solution, but inequalities usually have multiple solutions.

Supporting Students

Struggling students

Some students will have difficulty if negative coefficients are involved or if the solutions of the related equations are fractions. Allow them to use simpler equations until they are familiar with the process for solving inequalities. They might also feel more comfortable if they solve the related equation in pencil and then replace the equal sign (=) with the inequality symbol.

3.3.3 Solving Linear Equations Graphically

Curriculum Outcomes	Outcome relevance
9-C6 Single Variable Equations: solve algebraically and graphically • solve equations graphically • solve problems involving equations with coefficients that may be integers or rational numbers	Many students better understand what the solution of an equation means if they look at it graphically.

Pacing	Materials	Prerequisites
1 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• graphing a linear relation

Main Points to be Raised

• When you use a graph to solve an equation, the answer is often only an estimate since it is very difficult to be exact with a graph. However, the estimate provides a good starting point for the solution.

• The solution to a linear equation is the x- or y-coordinate of a point on the graph of the corresponding linear relation.

• You can solve an equation for one variable when you know the value of the other variable.

• You locate the point that has the coordinate you know on the graph of the corresponding linear relation.

• You can use the graph to provide the value for the missing coordinate of the ordered pair.

Try This—Introducing the Lesson

A. Assign the problem to individuals or to pairs of students. Observe while students work. You might ask:

• *The vertical and horizontal lines drawn on the graph intersect at a point on the line. What are the coordinates of this point and what do they mean?* (The coordinates of the point are about (2.4, 2000). This means that Nu 2000 can buy about 2.4 h of computer service labour.)

The Exposition—Presenting the Main Ideas

• You may wish to display the example graph on the board without the dashed horizontal and solid vertical lines. Draw these on the graph as you discuss the main ideas in the Exposition with your students.

Revisiting the Try This

B. This question allows an opportunity to understand how the graphical solution and the algebraic solution provide the same result. The students might decide which method, graphical or algebraic, they prefer to solve an equation and why.

Using the Examples

• Assign pairs to work through the examples. One student focuses on **example 1** and the other on **example 2** and then they share what they have learned with each other.

• You might ask why the graph in **example 1** actually provides an exact solution to the equation while the graph in **example 2** provides only an estimate.

Practising and Applying

Teaching points and tips

Q 3: Ask a follow-up question about the number of days to build 27 bicycles and watch to make sure students draw a horizontal line at $y = 27$ and not a vertical line at $x = 27$.

Q 5: You may need to support students in creating the equation. Students should realize that the constant term could be 3 since there are always the 3 sticks from the first triangle. Since 2 sticks are added each time, a term involving $2x$ would also make sense.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can determine the equation represented by a graphical display
Question 3	to see if students can use a graph to solve a linear equation
Question 5e	to see if students can explain why a graphical solution to a linear equation might be only an estimate and not exact
Question 5	to see if students can model a problem with an equation and solve it graphically

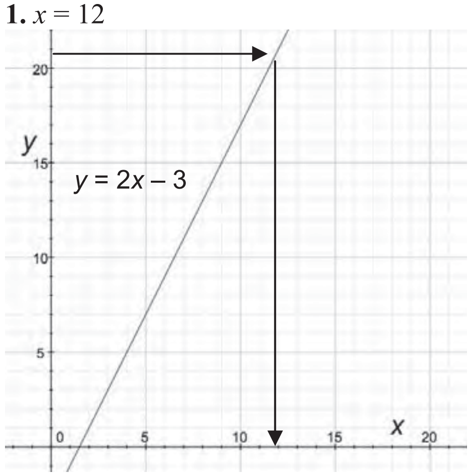
Answers

A. about 2.4 h

B. i) approximate equation is $1000 + 420t = 2000$

ii) Locate the point that has 2000 as the y-coordinate and use the graph to estimate the x-coordinate.

1. $x = 12$



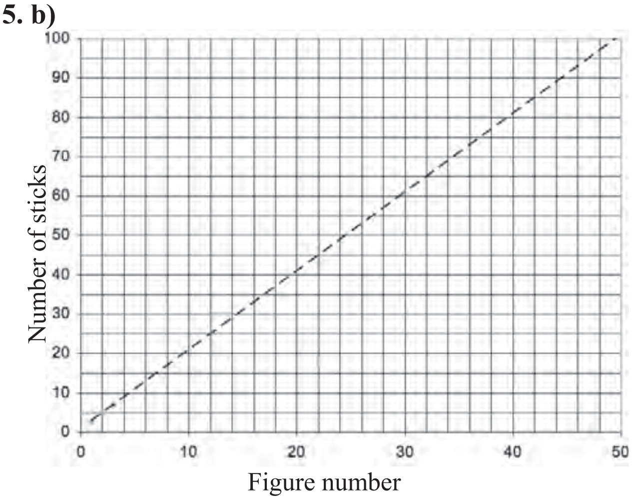
b) Unless coordinates are on labelled grid lines, you often gave to estimate between increments on the horizontal and vertical axes scales.

3. Day 8

4. $2x – 3 = 14 \rightarrow 14 = 2x – 3$, locate 14 on the y-axis and then find the corresponding point on the x-axis (6).

5. **a)** $s = 2n + 1$, where s is the number of sticks and n is the figure number.

5. b)



c) Locate 97 on the s-axis and then locate the corresponding point on the n-axis (48)

d) $s = 48$

e) When you use the graph you often estimate the position of points on the axes between scale increments. The larger the scale, the more likely you have to estimate.

6. You find the x-coordinate on the graph that goes with the given y-coordinate to get the solution.
Or, you find the y-coordinate on the graph that goes with the given x-coordinate.

Supporting Students

Enrichment

Students can create a pattern problem like **question 5**.

3.3.4 Solving a System of Linear Equations

Curriculum Outcomes	Outcome relevance
9-C7 Two Linear Equations: find solutions to a problem by graphing • solve problems by graphing pairs of linear equations	Many real-world problems involve more than one condition. Students will now be able to begin to address some of the simpler problems of this type.

Pacing	Materials	Prerequisites
2 h	• Grid paper (see page 232 in this *Guide*) • Rulers	• graphing linear relations

Main Points to be Raised

• Some problems are modelled by two linear relations that must be satisfied at the same time. The pair of equations that represent these relations is called a system of linear equations.

• The coordinates of the point of intersection of the graphs of two linear relations provide the only values of x and y that satisfy both relations simultaneously.

Try This—Introducing the Lesson

A. Students can solve the problem alone or in pairs. Observe while students work. You might ask:

• *Do you know what a commission is?* (It is a payment you get that is a percentage of your sales.)

• *How is it possible that Plan A sometimes pays more than Plan B and sometimes pays less than Plan B?* (If Meto does not sell very much, then the Nu 10,000 base salary might be much larger than the commission portion of the money he makes and Plan B would pay more. If Meto is very good at selling, then 5% of the sales could be more than Nu 10,000 greater than 3% of those sales.)

The Exposition—Presenting the Main Ideas

• You might ask students to work in pairs. One partner makes a table of values for $y = -x + 5$ and the other does the same for $y = x + 2$. Suggest they pick some non-integer values for x.

• Ask students to see if there are any points shared by their two tables of values.

• Ask students to read the exposition to see how their common solution relates to the presentation in the text.

• Point out that the sample graphs demonstrate that there is only one point in common.

Revisiting the Try This

B. This question shows that plotting the linear graphs of the two plans in the **Try This** can make it easier for students to see the sale values for which each plan pays the same amount. Meto can then determine which plan is better.

Using the Examples

You may ask students to read the example independently. They could discuss, in pairs, how they could determine whether or not the solution presented in the example is exact.

Practising and Applying

Teaching points and tips

Q 1: Make sure students understand that the only intersection involved is the intersection of the lines $y = 3$ and $y = 2x - 1$, not the intersection of the lines with the axes.

Q 2: Some students might be interested to consider how the x-coordinate of their solution to this equation is also the solution to the equation $2x - 7 = 5x - 4$.

Q 3: Some students will need help creating the equations. Suggest that they use words first and then replace the words with symbols. For example:

Hours in job 1 + hours in job 2 = 8

Pay for job 1 = 600 × number of hours

Pay for job 2 = 500 × number of hours

4500 = Pay for job 1 + Pay for job 2

Q 4: Emphasize that the quality of the graphs the students draw will determine the accuracy of the coordinates of the point of intersection.

Common errors

Students may draw graphs inaccurately or plot graphs with a scale too small to allow for accurate determination of the co-ordinates of the point of intersection. Encourage them to draw a rough sketch first and then decide what scale they really need.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the point of intersection of the graphs of two linear relations
Question 2	to see if students can create graphs to determine the solution to a system of linear equations
Question 3	to see if students can model and solve a problem involving a system of linear equations

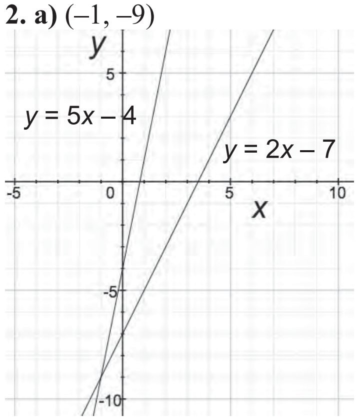
Answers

A. He needs to know what the predicted sales will be if he wants to choose the option that pays the most.

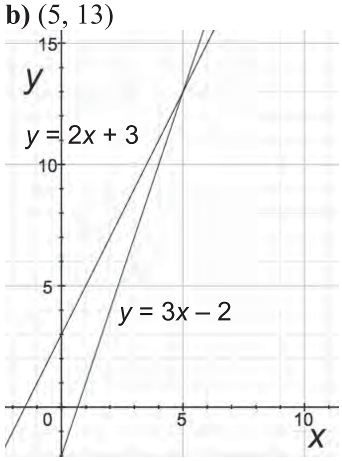
B. i) A: $y = 0.05x$ and B: $y = 0.03x + 10,000$
ii) It is the point at which both plans pay the same amount for the same sales.

1. a) (2, 3) **b)** (2, –1)

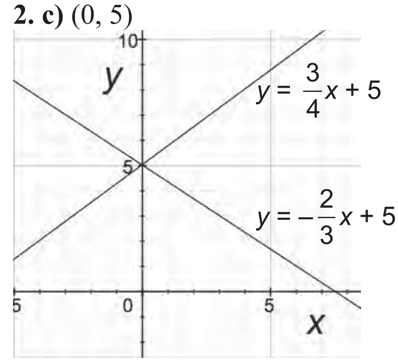
2. a) (–1, –9)



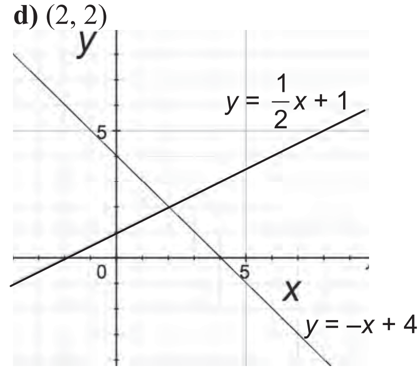
b) (5, 13)



2. c) (0, 5)



d) (2, 2)



3. Using a for the hours worked at Nu 600 and b for the hours worked at Nu 500:
a) $600a + 500b = 4500$
b) $a + b = 8$
c) (5, 3)

4. Using m to represent the mass of the vehicle and fuel and f to represent the volume of fuel: **a)** $m = 1295 + 0.737f$ **b)** $m = 1290 + 0.820f$ **c)** Approximately (60, 1340)	**5.** Alike: the solution involved determining the coordinate of a point on the graph of a linear relation Different: to solve a linear equation you determine one coordinate of a point on the graph when you know the other coordinate, to solve a linear system, you determine both coordinates

Supporting Students

Struggling students

Make sure students who are struggling are not required to deal with equations involving fractional coefficients.

Enrichment

Interested students could solve the equations in **question 2** by equating the right sides of the equations and solving for x, and then substituting to determine y.

UNIT 3 Revision

Pacing	Materials
2 h	• Grid paper • Rulers

Question(s)	Related Lesson(s)
1	Lesson 3.1.1
2	Lesson 3.1.3
3	Lesson 3.1.2
4–6	Lessons 3.1.1, 3.1.2, and 3.1.4
7	Lesson 3.2.1
8	Lesson 3.2.3
9, 10	Lesson 3.2.4
11–13	Lesson 3.2.5
14	Lessons 3.3.1 and 3.3.3
15	Lesson 3.3.2
16	Lesson 3.3.4

Revision Tips

Q 3: Some students might need help in finding the pattern. If necessary, assist with this so that they can continue with the rest of the problems.

Q 5: You might have to remind students what the term *extrapolate* means.

Q 7: Remind students that they do not have to use points with integer coordinates to determine the slope.

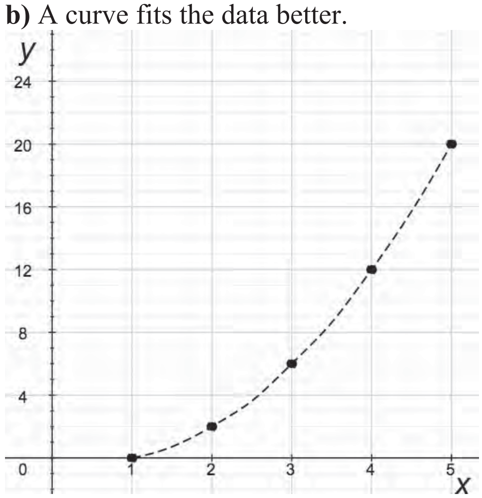
Q 9: Have students think about the appropriate scale before beginning the scatter plot.

Answers

1. a) Quadratic; second differences are equal and not zero
b) Exponential; the ratios of the first differences to the term numbers are equal
c) Linear; the first differences are equal

2. a) A **b)** C **d)** B

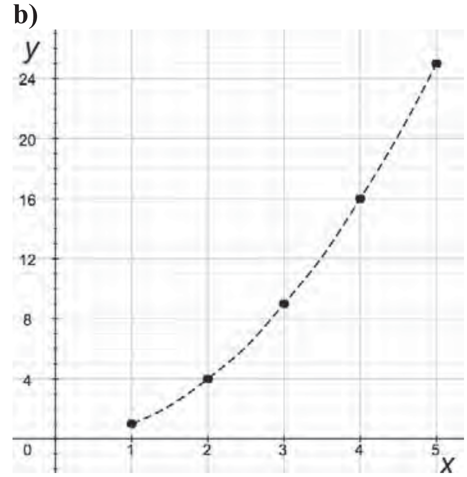
3. a) 0, 2, 6, 12, 20
b) A curve fits the data better.



c) Dashed; there are no fractional figure numbers

4. a) 1, 4, 9, 16, 25

b)

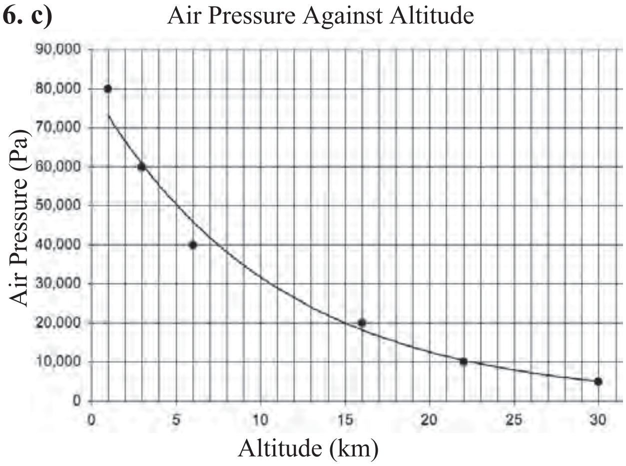


c) Cannot see a full U shape
d) Show that the second differences are equal but not zero

5. 56

6. a) *x*-coordinates are not equally spaced
b) Continuous

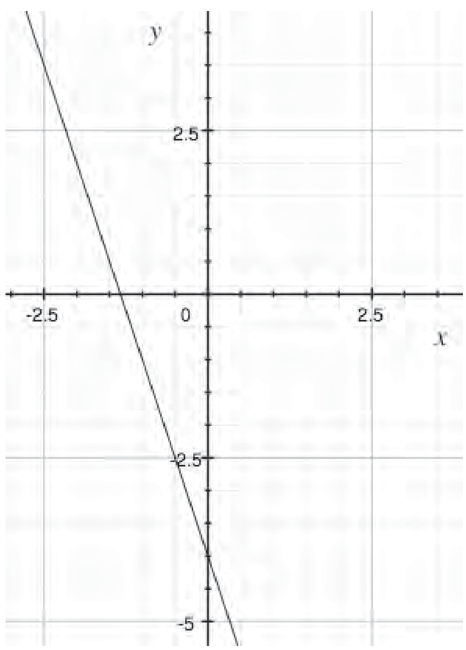
6. c)



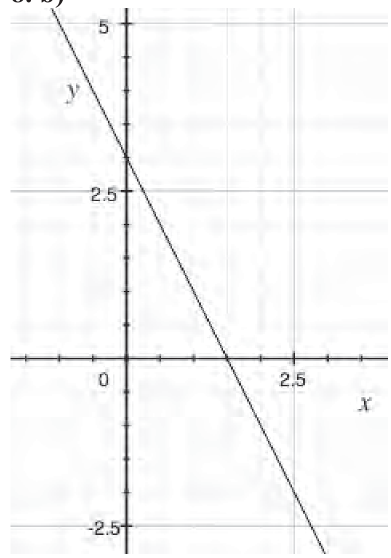
d) Looks exponential; it seems like the graph is becoming parallel to the x-axis

7. A: $y = -2x - 5$ **B:** $y = -2x + 5$ **C:** $y = \frac{2}{3}x + 5$

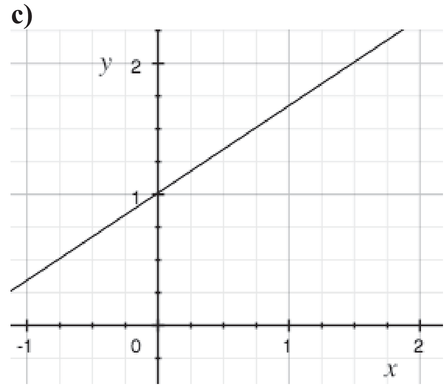
8. a)



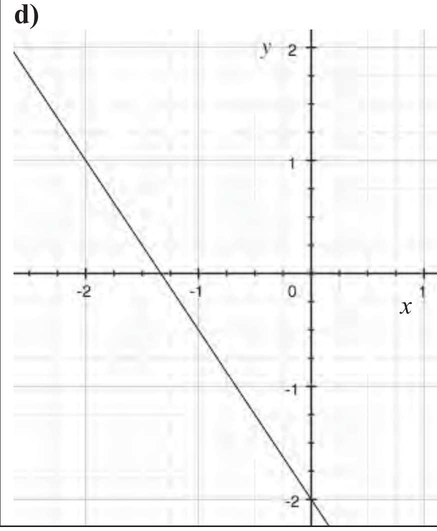
8. b)



c)

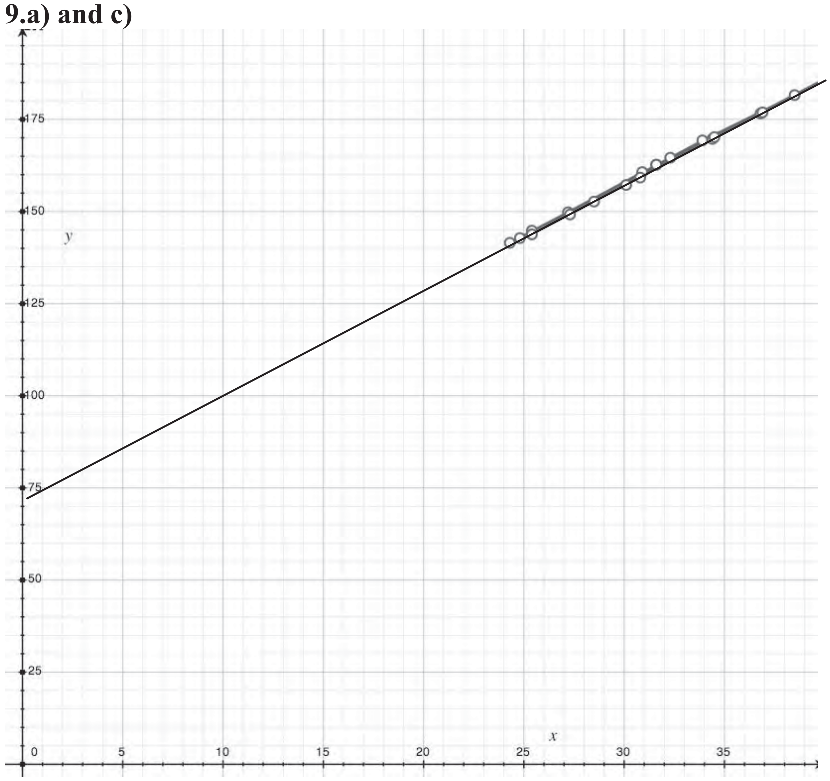


d)



Answers [Continued]

9.a) and c)



b) Strong positive correlation

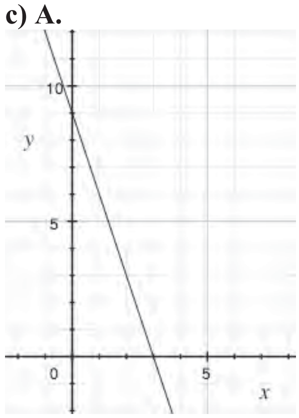
d) About 152 cm; about 192 cm

10. a) $y = 2.85x + 71.9$ **b)** 151.7 cm; 191.6 cm

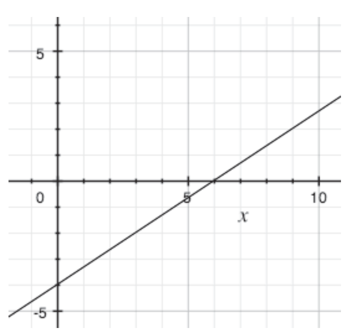
11. a) A: 3 B: 6 C: 10 D: –3

b) A: 9 B: 4 C: 2 D: 4

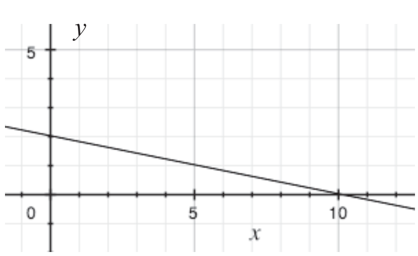
c) A.



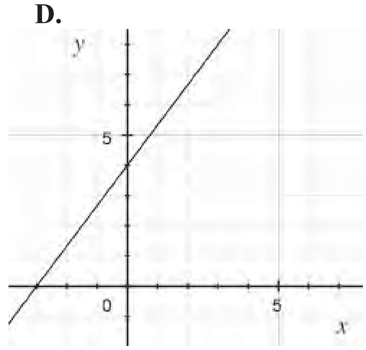
B.



C.



D.



12. a) A: –3 B: $\frac{2}{3}$

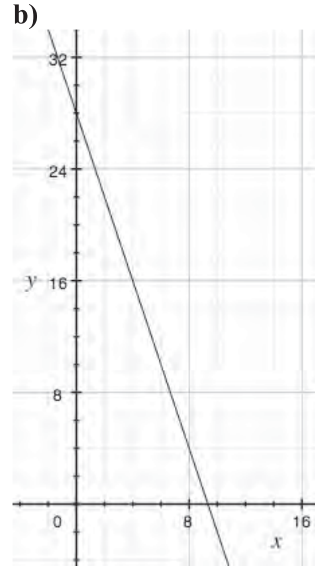
C: $-\frac{1}{5}$ D: $\frac{4}{3}$

b) A: $y = -3x + 9$ B: $y = \frac{2}{3}x - 4$

C: $y = -\frac{1}{5}x + 2$ D: $y = \frac{4}{3}x + 4$

13. a) $1500m + 500t = 14{,}000$

b)



c) (0, 28), (1, 25), (2, 22), (3, 19)

14. a) $a = 7$ **b)** $x = 6$ **c)** $y = \frac{7}{3}$

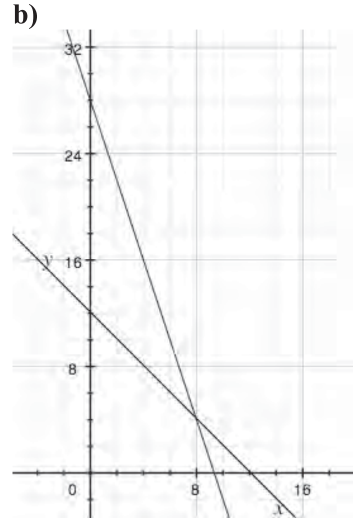
d) $x = 2$ **e)** $b = 20$ **f)** $x = -36$

15. a) $y < 2$ **b)** $6 \geq y$

c) $5 \leq a$ **d)** $n > -3$

16. a) $m + t = 12$; $1500m + 500t = 14{,}000$

b)



8 magazine and 4 technical art

UNIT 3 Linear Relations and Equations Test

1. Tell whether each relationship between x and y is linear, quadratic, or exponential.
How do you know?

a)

x	–3	–1	1	3	5	7
y	16	4	0	4	16	36

b)

x	–3	–1	1	3	5	7
y	0.0123	0.11	1	9	81	729

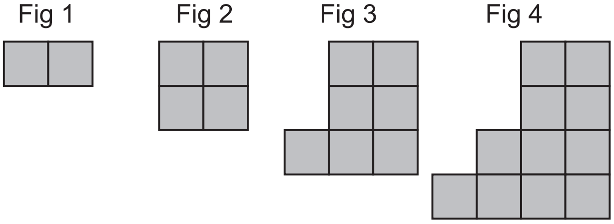
2. The second differences for a particular relationship are zero.

a) What do you know about the relationship?

b) Give an example of such a relationship and show that the second differences are zero.

3. a) Create a table of values based on the number of squares in each of the figures below.

Fig 1 Fig 2 Fig 3 Fig 4



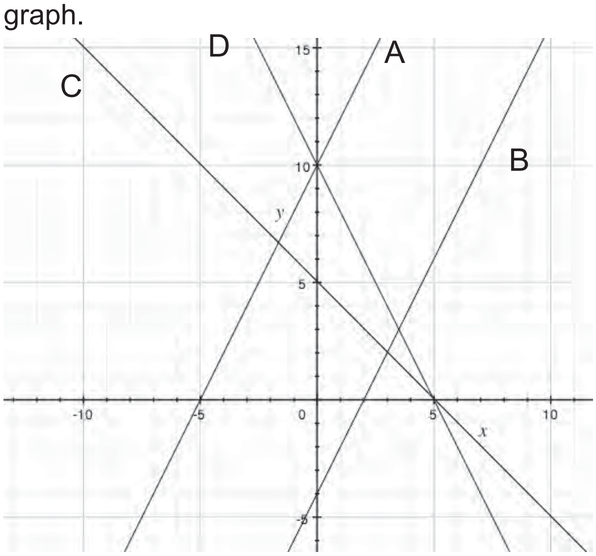
b) Draw a scatter plot of the data in **part a**.

c) Should you connect the points? Why?

d) Is the function linear or quadratic?

e) Estimate the number of squares in Figure 10.

4. Write the slope and intercept form for each graph.



5. a) Sketch the graph for each.

i) $y = 3x + 4$

ii) $2x - 4y = 5$

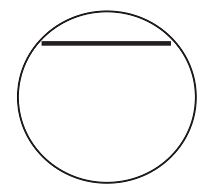
b) List the x-intercept and y-intercept for each equation in **part a**.

c) Change the equations in **part a** from slope and y-intercept form to standard form or from standard form to slope and y-intercept form.

6. Dechen measured the lengths of eight horizontal wires used to hang circular plates of different sizes

d is the circle's diameter

l is the length of the wire



d	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
l	1.6	3.1	4.9	6.0	7.9	9.2	11.2	12.1

a) Create a scatter plot to show the data.

b) Draw a line of best fit.

c) Write the equation of the line in slope and y-intercept form.

7. Solve each equation.

a) $2x + 5 = 18$

b) $-3a - 4 = 10.4$

c) $\frac{2}{3}m + 6 = \frac{4}{5}$

8. Solve each inequality.

a) $8 + 3n \le 10.9$

b) $2n - 3.8 > 2.7$

c) $15 < 3 - 3n$

9. A group of students raised money for sports equipment by holding a volleyball marathon. Mani and Nima played for a combined total of 38 h and raised Nu 4120. Mani was sponsored for Nu 100/h and Nima was sponsored for Nu 120/h.

a) Write a system of linear equations to represent the hours played and the money each has raised.

b) Solve the equations to determine how many hours each student played.

UNIT 3 Test

Pacing	Materials
1 h	• Grid paper • Rulers

Question	Related Lesson(s)
1	Lesson 3.1.1
2	Lesson 3.1.1
3	Lesson 3.1.2
4	Lesson 3.2.1
5	Lessons 3.2.3 and 3.2.4
6	Lesson 3.2.4
7	Lessons 3.3.1 and 3.3.3
8	Lesson 3.3.2
9	Lesson 3.3.4

Select questions to assign according to the time available.

Answers

1. a) Quadratic; I calculated the second differences. They are equal and not zero.

b) Exponential; The graph is close to horizontal on one side and close to vertical on the other side.

2. a) The relationship must be linear.

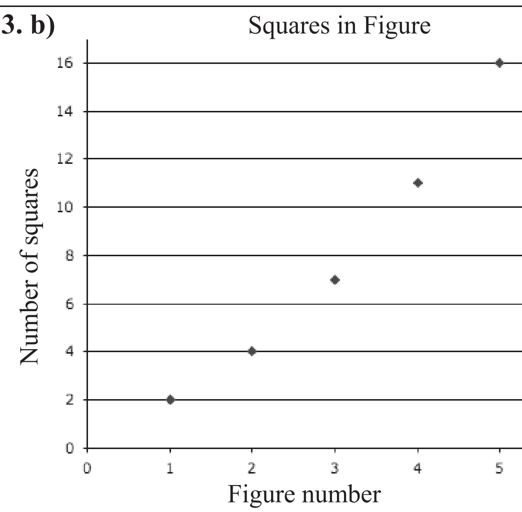
b) *Sample response*:
$y = 2x + 3$; first differences are 2 and second differences are 0.

x	1	2	3	4	5
y	5	7	9	11	13

3. a)

f	1	2	3	4	5
s	2	4	7	11	16

3. b)



c) No; there cannot be fractional figure numbers, so I cannot use a solid line to connect the points.

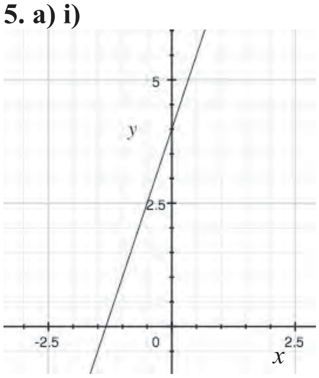
d) Quadratic

e) any value between 50 and 60

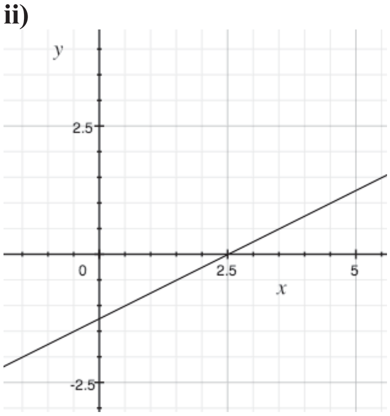
Answers [Continued]

4.
A: $y = 2x + 10$
B: $y = 2x - 4$
C: $y = -x + 5$
D: $y = -2x + 10$

5. a) i)



ii)



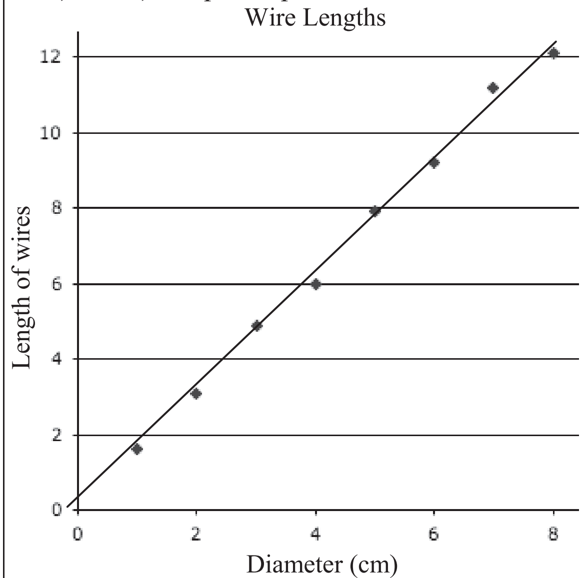
b) i) x-intercept is -1.333; y-intercept is 4

ii) x-intercept is 2.5; y-intercept is -1.25

c) i) $3x - y = -4$

ii) $y = \frac{2}{4}x - \frac{5}{4}$

6. a) and **b)** *Sample response*:



c) *Sample response*: $y = 1.6x$

7. a)
$$\begin{aligned} 2x + 5 - 5 &= 18 - 5 \\ 2x &= 13 \\ x &= 6.5 \end{aligned}$$

b)
$$\begin{aligned} -3a - 4 + 4 &= 10.4 + 4 \\ 3a &= -14.4 \\ a &= -4.8 \end{aligned}$$

c)
$$\begin{aligned} \frac{2}{3}m + 6 - 6 &= \frac{4}{5} - 6 \\ \frac{2}{3}m &= -\frac{26}{5} \\ m &= -\frac{39}{5} \end{aligned}$$

8. a) $n \leq \frac{2.9}{3}$ **b)** $n > 3.25$ **c)** $n < -4$

9. a) $m + n = 38$ and $100m + 120n = 4120$; n is Nima and m is Mani

b) Mani played for 22 h and Nima played for 16 h.

UNIT 3 Performance Task—Predicting Internet Use

A. The International Telecommunications Union (ITU) provides estimates of the number of Internet users in Bhutan each year since 1999. Before 1999, the ITU indicates there were none.

Year	1999	2000	2001	2002	2003
Number of Internet users	750	2250	5000	10000	15000

i) Use the ITU data to create a graph and an equation that describes the relation between the year and the number of Internet users in Bhutan.

ii) Use the equation to predict the number of Bhutanese Internet users in 2010.

B. The 2005 United Nations Human Development Report (UNHDR) provided these estimates of Bhutan's population.

Year	Population (1000s)	Year	Population (1000s)
1984	1447	1994	1785
1985	1486	1995	1814
1986	1528	1996	1851
1987	1573	1997	1896
1988	1619	1998	1948
1989	1661	1999	2004
1990	1696	2000	2063
1991	1724	2001	2125
1992	1745	2002	2190
1993	1763	2003	2257

i) Based on these estimates, create a graph and an equation that describes the relation between the year and the population of Bhutan.

ii) Use the equation to predict the population of Bhutan in 2010.

C. In 2005, the UNHDR reported that Bhutan had an Internet usage rate of about 20 users per 1000 people.

i) Use your prediction in **part B** and this Internet usage rate to predict the number of Internet users in Bhutan in 2010.

ii) Why might this prediction be different from your prediction in **part A**?

UNIT 3 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-C4 Equation of a Line: use graph to determine equation **9-C5** Lines of Best Fit: sketch and determine equations **9-F6** Displaying Data: draw inferences and make predictions	1 h	• Grid paper • Rulers

How to Use This Performance Task

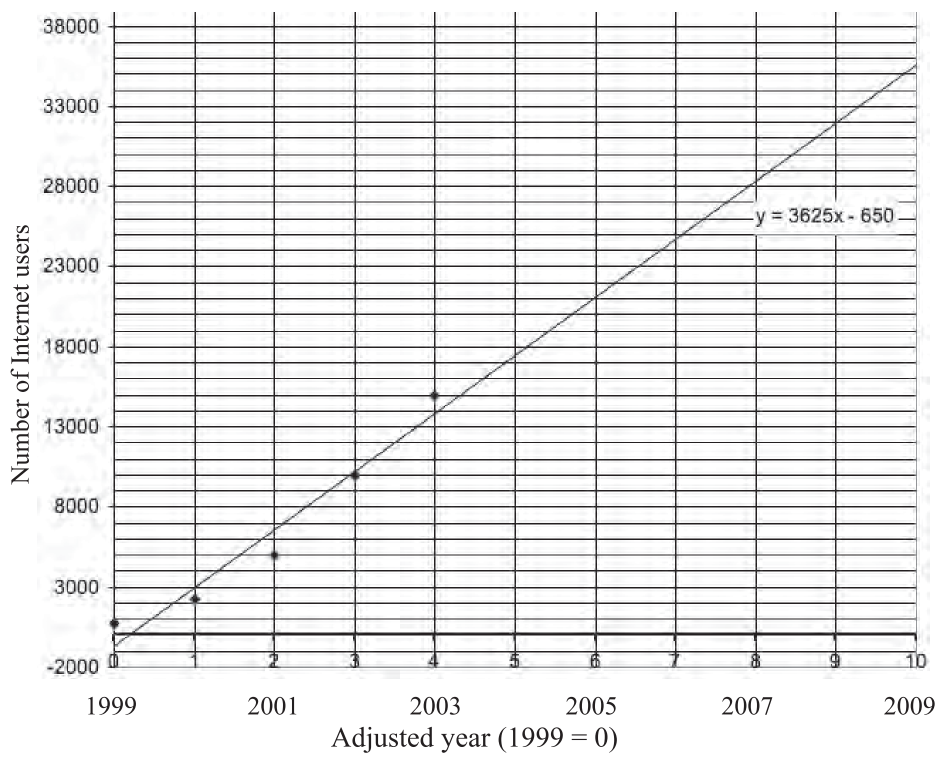
You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used as enrichment material for some students. You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Note that these population estimates may not be accurate but are in fact what the United Nations reported.

Sample Solution

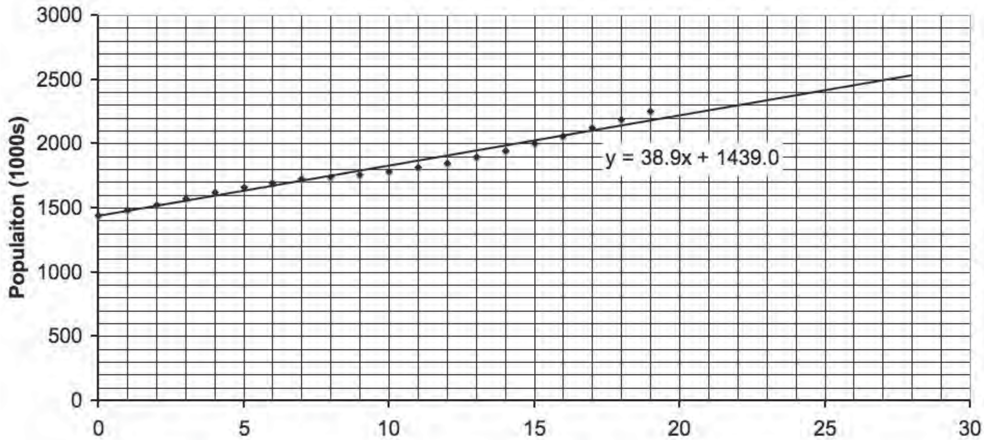
A. i)

Growth of Internet Users in Bhutan



ii) 2010 is year 11. Substitute $x = 11$; $y = 3625(11) - 650 = 39{,}225$.
This model predicts 39,225 Internet users in 2010.

B. i) Bhutan's Population



Adjusted year (1984 = 0)

B. ii) 2010 is year 27. Substitute $x = 26$; $y = 38.9(26) + 1439 = 1160.4$. This model predicts the population of Bhutan in 2010 will be 1,160,400.

C. i) Using the rate of 20 users per 1000 people, the model would predict $0.020 \times 1{,}160{,}400 = 23{,}208$ Internet users in 2010.

ii) The prediction using the second model assumes the percentage of the population using the Internet remains the same. The first model deals only with the number of Internet users and, as a result, does not take into account the fact that the actual population and the percentage of the population using the Internet are both increasing with time.

UNIT 3 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Creation of Scatter Plots and Drawing of Line of Best Fit	Completely accurate scatter plots and lines of best fit	Reasonably accurate scatter plots and lines of best fit	Reasonably accurate scatter plots or lines of best fit	Major errors in plotting points or in drawing lines of best fit
Determination of Equation of Lines of Best Fit	Determination of both equations is complete and correct	Determination of both equations is reasonably complete and correct	Some problems with the determination of one or both equations	Major errors with the determination of both equations
Solution of problem	Correct predictions for both models and insightful analysis of the differences between the two predictions	Correct or almost correct predictions for both models and reasonable analysis of the differences between the two predictions	Correct or almost correct predictions for one or both models and errors or omissions in the analysis of the differences between the two predictions	Incomplete or incorrect predictions for both models and errors or omissions in the analysis of the differences between the two predictions

UNIT 4 DATA AND PROBABILITY

UNIT 4 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, terminology, and pre-assessment	1 h	None	All questions
Chapter 1 Displaying and Analysing Data				
4.1.1 Constructing Familiar Data Displays	**9-F2 Displaying Data: most appropriate methods** • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	1 h	• Rulers, protractors, graph paper, and coloured pencils • Graphing programs such as spreadsheets (optional)	Q3, 5, 6, and 8
4.1.2 Using Graphs to Compare and Organize Data	**9-F4 Data Analysis: evaluate arguments and interpretations** • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness **9-F2 Displaying Data: most appropriate methods** • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	1 h	None	Q1, 2, and 6
4.1.3 Using Graphs to Examine Change	**9-F3 Data Analysis: evaluate arguments and interpretations** • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness **9-F2 Displaying Data: most appropriate methods** • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	1 h	None	Q1, 2, and 4
4.1.4 Misleading Graphs	**9-F4 Data Analysis: evaluate arguments and interpretations** • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness • examine how the choice of certain graphs can lead to errors in judgment	1 h	None	Q1, 3, and 4
4.1.5 Drawing Conclusions From Graphs	**9-F1 Displaying Data: draw inferences and make predictions** • draw inferences and conclusions from a number of data displays • interpolate and extrapolate from a data set	1 h	None	Q1, 3, 5, and 8

UNIT 4 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Probability				
4.2.1 Determining and Comparing Probabilities	**9-G1 Theoretical Probability: independent events** • determine the number of possible outcomes for independent events using outcome charts, organized lists, and tree diagrams	1 h	None	Q2, 5, 7, and 9
4.2.2 Calculating the Probability of Two Independent Events	**9-G1 Theoretical Probability: independent events** • calculate the probability of two independent events, A and B, as $P(A) \times P(B)$ • determine the number of possible outcomes for independent events using outcome charts and tree diagrams.	1 h	None	Q1, 2, 6, and 11
GAME: On a Roll	Practise probability skills and concepts in a game situation	30 min	• Dice	N/A
4.2.3 Randomness: Comparing Experimental and Theoretical Results	**9-G2 Simulations and experiments: dependent and independent events** • distinguish between theoretical and experimental probability	1 h	• Coins, spinners, cards, and dice	Q1, 2, and 5
4.2.4 Conducting a Simulation	**9-G2 Simulations and Experiments: dependent and independent events** • conduct and design simple simulations involving both dependent and independent events • distinguish between theoretical and experimental probability • determine experimental probabilities for simulations	1 h	• Spinners, dice, coins, and coloured blocks	Q1, 3, and 5
4.2.5 EXPLORE: Designing a Simulation	**9-G2 Simulations and Experiments: dependent and independent events** • conduct and design simple simulations involving both dependent and independent events • distinguish between theoretical and experimental probability • determine experimental probabilities for simulations	1 h	• Spinners, dice, coins, and coloured blocks	Observe and Assess questions
CONNECTIONS: Computer Simulations	Investigate the application of simulations in a real world context	30 min	None	N/A
UNIT 4 Revision	Review the concepts and skills in the unit	1 h	• Rulers, protractors, compasses, graph paper, dice or 6-section spinners	All questions
UNIT 4 Test	Assess the concepts and skills in the unit	1 h		All questions
UNIT 4 Performance Task	Assess the concepts and skills in the unit	1 h	• Probability devices such as spinners, dice, and cards	Rubric provided
UNIT 4 Blackline Master	Fraction Circles for Spinners			

Math Background

• Students have been exposed to many different types of graphical displays such as pictographs, bar graphs, circle graphs, line graphs, stem and leaf plots, box and whisker plots, histograms, and scatter plots. The focus in Class IX is on graphs that display data that does not need to be grouped into intervals. Drawing conclusions from graphs, rather than creating them, is emphasized in this unit.

• Until now, students have rarely encountered graphs that have been designed to influence the reader to draw certain conclusions. This unit will examine characteristics of graphs that can often lead to inappropriate conclusions.

• Students are already familiar with probability, both experimental and theoretical. Students will further their understanding of the connection between experimental and theoretical probability in this unit. They will learn how to design and conduct simulations and compare the experimental results to the corresponding theoretical probability. They will also focus on calculating theoretical probability for two independent events.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections. For example:

- Students use problem solving in **question 3** in **lesson 4.2.3** and in **questions 3 and 4** in **lesson 4.2.4** where they design their own experiments and simulations.

- They use communication frequently as they explain their thinking in answering questions, for example in **questions 2 and 3** in **lesson 4.1.2** where they explain why a certain type of graph was used or which type of graph they would use to represent given data. You might notice that the last question in a lesson often requires an element of communication.

- They use reasoning in answering questions such as **question 5** in **lesson 4.1.5**, where they think about how two different graphs compare with each other and judge the validity of conclusions drawn from each graph.

- They use representation and visualization and make connections as they represent numerous data sets graphically throughout the unit. Tree diagrams and outcome charts are also used frequently when dealing with probability concepts.

Rationale for Teaching Approach

• The unit is divided into two chapters. **Chapter 1** reviews various types of graphs that can be used to display sets of data that are not grouped into intervals. The chapter deals mainly with justifying the choice of which type of graph to use, interpreting these graphs and drawing conclusions from them. **Chapter 2** looks at the similarities and differences between experimental and theoretical probability and ways to determine theoretical probability. Students focus on designing simulations to determine probabilities of given events and carrying out the simulations.

• **Chapter 1** is designed to remind students how to construct graphs, to help them focus on how the choice of a graph relates to its purpose, and only then to focus on interpreting and drawing conclusions from graphs.

• In **Chapter 2**, students begin by using outcome charts, tree diagrams, and organized lists to find all possible outcomes of one and two events. Then, they are introduced to the formula for calculating probabilities of compound events, which is reserved for independent events only. The formula for dependent events is much more complicated.

• Students will have already explored the distinction between experimental and theoretical probability, and that learning is solidified in this chapter. The design of simulations is the most complex idea in the chapter and it comes last.

• There is one open exploration, **lesson 4.2.5**, where students are required to design and carry out a simulation for a given event. This is followed by a Connections feature that looks at the use of computers in simulations. Students are encouraged to investigate how, where, and why computer simulations are used.

Technology in This Unit

• If computer access is available, a spreadsheet program would be very useful for creating some of the graphs in **Chapter 1**. Note that spreadsheets do not create histograms and box and whisker plots. There are some useful Internet sites with graphing applets that can be used to create graphs. Do a search using the words "graphing applets" to find them.

• The use of a calculator should be encouraged in **lesson 4.2.2.** Some calculators can be put into fraction mode by pressing [MODE], then [3]. Then, to calculate with fractions, you use the [a $^{b}/_{c}$] key.

For example, to calculate $\frac{1}{2} \times \frac{1}{2}$ press:

1 [a $^{b}/_{c}$] 2 [×] 1 [a $^{b}/_{c}$] 2 [=].

Getting Started

Curriculum Outcomes	Outcome relevance
8 Theoretical Probability: single and complementary events **8** Simulations and Experiments: single and complementary events **8** Compare Results: theoretical and experimental **8** Circle Graphs: construct and interpret **8** Patterns and Relations: represent in a variety of formats **6** Bar and Double Bar Graphs: construct and interpret	Students will need to review their knowledge about different types of graphs as well as theoretical and experimental probabilities to be ready to move forward in this unit.

Pacing	Materials	Prerequisites
1 h	• Graph paper • Dice (optional)	• probability vocabulary and notation (theoretical, experimental, likely, P(x) notation) • double bar graphs • interpreting circle graphs • interpreting line graphs

Main Points to be Raised

• Experimental probability is only an estimate of the probability of an event. Its accuracy depends on the number of times the experiment is performed.

• A double bar graph is a good choice of graph to compare two different classifications of data. In this case you can easily compare the experimental probability with the theoretical probability for each difference.

• Theoretical probability makes the best predictions. It is determined by the ratio of the number of favourable outcomes to the total number of possible outcomes.

Use What You Know—Introducing the Unit

• Use this activity to re-introduce theoretical and experimental probability and the use of double bar graphs to compare probabilities for the possible outcomes. Students should already be familiar with experimental and theoretical probability from previous classes.

• Encourage students to work in pairs, where one student rolls the dice and the other records the differences. Both students should be responsible for recording the chart in their notes and creating the double bar graph so that they can compare their graphs and discuss any differences they find.

Observe students as they work. You might ask:

• *How are you determining the theoretical probability for each difference?* (Creating a fraction that divides the number of ways the specific difference can occur by the total number of ways any possible differences can occur.)

• *How are you determining the experimental probability for each difference?* (Creating a fraction that divides the number of ways that the specific difference occurred by the total number of trials that were conducted in the experiment.)

• *Why is a double bar graph a good choice to compare the experimental probability of each difference to the theoretical probability?* (Double bar graphs show the experimental and theoretical probability for each difference. Comparing the heights of each pair of bars is easy because they are drawn side by side.)

• *Which probability did you use to determine which difference was most likely?* (I used the theoretical probability because it is the expected probability that would be achieved if the experiment were run many, many times. It does not rely on the results of an experiment that was only tested a few times.)

• *Why is a difference of 1 more likely than a difference of 5?* (There are many combinations of numbers that have a difference of 1, but a difference of 5 is only achieved if one die shows 6 and the other shows 1.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign all of these questions.

• Students can work individually.

Answers

A. *Sample response*:
I expect that a difference of 2 is more likely because I can think of many pairs of numbers that have a difference of 2.

B. i)

–	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

ii) 36

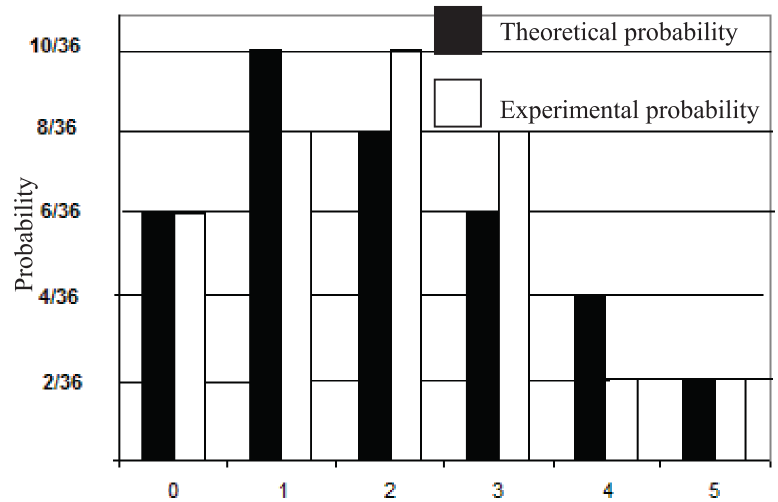
iii) 6; 10; 8; 6; 4; 2

iv) 0: $\frac{6}{36}$; 1: $\frac{10}{36}$; 2: $\frac{8}{36}$; 3: $\frac{6}{36}$; 4: $\frac{4}{36}$;
5: $\frac{2}{36}$

C. *Sample response*:
0: $\frac{6}{36}$; 1: $\frac{8}{36}$; 2: $\frac{10}{36}$; 3: $\frac{8}{36}$; 4: $\frac{2}{36}$;
5: $\frac{2}{36}$

D. *Sample response* (based on the sample experimental probabilities in answer to **part C)**:

Probability of Rolling Differences



E. *Sample response* (using experimental probabilities in answer to **part C**):
The theoretical and experimental results matched by the same amount for differences of 1 or 2 but in opposite ways.

F. *Sample response* (using experimental probabilities in answer to **part C**):
It is theoretically more likely that the difference is 1 but that is not what happened in my experiment.

1. a) *Sample response*:
There are more fishes than birds, mammals, reptiles, or amphibians. There are about five times as many fishes as mammals or amphibians. There are about twice as many birds as amphibians or mammals.

b) *Sample response*:
You can see at a glance what fraction of the whole each species represents; you can also see the relative sizes of the different species.

2. a) fishes **b)** amphibians **c)** about twice as many

3. a) part b) [the 9% and 10% are very close, it would be hard to judge if there were more amphibians or mammals without seeing the percentage values]

b) part c) [It is relatively easy to see whether one wedge is twice as large as another.]

4. a) 8 mm **b)** between 2 and 3 pm
c) 5 mm, 1.5 mm **d)** 5 pm

5. a) 0.6 **b)** 0.1 **c)** 0.5
d) 0.5 **e)** 1.0 **f)** 0.9

Supporting Students

Struggling students

If students are struggling with probability language, you should be prepared, in **Chapter 2**, to review probability vocabulary in simple situations.

Chapter 1 Displaying and Analysing Data

4.1.1 Constructing Familiar Data Displays

Curriculum Outcomes	Outcome relevance
9-F2 Displaying Data: most appropriate methods • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	Students revisit the necessary procedures to construct stem and leaf plots, box and whisker plots, circle graphs, and bar graphs. Before they can justify why a specific type of graph was used they must recall how and why these graphs are created.

Pacing	Materials	Prerequisites
1 h	• Rulers • Protractors • Graph paper • Coloured pencils • Graphing programs such as spreadsheets (optional)	• median • stem and leaf plots • box and whisker plots • circle graphs • line graphs • bar graphs

Main Points to be Raised

• The type of graph used depends on the nature of the data and the purpose of the graph.
• If you are given numerical data for a group that measures a single characteristic, then a stem and leaf plot or box plot may be appropriate.
• If you are given data that is sorted into categories and you want to compare the relative sizes of the categories, then a circle graph or bar graph may be appropriate.
• If you want to show how a single characteristic changes over time, then a line graph may be appropriate.
• All graphs should have a title and their axes should be labelled with the appropriate units of measure.
• There are software programs that can be used to create graphs from data. A spreadsheet is a good tool to use if it is available.

Try This—Introducing the Lesson

A. Allow students to try these alone and then compare their answers with a partner. These questions provide an opportunity to revisit the skill of finding some useful information from the data set that can be used to create a box and whisker plot. The actual step-by-step procedure is introduced in the lesson.

Observe while students work. You might ask:
• *How could you make it easier to find the minimum and maximum values?* (Arrange the data in order from the smallest value to the largest.)
• *What mathematical operation did you use to find the range?* (subtraction)

The Exposition—Presenting the Main Ideas

• You may have students read along with you as you introduce the terminology in the main ideas in the Exposition box. Or, you may use the given distance data to lead the class through the steps to create a stem and leaf plot and a box plot.

• You may want to show the students a circle graph and probe to see if they can recall the steps used to create this type of graph and when it would be appropriate to use such a graph. You could also do this for a line graph and a bar graph.

• You may wish to display a poster of each type of graph that also indicates what each type of graph is used for and from what kind of data each graph type is created.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** and the new graph types and procedures used to create them that have been presented in the main ideas in the Exposition. You might approach **part B** as a whole class.

Using the Examples

Allow students time to read the example:
• Ask them why a circle graph was a good choice in **example 1** and whether any other types of graphs could have been used to represent this set of data.
• In **example 2** point out that in a line graph the horizontal axis is always the time variable. The scale used for the vertical axis depends on the range of the data you are working with and the number of intervals you want to use.
• In **example 3**, students will see how a box and whisker plot is created from a set of data with an odd number of values. They will also see that the median can be included in the lower and upper quartiles (**solution 1**) or not (**solution 2**). When the data set is large, its makes very little difference whether the median is included or not. However, when the data set is small, it is more mathematically sound to include the median in the calculation of Q1 and Q3.

Practising and Applying

Teaching points and tips

Q 1: You might ask students to compare the graphs they have created. Ask them to explain how these graphs differ and why a person would use one over another.

Q 2: Observe strategies the students use to create their circle graphs and how, and whether, they use estimation to create them.

Q 5: Some students may need to be reminded to put the data in order before determining the median.

Q 8: Make sure students understand that often more than one type of graph is appropriate for the same data set.

Throughout the Practising and Applying questions, students should be encouraged to compare the graphs they create with a partner. They should identify any differences in their graphs and discuss the reasons for these.

Common Errors

Students often struggle with choosing an appropriate scale to use for the vertical axis of line and bar graphs. Make sure you emphasize that for a given set of data the scale used must be reasonable for the data. A scale that is too large can distort the graph and lead to errors in interpretation.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can use the required procedures to create a bar graph by hand
Question 5	to see if students can use the required procedures to create a box and whisker plot by hand
Question 7	to see if students can use the required procedures to create a multiple line graph by hand
Question 8	to see if students can decide which type of graph should be used for a given set of data and then construct it correctly by hand

Answers

Note that all box plots include the median in the calculation of Q1 and Q3.

A. i) 1000 min

ii) 60 min

iii) 940 min

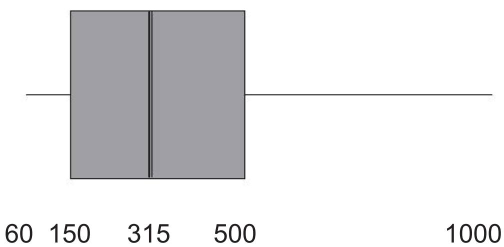
B. A stem and leaf plot could be used, with the data grouped into hundreds, showing the extreme values of the data at a glance and a median can be easily determined. For an even clearer representation of the median and the range, a box and whisker plot could be used. The median and extreme values can be read immediately off the box and whisker plot, which also gives information about the upper and lower quartiles.

Answers [Continued]

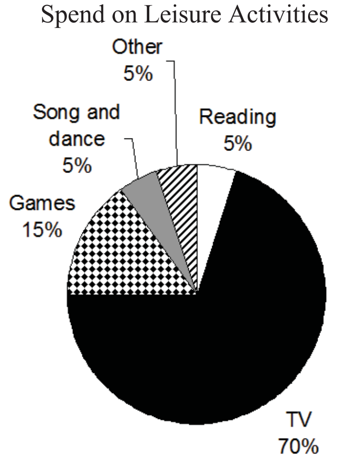
1. a)

10	00			
9				
8				
7	50			
6	00	50		
5	00	00		
4	00	20		
3	00	30	50	
2	00	50	85	
1	50	50		
0	60	75	90	90

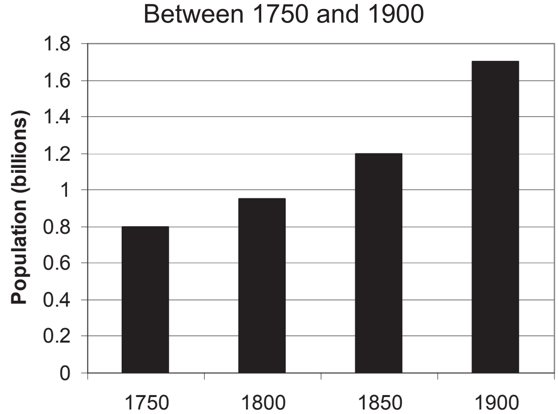
b)



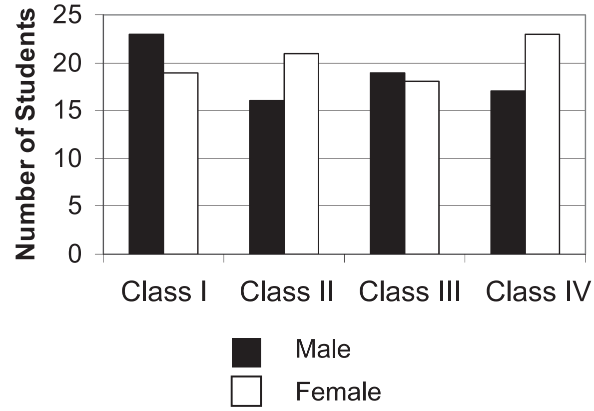
2. Percent of Time Teens in Thimphu Spend on Leisure Activities



3. Population of the Earth Between 1750 and 1900



4. Number of Students by Gender in Four Classes

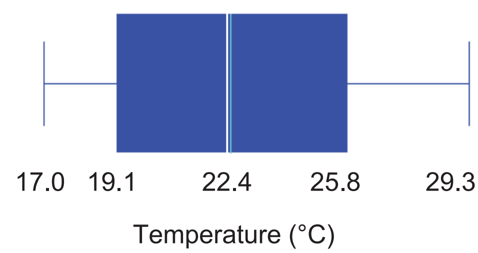


5. a) 22.4

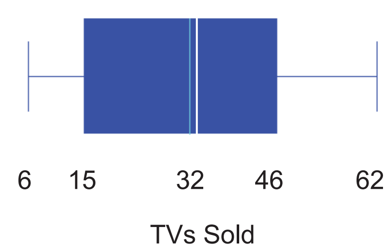
b) 12.3

c) Q1 = 19.1, Q3 = 25.8

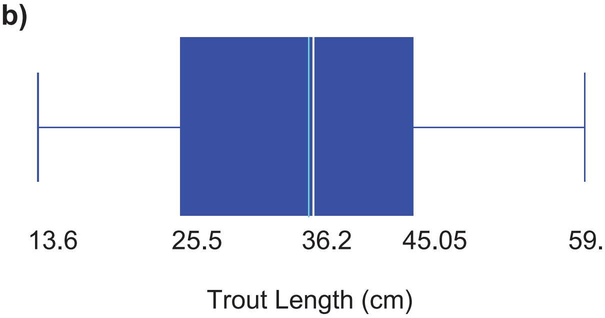
d)



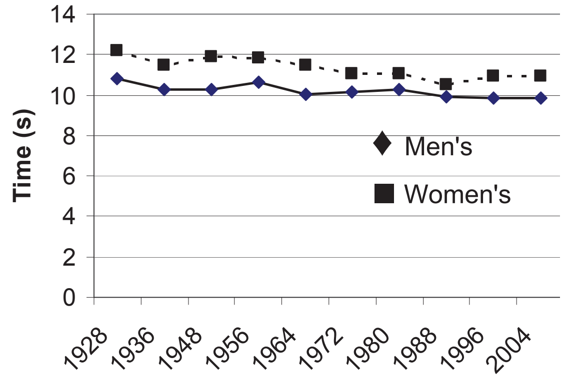
6. a)



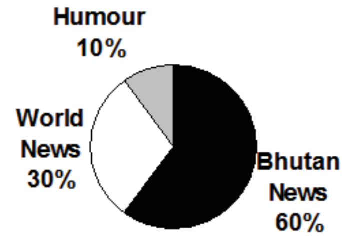
b)



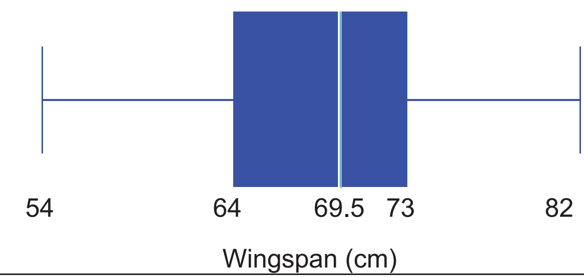
7. Olympic 100 m Winning Times



8. a) Amount of Space Devoted to Various Sections of the *Bhutan Times*



b) *Sample response*:



Supporting Students

Struggling students

If some students are struggling with creating certain types of graphs, it might be useful to provide additional practice problems with that type of graph to replace some of the other graphs that do not present difficulties for those students.

4.1.2 Using Graphs to Compare and Organize Data

Curriculum Outcomes	Outcome relevance
9-F3 Data Analysis: evaluate arguments and interpretations • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness **9-F2 Displaying Data: most appropriate methods** • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	Students not only need to know how to construct graphs from data, they also need to be able to organize data into categories so that comparisons can be made. Students need to be able to recognize when a circle graph or a bar graph is the best choice for displaying data. They also must be able to justify when and why a box and whisker plot is best.

Pacing	Materials	Prerequisites
1 h	None	• ability to construct and interpret circle graphs, bar graphs, and box and whisker plots

Main Points to be Raised

• Circle graphs and bar graphs are used when data values have been grouped into categories.

• Circle graphs should only be used when all possible categories of the situation are represented. Otherwise a bar graph should be used.

• Box and whisker plots, or box plots, are not used for comparing categories. They are used to show how the data values are distributed about the median and to show the range in which the bulk of the data values actually lie.

Try This—Introducing the Lesson

A and B. Begin with students working alone on this and then engage the class in a group discussion. Focus on why they prefer one graph to another and ensure that they give valid reasons for their choice.

Observe while students work. You might ask:

• *Which graph best indicates the most common source of drinking water?* (Both graphs show the most common source, but it is most obvious in the circle graph.)

• *Which graph best compares the sizes of the categories to each other?* (The bar graph is easier to read when comparing the sizes of the categories to each other.)

• *Which graph requires the most effort and analysis to interpret?* (The bar graph requires more effort. The percentages for each category must be determined using the vertical scale whereas on the circle graph all of the percentages for each category are displayed directly.)

The Exposition—Presenting the Main Ideas

• Before students read the presentation of the main ideas in the Exposition, display a copy of the Population in Europe and Asia graph and ask them to identify the type of graph. They can then draw some conclusions and make some comparisons based on the graph.

• When they have finished, ask them to read the main ideas in the Exposition and compare their answers with the results that are listed for the Population in Europe and Asia graph.

• Lead a class discussion about what they learned about circle graphs and box plots.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between what was done in **parts A and B** and the main ideas presented. It also allows the teacher to quickly assess whether or not students understand when it is best to use each type of graph.

Using the Examples

• Some students might benefit from solving the problem in **example 1** on their own. They could create the corresponding box plot for the data and explain in writing why a box plot is a good graph to use for this data set. They could then compare their work with what is shown in the text.
• You could also have students complete **example 2** prior to reading the text, and ask them to discuss their reasons and observations with a partner and compare them with the text.

Practising and Applying

Teaching points and tips

Q 1: Encourage students to explain their thinking as they select which statements apply.

Q 3 and Q 4: There may be more than one type of graph that is suitable for each data set. For example, in **3a**, a circle graph and a bar graph are both suitable.

Q 5: This question emphasizes that often more than one type of graph is appropriate since it asks for more than one type.

Common Errors

Students often think that if data is organized in categories, a circle graph can always be used. This is true for bar graphs, but not true for circle graphs. Emphasize that circle graphs can only be used when all possible categories for a situation are represented.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can recognize the characteristics of circle graphs, bar graphs, and box and whisker plots
Question 2	to see if students can justify why a specific graph was used and if they can make insightful observations from the graph
Question 6	to see if students can identify contextual situations that would provide appropriate data to be displayed by the various types of graphs

Answers

A. i) circle graph (pie chart) and a bar graph
ii) Both graphs tell the percentages for the different sources of drinking water, but for the bar graph you compare the values by looking at heights while for the circle graph you look at areas.

B. *Sample response*:
I like the circle graph because it does not use as much space.

C. *Sample response*:
There is no other country to compare with, so a double bar graph does not make sense. The median and upper/lower quartile values would not be significant in a box plot because there are only four pieces of data.

1. a) circle graph B, D
b) bar graph B, C, D
c) box and whisker plot A, B

2. a) *Sample response*:
Swimming is the most popular sport and cycling the least popular. This circle graph lets you see how each sport in the survey compares with each other in terms of its popularity. A bar graph could also be used.

2. b) *Sample response*:
Chocolate milk sales decrease through the week while white milk sales increase. This double bar graph lets you see how chocolate milk sales compare with white milk sales. For example, white and chocolate milk sales are similar in the middle of the week. A multiple line graph could also be used to represent the data.

c) *Sample response*:
The majority of families have more children than pets. This bar graph lets you compare the number of children with the number of pets in families. A circle graph could be used since all possible categories of the situation are represented.

Answers [Continued]

2. d) The students scored between 43 and 100 on the test and the median mark was about 67. This box plot lets you see how the data is clustered around the median mark. No other graph could be used.

3. a) *Sample response*:
A circle graph because it compares parts of a whole.

b) *Sample response*:
A bar graph because it would be easier to make a scale and plot the data than to calculate the percentages to make a circle graph.

c) *Sample response*:
A box and whisker plot because it will show the median, range, quartiles, and spread of the data.

4. a) A box and whisker plot because the middle 50% of the data is easy to see.

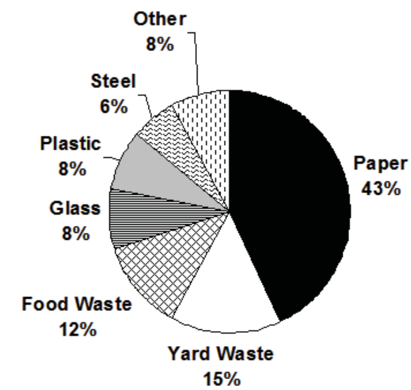
b) *Sample response*:
A double bar graph because there are discrete categories and it is easy to compare the heights of the bars.

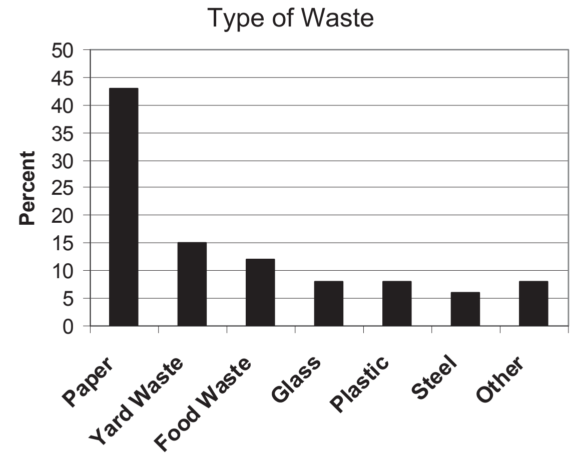
c) *Sample response*:
A box and whisker plot because it shows the median and you can compare the spread of the data to the median.

d) *Sample response*:
A box and whisker plot because it shows the median, extreme values, and the spread of the data.

e) *Sample response*:
A bar graph because you can compare the building heights easily.

5. a) Type of Waste





b) *Sample response*:
The bar graph because you only have to compare the heights of the bars

6. a) *Sample response*:
Compare the number of children in Bhutan to the number of adults at 5-year intervals starting in 1960

b) *Sample response*:
Compare the favourite foods of people in your class

c) *Sample response*:
Compare the median age of all the students in your school to the range of their ages

d) *Sample response*:
Compare the percent of each type of crop grown in Bhutan

Supporting Students

Enrichment

You might ask students to look for examples of different types of graphs in books, magazines, and newspapers. Have them identify the type of graph they have found and provide reasons why this type of graph was used. Ask them to make an observation about the data in the graph. If students have access to computers and the internet, they could do the same type of search, justification and observation using online sources.

4.1.3 Using Graphs to Examine Change

Curriculum Outcomes	Outcome relevance
9-F3 Data Analysis: evaluate arguments and interpretations • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness **9-F2 Displaying Data: most appropriate methods** • determine, discuss and justify, why a particular display is suited to a specific type of data, or to a given context or purpose	Students need to be able to recognize when a line graph or a scatter plot is the best choice for displaying data. With scatter plots, they also must be able to distinguish between when the set of data is discrete and when it is continuous.

Pacing	Materials	Prerequisites
1 h	None	• line graphs • scatter plots

Main Points to be Raised

• Line graphs are used to show how a quantity changes with time, more specifically from one time to the next, usually over a series of regular intervals.

• Scatter plots can involve the variable time but they do not need to. Scatter plots are used when you are looking for a relationship between two quantities in a data set. The points in a scatter plot are not joined point-to-point by a straight line.

Try This—Introducing the Lesson

A. Have students work in pairs. Students can take turns explaining to the other how they interpret Kinley's trip. If there are differences, they can discuss these in order to come to a consensus on how to interpret this graph. Once they have done this, choose a pair of students to give their interpretation to the class. Then have the class discuss the given interpretation and modify or correct as needed.

Observe while students work. You might ask:

• *When is Kinley travelling the fastest? the slowest?* (between 40 and 70 minutes and between 0 and 30 minutes)

• *When is he not travelling?* (between 30 and 40 minutes and between 70 and 90 minutes)

• *How far did he travel and how long did it take him?* (He travelled 4.5 km over a period of 90 minutes.)

• *How could you find his average speed?* (Divide the total distance he travelled by the total time.)

The Exposition—Presenting the Main Ideas

• Have students examine the five different graphs shown in the main ideas section before reading the section. Have a class discussion about their similarities and differences.

• Emphasize the different uses of a line graph and a scatter plot, pointing out that if there are several values of the dependent variable for a single value of the independent variable then a line graph is not appropriate. Line graphs are only used when there is a one–to-one relationship between the values of the independent and dependent variables.

• Stress the difference between continuous data and discrete data and how this affects the choice of a line to represent the relationship in a scatter plot.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** and the new, more formal graphing language and procedures that have been presented. Students can work in pairs.

Using the Examples

Put this question and graph on the board and ask students to close their textbooks. Have students answer the questions for this example individually in their notebook. Ask a student volunteer to share his or her solution with the class while the rest of the class listens and follows along with the solution presented in the book. Conclude with a class discussion to clarify any errors or omissions that were evident in the presented solution.

Practising and Applying

Teaching points and tips

Q 2: Some students may talk about the graph itself while others talk about the topic that the graph is addressing. For example, they might talk about why the population is growing and not just that it is growing.

Q 3c and d: A broken line graph or scatter plot are appropriate here since there is a unique value of the dependent variable for each value of the independent variable. Which graph is used depends on the purpose. A line graph is more suitable if you want to examine the change from year to year, but a scatter plot is more appropriate if you want to find a relationship between time and the dependent variables.

Common Errors

Some students will automatically join each point plotted with a solid line. Remind them to look closely at the purpose of the graph to see if it is looking at change (line graph) or looking for a relationship between the variables (scatter plot).

Suggested assessment questions from Practising and Applying

Question 1	to see if students can identify the characteristics of broken line graphs and scatter plots
Question 2	to see if students can justify why a specific graph was used and to see if they can make insightful observations from the graph
Question 4	to see if students can identify the most effective graph for showing differences between two sets of data

Answers

A. Kinley walked at a constant speed for the first 30 min. He then took a rest for 10 min; I know this because his distance from home did not change. He then walked faster than he did before for a period of 30 min. He stayed at this location for 20 min, but I do not know where he went after this because the graph ends.

B. This is a broken line graph. It makes sense to use this graph because it shows how his distance from home changes with time and broken line graphs are used to show how one quantity changes in relation to another.

1. a) broken line graph A, C, D

b) scatter plot A, B

c) multiple broken line graphs A, C, D

2. a) *Sample response*:
As time increases so does the population. This scatter plot lets you see if there is a relationship between time and population. A broken line graph could also be used.

b) *Sample response*:
Cell phone use is increasing faster among females than among males. This multiple broken line graph lets you see how cell phone use is changing over time in two different groups. A scatter plot could also be used with different types of dots for the male and female data.

3. a) *Sample response*:
A scatter plot to show whether a relationship exists between arm length and height.

b) *Sample response*:
A multiple broken line graph to compare exports with imports and compare how both these quantities change over this time period.

c) *Sample response*:
A broken line graph to show how the number of tourist arrivals has changed over this time period

4. a) *Sample response*: - Karma used two box plots on the same scale to compare the range, median, extremes, and spread of the data between men and women. - Lobzang used a multiple broken line graph to compare the trends in the heights for men and women over time.	**4. b)** The broken line graph is better at showing change over time. The box plot is better for looking at data spread.

Supporting Students

Struggling students

Some students may have difficulty drawing conclusions from multiple broken line graphs and scatter plots that compare two or more sets of data on the same graph. Make sure they understand the relationship between the overall general slope of the line and the rate at which the change is occurring for a single broken line graph or a scatter plot in which time is a variable. A steeper line indicates a faster rate of change.

Enrichment

Some students might collect their own data about something that changes over time and display the data in the appropriate type of graph.

4.1.4 Misleading Graphs

Curriculum Outcomes	Outcome relevance
9-F3 Data Analysis: evaluate arguments and interpretations • compare various methods of displaying data which does not require grouping into intervals and evaluate their effectiveness • examine how the choice of certain graphs can lead to errors in judgment	Examining how graphs can be used to influence how readers interpret the data will help students improve their ability to judge the validity of conclusions drawn from graphs.

Pacing	Materials	Prerequisites
1 h	None	• ability to read and interpret a variety of graphs

Main Points to be Raised

• Just because data is presented in a graph does not mean the graph is an accurate representation of the data. Graphs can be created to influence how the reader interprets the data, often with a specific purpose in mind.

• Some graphing errors are honest mistakes made by the graph's creator, while others are done deliberately to alter the display in a particular way.

• Care must be taken when analysing a graph to ensure it is free of any errors that could cause it to be misleading.

Try This—Introducing the Lesson

A. Students can work on **part A** with a partner.

Observe while students work. You might ask:

• *Why does the shape or arrangement of the bars in each graph look similar?* (The categories in all three graphs are the same. The rubber production graphs for each country represent the same set of data.)

• *What has caused the differences in rubber production for each country to be more pronounced in one graph and less pronounced in another?* (The scales are different; Dechen's graph minimizes the differences while Sonam's graph exaggerates the differences.)

The Exposition—Presenting the Main Ideas

• Before the students read the main ideas, hold a class discussion and brainstorm to list as many things as possible that could cause a graph to be misleading. Next, have students read the main ideas and compare the list to the causes listed in the textbook.

• Use each example to explain to the students how and why each cause contributes to a misleading graph.

Revisiting the Try This

B. i) This question allows an opportunity to make a formal connection between what was done in **part A** and some of the causes that contribute to misleading graphs presented in the main ideas.
ii) This question allows students to match a conclusion to the appropriate graph, with justification, regardless of whether the graph is misleading or not.

Using the Examples

For this example, it may be necessary to explain to students what an *auction house* is. An *auction house* is a company that sells merchandise through *live bidding*. *Live bidding* is a process by which interested customers identify themselves as willing purchasers of objects placed for auction. The auctioneer conducts the bidding process as customers bid against each other for the item up for sale. The item is sold to the person who bids the highest amount. This usually occurs live in a room with bidders present, but others who are not present may bid by phone or via the internet. Sotheby's and Christie's are two of the largest auction companies in the world.

For more information, see the following websites:
Sotheby's: http://www.sothebys.com
Christie's: http://www.christies.com
Before you discuss the question with the class, show them the graph and ask them to make some observations about what they see. Then ask them to read through the example. Finally, discuss the solution with the class.

Practising and Applying

Teaching points and tips

Q 1: Each graph shown has a different reason for being misleading. **1b** uses different-sized images to represent Petrol Sales in 1990 and 2000. The actual dimensions of the larger image have been approximately doubled when compared to the smaller image, but this results in an image that is four times larger in area. This exaggerates the difference between sales for these two years.

Q 5: Even though both graphs look the same, students must look closely at the vertical scales. The choice of a scale that goes to 40 de-emphasizes the difference between the maximum and minimum temperatures.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can identify the cause that contributes to a misleading graph
Question 3	to see if students can identify valid and invalid conclusions that are based on a misleading graph
Question 5	to see if students can identify the correct graph and use it to analyse the data, disregarding the features that make the graph misleading

Answers

A. i) Similarities–All three graphs are bar graphs with same categories on both axes. They all compare the amount of rubber production between countries.
Differences – The vertical scales are different on each graph. The heights of the bars differ from graph to graph.
ii) The differences in production look greater on some graphs compared to others.

B. i) Dechen's and Sonam's; Sonam's graph appears to show the greatest difference. This happens because the vertical scale does not start at zero. This exaggerates the difference from country to country. Dechen's graph appears to show the least difference. This happens because the vertical scale goes well beyond the highest value needed. This de-emphasizes the difference between countries.
ii) Sonam because the vertical scale does not start at zero and this exaggerates the difference; Dechen because the vertical scale goes well beyond the highest value needed and this de-emphasizes the difference; Sangay because the vertical scale starts at 0 and ends at 2,500,000.

1. a) *Sample response*:
The intervals on the horizontal axis are different, so it looks like the number of accidents increases until age 23 and then decreases, but that is not true because the three bars between 16 and 23 should be one bar showing about 42,000 accidents.

b) *Sample response*:
The difference between the years is shown by the difference in the heights of the bars, but the pump on the right is also wider, so it looks like it represents even more than it should.

c) *Sample response*:
The scale on the vertical axis is inconsistent, it doubles each time. It seems like the third bar is 1.5 times the height of the second but it really should be twice the height.

2. *Sample response*:
- Wrong use of scale
- Misplaced zero point
- Wrong choice of graph type

3. a) True **b)** False **c)** False

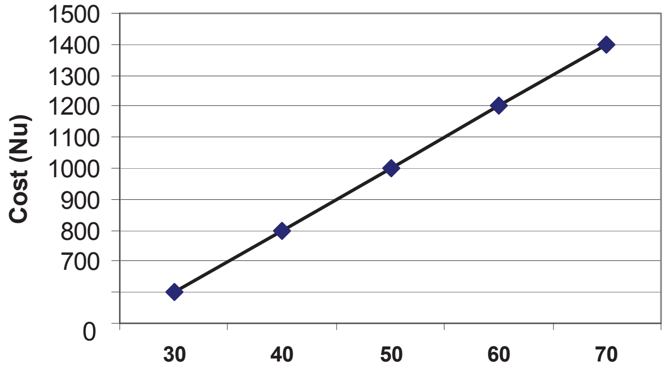
4. The graph does not start at zero, so the length of the bars from 0 to 20 is not accounted for. That means a bar that appears to be three times as long as another really isn't so you can't just compare the lengths of the bars without checking the numbers on the scale.

5. *Sample response*:
The vertical scale on the first graph goes much higher than the greatest plotted temperature, so the temperatures appear closer together than they really are.

Answers [Continued]

6. *Sample response*:

Cost to Ride the Bus



Number of passengers

Because an axis break was not used, the first two increments along the vertical scale look like they each have a value of Nu 350 but each really has a value of Nu 100. This might mislead the reader to think the cost for 30 passengers is Nu 350 instead of Nu 600.

Supporting Students

Enrichment

There are other techniques that are used to create misleading graphs. Ask students to do some research to find other ways in addition to those identified in the textbook. You could suggest they do an internet search using the words "misleading graphs."

4.1.5 Drawing Conclusions From Graphs

Curriculum Outcomes	Outcome relevance
9-F1 Displaying Data: draw inferences and make predictions • draw inferences and conclusions from a number of data displays • interpolate and extrapolate from a data set	Reading and correctly interpreting graphs is a lifelong skill. They will encounter and work with graphs in many aspects of their everyday lives. Making predictions from graphs that show a specific relationship is the primary purpose of mathematical modelling.

Pacing	Materials	Prerequisites
1 h	None	• ability to read and interpret a variety of graphs such as circle graphs, bar graphs, line graphs, scatter plots and box plots

Main Points to be Raised

• Conclusions drawn from graphs are valid as long as the graph accurately displays the data. Conclusions drawn from biased data or misleading graphs may be invalid. It makes sense to check your conclusion against what your common sense suggests should happen.

• Interpolation and extrapolation are usually limited to graphs that show relationships and trends: line graphs and scatter plots.

Try This—Introducing the Lesson

A. Students can work on **part A** with a partner. Each answers the question independently and then they compare their answers.

Observe while students work. You might ask:

• *What is the purpose of the guest speaker?* (to educate students about the advantages of eating a balanced healthy diet)

• *What do you think the guest speaker's message to the students would be?* (Make healthier choices when eating so you will feel better and increase your overall health.)

• *How did the amount of data in the different categories change in light of the message given by the speaker?* (After hearing the speaker, students began to eat healthy foods more than unhealthy foods and the amount of data in the healthy food categories became greater.)

The Exposition—Presenting the Main Ideas

• You may want to review with the class the idea of *bias* in data collection. When a response is affected by factors other than the concept a question is designed to measure, then the response is *biased*. A biased response provides inaccurate information. For example, if a test of history is given only to adults with degrees in history, the results cannot be generalized to apply to all learners.

• The day before you present this lesson, take some time to measure each student's height and forearm length (the distance from the wrist to the elbow). Record the data in a chart and share it with the class.

Forearm length (cm)	Height (cm)

• Ask students to create a scatter plot of the data. Ask them to draw some conclusions from their graph.

• On the day of the lesson, go over the scatter plot and discuss their conclusions. Use the graph to make some predictions by interpolating and extrapolating.

• Next, have students read through the exposition on their own.

Revisiting the Try This

B. This question gives students an opportunity to recognize that they already have experience drawing conclusions from graphs, as they did in **part A**. Students can answer this individually and then discuss responses.

Using the Examples

• Students may have not encountered stacked bar graphs. Read through **example 1** along with the students. Make sure they realize that stacked bar graphs can be used to make comparisons within a category as well as comparisons about the total size of the categories in relationship with each other. For example, the graph shows that in the USA there are about 60 million cats, 78 million dogs and 138 million cats and dogs.
• Ask students to look at the question in **example 2** and respond to it before reading through the solution. Once you have had a couple of students indicate their responses, have students read through the solution. Make sure students realize that when interpolating and extrapolating, the accuracy of the predictions made depends on how closely a line of best fit models the data in a scatter plot. If the points are widely scattered then any predictions made by interpolating or extrapolating will likely not be very accurate. On the other hand, if the points are closely grouped around a line, then predictions made by interpolating or extrapolating should be very accurate.
• It would be useful to draw the graphs on the board for each example so you can talk about the characteristics of the graphs as you discuss the conclusions that can be drawn from each of them.

Practising and Applying

Teaching points and tips

Q 5: It may be useful for students to redraw both graphs on the same axis using the same vertical scale in order to compare the two graphs.

Q 6: This question forces students to realize that more research needs to be done before the conclusion can be validated. Investigation into extinction trends over the past 100 years can help determine whether the conclusion is valid. The graph alone does not provide enough information.

Common errors

Students may base their conclusions on a very superficial examination of a graph. They must learn to look at the graph carefully and examine it for any features that might cause it to be misleading. Once they have drawn some conclusions they should take a minute to reflect on whether or not the conclusions seem reasonable.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can draw conclusions from a line graph and use it to make predictions
Question 3	to see if students can draw conclusions from a box plot
Question 5	to see if students can judge whether a conclusion is valid based on data presented in two or more graphs
Question 8	to see if students understand when interpolation and extrapolation are practical

Answers

A. After the visit from the guest speaker students began to eat healthier snacks. It seems reasonable because I think some students would have been affected by what the speaker had to say.

B. No, there is not sufficient data to make predictions. Also, the advice from the guest speaker may no longer affect the students in three months because they will have forgotten.

1. **a)** *Sample response*: 8 L, 13 L **b)** *Sample response*: 65 km, 90 km **c)** *Sample response*: - The faster you drive, the more gas you use over the same distance. T - They travelled at different speeds over the 100 km. **2. a)** *Sample response*: Most teens spend the majority of time at school and sleeping **b)** about 3 h **c)** homework **d)** about 29% **3. a)** about 65 km **b)** No; a box and whisker plot only shows the spread of the data and not the frequency of the actual data **c)** *Sample response*: The median distance to work is about 15 km. The closest distance is 1 km and the farthest is 66 km. **4. a)** About 4800 **b)** About 15 ^{o}C **c)** No. *Sample response*: 200^{o}C is well beyond the last data point and you cannot be certain that the trend will continue. Also, the number of bacteria appears to reach zero before 100 ^{o}C if the trend continues.	**4. d)** *Sample response*: There is a relationship between the number of bacteria and temperature — as the temperature increases, the number of bacteria decreases. **5.** No; she has based her conclusion on the shape of the second graph, which does not start at Nu 0, compared to the shape of the first graph which does start at Nu 0. The graphs actually show the same increase in profit over this time period for both years. **6.** No. *Sample response*: While the graph shows there are more terrestrial species in danger of extinction than marine, you do not know the total numbers of each type of species, including those that are not in danger of extinction. **7.** *Sample response*: On average, Brand B has more raisins per box than Brand A. But it is possible to get a Brand A box with more raisins, for example a Brand A box could have as many as 30 raisins and a Brand B box could have as few as 22 raisins. It would be important to know how many boxes of raisins were actually sampled in order to know whether to trust the data shown in the graph. **8.** *Sample response*: Circle graphs, bar graphs and double bar graphs. These are used for comparisons and do not show relationships between variables that could lead to extrapolation or interpolation.

Supporting Students

Struggling students

Some students who do not fully understand the information that each type of graph presents will struggle with this section because it mixes all the different types of graphs together. For these students it may be useful to organize their assigned problems around each type of graph. You may need to come up with some additional problems to do this. Look to the revision questions and the unit test in this Teachers' Guide as a source for some of these.

Chapter 2 Probability

4.2.1 Determining and Comparing Probabilities

Curriculum Outcomes	Outcome relevance
9-G1 Theoretical Probability: independent and dependent variables • determine the number of possible outcomes for independent events using outcome charts, organized lists, and tree diagrams	This lesson provides a brief review of probability concepts covered in earlier classes. These focus on a combination of two independent events to set the stage for the later comparison between dependent and independent events. Students continue to see the value of organizational tools to keep track of possible outcomes.

Pacing	Materials	Prerequisites
1 h	• Dice	• theoretical probability • experimental probability

Main Points to be Raised

Outcome charts, lists, and tree diagrams help organize outcomes when dealing with two independent events so that the determination of theoretical probabilities is possible or easier.

Try This—Introducing the Lesson

A and B. Students can work on **parts A and B** with a partner.

Observe while students work. You might ask:

• *What information about the situation did you use to make your prediction?* (My prediction is I: Both numbers will be even. You can only make 8 a few ways: 2 + 6, 3 + 5, and 4 + 4. There are more ways of getting two even numbers: 2 + 2, 2 + 4, 2 + 6, 4 + 4, 4 + 6, and 6 + 6.)

• *Was the experimental result what you predicted?* (No, I predicted that there would be more pairs of evens, but there were more pairs with sums of 8.)

• *Would your results be the same if you performed the experiment more times?* (No, I think they will be different. I just happened to get many sums of 8 in my first set of trials.)

The Exposition—Presenting the Main Ideas

• The main ideas section is written to remind students that experimental and theoretical results may be different, but also to focus on how the information is organized. You may want to use dice to actually perform the experiment, to show that your results may be different from those in the book, because of the randomness of experimental results.

• Have students read through the exposition to review the ideas presented.

Revisiting the Try This

C. This question allows an opportunity to make a formal connection between what was done in **parts A and B** and the concepts of the contrast between theoretical and experimental results, as well as the calculation of probabilities.

Using the Examples

• Display both problems, **parts a) and b)**, from the example on the board. Have pairs of students try to determine the results. Once students are done, have a discussion to compare results. Allow at least some students to explain how they achieved the results they did.
• Suggest to students that they might refer to the examples as they work through the exercises. Assist the students with the idea of setting up an outcome chart and then using it to determine the desired probability.

Practising and Applying

Teaching points and tips

Q 1d and Q 2d: Some students may forget to exclude situations where the quotient is exactly 2. Note that when the values are equal, the quotient is 1 and it does not matter which value is taken as being greater/lesser.

Q 3: Observe whether students recognize that situation **part d**) is a complementary event to the situation in **part b)** and so the probabilities add to 1.

Q 6: This question is designed to help students consider situations where outcomes are not all equally likely. They will need to find a way to create equivalent equally likely outcomes (for example, by creating a 5 by 5 chart with 2 columns and 2 rows labelled 2 and the remaining columns/rows each labelled 3, 4, and 5) or find a way to carefully weight the outcomes properly.

Q 7: Students can refer back to the earlier problem in **question 5** to recognize why they might associate a probability in 25ths with a spinner with 5 sections.

Common Errors

Many students will forget to include both cases when a situation occurs. For example, if they are rolling two dice and looking for a sum of 4, they will forget to count both 1, 3 and 3, 1. If students are encouraged to use charts, this is less likely to occur. Indicate that rolling a 1 and then a 3 is different from rolling a 3 and then a 1 (or rolling a 1 on a red die and a 3 on a blue die is different from rolling a 3 on the red die and a 1 on the blue die).

Suggested assessment questions from Practising and Applying

Question 2	to see if students can distinguish and reconcile between theoretical and experimental probability
Question 5	to see if students can organize all possible outcomes to determine probability
Question 7	to see if students can work backwards to create an event with a particular probability
Question 9	to see if students can communicate about the value of organizational tools to consider compound events

Answers

A. *Sample response*:
I, since one-quarter of the time you should get two even rolls, but a sum of 8 won't happen that often.

B. *Sample response*:
Both even: 10 times; Sum of 8: 4 times
My prediction was correct.

C. i) for both evens: $\frac{9}{36}$ or $\frac{1}{4}$; for sum of 8: $\frac{5}{36}$

ii) *Sample response*:
My results were fairly close to the theoretical ones since $\frac{10}{36} \approx \frac{9}{36} = \frac{1}{4}$ and $\frac{4}{36} \approx \frac{5}{36}$.

Answers [Continued]

1. a) $\frac{15}{36}$ or $\frac{5}{12}$

b) $\frac{16}{36}$ or $\frac{4}{9}$

c) $\frac{17}{36}$

d) $\frac{12}{36}$ or $\frac{1}{3}$

2. a) *Sample response*: $\frac{5}{30}$; very low

b) *Sample response*: $\frac{11}{30}$; a bit low

c) *Sample response*: $\frac{18}{30}$; a bit high

d) *Sample response*: $\frac{11}{30}$; a bit low

3. a) $\frac{16}{49}$

b) $\frac{25}{49}$

c) $\frac{9}{49}$

d) $\frac{24}{49}$

4. a) $\frac{7}{64}$

b) $\frac{52}{64}$ or $\frac{13}{16}$

c) $\frac{16}{64}$ or $\frac{1}{4}$

5. a) $\frac{9}{25}$

b) $\frac{10}{25}$ or $\frac{2}{5}$

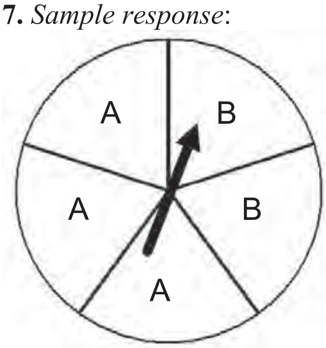
c) $\frac{3}{25}$

6. a) $\frac{4}{25}$

b) $\frac{9}{25}$

c) $\frac{3}{25}$

7. *Sample response*:



8. *Sample response*:
Outcome A might be a multiple of 3 because the probability of rolling a multiple of 3 twice is $\frac{4}{36}$.

9. A chart, tree diagram, or list makes it easier to be sure that each possible outcome is considered because it forces you to think systematically.

Supporting Students

Struggling students

For students who are struggling with compound events, spend more time working with simple events. Gradually change to a situation involving compound events, but with not too many possible outcomes for each event. For example, you might spin a spinner with only 3 equal sections twice.

4.2.2 Calculating Probability of Two Independent Events

Curriculum Outcomes	Outcome relevance
9-G1 Theoretical Probability: independent and dependent variables • calculate the probability of two independent events, A and B, as P(A) × P(B) • determine the number of possible outcomes for independent events using outcome charts and tree diagrams	Students need to have systematic ways (tree diagrams and outcome charts) to determine the total number of possible outcomes and the number of ways an event can occur. As the situations become more complicated, these methods become too large and cumbersome to be effective. It is more direct and efficient to calculate using the probability of each event.

Pacing	Materials	Prerequisites
1 h	None	• theoretical probability for a single event • dependent and independent events

Main Points to be Raised

• Multiplying the probabilities together for two events will give you the probability that they both occur if the two events are independent.
• In this lesson, the focus is on independent events. If the events are independent, then P (both A and B occur) = P(A) × P(B).
• Tree diagrams and outcome charts are useful because they display the total number of possible outcomes as well as the number of ways a particular event can occur.

Try This—Introducing the Lesson

A. Students can work on **part A** alone or with a partner.

Observe while students work. You might ask:

• *What constitutes an outcome in this case?* (The outcome is the pair of numbers that show up when the dice are rolled.)

• *What is the difference between an even number and an odd number?* (Even numbers are divisible by two but odd numbers have a remainder of one when divided by two.)

B. *Why would you expect the same probability for two odds as for two evens?* (Evens and odds each have the same chance of being rolled, so it makes sense that it should also be true for rolling two of them.)

The Exposition—Presenting the Main Ideas

Have the students read the exposition individually and then ask them to form pairs. Ask the pairs to create a tree diagram and an outcome chart for this situation: tossing a coin and spinning a spinner divided into 4 equal sections. One student should make the outcome chart and the other the tree diagram. Have them compare their results and discuss how they are the same and how they are different. Is one more useful than the other?

Revisiting the Try This

C and D. These questions allow opportunities to make a formal connection between outcome charts and tree diagrams and also to make the connection between the probability of two events occurring together and the product of their individual probabilities.

Using the Examples

Have each student in groups of three take responsibility for one of the solutions. Each student then explains his or her solution to the others.

Practising and Applying

Teaching points and tips

Q 2b and c: Do students notice that these parts describe the same situation?

Q 4: A 12-sided die behaves like a 6-sided die. The die is a dodecahedron with each side associated with a number from 1 to 12. Each number has a 1 in 12 chance of turning up on each roll.

Q 5 and Q 7: These can be solved without creating tree diagrams or outcome charts. The products of the probabilities of the separate events in each pair must be calculated.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can create an outcome chart for a described situation
Question 2	to see if students can create and apply a tree diagram for a described situation
Question 6	to see if students can calculate the probability of two independent events
Question 11	to see if students can recognize the error in a given calculation for a pair of dependent events

Answers

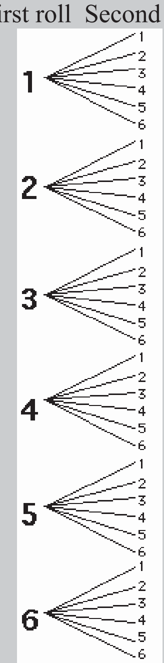
A. i) 36

ii) 9; 9

B. P(even, even) = $\frac{9}{36} = \frac{1}{4}$

P(odd, odd) = $\frac{9}{36} = \frac{1}{4}$

C. i) First roll Second roll



ii) They would be the same as the ones he did with the chart.

D. They are the same.

1.

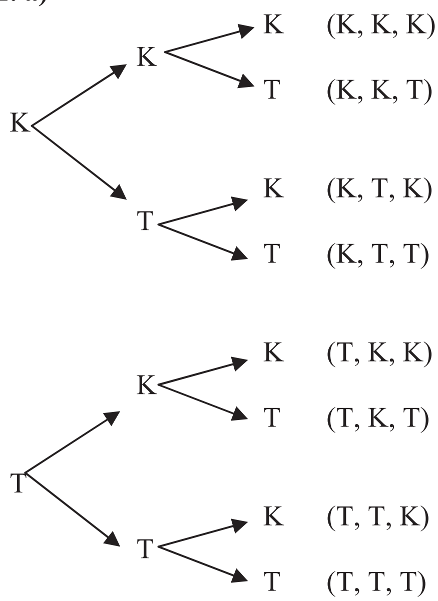
	1	2	3	4	5	6	7
K	(1,K)	(2,K)	(3,K)	(4,K)	(5,K)	(6,K)	(7,K)
T	(1,T)	(2,T)	(3,T)	(4,T)	(5,T)	(6,T)	(7,T)

a) $\frac{1}{14}$ **b)** $\frac{3}{14}$

1. c) $\frac{6}{14}$ or $\frac{3}{7}$

d) $\frac{2}{14}$ or $\frac{1}{7}$

2. a)



b) P(one K) = $\frac{3}{8}$ **c)** P(two Ts) = $\frac{3}{8}$

d) P(no Ts) = $\frac{1}{8}$

e) *Sample response*:
What is the probability of getting more Tashi Ta-gyes than Khorlos when tossing three coins? ($\frac{4}{8}$ or $\frac{1}{2}$)

3. a) $\frac{20}{81}$ **b)** $\frac{25}{81}$

4. a) P(two 8s) = $\frac{1}{144}$ **b)** P(two odds) = $\frac{36}{144}$ or $\frac{1}{4}$

c) P(a number greater than 3, a number less than 6) = $\frac{45}{144}$ or $\frac{15}{48}$

5. Game 1
In Game 1, the probability of not spinning a 1 on each spin is $\frac{2}{3}$ and the spins are independent. So, for

5. [Cont'd] Game 1, P(scoring) = $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ = 0.444.
In Game 2, the probability of spinning less than 6 on each spin is $\frac{5}{8}$ and the spins are independent. So, for Game 2, P(scoring) = $\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$ = 0.391.
He has a better chance with Game 1.

6. a) $\frac{1}{36}$ **b)** $\frac{1}{36}$ **c)** $\frac{9}{36}$ or $\frac{1}{4}$ **d)** $\frac{18}{36}$ or $\frac{1}{2}$

e) $\frac{3}{36}$ or $\frac{1}{12}$ (since the green die must show a 4, 5, or 6 to make the sum for the two dice greater than 7)

7. a) $\frac{1}{2704}$ **b)** $\frac{32}{2704}$ or $\frac{2}{169}$ **c)** $\frac{169}{2704}$ or $\frac{1}{16}$

8.

	50	100	200	1000
50	100	150	250	1050
100	150	200	300	1100
200	250	300	400	1200
1000	1050	1100	1200	2000

$\frac{2}{16}$ or $\frac{1}{8}$

9. a) *Sample response*:
a spinner with six equal sectors numbered 1 to 6
b) *Sample response*:
a spinner with ten equal sectors numbered 1 to 10
c) a spinner with no number 6 on it
d) *Sample response*: a spinner with two equal sectors with 6 on one and any number on the other.

10. $\frac{100}{900}$ or $\frac{1}{9}$

11. Since the events are dependent you cannot calculate by multiplying the probability of each event together as if they are independent events. When the red cube is not returned, that means that there are only four choices for the second cube and the probability of drawing a blue one is $\frac{2}{4}$.

Supporting Students

Struggling students

It may be useful for some students to create a tree diagram for each situation with the associated probabilities for each event written on each branch. This model can be used to answer any questions related to the given situation.

Enrichment

• You could extend these types of problems to three or more independent events. The process is the same and the products of the individual probabilities gives you the probability that all of the events will occur simultaneously.
• Students might be interested in how the probability of dependent events is calculated. If they are dependent then P (both A and B occur) = P(A) × P(B | A), where P(B | A) is the conditional probability (the probability that B will occur, given that A has occurred).

GAME: On a Roll

• The object of this game is to score points by making guesses based on probability. Each round begins with players rolling two dice and calculating the sum of the numbers they have rolled. Players then predict whether the sum of the next roll will be greater than, less than, or equal to this sum. A correct prediction scores a point. This continues until a player reaches ten points and is declared the winner.

• Students can play this game after they demonstrate an understanding of how probability works and, in particular, the probability of particular numbers coming up when two dice are rolled.

• Students may use a simple strategy not based on any theory. For example:
- If the sum of one roll of the dice is low, they may predict that the sum of the next roll will be greater than the current sum. If the sum is great, they may predict that the sum of their next roll will be less than the current sum. Finally, they rarely predict that the sum of the next roll will be equal to the current sum, thinking that the probability of two consecutive sums being equal is low.
- Other students may see that the game is based on theoretical probability. Only certain combinations of numbers can be rolled with two dice and, thus, only certain sums can result.
- Students can make educated guesses based on the number of sums that are less than or greater than a sum rolled.
- Students can also see the theoretical probability of rolling two consecutive equal sums and notice that the probability is greater for some numbers than for others.

Watch for students who do the following:

• create a table to help make their predictions

• are able to tell you on what basis they are making their predictions

• base their guesses on a "hunch," and cannot explain how probability affects their predictions

After the game, you might ask students the following:

• *Is the experimental probability of the result of the next turn the same as the theoretical probability? Explain.*

• *What part does chance or "a lucky roll" play in this game?*

• *For which sum is the probability the greatest that you would roll an equal sum on the next turn? Explain.*

• *For which sums is the probability only 1 out of 36? What are the numbers rolled on the dice for those sums?*

• *Which sums have the greatest probability of being rolled? What are the numbers rolled on the dice for those sums?*

4.2.3 Randomness: Experimental Versus Theoretical Results

Curriculum Outcomes	Outcome relevance
9-G2 Simulations and Experiments: dependent and independent events • distinguish between theoretical and experimental probability	Students will eventually conduct simulations to determine estimates of the likelihood that a series of events will occur. To do this, they need to understand the difference between theoretical and experimental probability and how they are related to each other.

Pacing	Materials	Prerequisites
1 h	Coins, spinners, cards, dice	• ability to calculate theoretical probability • ability to calculate experimental probability

Main Points to be Raised

• The theoretical probability and experimental probability of the same event are not always the same due to randomness. However, if you increase the number of trials over which you conduct an experiment, the experimental probability will approach the theoretical probability.

• Experimental probability can be deceptive. If only a few attempts or trials have been conducted the results may not give you a good indication of what is likely to happen. That is why it is important to conduct a large number of trials.

• The formulas for theoretical probability and experimental probability are similar. Theoretical probability is based on the ratio of favourable outcomes to possible outcomes. Experimental probability is the ratio of the number of times an event happens to the number of times the experiment is performed.

Try This—Introducing the Lesson

Students can work on **part A** alone or with a partner.

Observe while students work. You might ask:

• *What type of events are you dealing with in this situation?* (Spinning white and then spinning grey are independent events.)

• *What is the probability of spinning white for each spin? grey?* ($\frac{1}{2}$ for both)

• *What is the ratio of white then grey spins to the total number of spins?* ($\frac{10}{20}$ = 50%)

• *What could account for the difference between Maya's experimental probability and the theoretical probability?* (This is due to randomness and the small number of spins that were used in the experiment.)

The Exposition—Presenting the Main Ideas

• Divide the class into pairs. Give each pair of students a die. Have them roll the die ten times and record the number of times they get an even number. Using their results, have them determine the experimental probability of rolling an even number and compare this to the theoretical probability of rolling an even number.

• Ask each pair of students for their results and as a class record the number of times an even number came up for each group. Use the combined results to determine the experimental probability and compare it with the theoretical probability–discuss what has happened. If you have time, have each pair roll the die ten more times and add each group's results to the previously collected data. Recalculate the experimental probability and discuss the new results.

• Talk to the students about the difference between experimental and theoretical probability and ask them to read the exposition.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** and the main ideas in the Exposition. Increasing the number of spins will give Maya better results that more closely resemble the theoretical probability.

Using the Examples

As you discuss the example with the students, it might be useful to pass around some decks of cards for them to examine if they are unfamiliar with cards. Ask the students the following questions:

- *If the experiments were performed a second time in* ***part b*** *would they obtain the same results? Explain.*
- *How many students combined their data in* ***part c****? How do you know?*
- *Explain what would happen to the difference between the experimental probability and the theoretical probability if 20 students combined their data.*

Practising and Applying

Teaching points and tips

Q 3: Students may want to use the examples posed but they are also free to create their own situations.

Q 4: Comparing fractions with different denominators is difficult unless a common denominator is found. This is not necessary when probabilities are expressed as decimals and percentages.

Common Errors

Some students will think that the data collected will be very similar each time when an experiment is conducted. They need to understand that this is not the case. Use the strategy presented in the main ideas above to illustrate how the results differ from trial to trial by comparing the results of each group in the class. This will reinforce the need to conduct an experiment many times to gather enough data to make the experimental probability a reasonable indicator of the likelihood that an event will occur.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can correctly use the appropriate formulas for determining experimental and theoretical probability
Question 2	to see if students understand how theoretical and experimental probability are related
Question 5	to see if students can justify why theoretical probability is the best indicator of the likelihood that an event will occur

Answers

A. i) *Sample response*:
She knew that the events were independent so she could multiply the probabilities of the events together.
P(white) = $\frac{1}{2}$ and P(grey) = $\frac{1}{2}$ so P(white, then grey) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

ii) No. She spun white then grey ten times out of 20 spins, or 50% of the time.

iii) She got the results she did because where the spinner stops is up to chance or randomness. Just because she expects to get white then grey 25% of the time does not mean this will actually happen in a small number of trials.

B. i) They are different because the experimental probability changes each time you conduct the experiment. The results are up to chance. If she spun the spinner 20 times, then another 20 and so on, the number of times out of 20 she got white then grey could be different each time. Theoretical probability predicts what should happen over the long run, not what will happen in a small number of trials.
ii) She could increase the number of spins. The more times she spins the spinner, the more closely the experimental probability will approach the theoretical probability.

1. a) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ or 44.4% **b)** about 4 or 5 times **c)** The experimental probability is $\frac{4}{10}$ or 40%, which is close to but not the same as the theoretical probability. **d)** No; due to randomness, the experimental probability can vary every time she repeats the experiment. **2.** C and D C: Just because the experimental and theoretical probabilities are different does not mean they are incorrect. Experimental and theoretical probabilities often differ due to chance. D: It is likely that the experimental and theoretical probabilities would be closer if he rolled the dice 100 times because it's a greater number of trials. **3. a)** *Sample response*: My events are to draw a club and then draw a 6 from a deck of cards. The first card is replaced before the second draw. **b)** *Sample response*: I will draw 2 cards from a deck, with replacement, 100 times.	**3. c)** *Sample response*: $\frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$ or 1.92% so I predict drawing a club and then a 6 (with replacement) two times in 100 trials. **d)** *Sample response*: Everyone has different results and some are closer to the theoretical probability than others. **e)** *Sample response*: Our combined results are closer to what we predicted because more trials are being combined. **4.** *Sample response*: When you are comparing probabilities as fractions with different denominators, it is sometimes easier if both are either decimals or percentages. **5.** The theoretical probability is a much better indicator of what will happen because it is based on two known facts: the number of ways the event can occur and the total number of possible outcomes.

Supporting Students

Enrichment

Ask students to design and carry out an experiment to estimate the probability of a thumbtack landing with its tip facing up.

4.2.4 Conducting a Simulation

Curriculum Outcomes	Outcome relevance
9-G2 Simulations and Experiments: dependent and independent events • conduct and design simple simulations involving both dependent and independent events • distinguish between theoretical and experimental probability • determine experimental probabilities for simulations	Sometimes determining the theoretical probability of an event or series of events is impossible or extremely complicated. In these situations, an experiment can be conducted to approximate likelihood of these events occurring.

Pacing	Materials	Prerequisites
1 h	• Spinners • Dice • Coins • Coloured blocks	• ability to calculate theoretical probability • ability to calculate experimental probability

Main Points to be Raised

• A simulation is an experiment designed to model a problem.
• Materials such as spinners, dice, playing cards, and slips of paper can be used to simulate probability events. Different parts of these models are used to represent favourable outcomes.
• When determining an experimental probability, it is important to use a large number of trials to make the results more reliable.

Try This—Introducing the Lesson

A. Students can solve the problem alone or in pairs.

Observe while students work. You might ask:

• *In each fraction what numbers will you use in the numerator? the denominator?* (The numerator is the number of favourable outcomes and the denominator is the total number of matches.)

• *Are the fractions you have written theoretical or experimental probabilities? Explain.* (They are all experimental probabilities because they are based on data from 100 matches. They are not based on theoretical numbers but, instead, on the results of an "experiment" which in this case is a match.)

The Exposition—Presenting the Main Ideas

• Present to the class the runner problem from the exposition and use a die to demonstrate how you could conduct a simulation for this problem. Have the students follow the steps in the textbook as you carry out the simulation.

• Discuss why you chose a die. Ask: *Could you use any other probability devices?*

• Discuss the number of trials you will use in the simulation. Ask: *Is it better to use more than 25 or fewer than 25 trials? Why?*

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between the concepts presented in the main ideas and what was done in **part A**. It also gives the teacher some insight into the students' ability to design a simulation on their own.

Using the Examples

• Present the problem in **example 1** to the class and discuss the solution that is presented. Talk about alternative ways to carry out the simulation, such as placing ten slips of paper labelled with S, R, and C in a box, or, if available, using a random number generator on a spreadsheet or calculator.

• In **example 2** you may want to point out that multiplying $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ gives the probability that two girls and a boy will be born. Because this can happen in three different orders, as shown on the tree diagram, the theoretical probability is $3 \times \frac{1}{8} = \frac{3}{8}$.

Practising and Applying

Teaching points and tips

Q 1: Encourage students to be flexible and accept solutions where a student uses an unexpected device in a reasonable way, e.g. a die for **part a)** where rolling 1, 3, or 4 is rain and 2, 5, or 6 is no rain. Be equally alert to using a device incorrectly. For example, suggesting the use of a die for **part c)** where each name is associated with the sum when the die is rolled twice would be incorrect, since the sums are not equally probable.

Q 2: Make sure students discuss the difference between the experimental and theoretical probabilities. Ask them to give reasons for the difference.

Q 4: Discuss all the different probability devices students came up with for conducting a simulation with four equally likely outcomes (a deck of cards, a 4-section spinner, 4 slips of different coloured paper, a die if the outcomes of 5 and 6 are not used, etc.).

Common Errors

When students design a simulation they often do not use enough trials to make the experimental probability a good indicator of the likelihood of the event occurring. Stress that in simulations, the more trials you use the better your results.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can choose the correct probability device to conduct a simulation for a given situation
Question 3	to see if students can design a simulation on their own for a given situation
Question 5	to see if students can make the appropriate distinctions and connections between theoretical probability, experimental probability, and simulations

Answers

A. i) about $\frac{1}{3}$ **ii)** about $\frac{1}{2}$ **iii)** $\frac{1}{4}$

B. *Sample response*:
a spinner in fourths with three sections marked “0 to 15” and one section marked “over 15”; he would spin the spinner 100 times and see what fraction of the time over 15 was spun.

1. a) any of them could be used to model 50% or $\frac{1}{2}$

b) spinner, and cards if only cards numbered 1 to 8 were included

c) dice, or cards if only cards numbered 1 to 6 were included

d) spinner, or cards

2. a) $\frac{1}{10}$ or 10% **b)** $\frac{1}{16}$ or 0.0625% **c)** $\frac{1}{16} \neq \frac{1}{10}$

3. *Sample response*:
Use a die. If you roll a 1 or a 2, it represents a win. A roll of 3, 4, 5, or 6 is a loss. Roll the die five times for each trial and do 25 trials. Count the number of trials that had three rolls of 1 or 2 out of five rolls. Create a fraction with the number of trials that had a roll of 1 or 2 three times out of 5 as the numerator and 25 as the denominator.

Answers [Continued]

4. *Sample response*: Use a spinner with four equal sections numbered 1 to 4. A spin of 2, 3, or 4 represents a successful penalty kick. A spin of 1 means a miss. Spin five times for each trial and do 50 trials. Count the number of trials a 2, 3, or 4 is spun at least four times. Create a fraction with the number of trials where a 2, 3, or 4 was spun at least four times as the numerator and 50 as the denominator.	**5.** When you conduct a simulation, you use the theoretical probability of the probability device to model the probability, but the results are the experimental probability.

Supporting Students

Struggling students

• Some students may still have difficulty identifying favourable outcomes and total possible outcomes in real-life situations. Suggest a number of sample scenarios with equally likely outcomes (e.g., the probability that a student would arrive at school before a friend arrives assuming that they are equally likely to arrive first on any day). Have students identify the favourable outcomes and the total possible outcomes and then explain how they can use those two numbers to determine the probability for the sample scenario.

• If students have difficulty understanding how a simulation can model probability, discuss how the same six-sided die can be used to model different real-life events with various probabilities. For example:

- a situation where each outcome has a 1 in 6 chance of happening (use one number to represent the favourable outcome.)
- a situation where each outcome has a 1 in 2 chance of happening (Use the numbers 1–3 to represent one outcome and the numbers 4–6 to represent the other outcome, or use even numbers to represent one outcome and odd numbers to represent the other.)
- a situation where each outcome has a 1 in 3 chance of happening (Use two numbers to represent each outcome.)

4.2.5 EXPLORE: Designing a Simulation

Curriculum Outcomes	Outcome relevance
9-G2 Simulations and Experiments: dependent and independent events • conduct and design simple simulations involving both dependent and independent events • distinguish between theoretical and experimental probability • determine experimental probabilities for simulations	In this core exploration, students not only design but they also carry out a simulation for a given situation, which includes interpreting their results.

Pacing	Materials	Prerequisites
1 h	Spinners, dice, coins, coloured blocks	• ability to calculate theoretical probability for an event • ability to calculate experimental probability for an event • understanding of how a simulation can be used to model a situation

Main Points to be Raised

• In designing a simulation, the two most critical decisions are the choice of a proper probability device for the situation and the choice of the number of trials you will conduct.

• Each pair of students may design their simulation using a different probability device and a different number of trials. However, what they define as a trial in each group should be relatively similar.

Exploration

• Ask students to work on **parts A to C** with a partner or in a small group.

• Make sure they that they choose an appropriate probability device for the situation.

Observe while students work. You might ask:
• *Why did you choose the probability device that you did?* (I wanted a spinner with 10 equal parts so I could create three unequal parts to represent 50%, 40%, and 10%.)
• *What constitutes a trial in your simulation?* (Each spin is a trial.)
• *How did you decide on the number of trials to use?* (75 trials seems like enough to get a good idea.)

Observe and Assess

As students are working, notice:
• Do they persevere and learn from experience? If one approach fails, do they realize what they need to rectify or modify in their simulation design?
• Do they generalize appropriately from their experiences?
• Do they look for commonalities in the various simulations designed by their peers?
• Are they able to identify design flaws in other students' simulations and exclude the results obtained from these to help them answer **part B** appropriately?
• Do they use their own results or a variety of results from numerous groups to answer **part C?**

Common errors

The most common errors students make are:
• improper choice of a suitable probability device
• defining a sequence of events incorrectly for a typical trial
• using too few trials in the simulation to make the results reliable
These errors can be best addressed through probing questions, for example:
• *But are all of your outcomes equally likely?*
• *What will you do if…?*

Share and Reflect

After students have had enough time to answer **parts A, B and C**, encourage different groups to come forward and describe how they designed their simulation and what they learned.

Answers

A.

Step 1

Use a spinner in tenths: 2 tenths or 20% will be labelled “less than 5 min,” 5 tenths or 50% will be labelled “between 5 and 10 min,” and 3 tenths or 30% will be labelled “more than 10 minutes.” Each spin represents being stopped by the guard one time.

Step 2

Each spin is one trial. There will be 100 trials.

Step 3 and Step 4

Use a table and a tally chart

	Less than 5 min	Between 5 and 10 min	More than 10 min
Frequency	4	56	40

Step 5

The experimental probability that the stop will be less than 5 min is $\frac{4}{100}$ or 4%; between 5 and 10 min is $\frac{56}{100}$ or 56%; more than 10 min is $\frac{40}{100}$ or 40%.

B. The results will vary.

C. I think they will be stopped for between 5 min and 10 min because the experimental probability that they will be stopped for between 5 min and 10 min is greater than 50%.

Supporting Students

Enrichment

There are all sorts of different scenarios that can be presented for students to model through simulation. Here are some additional problems.

- A company randomly places one of four prizes in each box they sell. Design a simulation to predict how many boxes you are likely going to need to buy to collect at least one of each prize.
- Design a simulation that will predict the probability that two people in a random group of five people were born in June.
- Design a simulation to determine the probability that, on a true-false quiz, you get at least five out of ten questions correct by guessing all the answers.

CONNECTIONS: Computer Simulations

A computer simulation or computer model is a computer program that attempts to simulate a particular system using an abstract model. These are used in physics, chemistry, biology, economics, psychology, social science, and engineering. Traditionally, the formal modelling of systems has been via a mathematical model, which attempts to find analytical solutions to problems. The model enables the user to predict the behaviour of the system from a set of parameters and initial conditions. Computer simulations build on, and are a useful adjunct to, purely mathematical models.

Answers

1. Scientists conduct computer simulations to look at patterns of events and to make predictions. They are able to vary the parameters of a computer simulation to see how this will affect the outcome and thus discover how best to deal with situations that could be modified to have more favourable outcomes.

2. There are many possibilities. Here are a few:
- weather patterns, such as temperature trends (global warming and its impact on the ecosystems of the world)
- hurricane and typhoon models, to predict the intensity and the path of such storms

UNIT 4 Revision

Pacing	Materials
1 h	• Rulers • Protractors • Compasses • Graph paper • Dice or 6-section spinners

Question	Related Lesson
1	Lesson 4.1.1
2	Lesson 4.1.3
3	Lesson 4.1.2
4	Lesson 4.1.3
5	Lesson 4.1.4
6	Lesson 4.1.5
7	Lesson 4.2.1
8	Lesson 4.2.2
9	Lesson 4.2.2
10	Lesson 4.2.2
11	Lesson 4.2.2
12	Lesson 4.2.2
13	Lesson 4.2.3
14	Lesson 4.2.4
15	Lesson 4.2.4
16	Lesson 4.2.4
17	Lesson 4.2.4

Revision Tips

Q 1: Accept a variety of responses. For example, **1a** or **1b** could be modelled by a circle graph or a bar graph. If the student's choice is unexpected, ask him or her to explain. The explanation may show some logical thinking, even if it was an unusual approach.

Q 6: You might refer students back to **lesson 4.1.5** to recall a similar problem (also **question 6**) involving animal species.

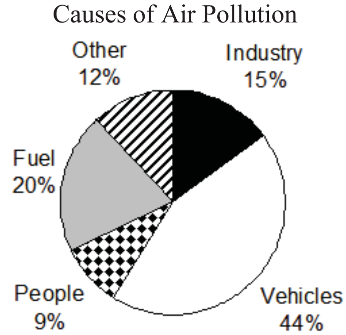
Q 12: Some students might want to use an outcome chart to help them see the situation more clearly.

Q 11: Students need to realize that all sections on the spinners are congruent.

Answers

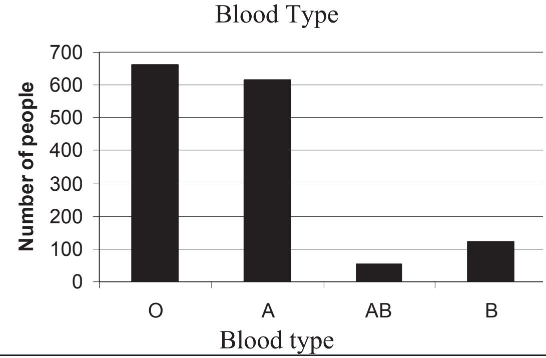
1. a) *Sample response*:

Causes of Air Pollution



1. b) *Sample response*:

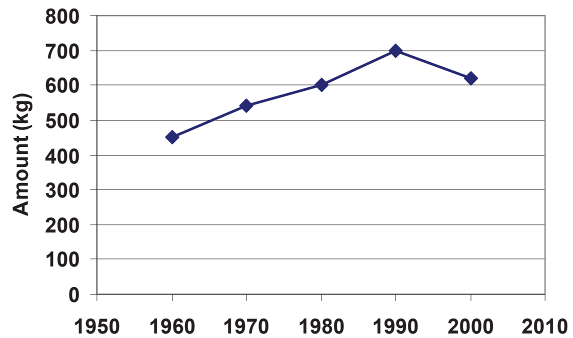
Blood Type



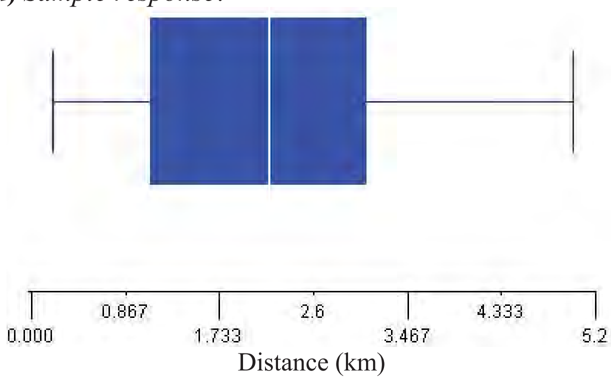
Answers [Continued]

1. [Cont'd] **c)** *Sample response*:

Amount of Garbage Produced Per Person

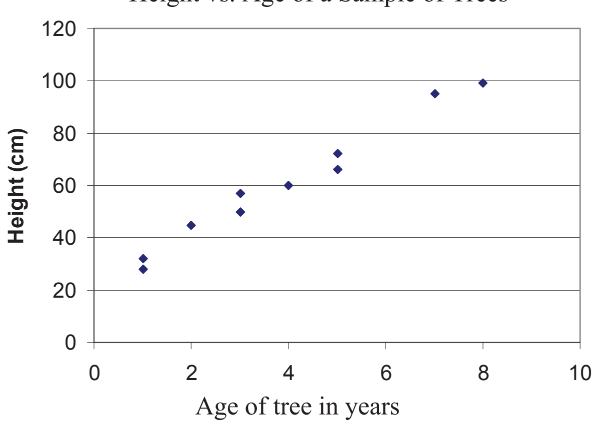


d) *Sample response*:



e) *Sample response*:

Height vs. Age of a Sample of Trees



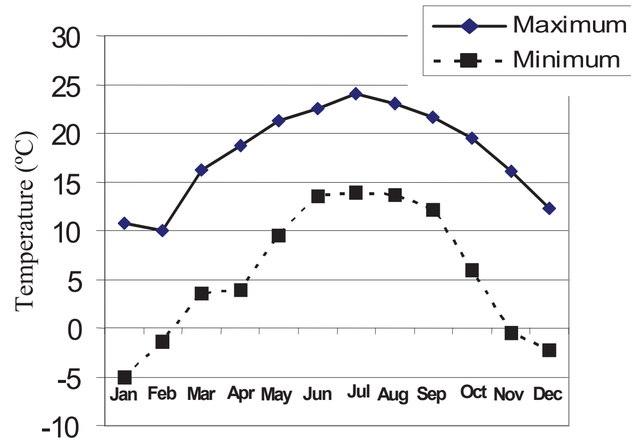
2. Disagree; a broken line graph is a poor choice because the data is in categories and it is not about a trend over time. A circle graph or bar graph would be better.

3. a) You are not given all countries of the world so you cannot represent the data as percentages or parts of a whole.

b) *Sample response*:

A bar graph would allow you to compare the number of vehicles per thousand people for the countries listed.

4. Average Maximum and Minimum Temperatures in Bumthang



5. *Sample response*:

- Wrong use of scale
- Misplaced zero point
- Wrong choice of graph type
- Improper use of shading and visual effects

6. No; he has interpreted the graph incorrectly. He can conclude that there are a relatively small number of people living in forest ecosystems in North America compared to other parts of the world but the graph gives no information about the amount of forest ecosystems in these regions.

7. a) C

b) Previous rolling of a die has no effect on the next roll. Each roll is independent of all other rolls.

8. 2 times because $P(7) = \frac{1}{6}$ and $P(10) = \frac{3}{36}$ or $\frac{1}{12}$

9. a) $\frac{1}{8}$

b) $\frac{1}{4}$

c) $\frac{1}{26}$

d) $\frac{1}{104}$

10. $\frac{1}{169}$

11. P(same) = $\frac{7}{30}$

12. a) $\frac{4}{45}$ **b)** $\frac{8}{75}$ **c)** $\frac{4}{45}$ **d)** $\frac{1}{9}$

13. C; the difference becomes less because the experimental probability approaches the theoretical probability as the number of trials increases

14. Yes, the experimental probability could be anything. If it turns out to be zero it means that the event has not occurred in any of the trials conducted. If you increase the number of trials then the experimental probability will approach the theoretical probability.

15. a) *Sample response*:
Toss a coin. Toss the coin to simulate a person walking into the room. If a K comes up count it as female. If a T comes up count it as male. Toss the coin four times for each trial and conduct ten trials. The experimental probability is the number of trials where K is tossed all four times divided by 10, the total number of trials.

b) $\frac{1}{16}$

16. a) *Sample response*:
Roll a die. Roll the die to simulate playing one game. If 1, 2, or 3 comes up, Lema wins. If 4, 5, or 6 comes up, Maya wins. Roll the die four times for each trial and conduct 25 trials. Count the number of trials where one of the players wins in all four rolls in a trial. The experimental probability is the number of trials where one player wins in all four rolls divided by 25, the total number of trials.

17. a) *Sample response*:
Roll a die. If a 1 comes up, it means she wins. If 2 to 6 come up, she loses. Roll the die twice for each trial and conduct 50 trials. The experimental probability is the number of trials where a 1 is rolled in both rolls divided by 50, the total number of trials.

b) P(wins 2 in a row) = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

c) The two probabilities will differ and the amount of difference will depend on the number of trials used in the simulation. The greater the number of trials, the closer will be the experimental probability to the theoretical probability.

UNIT 4 Data and Probability Test

1. For each set of data suggest an appropriate graph you could use to represent it. Justify your decision.

a) Types of Food Dorji Ate Last Week

Type of food	Amount (%)
Milk products	15
Grain products	42
Vegetables and fruits	30
Meat products	13

b) Average Monthly Rainfall (mm) in Kathmandu

Jan	Feb	Mar	Apr	May	Jun
15	41	23	58	122	246

Jul	Aug	Sep	Oct	Nov	Dec
373	345	155	38	8	3

c) Percent of World's Land and Population

Continent	Percent of land	Percent of population
Asia	30	60
Africa	20	12
N. America	16	8
S. America	12	5
Antarctica	9	0
Europe	7	14
Australia	5	0.3

d) Average Monthly Temperatures in Bumthang

	Jan	Feb	Mar	Apr	May	Jun
Max	10.8	10.0	16.2	18.7	21.3	22.5
Min	–5.1	–1.4	3.5	3.9	9.5	13.5

	Jul	Aug	Sep	Oct	Nov	Dec
Max	24.1	23.0	21.6	19.5	16.1	12.3
Min	10.9	13.7	12.1	5.9	–0.5	–2.3

e) Times (s) for 20 Runners to Run 100 m

12.8	12.1	13.5	11.8	13.2
12.6	12.3	13.0	11.9	11.5
12.5	12.7	13.9	14.0	13.2
11.8	12.0	13.1	13.8	12.4

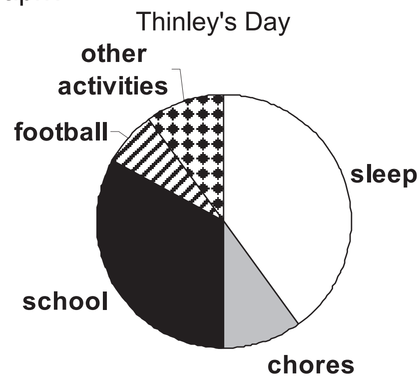
2. a) Use the data from **question 1b)** to construct a graph.

b). What conclusions can you make?

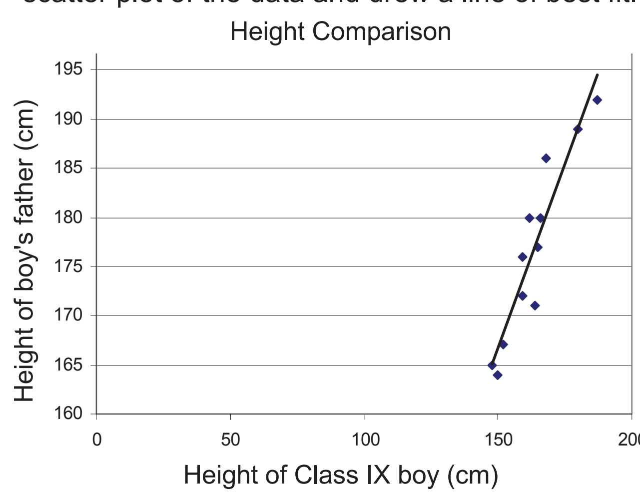
3. a) Use the data from **question 1e)** to construct a graph.

b) What conclusions can you make?

4. What conclusions can you make from the data in this graph?



5. Dodo measured the heights of a sample of Class IX boys and their fathers. He created a scatter plot of the data and drew a line of best fit.



a) What conclusions can you make?

b) Use the graph to predict the height of a boy's father if the boy's height is 175 cm.

c) Use the graph to predict the height of a boy if his father's height is 170 cm.

d) Use the graph to predict the height of a boy if his father's height is 200 cm.

6. a) Yuden claims that the graph in **question 5** is misleading. Is she correct? Explain.

b) Identify two additional causes that lead to misleading graphs.

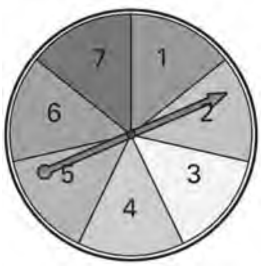
7. Seldon rolls two dice. He says that the probability that the difference between the two numbers rolled will be even is the same as the probability that the difference will be odd. Is he right? How do you know?

8. The spinner below is spun and the coin is tossed. Use an outcome chart to determine each probability.

a) P(6 and Tashi Ta-gye)

b) P(odd and Khorlo)

c) P(less than 5 and Khorlo)



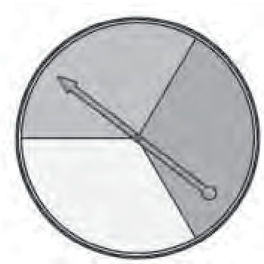
9. Create a tree diagram to determine the probability of rolling a sum of 9 on two rolls of a die if one of the rolls is a 4.

10. Determine the probability of tossing a Khorlu face up and rolling a number divisible by 3 on a die at the same time.

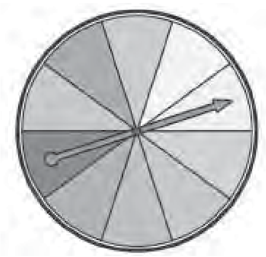
11. Kezang conducted a simulation to determine the experimental probability of an event. He also calculated the theoretical probability. He found the two results to be quite different. Explain why this happened.

12. Which spinner(s) could be used to simulate the experimental probability for each situation described below? Explain.

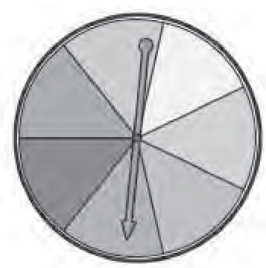
Spinner A



Spinner B



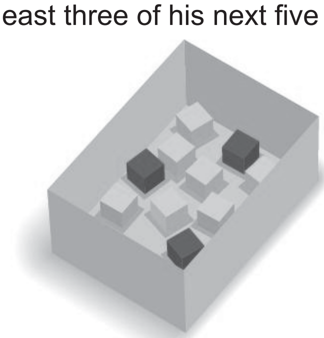
Spinner C



a) The probability that your new baby brother will be born on a Tuesday (assume every day is equally likely)

b) The probability that Nado wins three of the next five games if Leki has a 40% chance of beating Nado in a game of Khuru

13. Thinley knows from past archery competitions that he will win his match about 70% of the time. Explain how these cubes can be used to conduct a simulation to determine the probability that he will win at least three of his next five matches.



UNIT 4 Test

Pacing	Materials
1 h	None

Question(s)	Related Lesson(s)
1, 2, 3	Lessons 4.1.1 to 4.1.3
4, 5	Lesson 4.1.5
6	Lesson 4.1.4
7	Lesson 4.2.1
8, 9, 10	Lesson 4.2.2
11	Lesson 4.2.3
12, 13	Lesson 4.2.4 and 4.2.5

Select questions to assign according to the time available.

Answers

1. a) bar graph – he may have eaten other foods so a circle graph is not appropriate.

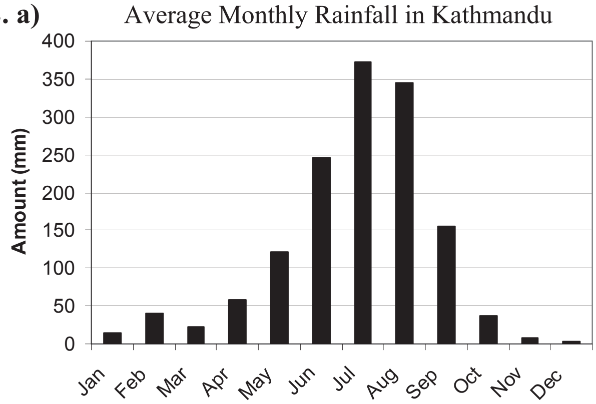
b) broken line graph – time is the independent variable and you want to look for trends

c) double bar graph – you want to compare the data and it is in categories

d) multiple broken line graph – you want to look for trends and compare the two sets of data

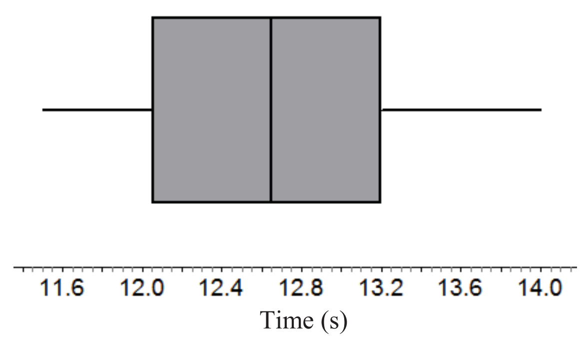
e) box plot – you can compare how the times are distributed about the median

2. a) Average Monthly Rainfall in Kathmandu



b) The highest monthly rainfall occurs in July and August. The driest months are November and December.

3. a)



b) The median time is about 12.6 s and the range is 2.5 s. About 50% of the data lies between 12.1 s and 13.2 s.

4. Most of his day is spent at school and sleeping. He spends about an equal amount of time between football, chores and other activities.

5. a) There exists a linear relationship between the height of class IX boys and the heights of their fathers. As the fathers' heights increase, so do the heights of their sons.

b) about 185 cm

c) about 155 cm

d) about 195 cm

6. a) Yes. The vertical scale does not start at zero. It starts at 160 cm. This overemphasizes the steepness of the line of best fit.

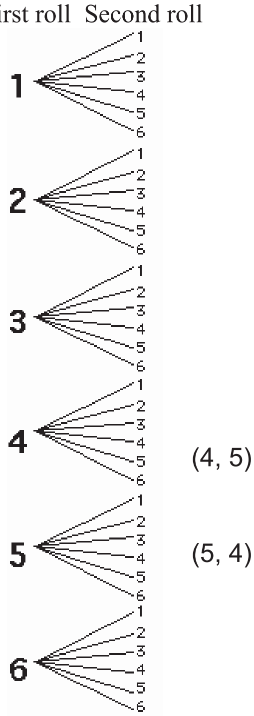
b) *Sample responses*: poor use of scale, inconsistent shading or visual effects, wrong choice of graph type

7. Yes; I drew an outcome chart and listed all the possible differences. There were the same number of differences that were even as there were differences that were odd.

8. a) $\frac{1}{14}$ **b)** $\frac{2}{17}$ **c)** $\frac{2}{7}$

9. $\frac{1}{18}$

First roll Second roll



10. $\frac{1}{6}$

11. *Sample response*:
A simulation only provides an estimate of the likelihood that an event will occur. This estimate is dependent on the number of trials conducted in the simulation. There is no guarantee that the experimental probability will be close to the theoretical probability even with a lot of trials.

12. a) C, since there are seven equal sections

b) B, since there are ten equal sections so you can show 40%

13. *Sample response*:
Draw a cube from the box to represent a match. If a white cube is selected he wins the match. If a black cube is selected he loses the match. Replace the cube and draw again. Repeat this process until you have drawn five cubes. Record the number of white cubes selected in the five draws. This constitutes one trial. Repeat for ten trials. From the ten trials count the number of times a white cube was drawn at least three times out of five. This number out of ten is an estimate of the probability he will win at least three of his next five matches.

UNIT 4 Performance Task — Free Throw

In basketball practice, Rishi has the following free-throw accuracy:

• Standing 5 m from the basket, he is able to sink the ball in the basket 50% of the time.

• From 10 m, he is able to sink the ball half as often as from 5 m.

• From 15 m, he is able to sink the ball with the same frequency he does from 10 m.

In the tryouts for the school team, Rishi will take one shot from each distance.

A. Predict the probability that Rishi will sink all three shots.

B. Design a simulation to model Rishi's chances of sinking all three shots. Explain why your choice of probability model is appropriate.

C. Use your simulation to calculate the experimental probability that all three shots will go in.

D. Create a suitable graph that compares the experimental probability you found with the given probabilities from the three different distances.

E. Compare your predicted probability with the experimental probability. Which do you think would be the more likely result if you did the simulation again?

UNIT 4 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-F5 Displaying Data: most appropriate methods **9-F6** Displaying Data: draw inferences and make predictions **9-G1** Theoretical Probability: independent and dependent events **9-G2** Simulations and Experiments: dependent and independent events	1 h	Probability devices such as spinners, dice, and cards

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used as enrichment material for some students.
You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

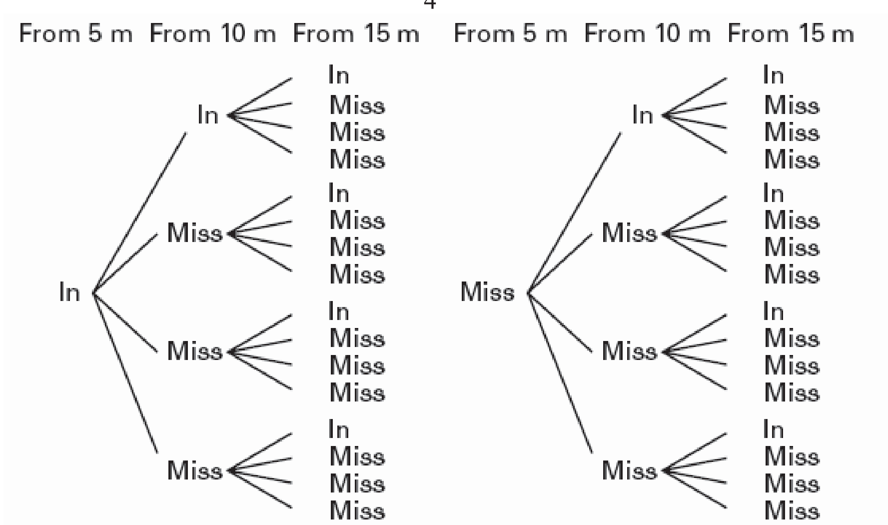
Sample Solution

A. *Using a tree diagram*:

- From 5 m he has a 1 in 2, or $\frac{1}{2}$ chance of sinking the shot.

- From 10 m he has a 1 in 4, or $\frac{1}{4}$ chance of sinking the shot.

- From 15 m he also has a 1 in 4, or $\frac{1}{4}$ chance of sinking the shot.



The tree diagram shows 32 possible outcomes and one favourable outcome, so the probability that Rishi will sink all three shots is $\frac{1}{32}$.

Calculating probability:
Assuming the three shots are all independent events, then the probability he sinks all three shots is
$\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$.

Sample Solution [Continued]

B. *Sample response*:
I can make three spinners.
- I will divide the first spinner into two equal sections, labelling one section “In” and the other section “Miss.”
- I will divide the second spinner into four equal sections, labelling one section “In” and the other three sections “Miss.”
- I will divide the third spinner into four equal sections, labelling one section “In” and the other three sections “Miss.”

I will spin each spinner in turn and do 20 trials altogether.
My simulation is appropriate because it correctly models the theoretical probability of sinking the shot from 5 m, 10 m, and 15 m. By spinning each spinner in turn, I can model the shots in turn.

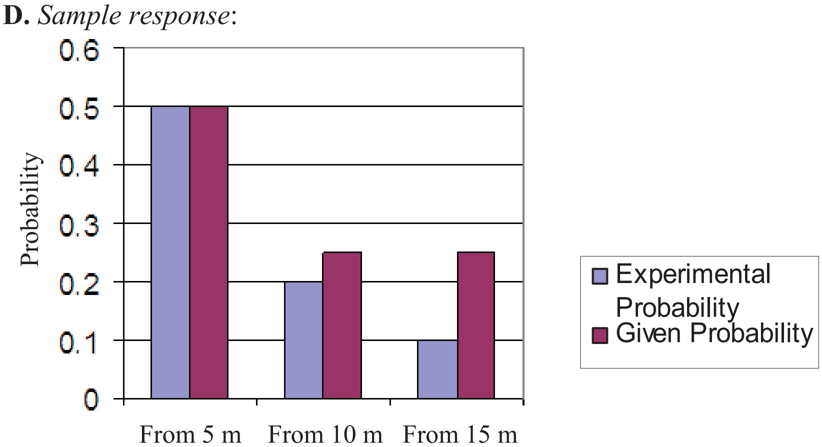
C. *Sample response*:

Trial	Spinner 1		Spinner 2		Spinner 3	
	In	Miss	In	Miss	In	Miss
1	✓		✓			✓
2		✓		✓		✓
3		✓	✓			✓
4	✓			✓		✓
5	✓			✓		✓
6	✓			✓		✓
7		✓		✓		✓
8	✓		✓			✓
9		✓		✓		✓
10	✓			✓	✓	

Trial	Spinner 1		Spinner 2		Spinner 3	
	In	Miss	In	Miss	In	Miss
11	✓		✓		✓	
12		✓		✓		✓
13	✓			✓		✓
14		✓		✓		✓
15	✓			✓		✓
16	✓			✓		✓
17		✓		✓		√
18		✓		✓		✓
19		✓		✓		✓
20		✓		✓		✓

The experimental probability of sinking all three shots is $\frac{1}{20}$.

D. *Sample response*:



E. *Sample response*:

The predicted or theoretical probability is $\frac{1}{32}$, which is lower than the experimental probability of $\frac{1}{20}$.

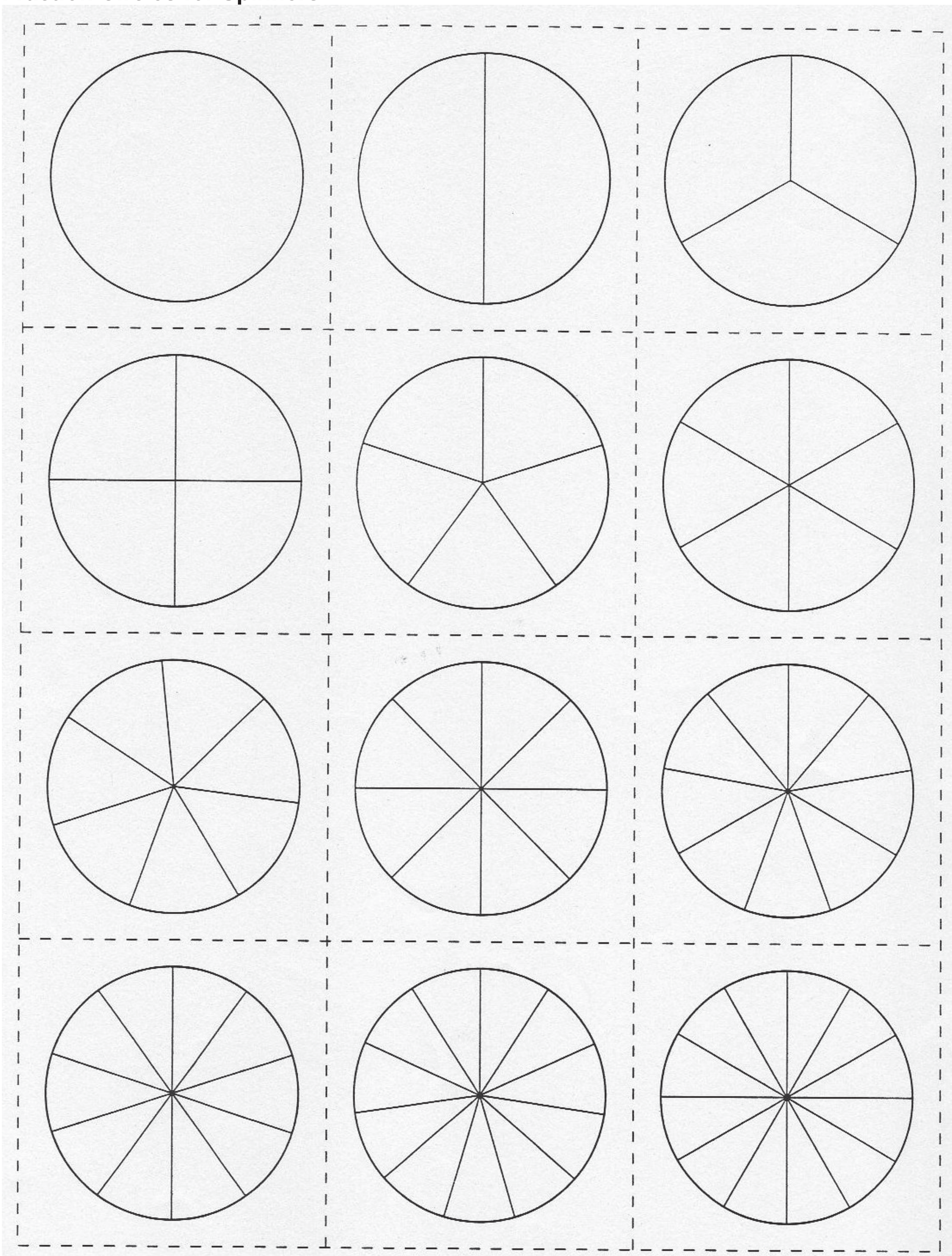
If I did the simulation again exactly as designed, the experimental results would not likely be the same because, with only 20 trials, it is reasonable that all three shots are made only on one occasion, or maybe not at all. If I increased the number of trials, the theoretical and experimental probabilities would probably be closer together.

UNIT 4 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Determining the Theoretical Probability	Completely accurate determination of the theoretical probability of sinking all three shots	Reasonably accurate determination of the theoretical probability of sinking all three shots	Some errors in the determination of the theoretical probability of sinking all three shots	Major errors in the determination of the theoretical probability of sinking all three shots
Designing the Simulation	Demonstrates sophisticated ability to apply mathematical knowledge and skills to design a simulation to calculate experimental probability	Demonstrates considerable ability to apply mathematical knowledge and skills to design a simulation to calculate experimental probability	Demonstrates some ability to apply mathematical knowledge and skills to design a simulation to calculate experimental probability	Demonstrates limited ability to apply mathematical knowledge and skills to design a simulation to calculate experimental probability
Comparing Experimental Probabilities with Given Probabilities	Chooses and creates an appropriate graph with no (or very minor) errors	Chooses and creates an appropriate graph with minor errors	Chooses and creates an appropriate graph with some errors	Incomplete or inappropriate choice of graph
Making Conclusions	Provides thorough, clear, and insightful explanations/ justification of predicted probability versus experimental probability using a range of words, pictures, symbols, and/or numbers	Provides complete, clear, and logical explanations/ justification of predicted probability versus experimental probability using appropriate words, pictures, symbols, and/or numbers	Provides partial explanations/ justification of predicted probability versus experimental probability using simple words, pictures, symbols, and/or numbers	Provides limited or inaccurate explanations/ justification of predicted probability versus experimental probability using minimal words, pictures, symbols, and/or numbers

UNIT 4 Blackline Master

Fraction Circles for Spinners



UNIT 5 GEOMETRY

UNIT 5 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Square dot paper or grid paper • Rulers (mm) • Protractors • Compasses	All questions
Chapter 1 Similarity and Congruence				
5.1.1 EXPLORE: Unique Triangles	**9-E1 Congruent Triangles: properties and minimum sufficient conditions** • understand, through investigation, that if two triangles are congruent through: SSS, SAS, ASA, or AAS, then the other corresponding parts of the triangle are also congruent **9-E2 Unique Triangles: minimum sufficient conditions** • examine what pieces of information are needed to guarantee a unique triangle • understand that the following are necessary in order to produce unique triangles: three sides; two sides and a contained angle; two angles and a contained side; two angles and a non-contained side • discover that AAA and SSA do not result in a unique triangle	1 h	• Rulers (mm) • Protractors • Compasses	Observe and Assess questions
5.1.2 Congruent Triangles	**9-E1 Congruent Triangles: properties and minimum sufficient conditions** • understand that if two triangles are congruent through SSS, SAS, ASA, or AAS, then the other corresponding parts of the triangle are also congruent • interpret and use the symbol $\cong$, which is read as “is congruent to” **9-E2 Unique Triangles: minimum sufficient conditions** • examine what pieces of information are needed to guarantee a unique triangle • understand that the following are necessary in order to produce unique triangles: three sides; two sides and a contained angle; two angles and a contained side; two angles and a non-contained side	1 h	• Rulers (mm) • Protractors	Q3, 6, and 7
5.1.3 Similar Triangles [Continued]	**9-E3 Similar Triangles: understand and apply proportions** • understand that, in similar triangles, the ratios of side lengths of one triangle are equal to the ratio of the corresponding side lengths of the second triangle • understand that the ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides	1 h	• Rulers (mm) • Protractors • Compasses	Q1, 2, 7, and 8

UNIT 5 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
5.1.3 Similar Triangles [Cont'd]	**9-E4 Similar Triangles: apply properties** • understand properties of similar triangles: that the corresponding angles are congruent (AAA) and the corresponding sides are in proportion (SSS) • understand that two triangles are similar when two pairs of corresponding sides are in proportion and the pair of included corresponding angles are congruent (SAS) • understand that two triangles are also similar when two angles of one triangle are congruent to two corresponding angles of another triangle (AAA) **9-E5 Triangles: relate congruency and similarity** • compare and contrast congruence and similarity as they relate to triangles			
5.1.4 Solving Problems with Similarity	**9-E3 Similar Triangles: understand and apply proportions** • understand that, in similar triangles, the ratios of side lengths of one triangle are equal to the ratio of the corresponding side lengths of the second triangle • understand that the ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides	1 h	• Rulers (mm) • Protractors • Compasses	Q1c, 5, and 6
Chapter 2 Transformations				
5.2.1 Translations	**9-E6 Transformations (mapping notation): represent and interpret** • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping **9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation: ■ *Translations* • line segments joining points on the original shape to their images are parallel and equal in length • translation image of any shape is congruent • orientation of a translation image is the same as that of the original shape • translation images of lines or line segments are parallel or collinear to the original lines and line segments	2 h	• Grid paper • Rulers (mm)	Q1, 3, 9, and 10

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
5.2.2 Reflections and Rotations	**9-E6 Transformations (mapping notation): represent and interpret** • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping **9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation ■ *Reflections* • line segments joining points on the original shape to their images are perpendicular to the reflection line and have their midpoint on the reflection line • reflection image of any shape is congruent to the original shape • orientation of a reflection image is the opposite of the original shape [Reflections are limited to reflections in the axes and the line $x = y$.] ■ *Rotations* • for a rotation of $a°$ about point X, a line segment joining a point to X and a line segment joining its image to X are equal in length and form an angle of $a°$ • rotation image of any shape is congruent • orientation of a rotation image is the same as that of the original shape [Rotations are limited to 90 and 180 degrees, clockwise (cw) and counterclockwise (ccw), around (0, 0).] • for 90° rotations, horizontal line segments become vertical, vertical line segments become horizontal, and any line segment and its image are perpendicular • for 180° rotations, line segments are parallel or collinear to their images	2 h	• Grid paper • Rulers (mm)	Q6, 8, 9, and 10
GAME: Shards	Practise skills in coordinate geometry and transformations		• Grid paper	N/A
5.2.3 Dilatations [Continued]	**9-E6 Transformations (mapping notation): represent and interpret** • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping	2 h	• Grid paper • Rulers (mm)	Q2, 6, 7, and 10

UNIT 5 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
5.2.3 Dilatations [Cont'd]	**9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation: ■ *Dilatations* • dilatation centre (a point) and its image form a line • ratio of the distance between the dilatation centre and a point on the shape, to the distance between the dilatation centre and a corresponding point on the image, is the same as the scale factor • ratio of the length of a line segment in the original shape to the length of a corresponding line segment in the image is the same as the scale factor • dilatation image of any shape is similar to the original shape • angle measures in the original shape are the same as corresponding angle measures in the image [Dilatations are limited to positive scale factors around centre (0, 0).]			
CONNECTIONS: Making an Animated Movie	To explore animation, a real life application of transformations		• Stiff paper or light card-board • Scissors	N/A
5.2.4 Combining Transformations	**9-E8 Transformations (mapping notation): analyse and represent composite transformations** • analyse a transformation given in mapping notation and represent a transformation using mapping notation • identify the transformations when given the original shape, and the image after a combination of transformations	2 h	• Grid paper • Rulers (mm)	Q5 and 7
UNIT 5 Revision	Review the concepts and skills in the unit	1 h	• Grid paper • Rulers (mm)	All questions
UNIT 5 Test	Assess the concepts and skills in the unit	1 h		All questions
UNIT 5 Performance Task	Assess the concepts and skills in the unit	1 h	• Grid paper • Rulers (mm) • Coloured pencil crayons (optional)	Rubric provided
UNIT 5 Blackline Masters	• Centimetre Square Dot Paper • Grid Paper (0.5 cm by 0.5 cm)			

Math Background

• Work with congruency allows students to see that to determine whether two triangles are congruent, sometimes less information is needed than one might expect. Although six pieces of information must match, they are interrelated and so less information is actually needed. A similar idea is useful in finding the perimeter of a square. For example, you need to add four side lengths to find the perimeter, but you actually only need one piece of information – the length of one side

• Students learn to be careful to collect the required information. For example, by SAS you can determine that two triangles are congruent but by SSA (still two sides and an angle) you cannot.

• As students work through this unit, they learn about connections between similarity and congruence as well as between similarity and dilatations.

• Students also learn that any sequence of reflections, rotations, or translations results in a congruent figure. Dilatations, however, result in figures that are similar but not necessarily congruent.

• Thinking about "inverse" operations, such as addition and subtraction, may help students compare transformations that move A to B with those that move B to A.

• The use of mapping notation in this unit supports students' work with the mapping notation used for describing functions.

• Students use many mathematical processes in their work for this unit, including problem solving, reasoning, communication, representation, visualization, and making connections. For example:

• Students make connections by relating dilatations to the concept of scale factor and similarity in **question 10** of **lesson 5.2.3**.

• They use reasoning to figure out which sequence of transformations might have moved shape A to shape B by examining properties of transformations in **question 7** of **lesson 5.2.4**.

•They use visualization to imagine where the rest of the square could be when only two vertices are given in **question 9** of **lesson 5.2.1**.

• Communication is essential for students in **question 7** of **lesson 5.1.2** as they synthesize what they have learned.

• Students use representation as they describe a translation in different ways in **question 3** of **lesson 5.2.1**.

• By using a strategy for determining the necessary information in **question 4** of **lesson 5.1.4**, students use the process of problem solving.

Rationale for Teaching Approach

• This unit is divided into two chapters. **Chapter 1** focuses on concepts of congruence and similarity while **Chapter 2** focuses on transformations. The segmenting of the unit is meant to help students understand congruence and similarity before they have to distinguish transformations that preserve congruence from those that only preserve similarity.

• The **Getting Started** allows students to explore the creation of triangles that meet particular conditions and result in unique triangles. They will draw on this experience when they begin learning about what information is required to assure congruence.

Getting Started

Curriculum Outcomes	Outcome relevance
8 Pythagorean Relationship: application **8** Angle Pair Relationships: parallel and non-parallel lines **8** Proportion: solve problems	Students will need to review some drawing, measurement, and calculation skills from previous grades to help them work with shapes and transformations.

Pacing	Materials	Prerequisites
1 h	• Square dot paper (see blackline master at the end of this unit) or grid paper • Rulers (mm) • Protractors • Compasses	• solving proportions • facility with a protractor, ruler, and compass for geometric constructions and measurements • Pythagorean theorem • familiarity with the properties of parallel lines and transversals • coordinate plotting

Main Points to be Raised

• For the shapes in **part A**, the angles did not change and the side lengths are proportional.

• The area of the shape in **part A i)** is not doubled (it is actually quadrupled) even though the side lengths are doubled.

• The sum of the angles of a triangle is equal to 180 degrees.

• Instead of measuring directly, some angle measurements and side lengths of triangles can be calculated by using the properties of parallel lines and triangles.

NOTE: The task in **part A** should be re-examined in the context of similarity in **lesson 5.1.3**.

Use What You Know—Introducing the Unit

• Make sure students have the tools they need.

• Students should make careful drawings and measurements.

• Encourage students to work in pairs and to talk with each other as they work. In particular, students should discuss their strategies for doubling, as there are many ways to think about this question.

Observe students as they work. You might ask:

• *How can you be sure it is twice as long?* (I measured it with my ruler.)

• *How do you know that you need to go over 4 and down 2 to make a segment twice as long?* (Because it starts by going over 2 and down 1 so you would have to double both.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions. You might also review geometry terminology for transformations (translations, reflections, rotations, and dilatations).

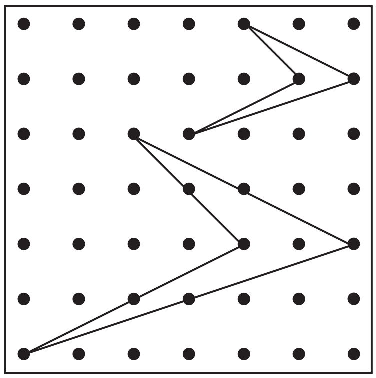
• Students can work individually.

• The symbol ║ can be used to show that two lines are parallel. For example, AB ║ FC means that the line segments AB and FC are parallel to one another.

Answers

A. *Sample responses*:

i) and **ii)**



iii) I counted the spaces across and down and doubled them.

B. *Sample responses*:

i) The angles are the same in both polygons: 8º, 45º, 19º, and 288º.

ii) Corresponding line segments are parallel.

iii) All classmates should have the same result: equal angles, parallel sides.

1. b) PR ≈ 5.55 cm, PQ ≈ 4 cm, RQ ≈ 6.8 cm;
$\angle R \approx 35º$, $\angle P \approx 90º$, $\angle Q \approx 55º$
c) 180º
d) $PR^2 + PQ^2 = RQ^2$; $5.55^2 + 4^2 \approx 30.8 + 16 = 46.8$
$6.8^2 = 46.24$
(The results of part d) are not exactly the same as **part b)** because of measurement imprecision.)

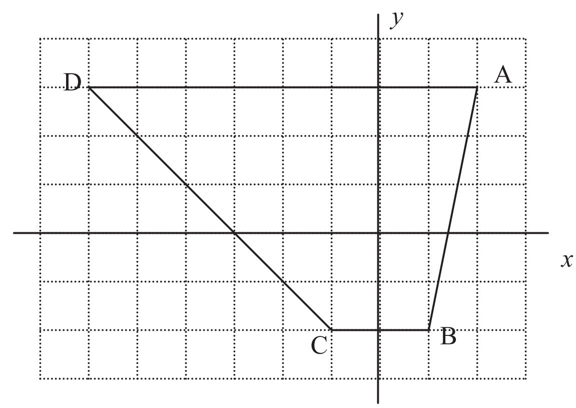
2. a) 6 **b)** 30 **c)** 4.52

3. a) AB ∥ FC ∥ ED, AF ∥ EB ∥ CD, BC ∥AD ∥ EF

b) rotational symmetry of order 6 about O;
6 lines of reflective symmetry intersecting at O

c) $\angle AOB = \angle EOD$ because they are vertically opposite angles, $\angle OAB = \angle AOF$ because they are alternate angles formed by transversal AO intersecting parallel lines AB and OF, $\angle BOC = \angle BED$ because they are corresponding angles formed by transversal BE intersecting parallel lines CF and ED, and $\angle COE + \angle DEO = 180º$ because they are interior angles formed by transversal BE intersecting parallel lines CF and ED.

4. a)



b) trapezoid

c) $\angle D = 45º$

5. Subtract 8

Supporting Students

Struggling students

If some students are struggling with measurement and geometric construction, you may need to re-teach those skills to those individuals. For certain students, you might choose to start with a right triangle instead of the polygon shown in **part A**. Students will still learn the underlying concepts and skills related to transformations.

5.1.1 EXPLORE: Unique Triangles

Curriculum Outcomes	Lesson relevance
9-E1 Congruent Triangles: properties and minimum sufficient conditions • understand, through investigation, that if two triangles are congruent through: SSS, SAS, ASA, or AAS, then the other corresponding parts of the triangle are also congruent **9-E2 Unique Triangles: minimum sufficient conditions** • examine what pieces of information are needed to guarantee a unique triangle • understand that the following are necessary in order to produce unique triangles: three sides; two sides and a contained angle; two angles and a contained side; two angles and a non-contained side • discover that AAA and SSA do not result in a unique triangle	In this core lesson, the symbolic language of congruency (such as SSS and SAS) becomes more meaningful to students through hands-on exploration with triangles.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Protractors • Compasses	• facility with a protractor, ruler, and compass for geometric constructions and measurements • Pythagorean theorem • properties of triangles (angles sum to 180º) • properties of isosceles triangles (two equal angles opposite two equal sides)

Main Points to be Raised

• The measurements of a triangle are related, so it is possible to describe a triangle without telling all the angle sizes and all the side lengths.

• Comparing certain information about a pair of triangles (such as SSS or SAS, ASA, or AAS) forces congruence, but other combinations of two or three pieces of information about the triangles do not (SSA or AAA).

• When they check for uniqueness, some students might think that because they found the same triangle as a friend or because they could find only one triangle, this means the triangle is unique. An example triangle is not enough to conclude that a given set of measurements results in a unique triangle.

Exploration

• Ask students to work on **parts A to C** with a partner or in a small group. Once they agree on the dimensions to try, they can work independently and then come together to see if they found the same triangle or a different one.

Observe while students work. You might ask:

• *If you know two sides are 20 m long and two angles are 45°, how do you know it is an isosceles right triangle and that the hypotenuse is longer than 20 m?* (the third angle must be 90° because the sum of the angles of a triangle is always 180°; the hypotenuse is always longer than the other two sides of a right triangle)

• *How could you calculate the length of the hypotenuse?* ($a^2 + b^2 = c^2$, using a = b = 20)

• *Why can your two angles NOT be 135° and 120°?* (the angles would add to more than 180°)

• *Why is it irrelevant to know the third angle?* (you can find it by subtracting the two known angles from 180°)

Observe and Assess

As students are working, notice the following:

• When answering **part A**, do they try to eliminate each piece of information in the order of presentation or do they recognize that there is a right triangle (item 3) before trying to eliminate item 2?

• Do they recognize that to get a different triangle with the same two angles, it is essential to make the side between them a different length?

• Do they have more difficulty trying to set up an AAS situation than an ASA one or do they realize they can just calculate the third angle and then use an ASA situation?

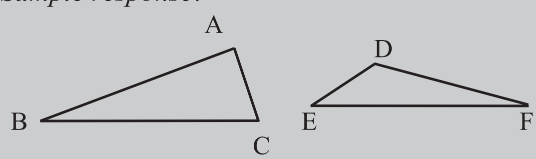
Share and Reflect

After students have had enough time to answer **parts A, B, and C**, encourage different groups to come forward and describe how they approached the exploration and what they learned.

Answers

A. There are two equal side lengths (item 1) and two equal angles (item 4), so the triangle must be isosceles. Since two of the angles are 45°, the other one must be 90°, so it is a right triangle. That is why I do not need item 3. Since the hypotenuse is longer than the two other sides, the two short sides must be 20 m and the long side can be calculated using the Pythagorean theorem. Since $20^2 + 20^2 = 800$, I calculate $\sqrt{800}$ for the length of the third side. It is about 28 m. That is why I do not need item 2.

B. *Sample response*:



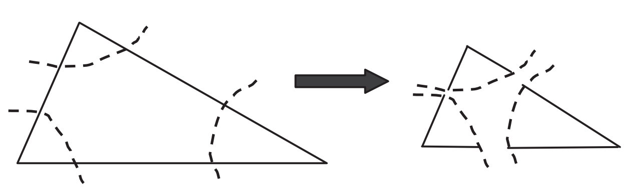
AC = DE; BC = EF

C. Unique triangles result with **iii)** two sides and the angle between them, **v)** two angles and the side between them, **vi)** two angles and a side not between them, and **vii)** three sides.
Sample responses:
i) two angles: Triangle 1 has sides measuring 1.7 cm, 1 cm, and 2 cm with angles of 30° and 60°. Triangle 2 has sides of 3.5 cm, 2 cm, and 4 cm and angles of 30° and 60°.
ii) three angles: Triangle 1 has sides of 1.7 cm, 1 cm, and 2 cm with angles of 30°, 90°, and 60°. Triangle 2 has sides of 3.5 cm, 2 cm, and 4 cm with angles of 30°, 90°, and 60°.
iii) two sides and the angle between them: You can make only one triangle, for example with side lengths of 4 cm and 6 cm and a 60° angle between them. The other side is 5.1 cm and other angles are 87° and 33°.
iv) two sides and an angle not between them: For example, ΔPQR with PQ = 5 cm and QR = 4 cm with ∠P = 30°. Two triangles are possible: Triangle 1 has PR = 1.6 cm while Triangle 2 has PR = 7.1 cm.
v) two angles and a side between them: You can make only one triangle. For example, ΔABC with AB = 4 cm, ∠A = 30°, and ∠B = 90° must have AC = 4.6 cm and BC = 2.3 cm.
vi) two angles and a side not between them: You can make only one triangle. For example, ΔGHI with ∠G = 25°, ∠H = 57°, and GI = 23.5 cm must have ∠I = 98°, HI = 10 cm, and GI = 19.9 cm.
vii) three sides: A triangle with sides measuring 5 cm, 6 cm, and 5.9 cm must have angles of 45°, 65°, and 70°.

Supporting students

Struggling students

Some students might benefit from drawing one triangle and then cutting wedges from the corners so that they can move the wedges around to make a new triangle with the same angles but different side lengths.



Students can also cut narrow strips of paper to represent side lengths and move them around in various combinations to form different angles and triangles.

5.1.2 Congruent Triangles

Curriculum Outcomes	Outcome relevance
9-E1 Congruent Triangles: properties and minimum sufficient conditions • understand that if two triangles are congruent through SSS, SAS, ASA, or AAS, then the other corresponding parts of the triangle are also congruent • interpret and use the symbol ≅, which is read as "is congruent to" **9-E2 Unique Triangles: minimum sufficient conditions** • examine what pieces of information are needed to guarantee a unique triangle • understand that the following are necessary in order to produce unique triangles: three sides; two sides and a contained angle; two angles and a contained side; two angles and a non-contained side	The tools for determining congruence of triangles will be valuable in later investigations in trigonometry.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Protractors	• facility with a protractor, ruler, and compass for geometric constructions and measurements • properties of triangles (angles sum to 180º)

Main Points to be Raised

• Some combinations of three pieces of information determine congruence of triangles (SSS, SAS, ASA, or AAS).

• The tools for congruence in this chapter apply only to triangles and not to other polygons, although some of the underlying ideas are the same.

• Proportions can be established once the congruence statement has been written. For example, if ABC ≅ FGH you can set up ratios such as $\frac{AB}{FG}$ (using the first two terms) = $\frac{AC}{FH}$ (using the first and third terms) = $\frac{BC}{GH}$ (using the second and third terms).

• Some students may wonder about the scale of the drawings. In some cases, such as the **Try This**, the drawing is a sketch. You can emphasize that sketches are good for keeping track of thinking (visualizing the problem, recording results, and arranging things in different ways). Students should not try to measure angles and side lengths of sketches. Careful drawings and measurement can be used if there is no way of determining the result through reasoning.

Try This—Introducing the Lesson

A. and B. Students can work alone or with a partner.

Observe while students work. You might ask:

• *Why could Seldon and Nima measure AB, BC, and CE?* (because they are land measurements)

• *Why could they NOT measure the distance across the lake?* (They could not stand in the lake to hold the tape measure.)

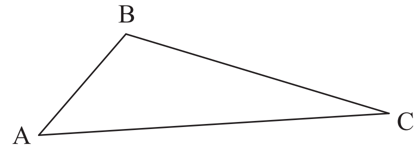
• *Why do you think they measured horizontal and vertical distances?* (they can make sure they are making right triangles that way)

• *Is there a different set of triangles they could have drawn?* (Yes, they could have made a triangle on the other side of the river.)

The Exposition—Presenting the Main Ideas

• Students have had enough experience with the earlier investigation to read through the box on their own. Make sure they are familiar with the notation (e.g. if the same symbol is used in two triangles, it means the angles labelled with that symbol are equal; the same number of hatch marks means the corresponding sides are equal).

• Make sure students can distinguish between AAS and ASA by drawing a triangle on the board and asking them to give you various combinations for each. For example, in the drawing below, $\angle A$, $\angle B$, and BC or $\angle A$, $\angle B$, and AC could be given for AAS. An example of ASA would be $\angle A$, $\angle B$, and AB.



• Though different terms are used for AAS and ASA in this unit, the two are essentially the same. Since the angles of a triangle sum to 180º, if two of the angles are known, the third can be calculated. For the triangle above, if $\angle A$, $\angle B$, and BC are known (AAS), $\angle C = 180^\circ - (\angle A + \angle B)$. Thus when $\angle B$, $\angle C$, and BC are known, congruence can also be established by ASA.

Revisiting the Try This

C. and D. Some students might suggest SAS by using the 54 m as one of the known elements. You can remind them that they do not know the length is 54 m on the triangle spanning the lake.

Using the Examples

• Encourage students to predict which triangles are congruent by looking at the drawings.
• Allow students to read through the example alone. Then they can talk to each other about which other angles could have been measured instead of the largest and smallest. Any angles can be measured as long as they match the angles on the second shape.
• Ask students to consider how the student in the example could have used SSA or some other strategy.

Practising and Applying

Teaching points and tips

Q 1: You might have to point out the two triangles in the first drawing and the two in the second. Make sure students write the corresponding vertices in the right order. This may be done with the whole class as another example, if necessary.

Q 2: Suggest that students re-sketch all the triangles in the same direction if they are having difficulty determining congruency.

Q 3: This question introduces another condition for congruency that applies only to right triangles — Right angle-Hypotenuse-Side (RHS).

Q 4: Many students may think it is obvious that triangles are congruent as they look at them. Encourage them to measure to be certain.

Common errors

Often students write corresponding vertices out of order, which can lead to false congruency and similarity statements (as well as incorrect measures and ratios). Writing the corresponding vertices in the right order will also be important in identifying transformations in the next chapter.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can use properties of congruent figures to reason about congruence
Question 6	to see if students select alternative strategies for determining congruence
Question 7	to see if students understand minimum conditions for determining congruence

Answers

A. i) the length of ED
ii) AB = 54 m, BC = 15 m, EC = 15 m, $\angle B = \angle E = 90^\circ$, and $\angle BCA = \angle ECD$ because they are vertically opposite angles

B. Yes, because I know that two angles and the side between them gives a unique triangle, and the angles and the side between them are the same for the two triangles. ΔABC and ΔDEC are identical.

C. ASA

D. 54 m because the triangles are congruent, so ED must equal AB

Answers [Continued]

1. a) SAS for ΔWXY and ΔWZY; $\Delta WXY \cong \Delta WZY$

b) ASA for ΔSVR and ΔSTU; $\Delta SVR \cong \Delta STU$

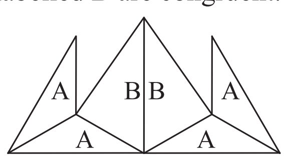
2. a) $\Delta ABC \cong \Delta KML$ using SSS or SAS

b) $\Delta ABC \cong \Delta FDE$ using AAS

c) $\Delta ABC \cong \Delta JGH$ using SAS

3. Given the hypotenuse and another side of a right triangle, you can use the Pythagorean theorem to find the third side. So, being given two sides of a right triangle is the same as being given all three sides (SSS).

4. The four triangles labelled A are congruent and the two triangles labelled B are congruent.



5. a) ASA

b) 78 m

c) AB ≈ 34.41 m;
$DE^2 + CE^2 = CD^2$; $DE^2 + 70^2 = 78^2$,
$DE^2 = 6084 - 4900$, $DE = \sqrt{1184}$ so $DE \approx 34.41$.
Since $\Delta ABC \cong \Delta DEC$ by ASA, AB = DE and AB ≈ 34.41 m.

6. a) Knowing $\angle ACB = \angle ACD$ would allow ASA, Knowing $\angle ABC = \angle ADC$ would allow AAS, and Knowing AB = AD would allow SAS

b) Knowing FG = HG would allow SSS and knowing $\angle FEG = \angle HEG$ would allow SAS

c) Knowing any corresponding sides are equal would allow AAS or ASA

7. You do not need to know all the measurements to know if triangles are congruent. Knowing SSS, SAS, ASA, or AAS is enough;
Sample response: For example, there is only one triangle possible if you know all three side lengths, so if two triangles have the same side lengths (SSS), they must be congruent. It works the same with SAS, ASA, and AAS.

Supporting students

Enrichment

Students could further explore congruence. How would you test the congruence of two quadrilaterals? Determine one or more sets of conditions that would establish congruence.

5.1.3 Similar Triangles

Curriculum Outcomes	Outcome relevance
9-E3 Similar Triangles: understand and apply proportions • understand that, in similar triangles, the ratios of side lengths of one triangle are equal to the ratio of the corresponding side lengths of the second triangle • understand that the ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides **9-E4 Similar Triangles: apply properties** • understand properties of similar triangles: that the corresponding angles are congruent (AAA) and the corresponding sides are in proportion (SSS) • understand that two triangles are similar when two pairs of corresponding sides are in proportion and the pair of included corresponding angles are congruent (SAS) • understand that two triangles are also similar when two angles of one triangle are congruent to two corresponding angles of another triangle (AAA) **9-E5 Triangles: relate congruency and similarity** • compare and contrast congruence and similarity as they relate to triangles	Students use measurement and proportions to find similar triangles. This is in preparation for further trigonometry work in subsequent grades. Students also work with scale drawings.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Protractors • Compasses	• relationship between kilometres and centimetres • facility with a protractor, ruler, and compass for geometric constructions and measurements • solving proportions • properties of triangles (angles sum to 180º)

Main Points to be Raised

• In similar triangles, the ratio between side lengths of one triangle is equal to the ratio between the corresponding side lengths of the second triangle.

• The ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides.

• The properties of similar triangles include: corresponding angles are congruent (AAA) and corresponding sides are in proportion (SSS).

• Two triangles are similar when two pairs of corresponding sides are in proportion and the included corresponding angles are congruent (SAS).

• Two triangles are also similar when two angles of one triangle are congruent to two corresponding angles of another triangle (AAA).

• Congruency is a special form of similarity.

• Scale diagrams are based on concepts of similarity.

Try This—Introducing the Lesson

A. Some students will simply work with the numbers and attach the proper unit whereas others will use the relationship between cm and km.

Observe while students work. You might ask:

• *How did you know that the number of km would always be more than 50?* (because all the distances were greater than 1 cm)

• *How did you know which distance would be longest?* (the longest distance on the map is the longest real distance and the distance from Trashigang to Phuntsholing is longer than from Thimphu to Trashigang)

• *How could you have predicted the longest distance would be about three times as long as the shortest distance?* (because on the map the distance from Trashigang to Phuntsholing is about three times as long as the one from Thimphu to Phuntsholing)

The Exposition—Presenting the Main Ideas

- Draw a pair of similar triangles on the board. Show how the triangles should be named and labelled in corresponding order and how the ratios are created using corresponding pairs of letters in the names.
- Show what AAA, SSS, and AAS mean with your triangles.
- Demonstrate to students how the perimeters have to be proportional because they are made up of side lengths. If each side length is multiplied by a factor to get the corresponding side length, then the sum of the three sides is also multiplied by that factor to get the corresponding sum. Students might benefit from seeing an example such as a triangle with doubled sides. Make the connection to the **Getting Started: Use What Your Know** activity.
- Allow students to read through the exposition after the **Try This** to see another example.

Revisiting the Try This

B. To create the scale factor, students have to use the relationship between cm and km. They may find it easier to go from km to cm: 50 km = 50,000 m = 5,000,000 cm.

Using the Examples

- Students can work with a partner. Assign one partner to read the first solution and the other to read the second solution. Then they can explain what they learned to each other.
- Ask students how they could have used SSS instead.

Practising and Applying

Teaching points and tips

Q 1: Students can trace and reposition the triangles so they are pointed the same way. For **1d)**, students may not get exactly the same ratio for each pair of corresponding sides due to measurement imprecision.

Q 3: Some students will need help with constructing the first triangle. They might benefit from cutting three strips of the right length and moving them around. Alternatively, they could use a compass for the construction.

Q 6: The number of triangles must be a square number for this pattern to continue. Each subsequent row has the next odd number of triangles in it. (The sum of consecutive odd numbers is a square number.)

Common errors

Often students are hasty in writing the order of the vertices and may, as a result, set up wrong proportions. It is important to reinforce the need to have matching angles in matching positions.

Suggested assessment questions from Practising and Applying

Question 1	to see if students measure to determine similarity and use corresponding ratios appropriately
Question 2	to see if students know the properties of similar figures
Question 7	to see if students understand minimum conditions for determining similarity
Question 8	to see if students understand the relationship between congruence and similarity

Answers

A. i) Thimphu to Phuntsholing is about 1.4 cm; Thimphu to Trashigang is about 3.8 cm; Trashigang to Phuntsholing is about 4.3 cm.

ii) Actual distances are about 70 km, 190 km, and 215 km respectively. I multiplied each distance by 5,000,000 to find the number of centimetres and then divided by 100,000 to convert from cm to km. (I could have multiplied the distance by 50 instead to get kilometres.)

B. i) 5,000,000 **ii)** corresponding sides have the same ratio, the scale factor (SSS)

1. a) $\angle BAC = \angle CAD \approx 37°$, $\angle ABC = \angle ACD \approx 87°$

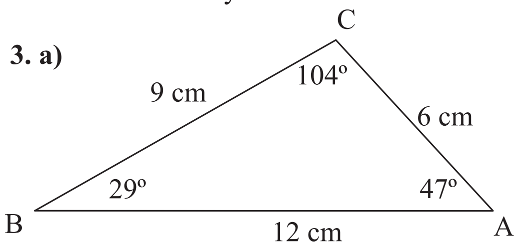
b) A corresponds with A, B corresponds with C, and C corresponds with D

c) *Sample response*: $\Delta CDA \sim \Delta BCA$

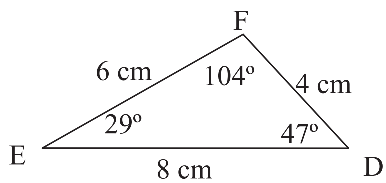
d) AD ≈ 6.4 cm, AC ≈ 5.3 cm, AB ≈ 4.4 cm, BC ≈ 3.2 cm, CD ≈ 3.9 cm; $\frac{AD}{AC} = \frac{AC}{AB} = \frac{CD}{BC} \approx 1.2$ (The ratios might not be exactly the same because of measurement imprecision.)

2. b and c are always true

3. a)



b)



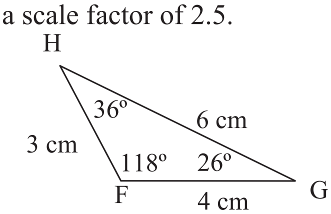
c) Angles are the same in both triangles

d) 27 cm and 18 cm

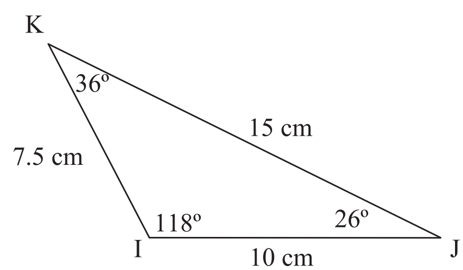
e) AAA

4. *Sample response:*
ΔFGH with FG = 4 cm, GH = 6 cm and FH = 3 cm, with a scale factor of 2.5.

a)



b)



4. c) Angles are the same in both triangles

d) 13 cm and 32.5 cm

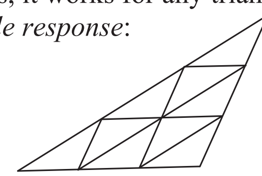
e) AAA

5. a) You need only two angles because the third angle in any triangle can be found by subtracting the other two from 180º.

b) You need only one angle, the one between the pairs of corresponding sides (for SAS)

6. a) 13 similar triangles: 9 small ones, 3 middle-sized ones, and the large triangle.

b) Yes, it works for any triangle.
Sample response:



c) It cannot be done with two or three triangles, but it can be done with four.

7. a) If ∠B = ∠D or ∠A = ∠E, there would be two equal angles, or if CD = 6.75 m ($\frac{9}{6.75} = \frac{12}{9}$), there would be two pairs of corresponding sides in equal ratio with an equal angle between them

8. Congruent triangles have more requirements and are a subgroup of similar triangles.

Supporting students

Struggling students

Some students find it easier to build triangles from SAS or ASA descriptions than from SSS or AAS. The instructions in **questions 3 and 4** could be modified to suggest SAS or ASA by including one angle.

Enrichment

• Students might consider how the scale factor for area of similar figures is determined (it is the square of the linear scale factor).

• Students might also be interested in exploring the patterns in **question 6** by predicting the number of different similar triangles. For example, for three rows of small triangles (9 small triangles forming a row of one triangle, a row of three triangles, and a row of five triangles as shown in the diagram), there are 13 different similar triangles. What if another row of triangles were added so that there were 16 small triangles (add a row of 7 triangles)? How many similar triangles would there be? Find a pattern to describe how the number of similar triangles is related to the number of rows of small triangles.

5.1.4 Solving Problems with Similarity

Curriculum Outcomes	Outcome relevance
9-E3 Similar Triangles: understand and apply proportions • understand that, in similar triangles, the ratios of side lengths of one triangle are equal to the ratio of the corresponding side lengths of the second triangle • understand that the ratio between the perimeters of similar figures is equivalent to the ratio between any pair of corresponding sides	Students use similarity to solve problems involving triangles. Similar triangles will form the basis for trigonometry and measurement applications.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Protractors • Compasses	• solving proportions

Main Points to be Raised

• In all these applications, recognizing which triangles are similar and setting up the ratios properly is key to solving the problems.

• Encourage students to see that there are always many ways to set up a proportion to be solved. For example, if $\frac{a}{b}=\frac{c}{d}$, then you could also write $\frac{b}{a}=\frac{d}{c}$ or $\frac{a}{c}=\frac{b}{d}$.

Try This—Introducing the Lesson

A. and B. Students can work alone or in pairs.

Observe while students work. You might ask:

• *Why does the diagram show a length of 6 m?* (because the full length is 8 m and 2 m is already marked)

• *What does the 1.5 m in the diagram mean?* (It is the height of the person.)

• *How does 1.5 relate to 2? Why is that relevant?* (It is $\frac{3}{4}$ of it, so the wall is only $\frac{3}{4}$ of 8.)

The Exposition—Presenting the Main Ideas

• Students can read through the explanation on their own. Encourage them to think of a different situation where they could use similar triangles to find a difficult measurement and record it in their notebooks as a reminder.

Revisiting the Try This

C. Students may want to bypass writing down the proportion in favour of simply finding the height of the wall, but there is benefit in having them set up the proportion as another reminder of how to use two similar triangles

Using the Examples

• Allow the students to read through both solutions for **example 1**. Answer any questions they might have. Ask which method of calculation they prefer and why.
• Students can read the **example 2** independently.
• Ask students to think about what the double arrows on DE and CA of **example 2** indicate (the sides are parallel).

Practising and Applying

Teaching points and tips

Q 1: Similarity can be assumed from the wording of the question (hence the parallel lines in **1b**).

Q 2: You might set up a mirror to show students how the angle of incidence relates to angle of reflection (the angles are the same). Because the angles of incidence and the right angles are equal, the triangles are similar using AAA. This question relates to the scientific idea of reflection.

Q 2, 4, and 5: These questions provide examples of real-world applications of similarity.

Q 3: Suggest that students darken or colour the inner triangles with a pencil to make them more visible. You might also talk about the orientation of the inner triangles as compared to the original triangles.

Q 4: Encourage students to try this with a partner or in a small group. You might also provide a drawing.

Q 5b: Some students might have trouble interpreting this question. You could show how the base would be bumpy and not straight and therefore a triangle is not possible.

Q 6: If students struggle to come up with a problem of their own, suggest that they adapt a context from the exercises.

Suggested assessment questions from Practising and Applying

Question 1c	to see if students can solve a simple proportion problem based on similarity
Question 4	to see if students can solve a more complex and real world proportion problem using similarity
Question 6	to see if students show an understanding of the properties of similar triangles by creating and solving a similarity problem

Answers

A. i) The two triangles are the small triangle formed by Roshan and his shadow and the larger triangle formed by the wall and its shadow.
ii) They share an angle, the bottom left angle, and they both have a right angle, which means they have two angles the same, making them similar (using AAA).

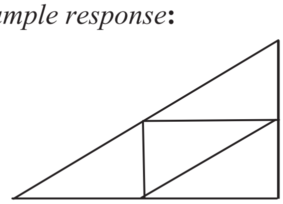
B. Pema might notice that the larger triangle is 4 times as large as the smaller triangle (8 m long instead of 2 m long). He might realize that this information could be used to find the height of the wall.

C. *Sample responses*: **i)** $\frac{1.5}{2} = \frac{h}{8}$ **ii)** 6 m

1. a) $x = 3.2$ cm **b)** $h \approx 8.8$ m **c)** $y = 9$ m

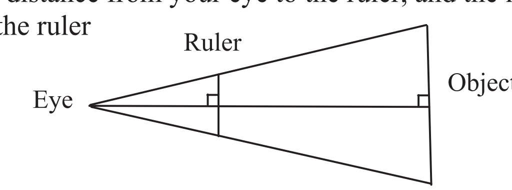
2. $\frac{7.8}{2.4} = \frac{x}{1.4}$; $x = 4.55$ m

3. a) *Sample response*:



b) Each side in the smaller triangle is exactly half the length of its corresponding side in the larger triangle. This means the triangles are similar.
c) This is true for all triangles.

4. The distance from your eye to the object (or person), the distance from your eye to the ruler, and the length of the ruler



5. a) The angle between the ground and the sun's rays is constant and the angle between vertical things and the ground is constant, so there will still be two equal angles in the two triangles, making them similar (AAA).
b) If the ground is hilly or uneven, triangles cannot be formed.

6. *Sample response*: A triangle with one side length of 2.3 cm is drawn on a map with the scale 1 cm represents 80 km. What is the actual distance of the side length? (184 km)

Supporting students

Enrichment

Similarity of parallelograms can be explored by extending **question 3**. Replace "triangle" with "parallelogram" in each part of the question.

Chapter 2 Transformations

5.2.1 Translations

Curriculum Outcomes	Outcome relevance
9-E6 Transformations (mapping notation): represent and interpret • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping **9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation: ▪ Translations • line segments joining points on the original shape to corresponding points on the image are parallel and equal in length • translation image of any shape is congruent • orientation of a translation image is the same as that of the original shape • translation images of lines or line segments are parallel or collinear to the original lines and line segments	Translations are one of the simplest and most useful types of transformation. It is important to recognize the properties of translations. Work with translations enhances students' ability to use mapping notation, which supports later work with functions.

Pacing	Materials	Prerequisites
2 h	• Grid paper • Rulers (mm)	• coordinate plotting in four quadrants • addition and subtraction of integers

Main Points to be Raised

• The line segments joining points on the original shape to corresponding points on the image are parallel and equal in length. This is why only one translation arrow is required to describe the transformation of the entire shape and why you can translate from any point on the original shape to a corresponding image to create the new shape.

• Since the translations are in two dimensions (2D), only two numbers are needed (the change in x and the change in y) to describe any translation. If a three-dimensional (3D) shape were translated in three dimensions, we would need three numbers to describe the translation.

• The various representations of a translation are interconnected.

• Translations are described by movement right or left and up or down.

• The translation image of any shape is congruent to the original shape because there are no changes to length or angle as a result of the move.

• The orientation of a translation image is the same as the orientation of the original shape. The images of lines or line segments are parallel or collinear to the original lines and line segments.

• Translations can change one coordinate (if the move is only horizontal or vertical) or both coordinates (if the move is diagonal).

• The convention of prime notation (A to A') should be described as something mathematicians do to help connect the original shape to the new one. Since they are both named with an A, it is clear that A and A' are connected. If you used new letters for the new shape, the connection would not be as clear.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Mindu's technique is one way to draw 3D shapes. It happens to be related to the topic of transformations. These questions provide an opportunity to use a transformation in an informal way, before formal mapping notation and translation rules are introduced in the lesson. Observe while students work. You might ask:

• *How did you know the y-coordinate would be greater?* (because I moved the shape upwards)

• *How did you know the x-coordinate would be greater?* (because I moved the shape to the right)

• *Did any point move more than the others?* (No, each point moved by the same amount up and to the right.)

The Exposition—Presenting the Main Ideas

• You might display a poster of the new vocabulary (the bolded words in the student book) or ask students to define these terms in their own words in a personal glossary.

• As an introduction to the vocabulary of translations, it might be useful for you to present the ideas from the exposition using a particular example. For example, draw a shape on the board and apply the translation [2, –3]. In the course of the translation, introduce the vocabulary from the exposition. At this time, you could also point out how the same translation could be similarly described in words and by using mapping notation.

• You could then ask students to read the exposition from the student book on their own.

Revisiting the Try This

C. and D. These questions give students an opportunity to use the new vocabulary and concepts of translations in the context of Mindu's diagram as well as each student's diagram of a rectangular prism. This is an occasion to make a formal connection between what was done in **parts A and B** and the new, more formal language and notation after the main ideas in the Exposition have been presented. You might approach **part C** as a whole class, while **part D** could be done individually or in pairs.

Using the Examples

• **Example 1** could be presented to students as a problem to try on their own, before reading the solution. Once they have tried the translation and given an explanation, they can compare their work to the solution.
• Students can read **example 2** individually and then, in pairs, one partner can explain the student thinking in the first solution using his or her own words. The partners would then reverse roles for the second solution.

Practising and Applying

Teaching points and tips

Q 2: Some students will find coordinates of the image and do the subtraction to get the translation rule whereas others will physically or mentally draw a translation arrow without finding the second set of coordinates.

Q 3: This question can be used to help students see more of the properties of translations.

Q 6: Asking students to write out the mapping notation can help them see that any combination of translations is a translation or, inversely, that any translation can be accomplished through a sequence of other translations.

Q 9: You might suggest that students plot the points to see that the second set of points could be to the right or the left. A square could also be formed by a rotation but here the focus is on translations.

Common errors

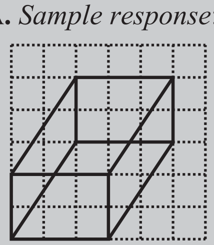
Students might use square brackets [ ] instead of round brackets ( ) when recording transformations in mapping notation. Or, they might use round brackets instead of square brackets when recording translation rules.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can perform a translation
Question 3	to see if students can express a translation with the translation rule and with mapping notation
Question 9	to see if students understand the important properties of transformations
Question 10	to see if students understand representations of transformations

Answers

A. *Sample response*:



B. For the example in **part A**: 2 units to the right and 3 units upward

C. i) from A to A', from B to B', or from C to C'
ii) any two of: [1, 3], $(x, y) \rightarrow (x + 1, y + 3)$, 1 unit right and 3 units up, or using a translation arrow

D. for the example in **part A**: 2 units right and 3 units up, and any two of: [2, 3], $(x, y) \rightarrow (x + 2, y + 3)$, or a translation arrow

Answers [Continued]

1. a) (7, 3) **b)** (3, 0) **c)** (0, 0)

2. a) (–2, –2) **b)** (–3, 6)

3. a) [–2, 3] **b)** $(x, y) \rightarrow (x - 2, y + 3)$

4. PQ ║ P'Q', PR ║ P'R', QR ║ Q'R'

5. a) [5, – 3] **b)** [6, 1]
c) [– 3, – 6] **d)** [– 3, 5]

6. a) $(x, y) \rightarrow (x + 5, y - 3)$

b) $(x, y) \rightarrow (x + 6, y + 1)$

c) $(x, y) \rightarrow (x - 3, y - 6)$

d) $(x, y) \rightarrow (x - 3, y + 5)$

7. a) A'(–4, 0), B'(–3, 2), C'(1, 0), D'(0, –2)

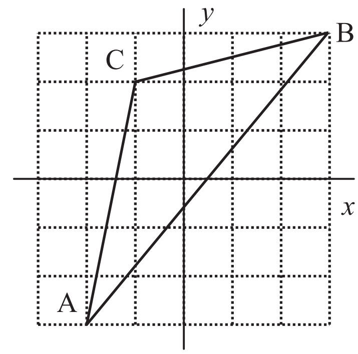
b) A"(0, 1), B" (1, 3), C"(5, 1), D"(4, –1)

c) A translation of [3, 2] and then a translation of [4, 1] is the same as one translation of [7, 3].

d) [–7, –3]

e) To return to the original position, you need the opposite translation, [–7, –3].

8. a)



b) [3, –6] or $(x, y) \rightarrow (x + 3, y - 6)$

c) A'(1, –9), B'(6, –3)

d) Yes; the sides are the same lengths because a translation does not affect size

e) AB ║ A'B', AC ║ A'C', and BC ║ B'C'

f) AA" ║ BB" ║ CC"

g) same lengths

9. a) [2, 2] or [–2, –2]

b) Because the other vertices could be on either side of the given vertices

c) [–3, 1] or [3, –1]

10. *Sample response*: For a translation, all vertices move the same distance. The top vertex of A would have to move a different distance than the other vertices.

11. All the vertices move the same distance and in the same direction.

Supporting Students

Struggling Students

You can help individual students by working only with translation rules or mapping notation, depending on which is easier for them to use. However, later work will require facility with both forms.

Enrichment

Students could explore drawing three-dimensional figures by using the techniques described in the **Try This**. They might also investigate the use of a vanishing point in art and the connections to dilatations.

5.2.2 Reflections and Rotations

Curriculum Outcomes	Outcome relevance
9-E6 Transformations (mapping notation): represent and interpret • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping **9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation: ■ *Reflections* • line segments joining points on the original shape to their images are perpendicular to the reflection line and have their midpoint on the reflection line • reflection image of any shape is congruent to the original shape • orientation of a reflection image is the opposite of the original shape [Reflections are limited to reflections in the axes and the line $x = y$.] ■ *Rotations* • for a rotation of $a°$ about point X, a line segment joining a point to X and a line segment joining its image to X are equal in length and form an angle of $a°$ • rotation image of any shape is congruent • orientation of a rotation image is the same as that of the original shape [Rotations are limited to 90 and 180 degrees, clockwise (cw) and counterclockwise (ccw), around (0, 0).] • for 90° rotations, horizontal line segments become vertical, vertical line segments become horizontal, and any line segment and its image are perpendicular • for 180° rotations, line segments are parallel or collinear to their images	This introduction to reflections will support students' later work with transformation of functions. The introduction to rotations will also support future work in trigonometry. In addition, students will enhance their ability to use mapping notation, which is also important in working with functions.

Pacing	Materials	Prerequisites
2 h	• Grid paper • Rulers (mm)	• ability to construct 90° angles • recognition that a turn of 180° has the effect of moving an object to the opposite side of the origin

Main Points to be Raised

• The focus is on reflecting in the axes or in the line $y = x$ because the coordinate descriptions for these reflections are easier.

• The line of reflection can be found by constructing the perpendicular bisector of the segment connecting any point to its image point.

• An original point and its image must be located the same distance from the reflection line.

• The reflection image of any shape is congruent to the original shape, but the orientation is reversed.

• The rotations in this unit are limited to 90° and 180° rotations around the origin so that the mapping descriptions are easy enough for students to use. For other rotations they would need to be familiar with trigonometric functions.

• For rotations of 90° or 180° around the origin, a line segment joining a point to the origin and a line segment joining its corresponding image point to the origin are equal in length and form an angle of 90° or 180°, respectively. This angle is called the angle of rotation.

• The rotation image of any shape is congruent to its original shape and has the same orientation.

• In 90° rotations, each line segment is perpendicular to its image. In 180° rotations, each line segment is parallel or collinear to its image.

• For a 90° rotation, the coordinates are reversed and one coordinate changes its sign. For a 180° rotation, the coordinates do not reverse but both change signs.

Try This—Introducing the Lesson

A. and B. Students can work alone or with a partner. To prepare students for working on **part A**, you might ask them to trace and cut out a copy of ΔPQR. They could then move the copy from the original position to the image coordinates in different ways.

Observe while students work. You might ask:

• *Are the two triangles congruent? How do you know?* (The three sides are the same length, SSS.)

• *Which point would you call P' if you are Novin? Lobzang? Explain your thinking*. (P' is (–2, –1) because the transformation is a reflection; P' is (2, –1) because the transformation is a rotation)

The Exposition—Presenting the Main Ideas

• With students working in pairs, have one partner read the material on reflections while the other partner reads the material on rotations. They could then "teach" each other the ideas about reflections and rotations. Finally, as a class you could go over the material in the second half of the exposition.

• Ask students to predict the mapping notation for a reflection in the *y*-axis or for a 90° rotation around (0,0).

• The focus in the exposition is not on presenting rules for students to follow in doing transformations, but on providing some background and vocabulary for their upcoming work with reflections and rotations.

Revisiting the Try This

C. This question allows students the opportunity to see that different mapping rules might be used for an image and its original shape. Having different mapping rules suggests that the bottom triangle may be labelled differently, depending on the transformation(s).

Using the Examples

• Allow students time to read through the examples on their own.
• To check for understanding you might ask:
- *Which coordinate changed with the reflection in the y-axis?* (the *x*-coordinates changed sign)
- *Why does that make sense?* (the *x*-coordinates are still the same distance from the *y*-axis, but in the opposite direction)
- *Why are the y-coordinates of the image points in **example 1** all the same? Why have the x-coordinates changed?* (The shape has been flipped horizontally, so the *y*-coordinates should stay the same. Since the shape has moved horizontally to a new quadrant, the *x*-coordinates should be opposite.)
- *How could you draw the 90° angle that shows where C moves in **example 2**? Where B moves?* (Draw a line from C (or B) to the origin and another line from C' (or B') to the origin. The angle between the two lines should be 90° counterclockwise (ccw), the angle of rotation.)

Practising and Applying

Teaching points and tips

Q 2: For these early problems with rotation, many students will benefit from using tracing paper to trace the triangle along with the origin and rotating the tracing paper while holding down the origin with a pencil. Others will be more comfortable drawing some or all of the 90° or 180° angles they need to show the rotation.

Q 3: This question is important to help students see further properties of rotations.

Q 4: The corresponding sides PQ and P'Q' as well as PR and P'R' are not parallel to one another (although non-corresponding sides PQ ∥ P'R' and PR ∥ P'Q', because the triangle is isosceles).

Introduce the term *collinear* to describe two line segments which lie along the same line, as is the case with QR and Q'R'. Depending on how the term parallel is defined, these line segments might be described as parallel. If the definition is that parallel lines have the same slope, they could be considered parallel. However, if parallel is defined as two lines that never meet, then QR and Q'R' are not parallel.

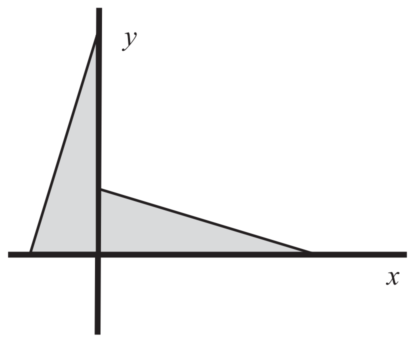
Q 5 and 6: Students can synthesize the differences between the 90° and 180° rotation mappings.

Q 7: By looking for patterns, students can see how easy it is to predict coordinates for reflections in the line $y = x$. Their results for the patterns will be comparable, but not identical.

Common errors

• Many students mix up cw and ccw. Given mapping notation, encourage students to predict the quadrant where the image should end up before deciding if the turn is cw or ccw. For example, if they are given $(x, y) \rightarrow (-y, x)$, they can predict that if the original shape was in the first quadrant, the transformation will produce an image in the second quadrant (a 90° turn ccw).

• To help students remember how the mapping notation is different for 90° cw and 90° ccw, students might use a "model" triangle. For a transformation from the triangle on the right to the triangle on the left, go ccw and notice that $(0, y)$ becomes $(-y, 0)$ so (x, y) becomes $(-y, x)$. For a transformation from the triangle on the left to the one on the right, go cw and notice that $(-x, 0)$ becomes $(0, x)$ so (x, y) becomes $(y, -x)$.



Suggested assessment questions from Practising and Applying

Question 6	to see if students can distinguish between reflections and rotations by looking at the mapping rules
Question 8	to see if students can predict the mapping rules for various familiar rotations and reflections
Question 9	to see if students can visualize and describe the properties of reflections
Question 10	to see if students can visualize and describe the properties of rotations

Answers

A. If you reflected ΔPQR across the x-axis, you would get the bottom triangle. If you rotated ΔPQR 180° around the origin (0, 0), you would get the same bottom triangle.

B. i) P(–2, 1) becomes P'(–2, –1), Q(0, 3) becomes Q'(0, –3), and R(2, 1) becomes R'(2, –1)
ii) P(–2, 1) becomes P'(2, –1), Q(0, 3) becomes Q'(0, –3), and R(2, 1) becomes R'(–2, –1)
iii) The vertices for P' and R' are different.

C. i) Novin: $(x, y) \rightarrow (x, -y)$ **ii)** Lobzang: $(x, y) \rightarrow (-x, -y)$

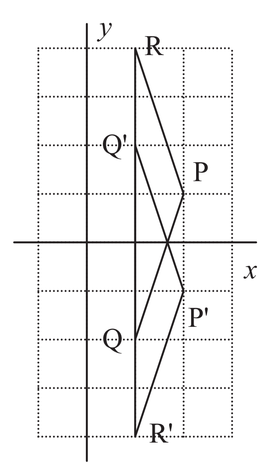
1. a) X'(–1, –2), Y'(–3, 0), Z'(–1, –4)
b) X'(1, 2), Y'(3, 0), Z'(1, 4)

2. a) X'(–2, –1), Y'(0, –3), Z'(–4, –1)
b) X'(2, 1), Y'(0, 3), Z'(4, 1)
c) X'(1, –2), Y'(3, 0), Z'(1, –4)
d) X'(1, –2), Y'(3, 0), Z'(1, –4)

3. a) XY ∥ X'Y' only in the 180º rotations. A line segment that is rotated 90º will be perpendicular to the original.

b) The image ends up in the same place as the original shape in a 180º ccw rotation; and 180º ccw = 180º cw

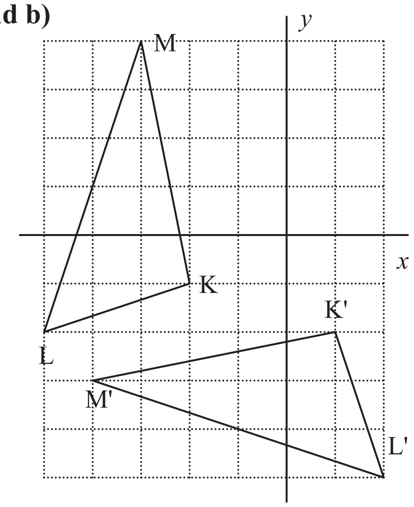
4. a) and b)



Answers [Continued]

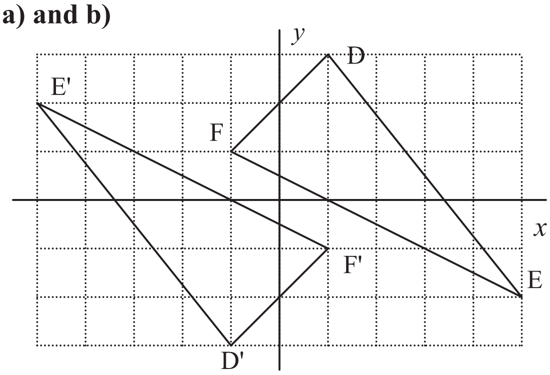
4. c) reflection in the x-axis
d) No, PQ is not parallel to P'Q' and PR is not parallel to P'R' but QR and Q'R' are collinear (see *Teaching Tips*)
e) reflection in the y-axis; corresponding sides would still not be parallel

5. a) and b)



c) rotation around the origin 90º ccw

6. a) and b)



6. c) 180º rotation around origin
d) rotation about the origin; the angle would be 90º ccw or 270º cw

7. a) A(–2, 1), B(–1, 3), C(3, 4), D(2, 2) and A'(1, –2), B'(3, –1), C'(4, 3), D'(2, 2)
b) $(x, y) \rightarrow (y, x)$

8. a) For a 90º cw rotation, one coordinate changes sign, but for a 180º cw rotation, both change sign. For a 90º cw rotation, the x- and y-coordinates switch positions, but for a 180º cw rotation, the coordinates stay in the same positions.
b) For a reflection in the x-axis, the y-coordinate changes sign, but for a reflection in the diagonal line, the x- and y-coordinates switch positions.
c) For a reflection in the y-axis, the x-coordinate changes sign, but for a reflection in the diagonal line, the x- and y-coordinates switch positions.

9. a) Both, because the lengths of the sides and the angle measures do not change, and if triangles are congruent, they are also similar.
b) The orientation is reversed. If the vertex labels are cw on the original shape, they will be ccw on the image.

10. a) Both, because the lengths of the sides and the angle measures do not change, and if triangles are congruent, they are also similar.
b) The orientation does not change. If the vertex labels are cw on the original shape, they will still be cw on the image.

Supporting Students

Struggling students

• Some students might concentrate on axis reflections and 180° rotations first. Later, you could introduce reflections in $y = x$ and 90° rotations.

• Some students will benefit from describing the transformations in their own words as well as in mapping notation in order to make the connection between what they see and how it is written in formal notation.

Enrichment

Students could consider reflections in lines such as $y = a$ or $x = a$ and how those mappings could be described. For example, if you reflect in the line $x = 4$, the mapping (x, y) becomes $(8 – x, y)$. Some students might also be interested in describing the mapping for a reflection in the line $y = –x$.

GAME: Shards

• Player A has the freedom to start with a smaller or larger triangle than the one in the example.
• Players should play the game at least twice, taking turns playing first.

5.2.3 Dilatations

Curriculum Outcomes	Outcome relevance
9-E6 Transformations (mapping notation): represent and interpret • apply translations, reflections, rotations, and dilatations to shapes on the coordinate plane, using mapping notation • describe the nature of a transformation based on a given mapping **9-E7 Transformations: investigate and apply effects on congruence, similarity, and orientation** • understand, through hands-on investigation, properties of each transformation: ■ *Dilatations* • dilatation centre (a point) and its image form a line • ratio of the distance between the dilatation centre and a point on the shape, to the distance between the dilatation centre and a corresponding point on the image, is the same as the scale factor • ratio of the length of a line segment in the original shape to the length of a corresponding line segment in the image is the same as the scale factor • dilatation image of any shape is similar to the original shape • angle measures in the original shape are the same as corresponding angle measures in the image [Dilatations are limited to positive scale factors around centre (0, 0).]	Work with dilatations enhances students' ability to use mapping notation, which supports later work with functions. It also helps students to create scale drawings.

Pacing	Materials	Prerequisites
2 h	• Grid paper • Rulers (mm)	• multiplication of integers • multiplication of fractions

Main Points to be Raised

• The effect of a dilatation is to enlarge or reduce. However, if the ratio is 1, the result is a congruent shape. (Note that the term dilation refers to an enlargement only. Dilatation came mean an enlargement or reduction. The term contraction is sometimes used to describe a reduction.)

• This lesson examines dilatations centred at the origin to make it easier to describe mapping notation. This way, you multiply both coordinates by the same factor to produce the dilatation.

• Dilatations can be described with negative scale factors, which not only reduce or enlarge the original shape but also place the image on the opposite side of the dilatation centre and upside down. These dilatations will not be used in this lesson.

• The line connecting a point to its image point goes through the dilatation centre.

• The angle measures in the original shape are the same as corresponding angle measures in the image shape. The two shapes are similar.

• There are many ways to find the scale factor of a dilatation. For example, the scale factor can be the ratio of the distance between the dilatation centre and a point on the original shape to the distance between the dilatation centre and the corresponding image point. Alternatively, the scale factor can be the ratio of the length of a line segment in the original shape to the length of a corresponding line segment in the image.

• Some people use the word dilation instead of dilatation, but dilation usually refers only to enlargement and not to reduction. Dilatation describes both enlargement and reduction.

Try This—Introducing the Lesson

A. and B. Students can work alone or with a partner. A dilatation can be a reduction or an enlargement. The triangle formed by Devika and her shadow can be dilatated to the triangle formed by the statue and its shadow or vice versa.

Observe while students work. You might ask:

• *How do you know the triangles are similar?* (SAS or SSS)

• *Why did you think of (6, 3) as triple (2, 1) instead of as 4 more in the x-coordinate and 2 more in the y-coordinate?* (Because the statue is not just a translated image. It is bigger.)

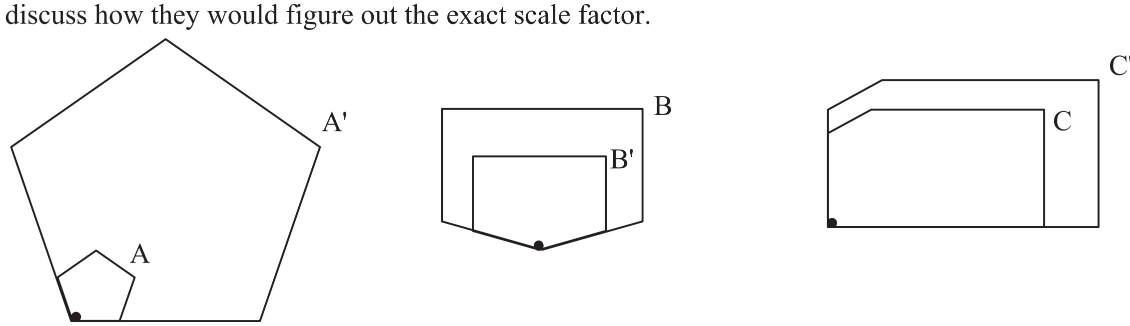
• *How would she use the picture to figure out the height of the statue?* (if she knows the scale factor of her own height to the picture, she can use it to figure out the height of the statue)

The Exposition—Presenting the Main Ideas

• Students can read the exposition on their own.

• Find out if students understand dilatations by asking, for example, what the image triangle would look like if a scale factor of 4 was used (slightly larger but similar to A'B'C), a scale factor of $\frac{1}{4}$ (a lot smaller than the original image), a scale factor of $\frac{2}{3}$ (slightly smaller than the original image), and a scale factor of $\frac{5}{4}$ (slightly larger than the original image).

• You might also provide students with various dilatations, such as those below (the origin is indicated by a dot), and ask them to identify the dilatation first by enlargement or reduction and then estimate the scale factor, and then discuss how they would figure out the exact scale factor.



Enlargement by a SF of about 4 — Reduction by a SF of about $\frac{2}{3}$ — Enlargement by a SF of about $1\frac{1}{3}$

Revisiting the Try This

C. The questions allow an opportunity to make a formal connection between what was done in **parts A and B** and mapping notation. Ask students how they could have predicted that the scale factor would be greater than 1.

Using the Examples

• Ask students to try the example before reading the solution. They can then check their work against the solution.

• You may have to point out that the *x* that identifies the coordinate and the × that identifies multiplication are similar but have very different meanings.

Practising and Applying

Teaching points and tips

Q 2: This question helps students to see more properties of dilatations.

Q 4: This question helps students to see that any one triangle can be seen as a dilatation in many ways.

Q 5: It is important to see that similarity and dilatation are related but not identical. Students should begin to recognize that a picture such as **pair B** with nested triangles is always a dilatation. Some students might use coordinates to help them decide and others might draw lines connecting original points with their images.

Q 8: Once students can explain a method for finding the centre of dilatation, ask them to use that method to find the actual centre.

Q 9 and 10: Together, these questions help students synthesize their understanding of dilatations.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can perform a dilatation
Question 6	to see if students recognize that the only point that does not move in a dilatation (unless the ratio is 1) is the dilatation centre
Question 7	to see if students recognize the need for both coordinates to be scaled in the same way for a dilatation
Question 10	to see whether students understand the effect of both position and scale factor in a dilatation

Answers

A. i) B(2, 0), A(0, 0), E(2, 1) **ii)** C(6, 0), A(0, 0), D(6, 3)

B. The coordinates of ΔCAD are all three times the coordinates of ΔBAE.

C. i) They share ∠A. As well, ∠B and ∠C are equal right angles. (AAA similarity)

ii) The coordinates of ΔCAD are all three times the coordinates of ΔBAE; the triangles are similar. You can draw a straight line from (0, 0) through each pair of corresponding vertices.

iii) $(x, y) \rightarrow (3x, 3y)$

1. a) P'(2, –4), Q'(6, 4), R'(10, 0)

b) S'(2, 2), T'(–2, 2), U'(0, –4)

2. a) W'(2, 2), X(3, 0), Y$(0, -\frac{3}{2})$, Z$(-1, \frac{1}{2})$

b) $\frac{1}{2}$; coordinates were all multiplied by $\frac{1}{2}$

c) Draw a line through each original vertex and its image vertex and the lines should meet at (0, 0).

d) Yes, because there are no rotations or reflections to change them.

3. a) 2 **b)** $\frac{1}{3}$ **c)** $\frac{2}{3}$

4. a) A(1, 2), B(1, –1), C(–2, 0)

b) *Sample response*: using scale factor 4: A(4, 8), B(4, –4), C(–8, 0), or using scale factor 0.5: A(0.5, 1), B(0.5, –0.5), C(–1, 0)

c) You can use any scale factor.

5. a) The angles in each pair are the same (AAA) because of the parallel lines; or, the ratios of pairs of corresponding sides are equal (SSS).

5. b)
pair A: No; because if I draw lines through pairs of corresponding vertices, they do not all meet at the origin;
pair B: Yes; because if I draw lines through pairs of corresponding vertices, they all meet at the origin

6. One vertex was at the centre of dilatation.

7. The *x*- and *y*-coordinates must be multiplied by the same scale factor.

8. Draw lines through pairs of corresponding vertices. They all meet at the centre of the dilatation.

9. a) Always similar and only congruent if the scale factor is 1

b) Corresponding angles are the same.

10. Dilatation images are similar because the ratio between any side length in the original shape and its corresponding side length in the image is the same for all pairs of sides. Not all similar triangles are dilatations because position is important in a dilatation. An example of similar triangles that are not a dilatation can be seen in **question 5, pair A**.

Supporting Students

Struggling students

Some students might benefit from performing more dilatations before you ask them to figure out the centre of the dilatation or the scale factor. You might not assign **questions 6** or **8** to these students.

Enrichment

Some students might explore negative scale factors. They could also try using dilatation centres other than the origin by translating the centre to the origin (and then translating the shape using the same translation rule), dilatating, and then translating back to the centre not at the origin.

CONNECTIONS: Making an Animated Movie

Modern-day cartoons and computer animation have their roots in transformations. Animations are essentially transformations of images with slight changes. The predecessor to the flipbook was the phenakistoscope, which used a circular sequence of images. In 1868 John Barnes Linnet patented the first flipbook, then called a kineograph ("moving picture"). The kineograph employed a linear series of images shown in rapid succession, creating the optical illusion of movement. In 1894, a German inventor named Max Skladanowsky made a flipbook using photographs instead of drawings. A year later, he and his brother Emil developed a movie projector for displaying series of photos. Thomas Edison invented a mechanized variant of the flipbook, the mutoscope, that same year. The pages were mounted on a rotating cylinder instead of being bound in a book. The mutoscope had popular appeal, appearing as a coin-operated machine. A hand-held version of the mutoscope appeared in 1897, marketed as a "Filoscope" by the English filmmaker Henry William Short. It was a toy flip book in a metal holder with a lever to facilitate flipping.

Adapted from: http://en.wikipedia.org/wiki/Flipbook

5.2.4 Combining Transformations

Curriculum Outcomes	Outcome relevance
9-E8 Transformations (mapping notation): analyse and represent composite transformations • analyse a transformation given in mapping notation and represent a transformation using mapping notation • identify the transformations when given the original shape, and the image after a combination of transformations	This lesson uses combinations of transformations to help students develop flexibility in visualizing what happens to a shape as a result of a variety of motions. This flexibility provides a strong basis for geometric problem solving.

Pacing	Materials	Prerequisites
2 h	• Grid paper • Rulers (mm)	• familiarity with translations, rotations, and dilatations

Main Points to be Raised

• The focus of this lesson is on recognizing the relationships between transformations. For example, every double translation can be described as a single translation and vice versa; every double rotation around the same centre can be described as a single rotation and vice versa; and combined reflections in the two axes could also be described as a 180° rotation.

• The mapping notations for combinations of transformations can be combined into one mapping notation. Alternatively, one mapping notation can be decomposed into two or more different notations.

Try This—Introducing the Lesson

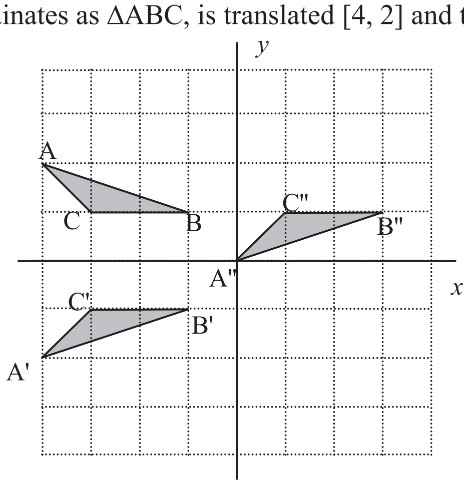
A. Students can work alone or with a partner. Some students will struggle with how to use two folds. You might give the hint of putting down another point.

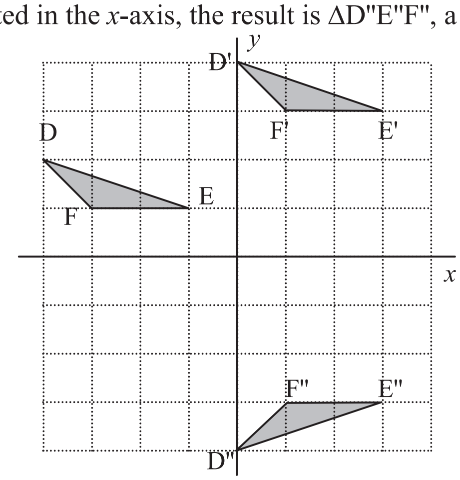
The Exposition—Presenting the Main Ideas

• It is important that combinations of transformations be applied in the given sequence.

• Composite transformations can be described either by a sequence of mapping rules or by a single mapping rule.

• Sometimes the order of transformations has an effect on the result and sometimes it does not. For example, the order is irrelevant with two translations, two rotations around the same centre, or reflections in two axes. However, the order can be important if the combination is a translation followed by a reflection. For example, if ΔABC below is reflected in the *x*-axis and then translated [4, 2], the result is ΔA"B"C". If ΔDEF, with the same coordinates as ΔABC, is translated [4, 2] and then reflected in the *x*-axis, the result is ΔD"E"F", a different result.





• A different number of transformations can have the same effect; a double transformation might also be described as a quadruple transformation. For example, if a shape is translated [2, 4] and [4, 2], that same transformation can be described as [1, 1] followed by [2, 2], followed by [1, 1], followed by [2, 2].

Revisiting the Try This

B. and C. The transformation is only composite if the two folds are equivalent to two reflections, mapping one point onto the other.

Using the Examples

• As you work through **solution 1** with the students, paraphrase the student thinking. You might need to model the translation and its opposite in **part c**. Show how the extra translation and its opposite were arbitrary and that any translation (and its opposite) could have been used.
• Students can read **solution 2** on their own. You could ask one student to explain how the final translations might have been created.

Practising and Applying

Teaching points and tips

Q 1: It is important to realize that the order of the transformations can make a difference in the location of the final image.

Q 2: Students could compare answers to see that it is possible to do this in different ways. For example, you could translate 2 left and 4 up and then dilatate around the origin with a scale factor of $\frac{1}{2}$.

Q 4: This question requires students to use a number of ideas about transformations to solve the problem.

Q 5: By creating their own triangles, students may feel more involved in the process of problem solving.

Common errors

Students often have trouble combining mapping notations. For example, to combine the rotation $(x, y) \rightarrow (-y, x)$ and the translation $(x, y) \rightarrow (x + 2, y + 1)$, many students will write $(x, y) \rightarrow (-y + 1, x + 2)$ instead of $(x, y) \rightarrow (-y + 2, x + 1)$, overlooking the fact that the x and y values had been switched in the first transformation. It is a good idea to ask students to check their work by using actual x and y values.

Suggested assessment questions from Practising and Applying

Question 5	to see if students can represent a mapping notation with a series of transformations
Question 7	to see if students can visualize the effects of transformations and use mapping notation to describe a series of transformations

Answers

A. i) The fold is along the perpendicular bisector of the segment joining the two points.

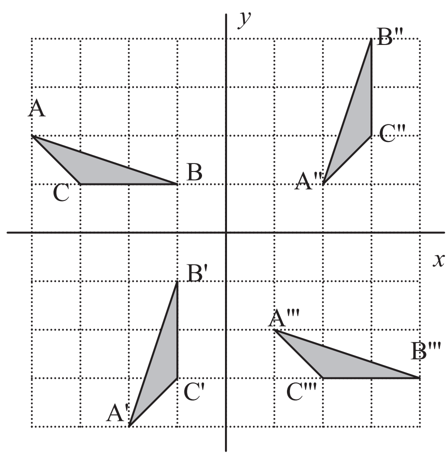
ii) Make a third point anywhere else on the page. Fold along the perpendicular bisector of the segment joining one of the original points and the new point. Then fold along the perpendicular bisector of the segment joining the new point and the remaining point.

B. It is one reflection followed by another reflection.

C. *Sample response*: Two translations

1. a) X'(3, –4), Y'(0, –3), Z'(4, –1) **b)** $(x, y) \rightarrow (x, -y)$ is the reflection, and $(x, y) \rightarrow (x + 1, y - 2)$ is the translation **c)** $(x, y) \rightarrow (x + 1, -y - 2)$ **d)** The coordinates would be X'(3, 0), Y'(0, 1), and Z'(4, 3).	**2.** *Sample responses*: **a)** Dilatate larger triangle using centre (0, 0) and scale factor $\frac{1}{2}$, then translate [–1, 2] **b)** $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$ and $(x, y) \rightarrow (x - 1, y + 2)$ **c)** $(x, y) \rightarrow (\frac{1}{2}x - 1, \frac{1}{2}y + 2)$

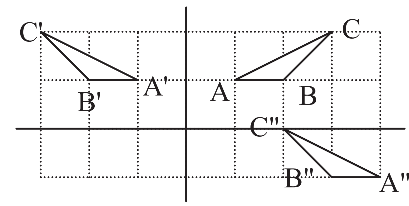
3. a) and b) *Sample response*: ΔABC has vertices A(–4, 2), B(–1, 1), and C(–3, 1).
- Rotate 90º ccw around the origin so the image (ΔA'B'C') is in the 3rd quadrant
- Translate [4, 5] so the image (ΔA"B"C") is in the 1st quadrant
- Rotate 90º cw around the origin so the image (ΔA'''B'''C''') is in the 4th quadrant
- Translate [–5, 4] so the image (ΔA''''B''''C'''') maps onto the original shape (ΔABC) in the 2nd quadrant



4. *Sample responses*:
a) Pair A: ΔPOQ ≅ ΔSRQ using SSS;
Pair B: ΔCOD ≅ ΔOCB using AAS

b)
Pair A: Rotate ΔPOQ 90º ccw around O and then translate [1, –5]. The final image is ΔSRQ.
Pair B: Rotate ΔCOD 180º cw around O and then translate up 4 units. The final image is ΔOCB.

5. *Sample responses*:
a) ΔABC maps onto ΔA"B"C" using $(x, y) \rightarrow (5 - x, y - 2)$



b) A reflection in the y-axis maps ΔABC onto ΔA'B'C', followed by the translation [5, –2], which maps ΔA'B'C' onto ΔA"B"C".

c) $(x, y) \rightarrow (-x, y)$ followed by $(x, y) \rightarrow (x + 5, y - 2)$

6. Yes; a reflection in the x-axis is mapped with $(x, y) \rightarrow (x, -y)$ and a reflection in the y-axis is mapped with $(x, y) \rightarrow (-x, y)$. If you combine them, you get $(x, y) \rightarrow (-x, -y)$, which is the mapping for a 180º rotation.

7. a) Rotate 90º ccw around the origin $(x, y) \rightarrow (-y, x)$, then translate $(x, y) \rightarrow (x + 1, y + 5)$.

b) Translate $(x, y) \rightarrow (x - 1, y - 5)$, then rotate 90º cw around the origin $(x, y) \rightarrow (y, -x)$

c) Reflect in the y-axis $(x, y) \rightarrow (-x, y)$, then translate $(x, y) \rightarrow (x, y - 5)$

d) Translate $(x, y) \rightarrow (x, y + 5)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$

Supporting students

Struggling Students

Some students might find it easier to start by focusing on problems that use only translations or a combination of reflections and translations.

Enrichment

Students could explore the effect of double reflections in pairs of horizontal or vertical lines other than the axes.

UNIT 5 Revision

Pacing	Materials
1 h	• Grid paper • Rulers (mm) • Compasses • Protractors

Question(s)	Related Lesson(s)
1	Lesson 5.1.1
2, 4, 6	Lessons 5.1.2 to 5.1.4
3	Lesson 5.1.2
5	Lesson 5.2.1
7, 8	Lesson 5.2.2
9, 10	Lesson 5.2.3
11, 14	Lessons 5.2.1 to 5.2.3
12, 13	Lesson 5.2.4

Revision Tips

Q 6: It is important to write the similarity relationships to help see which proportions can be used to solve the problem.

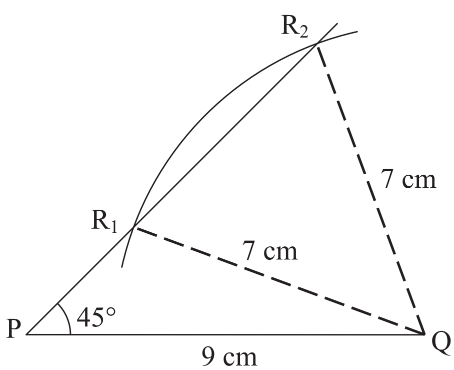
Q 8: The focus here is on the difference in mapping notations for 90° cw and ccw rotation.

Q 13: Students should recognize that there is more than one possibility for each transformation.

Q 14: Encourage students to use a specific example if it helps them describe the relationships.

Answers

1. In ΔPQR_1, $\angle R_1 = 115°$, $PR_1 \approx 3.5$ cm
In ΔPQR_2, $\angle R_2 = 62°$, $PR_2 \approx 9.3$ cm



b) When establishing congruence, knowing the lengths of two sides and any angle is not enough. The angle must be contained between the two sides.

2. a) $\Delta ABD \sim \Delta CBD$ because two angles are the same (AAA). $\Delta ABD \cong \Delta CBD$ because of AAS.

b) BC = 13 m because it corresponds to AB in the congruent triangle.

3. *Sample response*: I constructed three triangles, each with sides 7 cm, 8 cm, and 5 cm and measured all the angles. The corresponding angles in each triangle were the same. I repeated this with other sets of triangles and the results were the same. So SSS also means AAA.

4. VZ = VX would show SAS; $\angle Y = \angle W$ would show ASA; and $\angle Z = \angle X$ would show AAS

5. a) P'(0, 4), Q'(2, 5), R'(1, 12)

b) $(x, y) \rightarrow (x - 3, y + 5)$

6. a) $\Delta HFE \sim \Delta HJG$ because two angles are the same (AAA)

b) 6 cm;
$\frac{HE}{HG} = \frac{HF}{HJ}$ because the triangles are similar, so $\frac{HE}{5} = \frac{8}{4}$.
Thus HE = 10 cm and JE = 10 cm – 4 cm = 6 cm.

7. a) K'(–3, –1), L'(3, 1), M'(–3, 7)

b) $(x, y) \rightarrow (x, -y)$

c) K'(1, –3), L'(–1, 3), M'(–7, –3)

d) $(x, y) \rightarrow (y, x)$

e) In both cases the orientation is reversed.

8. a) S'(1, –2), T'(4, –2), U'(4, –4)

b) S'(–1, 2), T'(–4, 2), U'(–4, 4)

c) Part a) is a rotation of 90º ccw around the origin.
Part b) is a rotation of 90º cw around the origin

d) In both cases the orientation is the same.

9. *Sample response*:

a) triangle with vertices (0, 0), (1, 0), (0, 3)

b) triangle with vertices (0, 0), (3, 0), (0, 9)

c) Similar because of SSS

10. a) A'(0, 2), B'(3, 1), C'(1.5, –1)
b) $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$
c) The orientation is the same.

11. a) reflection in y-axis

b) dilatation, centre (0, 0), scale factor 2

c) rotation of 180º around the origin

d) rotation of 90º cw around the origin

e) translation [–3, 2]

12. a) ΔOJI ≅ ΔFGH using SAS or SSS.

b) *Sample response*:
Reflect ΔFGH in the y-axis and then translate [0, 1].
The final image is ΔOJI.

13. *Sample response*:
a) Dilatate with centre at the origin $(x, y) \rightarrow (2x, 2y)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$, and finally translate $(x, y) \rightarrow (x + 1, y - 8)$

b) Translate $(x, y) \rightarrow (x - 1, y + 8)$, then reflect in the y-axis $(x, y) \rightarrow (-x, y)$, and finally dilatate with centre at the origin $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$

c) Rotate 90º ccw around the origin $(x, y) \rightarrow (-y, x)$, then translate $(x, y) \rightarrow (x + 5, y)$

d) Translate $(x, y) \rightarrow (x - 5, y)$, then rotate 90º cw around the origin $(x, y) \rightarrow (y, -x)$

14. a) congruent, similar, same orientation

b) congruent, similar, opposite orientation

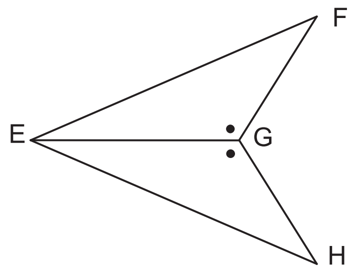
c) congruent, similar, same orientation

d) not congruent, similar, same orientation

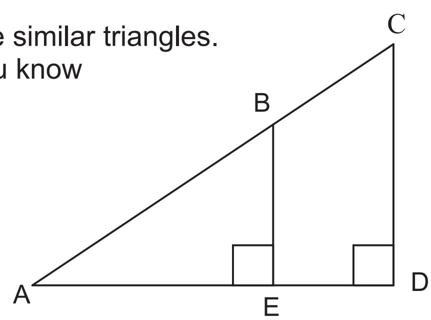
UNIT 5 Geometry Test

1. How would you use examples to show that two triangles are congruent if two sides and the angle between them are equal in both triangles?

2. What further information about side lengths would you need to explain that $\triangle EFG \cong \triangle EHG$? Explain how this information would help.



3. a) Identify the similar triangles. Explain how you know they are similar.



b) If AE = 3 cm, BE = 2 cm, and CD = 3 cm, find the lengths of AD and ED. Show your work.

4. $\triangle PQR$ has vertices P(–3, 4), Q(0, 3), and R(–3, 0). Describe the transformation that would give each image:

a) $\triangle P'Q'R'$: P'(3, –4), Q'(0, –3), R'(3, 0)

b) $\triangle P'Q'R'$: P'(0, 3), Q'(3, 2), R'(0, –1)

c) $\triangle P'Q'R'$: P'(4, –3), Q'(3, 0), R'(0, –3)

5. $\triangle ABC$ has these three vertices: A(1, 3), B(3, –1), and C(7, 6).

a) Locate the vertices of the image of $\triangle ABC$ after the translation [2, –5].

b) Write the mapping that represents the translation in **part a)**.

c) Locate the vertices of the image of $\triangle ABC$ after a reflection in the *y*-axis.

d) Write the mapping that represents the reflection in **part c)**.

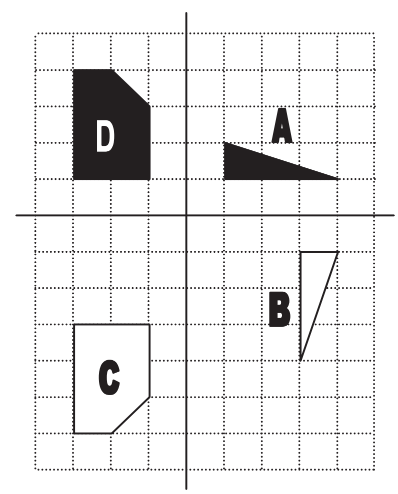
e) Locate the vertices of the image of $\triangle ABC$ after a rotation using the mapping $(x, y) \rightarrow (y, -x)$.

f) Describe the rotation in **part e)**.

g) Locate the vertices of the image of $\triangle ABC$ after a dilatation with centre (0, 0) and scale factor 2.

h) Write the mapping that represents the dilation in **part g)**.

6. For each shape, describe a composite transformation that would map it onto its image. Use mapping notation.



7. a) Fill in the chart. For each statement, write *A* for *Always true*, *S* for *Sometimes true*, or *N* for *Never true* to describe what happens in each kind of transformation.

	Translation	Reflection	Rotation	Dilatation
i) The original shape and its image are congruent.				
ii) The original shape and its image are similar.				
iii) Line segments in the original shape are parallel to corresponding line segments in the image.				
iv) The orientation of the original shape is the same as the orientation of its image.				

b) For each *S* (for *sometimes true*), describe a situation in which the statement is true.

UNIT 5 Test

Pacing	Materials
1 h	• Grid paper • Rulers (mm)

Question(s)	Related Lesson(s)
1, 2, 3	Lessons 5.1.1 to 5.1.4
4	Lessons 5.2.1 and 5.2.2
5	Lessons 5.2.1 to 5.2.3
6	Lesson 5.2.4
7	Lessons 5.2.1 to 5.2.3

Select questions to assign according to the time available.

Answers

1. I would ask everyone in the class to construct any triangle with two sides measuring 7 cm and 8 cm and a 40º angle between these sides. I would then get them to measure the other angles and the other side. We would see that all the triangles are congruent. I would do this again for triangles with different original side lengths and angles between these sides.

2. Knowing FG = HG would show SAS congruence.

3. a) $\Delta ABE \sim \Delta ACD$ because two angles are the same— the right angles and the shared angle.

b) $\frac{AD}{AE} = \frac{CD}{BE}$ because the triangles are similar, so $\frac{AD}{3} = \frac{3}{2}$. Thus AD = 4.5 cm and ED = 4.5 – 3 cm = 1.5 cm.

4. a) rotation of 180º around the origin

b) translation [3, –1]

c) reflection in the line that passes through the origin and (1, 1)

5. a) A'(3, –2), B'(5, –6), C'(9, 1)

b) $(x, y) \rightarrow (x + 2, y - 5)$

c) A'(–1, 3), B'(–3, –1), C'(–7, 6)

d) $(x, y) \rightarrow (-x, y)$

e) A'(3, –1), B'(–1, –3), C'(6, –7)

f) rotation of 90º cw with centre (0, 0)

5. g) A'(2, 6), B'(6, –2), C'(14, 12)

h) $(x, y) \rightarrow (2x, 2y)$

6. *Sample response*:

To map A onto B, rotate $(x, y) \rightarrow (y, -x)$ and then translate $(x, y) \rightarrow (x + 2, y)$.

To map B onto A, do the opposite: translate $(x, y) \rightarrow (x - 2, y)$ and then rotate $(x, y) \rightarrow (-y, x)$.

To map C onto D, translate $(x, y) \rightarrow (x, y + 2)$ and then reflect $(x, y) \rightarrow (x, -y)$.

To map D onto C, do the opposite: reflect $(x, y) \rightarrow (x, -y)$ and then translate $(x, y) \rightarrow (x, y - 2)$.

7. a) i) A A A S

ii) A A A A

iii) A S S A

iv) A N A A (note that this is true for positive dilatation scale factors only)

b) i) *Sample response*:

A dilatation with a scale ratio of one results in a congruent shape.

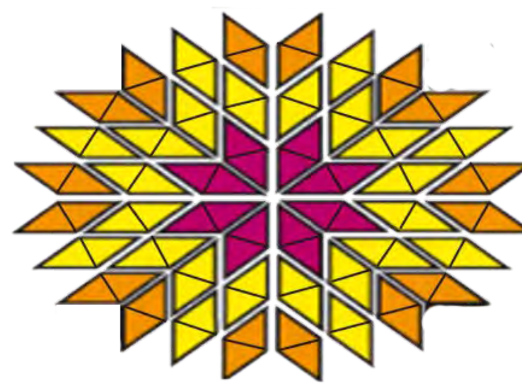
In a reflection, a line segment in the original shape is parallel to its image if the segment is parallel to the mirror line.

In a rotation, a line segment in the original shape is parallel to its image if it is a 180º rotation.

UNIT 5 Performance Task—Designing a Mosaic

Mosaics are used in many cultures to decorate floors and walls. A mosaic is usually made up of many small tiles. These two photographs of mosaics show a tile design on a floor and a wall covered with hand-painted tiles.

In this task, you will perform transformations on a triangle to construct a mosaic. Read through all the instructions and sketch out a plan before you begin your mosaic. *Your writing about your constructions should be done on a separate piece of paper.*



Part A

a) Start your mosaic by drawing any triangle on grid paper. Label the vertices A, B, and C. Decide which point on the grid you will use as the origin of your coordinate grid and label it O. Draw very faint lines (which you will erase when you are done) to show the *x*- and *y*-axes. *Use words to list the coordinates of the vertices of your triangle*.

b) Choose a translation rule and use it to translate your original triangle. Label the vertices of your image triangle. *Write about the translation using words, a translation rule, and mapping notation.*

c) Reflect the original triangle in the *x*- or *y*-axis. Label the vertices of your image triangle. *Use words and mapping notation to describe the reflection*.

d) Rotate the original triangle 90º or 180º using your origin as the centre. Label the vertices of your image triangle. *Use words and mapping notation to describe the rotation*.

e) Dilatate the original triangle using a scale factor of your choice and your origin as the centre. Label the vertices of your image triangle. *Use words and mapping notation to describe the dilatation*.

f) Choose two of your image triangles from **parts a), b), c), and d)**. *Use mapping notation and words to explain how you could map one onto the other*.

g) Add other triangles, polygons, and colour to your design to make it look attractive.

Part B

a) Choose two congruent triangles in your mosaic. *Describe four different ways you could establish that they are congruent.*

b) Choose two triangles in your mosaic that are similar but not congruent. *Describe three different ways you could establish that they are similar*.

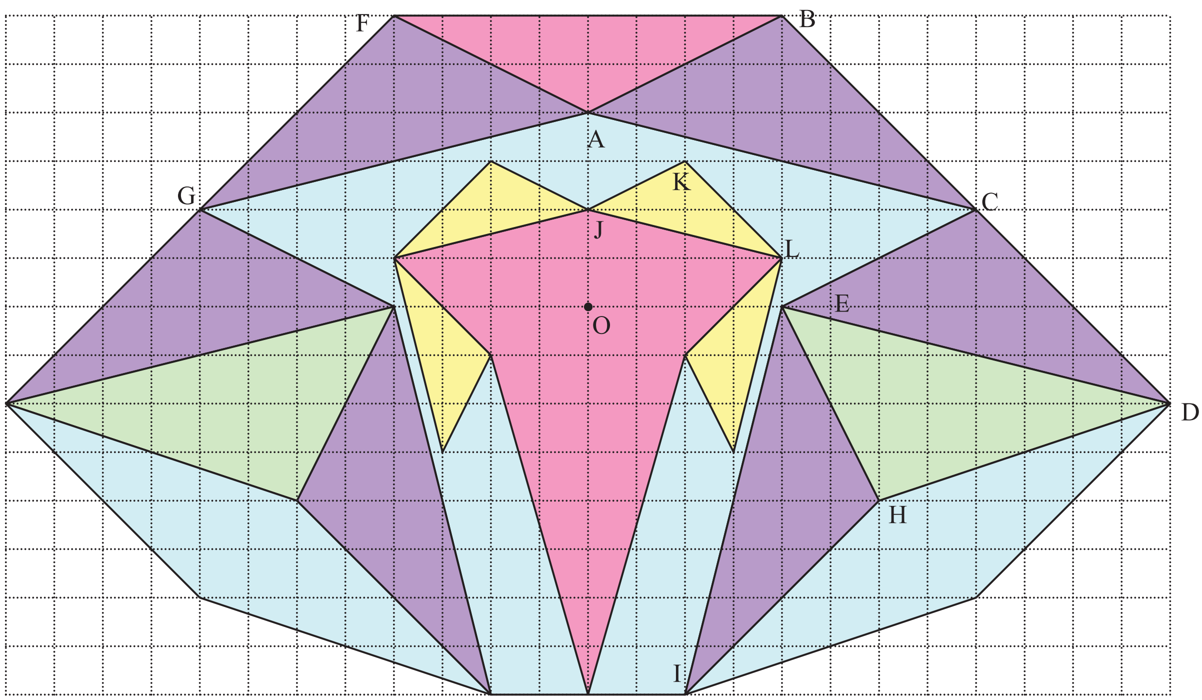
UNIT 5 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-E1 Congruent Triangles: properties and minimum sufficient conditions **9-E3** Similar Triangles: understand and apply proportions **9-E4** Similar Triangles: apply properties **9-E5** Triangles: relate congruency and similarity **9-E6** Transformations (mapping notation): represent and interpret **9-E7** Transformations: investigate and apply effects on congruence, similarity, and orientation **9-E8** Transformations (mapping notation): analyse and represent composite transformations	1 h	• Grid paper • Rulers (mm) • Coloured pencil crayons (optional)

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used as enrichment material for some students.
You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution



Part A

a) I marked the centre, or origin, of my coordinate grid with the letter O. My original triangle is ΔABC with vertices A(0, 4), B(4, 6), and C(8, 2).

b) I translated ΔABC using the rule [4, –4], which means 4 units to the right and 4 down, which is $(x, y) \rightarrow (x + 4, y - 4)$. I named the new triangle ΔECD.

c) I reflected ΔABC in the y-axis and named this triangle ΔAFG. This reflection is $(x, y) \rightarrow (-x, y)$.

d) I rotated ΔABC around the origin 90º clockwise and named the new triangle ΔEHI. This rotation is $(x, y) \rightarrow (y, -x)$.

Sample Solution [Continued]

e) I dilatated ΔABC with centre at the origin and scale factor $\frac{1}{2}$ and named the dilatated image ΔJKL.

This dilatation is $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$.

f) I could map ΔEHI onto ΔJKL using a rotation and a dilatation:

- First, a counterclockwise rotation around the origin $(x, y) \rightarrow (-y, x)$ maps ΔEHI onto ΔABC.
- Second, the dilatation with centre at the origin and scale factor $\frac{1}{2}$ maps ΔABC onto ΔJKL.

I could write the series of transformations with one mapping: $(x, y) \rightarrow (-\frac{1}{2}y, \frac{1}{2}x)$.

For this composite transformation, it would not matter if I switched the order.

Part B

a) There are many congruent triangles in my mosaic, but I chose ΔABC and ΔAFG:

- I could show that $\Delta ABC \cong \Delta AFG$ using SSS, where AB = AF, AC = AG, and BC = FG.
- Or, I could show that $\Delta ABC \cong \Delta AFG$ using ASA, where CB = GF, $\angle ABC = \angle AFG$, and $\angle ACB = \angle AGF$.
- Or, I could show that $\Delta ABC \cong \Delta AFG$ using AAS, where AB = AF, $\angle ABC = \angle AFG$, and $\angle ACB = \angle AGF$.
- Or, I could show that $\Delta ABC \cong \Delta AFG$ using SAS, where AB = AF, CB = GF, and $\angle ABC = \angle AFG$.

b) There are also many pairs of similar triangles in this mosaic, but I chose ΔABC and ΔJKL:

- I could show $\Delta ABC \sim \Delta JKL$ using AAA by measuring just two angles: $\angle ABC = \angle JKL$ and $\angle ACB = \angle JLK$.
- I could show that $\Delta ABC \sim \Delta JKL$ using SAS, where $\angle ABC = \angle JKL$ and $\frac{JK}{AB} = \frac{KL}{BC} = \frac{1}{2}$.
- I could show that $\Delta ABC \sim \Delta JKL$ using SSS, where $\frac{JK}{AB} = \frac{KL}{BC} = \frac{JL}{AC} = \frac{1}{2}$.

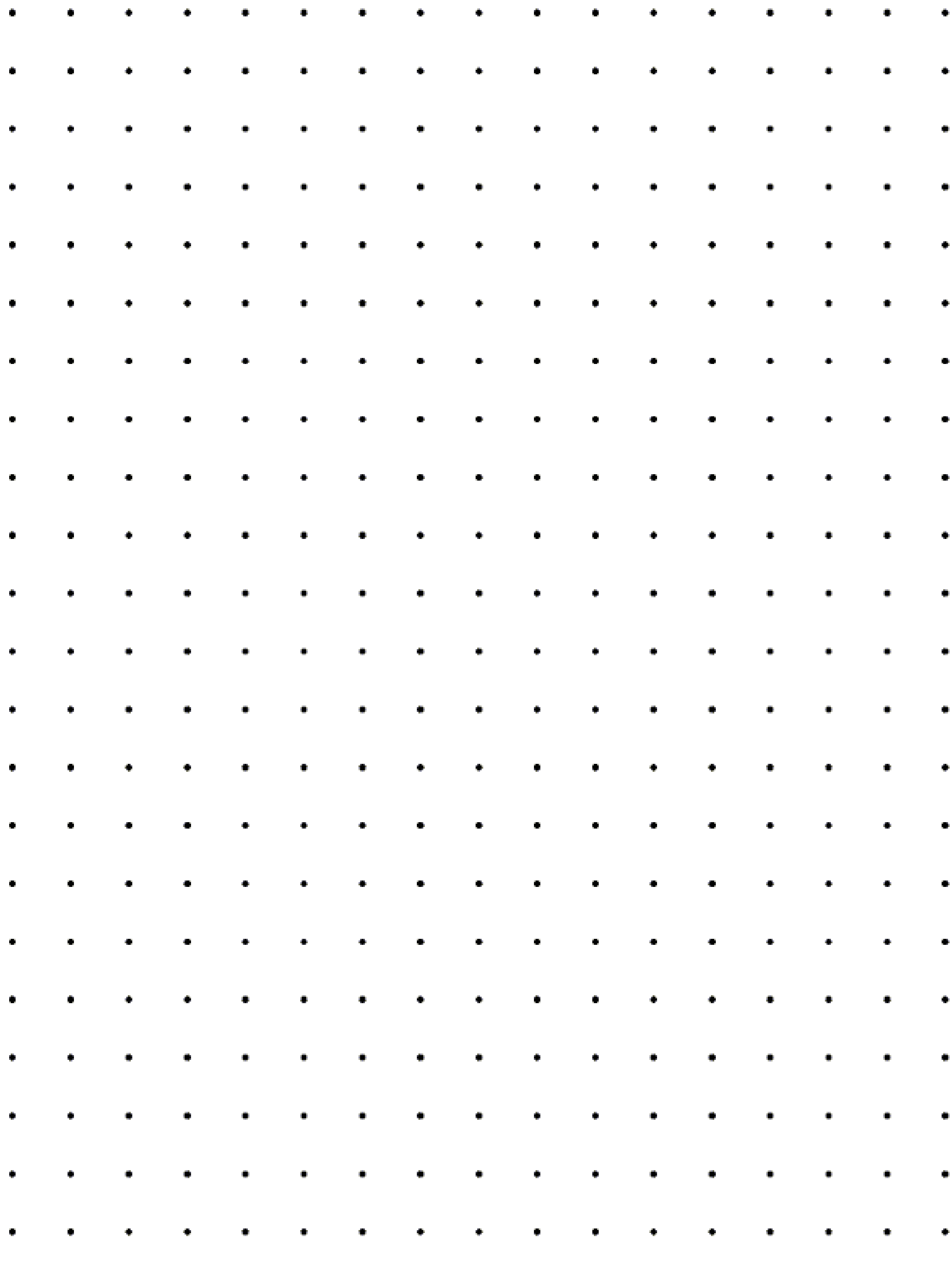
UNIT 5 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Transformations in the Mosaic	Completely accurate transformations	Reasonably accurate transformations (errors do not suggest misconceptions)	Reasonably accurate transformations for most of the mosaic	Major errors in transformations
Description of Transformations	Complete and accurate descriptions of all the transformations	Reasonably complete descriptions for all the transformations (errors or missing items do not suggest misconceptions)	Reasonably complete descriptions for most of the transformations	Major errors in descriptions
Descriptions of Similar and Congruent Triangles	Correct explanations with proper representations	Reasonable explanations with no major representation errors	Reasonable explanations for some of the representations	Major flaws in the explanations

See the UNIT 5 Assessment Interview on **page 274** of this teacher's guide (following the UNIT 6 Performance Task Assessment Rubric).

UNIT 5 Blackline Master 1

Centimetre Square Dot Paper (for Getting Started)



UNIT 5 Blackline Master 2

Grid Paper (0.5 cm by 0.5 cm)

UNIT 6 MEASUREMENT

UNIT 6 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Calculators	All questions
Chapter 1 Volume and Capacity				
6.1.1 Volume of Prisms and Cylinders	**9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders** • estimate and calculate the volume of prisms and cylinders [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: estimate and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes **9-D4 SI Units: solve measurement problems involving conversion** • apply prior measurement skills • solve problems involving mass and capacity units, as well as linear, area, and volume units	2 h	• Many small cubes • Cylinder made of clay • Small knife to cut cylinder • Calculators	Q1, 2, 4a, 5, and 11
GAME: Bean Counting	Explore estimation of volume and capacity in a game-like situation	30 min	• Glass jar • Dried beans	
6.1.2 EXPLORE: Comparing Pyramid and Prism Capacities	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate the volume of pyramids	1 h	• Dried beans (or substitute) • paper (preferably stiff) • Tape or glue • Rulers (mm) • Compasses • Protractors • Calculators	Observe and Assess questions
6.1.3 Volume of Pyramids and Cones	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate and calculate the volume of pyramids and cones • solve problems that involve finding the dimensions of a shape when the volume is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: estimate and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes	2 h	• Model of a pyramid • Model of a cone • Calculators	Q1, 6, 9, and 10

UNIT 6 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
6.1.4 Volume of Spheres and Composite Shapes	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate and calculate the volume of spheres • solve problems that involve finding the dimensions of a shape when the volume is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes	1 h	• Measuring cup • A ball that fits in the measuring cup • Enough water to fill the measuring cup • Calculators	Q1, 4, 7, and 10
CONNECTIONS: Perspective	Explore 3-D perspective		• A sheet of glass • Cube block • Cylinder block	
Chapter 2 Surface Area				
6.2.1 Surface Area of Prisms	**9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders** • estimate and calculate the surface area of prisms [memorization of formulas is not intended at this level **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	1 h	• Calculators	Q2, 3, 5, and 9
6.2.2 Surface Area of Pyramids	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate and calculate the surface area of pyramids • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	1 h	• Paper • Scissors • Calculators	Q1, 5, and 7
6.2.3 Surface Area of Cylinders	**9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders** • estimate and calculate the surface area of cylinders [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	1 h	• Paper (A4 or larger) • Rulers (mm) • Scissors • Calculators	Q2, 5, 6, and 9

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
6.2.4 Surface Area of Cones	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate and calculate the surface area of cones • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	1 h	• Paper • Compasses • Rulers (mm) • Scissors • Calculators	Q1, 2, 5, and 6
6.2.5 Surface Area of Spheres	**9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres** • estimate and calculate the surface area of spheres • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	1 h	• A soccer ball and a large rectangular piece of paper to wrap around it (22 cm × 69 cm) • Calculators	Q1, 5, and 7
UNIT 6 Revision	Review the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 6 Test	Assess the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 6 Performance Task	Assess the concepts and skills in the unit	1 h	• Calculators	Rubric provided

Math Background

• This unit builds on students' previous experience measuring the volume and total surface area of rectangular prisms. It extends that learning to pyramids, cylinders, cones, and spheres.

• Students should be encouraged to estimate volumes rather than always calculating first. This will build their sense of the relative sizes of objects. This is best accomplished by using real objects like food packages whenever possible.

• It is important not only that students know the formulas, but that they also understand, where possible, their derivation. This unit attempts to show students how the formulas are created whenever the derivations use concepts and skills accessible to these students.

• As students work through this unit, they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

• For example, they use problem solving in **question 9** of **lesson 6.1.4** where they have to consider the dimensions of a cube into which a ball fits and in **question 8** of **lesson 6.2.3** and in **question 6** of **lesson 6.2.5**, where they have to develop strategies for determining the surface areas of composite shapes.

• They use communication frequently as they explain their thinking in answering questions, for example, **question 10** of **lesson 6.2.1** or **question 7** of **lesson 6.2.2**. You might notice that the last question in most lessons requires an element of communication.

• Students make connections in each **Try This** problem, where the new learning is connected to a problem they can already solve. There are also specific connections made. For example, connecting mass to capacity in **question 6** of **lesson 6.1.1**, connecting volume to capacity in **question 6** of **lesson 6.1.4**, relating the concept of a pyramid to a real-world historical situation in the **Try This** of **lesson 6.1.3** and relating 3-D to 2-D in the **Connections** on perspective.

• Students use reasoning in relating the volumes of various shapes in **question 11** of **lesson 6.1.1** and in the **Try This** of **lesson 6.1.4.** They also use reasoning when they attempt to calculate the dimensions of two cups with the same total capacity as a given cup in **question 11** of **lesson 6.1.3**.

• They represent formulas in more than one ways as they consider total surface area of a prism in **lesson 6.2.1** and as they represent the volume of a prism as a group of layers of the base area in **lesson 6.1.1.**

• They frequently use visualization in the unit, whether to see the area of a polygon as the sum of triangular areas throughout the unit, to visualize a truncated cone in terms of the original and removed cones in **question 6** of **lesson 6.1.3**, or to visualize the lateral faces of a pyramid as compared to the base in **question 7** of **lesson 6.2.2**.

Rationale for Teaching Approach

• The unit is divided into two chapters. **Chapter 1** focuses on volume, whereas **Chapter 2** focuses on surface area. Although measurement formulas are a central focus of the chapter, the most important thing is to understand and apply the formulas. Students are not expected to memorize the formulas.

• Many of the answers are given exactly in terms of square roots, fractions, and using π, or as numerical approximations, for example,

$5\frac{1}{3}\pi$ cm or 16.76 cm.

• There is one **Explore** lesson, **lesson 6.1.2**, that makes sense of the relationship between the volume of a prism and the related pyramid and the formula for the volume of a pyramid. This particular exploration sets the stage for later work that encourages the use of a net to determine total surface area.

• The **Connections** feature encourages students to consider the notion of perspective.

Technology in This Unit

It is expected that students will have access to a calculator for applying the measurement formulas. Students will need to know how to do the following on scientific calculators:

• use pi (π)

• square a number and take the square root of a number

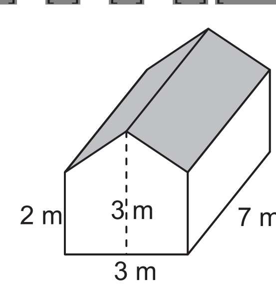
• cube a number and take the cube root of a number

In order to report final calculated answers as accurately as possible, students should be taught how to use the [ANS] key and/or the bracket keys, [( ] and [ )]. For example, the following keystrokes could be used to calculate the volume of the prism below:

[( ] 3 [×] 1 [÷] 2 [ )] [+] [( ] 3 [×] 2 [ )] [=] [×] 7 [=] 52.5

OR

3 [×] 1 [÷] 2 [=] 3 [×] 2 [+] [ANS] [=] [×] 7 [=] 52.5



Getting Started

Curriculum Outcomes	Outcome relevance
8 Square root: exact square root **8** Pythagorean relationship: application **8** Volume and surface area: estimate and calculate for right prisms **8** Area (circles): develop formula **8** SI units: solve measurement problems **8** Proportion: solve indirect measurement problems **7** Large numbers: model **7** Circles: solve problems with diameter, radii, and circumference **7** Area: composite 2-D shapes **6** Area of a triangle: relate to area of a parallelogram	Students have an opportunity to review skills involving SI units and the measurement of polygons and circles in preparation for the work of the unit.

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Calculators	• shape vocabulary • area formulas for triangles, circles, rectangles • circumference formula for a circle • SI unit relationships • exponents and square roots

Use What You Know—Introducing the Unit

A. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *How might you find the area of the trapezoid without using the formula?* (Divide it into a rectangle and a triangle.)

• *Explain why doubling the measurements quadruples (multiplies by four) the areas.* (For a rectangle, you double the width and double the length, and doubling a double is the same as multiplying by four.)

B. This must be done with a partner. Observe while students work. You might ask:

• *Did you get the same measurements as your partner? If not, why not?* (Not all books are exactly the same size. Measurement involves some judgment, e.g., how do you measure a rounded edge?)

Main Points to be Raised

• When linear measurements double, areas quadruple.

• All measurements are approximations. Two people might measure the same item and come up with slightly different results for a variety of reasons, one of them being precision.

• Hatch marks on sides and edges of shapes denote that the side lengths are equal.

Skills You Will Need

• To ensure students have the required skills for this unit, assign all of these questions.

• Students can work individually.

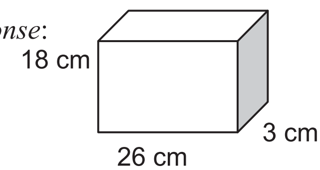
Answers

• In this and every lesson in this unit, there may be minor discrepancies between student responses and answers given in the answer guide due to rounding.

• Point out to students that any measurement is only an approximation, and thus any calculations using measurements are also approximations. If students' answers are close to the provided answers and their work resembles the work shown in the answers, then they should be satisfied with their accuracy.

• The use of the ≈ symbol denotes a rounded approximation.

• For many questions the calculations for reaching the final answer are shown. In these instances, the final answer has been bolded so it is easy to find. The answers in the student book have only the final answer.

A. i) trapezoid-based prism **ii)** two (congruent) trapezoids and four (different) rectangles **iii)** Each trapezoid is 66 cm^2, and the rectangles are 27 cm^2, 72 cm^2, 108 cm^2, and 117 cm^2. **iv)** The areas quadruple (increase by a factor of 4) because the length is multiplied by 2 and the width is multiplied by 2, so when you multiply them together, it is like multiplying by 4 because 2 × 2 = 4.	**B. i)** rectangular prism **ii)** Each face is a rectangle. **iii)** *Sample response*:  **iv)** *Sample response*: The two 18 cm × 26 cm rectangles are 468 cm^2 each. The two 18 cm × 3 cm rectangles are 54 cm^2 each. The two 26 cm × 3 cm rectangles are 78 cm^2 each.

1. a) cube	**b)** square-based pyramid	**c)** octagon-based prism		
2. a) 26 cm^2	**b)** 4.16 m^2	**c)** 13.85 cm^2 or 4.41π cm^2		
3. a) 125	**b)** 17.58	**c)** 3.66		
4. a) 5.22 m	**b)** 10 mm or 1 cm	**c)** 9.80 cm	**d)** 9 cm	**e)** 5.08 cm
5. a) 1897.5 mm^2	**b)** 374.12 m^2			
6. a) 50.27 cm or 16π cm	**b)** 4.77 m			
7. a) 2400 g	**b)** 0.23 L	**c)** 30,000 cm^2	**d)** 5 cm^3	**e)** 4300 cm^3

Supporting Students

Struggling students

It may be helpful to display formulas for the area of a triangle, the area of a circle, and for the circumference of a circle, as well as the statement of the Pythagorean theorem, along with visuals to support them. Some students may also need visual reminders of the relationships between cubic centimetres and millilitres and between millilitres and grams.

Enrichment

Students can create a different shape with the same total surface area as the one in the **Use What You Know** task.

Chapter 1 Volume and Capacity

6.1.1 Volume of Prisms and Cylinders

Curriculum Outcomes	Outcome relevance
9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders • estimate and calculate the volume of prisms and cylinders [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: estimate and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes **9-D4 SI Units: solve measurement problems involving conversion** • apply prior measurement skills • solve problems involving mass and capacity units, as well as linear, area, and volume units	It is important to have a sense of volume both to be able to estimate what might fit in restricted spaces, and to be able to compare the sizes of two items. Prisms and cylinders are commonly used for packaging food and other items, so it is particularly useful to know about their measurements.

Pacing	Materials	Prerequisites
2 h	• Many small cubes • Cylinder made of clay • Small knife to cut cylinder • Calculators	• area of common 2-D shapes • volume of a rectangular prism

Main Points to be Raised

• Students should recall that to calculate the volume of a rectangular prism made of cubes, you count the number of cubes in one layer of the prism and multiply by the number of layers.

• The volume of a prism can be calculated as the product of the area of its base and its height, $V = Ah$. It does not matter what polygon forms the base. This can be explained by imagining a stack of copies of the base, each with a height of one unit, stacked to the actual height of the prism.

• This lesson focuses on right prisms, prisms where the lateral faces joining the bases are rectangles. There are also oblique prisms where the faces joining the bases are parallelograms without right angles.

• A cylinder is similar to a prism in that it has two parallel bases; however, those bases are circles rather than polygons. It is for this reason that the formula for the volume of a cylinder is based on the same principles as the formula for the volume of a prism.

• To calculate the area of the base of a prism with a regular polygon base, it is often helpful to divide it up into congruent isosceles triangles. The base of each triangle is an edge of the base of the prism and the height of each triangle is the apothem, measured from the centre of the base.

• You can determine either the height or the area of the base of a prism or cylinder if you know the volume and the other dimension, height or area of base.

• A cubic centimetre of volume displaces a millilitre of liquid. One way to calculate volume is to determine the number of millilitres of liquid displaced.

•1 mL of water has a mass of 1g.

Try This—Introducing the Lesson

A. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *How did you figure out the number of blocks in the bottom layer, when you could not see it completely?* (It is the same as the top layer, and each other layer)

• *Do you have to count all the blocks in one layer?* (No, you can count one row and then add one more block for each successive row: 6 + 7 + 8 + 9 = 30.)

The Exposition—Presenting the Main Ideas

• Hold up a rectangular prism made up of five layers of six cubes, arranged in two rows of three. Hold it so that the students see the top layer. Ask for the number of cubes in the top layer of the prism. Ask how many layers there are. Then ask how many cubes make up the prism. Talk to the students about why the volume of the prism, which tells the number of cubes required to make it, is the area of the base (in this case, six) multiplied by the height (the number of layers, in this case, five).

• Hold up a cylinder made of clay. Again, talk about the volume as the amount of material it takes to make the cylinder. Ask:

- *Would the volume increase if the base had a greater area?*
- *Would the volume increase if the height were greater?*

Cut off the top layer of the cylinder. Ask the students why it would be reasonable to multiply the area of that layer by the height of the cylinder to determine the volume.

• Ask the students to read the exposition. Check their understanding by asking:

- *What would be the volume of a prism or cylinder with a base of area 20 cm^2 and a height of 5 cm?*
- *Would it matter if it were a prism or a cylinder?*

• Make sure students understand the relationship between cubic centimetres and millilitres, which is required for some of the problems in the exercises.

Note that the exposition (**Presenting the Main Ideas**), **Examples**, and **Practicing and Applying** questions involve only right prisms and right cylinders, where the lateral faces are perpendicular to the bases. Note that if a prism or cylinder is oblique (the bases are still congruent and parallel to each other, but the lines connecting corresponding vertices do not meet the bases at a right angle), the volume formula is still true: $V = Ah$. The height is measured perpendicular to the bases. This point does not need to be raised with students, but they might ask about it.

Revisiting the Try This

B. This question helps students understand the broad definition of a prism (it is does not necessarily have a traditional base like a rectangle or hexagon). It also reinforces the meaning of the volume formula.

Using the Examples

• Group students in threes. Have each student read one of the examples. Then each student explains his or her example to the other students in his or her group. Give an opportunity for students to ask any questions they might still have about the examples.

• **Example 3** uses the volume formula to find the diameter. Ask students what other measurements might be found when volume is given (radius, circumference).

Practising and Applying

Teaching points and tips

Q 3, 7, and 8: Whenever you calculate the volume of a real object, you have to use estimation because of rounded corners and incomplete parts. These small deviations from perfect prisms and cylinders can be ignored for calculation, but it is important for students to be aware that their calculations are only estimates.

Q 4b: This is neither a prism nor a cylinder, but the $V = Ah$ formula still holds.

Q 6: Students will need to make the connection between mass and capacity. In order to determine the volume, they will also need to remember to subtract the empty mass from the full mass to account for the mass of the trough itself.

Q 10: It is essential that students recognize that the diameter of the cylinder is the length of the diagonal of the square base of the cube.

Common errors

Students often struggle to figure out which face of a prism is the base, particularly in a problem like **question 6,** which features a less familiar shape. Students also commonly mistake the diameter of the cylinder in **question 10** as the side length of the square face instead of its diagonal. It may be helpful to observe students as they work on these questions and to watch for these errors so you can intervene.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the volume of a prism when given all required measurements
Question 2	to see if students can determine the volume of a cylinder given the height and either its radius or diameter
Question 4a	to see if students can apply volume formulas to composite shapes
Question 5	to see if students can use volume formulas to determine linear dimensions of a shape
Question 11	to see how students interpret volume formulas in a more abstract situation

Answers *See the note at the beginning of the answers to Getting Started*

A. i) 30

ii) 8

iii) *Sample response*: 30 in the base layer × 8 layers

iv) 240

B. i) it has two congruent opposite polygons joined by rectangle faces

ii) Multiplying the number of blocks in the base by the number of layers is like multiplying the area of the base by the height.

1. a) $A = 6 \times 6 = 36$; $V = Ah = 36 \times 15 =$ **540 cm²**.

b) Dividing the base into six triangles, each with $A = 8.7 \times 7.5 \div 2 = 32.625$ cm², so $V = Ah = (6 \times 32.625) \times 5.5 \approx$ **1076.63 cm³**.

c) Dividing the base into five triangles, each with $A = 5.8 \times 4 \div 2 = 11.6$ cm², so $V = Ah = (5 \times 11.6) \times 6 =$ **348 cm³**.

2. a) $A = \pi r^2 = \pi(6)^2 = 36\pi \approx 113.10$ m²; $V = Ah = 36\pi \times 10 = 360\pi \approx$ **1130.97 cm³**.

b) $A = \pi r^2 = \pi(3.8 \div 2)^2 = 3.61\pi \approx 11.34$ cm²; $V = Ah = 3.61\pi \times 0.7 \approx$ **2.53π or 7.94 cm³**.

3. Dividing the base into a rectangle and a semicircle, the area of the rectangle is $13.0 \times 6.5 = 84.5$ cm² and the area of the semicircle is $A = \pi r^2 \div 2 = \pi(13.0 \div 2)^2 \div 2 \approx 21.13\pi$ or 66.37 cm², so $V = Ah \approx (84.5 + 66.37) \times 22 \approx$ **3319.06 cm³**.

4. a) The volume of the cylinder is $V = \pi r^2 h = \pi(1)^2 \times 5 = 5\pi \approx 15.71$ cm³;
The volume of the rectangular prism is $V = 3 \times 4 \times 5 = 60$ cm³;
Total volume is $V = 16 + 60 =$ **75.71 cm³**.

4. b) $V = Ah = 285 \times 3 \approx$ **855 cm³**

5. a) $V = \pi r^2 h$, $450 = \pi(5.5)^2 h$, $h =$ **4.74 cm**.

b) 18 L = 18,000 cm³;
$V = \pi r^2 h$, $18{,}000 = \pi r^2(54)$, $r =$ **10 cm**.

6. a) Mass of water in trough = 32.6 – 3.7 = 28.9 kg; 28.9 kg = **28.9 L** or **28,900 mL**

b) The area of the base triangle is $A = 36 \times 15 \div 2 = 270$ cm²;
The volume is 28,900 cm³;
Substituting into the volume formula
$V = Ah$, $28{,}900 = 270h$, $h \approx$ **107.04 cm** or **1.07 m**.

7. To find the area of the base, subtract the area of the small circle from the area of the large circle.
The area of the large circle is $A = \pi r^2 = \pi(1.5)^2 = 2.25\pi$ or 7.07 cm²;
The area of the small circle is $A = \pi r^2 = \pi(1.25)^2 = 1.56\pi$ or 4.91 cm²;
The base area is $2.25\pi - 1.56\pi = 0.69\pi$ or 2.16 cm²;
The height is 500 cm;
$V = Ah \approx 2.16 \times 500 \approx$ **1079.92 cm³**.

Answers [Continued]

8. The base is a hexagon with a circle cut out of it. The area of the hexagon is the area of the six triangles that make up the hexagon with $A = 7.5 \times (13 \div 2) \div 2 = 24.375$ mm^2. The area of the circle is $A = \pi r^2 = \pi(7 \div 2)^2 = 12.25\pi$ or ≈ 38.48 mm^2. The base area is $6 \times 24.375 - 38.48 \approx 107.77$ mm^2. $V = Ah = 107.77 \times 6 \approx 646.59$ mm^3. One thousand nuts need $646 \times 1000 =$ **646,592 mm^3** or **646.59 cm^3** of steel. **9.** The diameter of the cylinder is the length and width of the container. To find the diameter, substitute into the volume formula for a cylinder using the volume of 1 L = 1000 cm^3: $V = \pi r^2 h$; $1000 = \pi r^2(15)$; $r \approx 4.61$ cm. The diameter is $4.61 \times 2 = 9.21$ cm. The dimensions of the container are 15 cm × 9.21 cm × 9.21 cm, so the volume is about **1273.24 cm^3**.	**10.** Each edge of the cube is 10 cm because $1000 = s^3$. The square on the base can be divided in half by drawing a diagonal/diameter through opposite vertices. This diameter can be determined using the Pythagorean theorem: $10^2 + 10^2 = d^2$; $d \approx 14.14$ cm. So the radius is $14.14 \div 2 = 7.07$ cm. The volume of the cylinder is $V = \pi r^2 h = \pi(7.07)^2(10) \approx$ **1570.78 cm^3**. **11.** *Sample response*: - You could quadruple the height because $V = Ah$ so $4 \times V = A \times 4 \times h$. - You could quadruple the area because $4 \times V = 4 \times A \times h$. - You could double the area and the height because $4 \times V = 2 \times A \times 2 \times h$. - You could double each base dimension because $4V = 2l \times 2w \times h$.

Supporting Students

Struggling students

• If students are struggling to interpret diagrams of 3-D shapes, it is a good idea to have actual shapes that are similar to those in the drawings available to be manipulated or viewed.

• If students have difficulty with composite shapes (**questions 7 and 8**), ask them to describe with words the 3-D shapes they see in the question.

• If students are struggling with accuracy (because of the many calculations they have to make), encourage or require them to structure their work neatly and to always write the formulas.

Enrichment

• You might challenge students to design a 3-D shape for which their classmates could find the volume.

• Students might be interested in estimating the volume of objects found in the classroom or in the schoolyard.

GAME: Bean Counting

This game has students measuring a cylinder (the jar) to calculate its volume, estimating the volume occupied by a small amount of beans, and comparing these to estimate the number of beans that would fit in the jar. Instead of beans, you can substitute any small objects that are similar in size and shape, such as small sweets or small pebbles.

As an alternative to this game, you could fill the jar with the beans in advance and have students estimate the number of beans in it. In North America this game is commonly used as a carnival game, where a prize is awarded to the person whose guess is closest to the actual number of beans in the jar.

Either form of the game invites students to talk about their volume calculation and estimation strategies.

Note that, in North America, accountants are sometimes called "bean counters." Thus the name of the game is a bit of a pun.

6.1.2 EXPLORE: Comparing Pyramid and Prism Capacities

Curriculum Outcomes	Lesson relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate the volume of pyramids	This core lesson prepares students for the next lesson. Although students can see that a pyramid is less than half the volume of its related prism, it is not apparent that it is exactly one third the volume of the related prism. This lesson gives students hands-on experience with that relationship. Because students work with nets in this lesson, the lesson also prepares them for the emphasis on surface area in the next chapter.

Pacing	Materials	Prerequisites
1 h	• Dried beans (if you do not have access to dried beans, you can replace them with sand, uncooked rice, small pebbles, or any very small solids that can be poured) • paper (preferably stiff) • Tape or glue • Rulers (mm) • Compasses • Protractors • Calculators	• ability to construct a triangle or rectangle with given dimensions

Main Points to be Raised

• The volume of a prism is three times that of the pyramid with the same base and height. (It is *exactly* 3 times, but this exploration is only an approximation because of inaccuracies in making the shapes and measuring.)

• It is best to use the term *lateral face* to describe the faces that are not bases when you talk about the shapes and the nets. This will help students become accustomed to the word, as it is used frequently in the next chapter.

Exploration

A, B, and C. Ask students to work with a partner or in a small group. Observe while students work. You might ask:

• *How did you find the volume of the prism?* (I found the area of the base and then I multiplied by the height.)

• *Are there other ways of making nets for these shapes? Explain how.* (I could have put the back and front rectangles on the same side of the net.)

• *Why does the net for the pyramid have only four and not five faces?* (The prism had the bottom, so the pyramid did not need one.)

Observe and Assess

• Are students able to visualize the surfaces to construct nets for **part B**?

• Do they use what they learn in **part A** to predict what is likely to happen in **part B**?

Share and Reflect

• Ask each group to record their results on the board (How many pyramids are needed to fill the prism in **part A** and in **part B**?)

• Ask one student to sketch one of his or her nets (from **part B**) on the board. Then ask another student to explain how it would fold together to make the prism or pyramid. Then ask for an example of a different net from another student, and ask yet another student to explain that net. Repeat until you have sketches of several of the different nets students have designed.

Answers

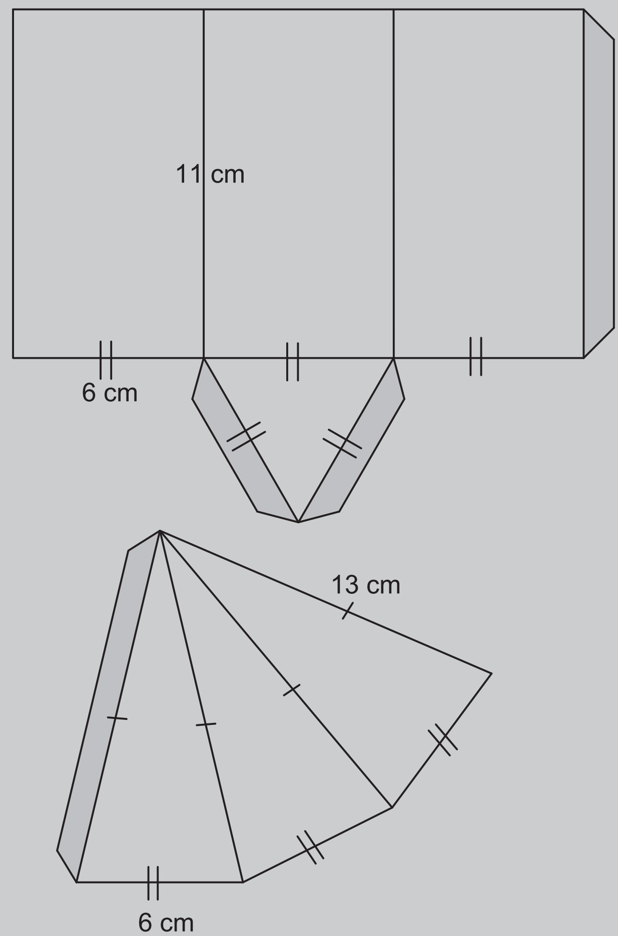
See the note at the beginning of the answers to Getting Started.

A. i) 306 cm^3 (6 cm × 6 cm × 8.5 cm)

ii) it should be three times

iii) 102 cm^3

B. *Sample response*:



B. i) ≈ 171.47 cm^2 (The area of the triangle is about 15.59 cm^2, $V = Ah$ = 15.59 × 11 cm.)

ii) about three times

iii) 57 cm^3

C. The volume of a pyramid is one third ($\frac{1}{3}$) the volume of a prism with same base and height.

Enrichment

You might challenge students to design nets of other pyramid-prism pairs, for example, using a regular hexagonal base.

6.1.3 Volume of Pyramids and Cones

Curriculum Outcomes	Outcome relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate and calculate the volume of pyramids and cones • solve problems that involve finding the dimensions of a shape when the volume is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: estimate and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes	This lesson connects **lessons 6.1.1 and 6.1.2** by presenting the exact volume relationship between prisms and pyramids, and between cylinders and cones. Students will need the formulas in this lesson to solve future problems that involve pyramids and cones.

Pacing	Materials	Prerequisites
2 h	• Model of a pyramid • Model of a cone • Calculators	• ability to calculate the area of basic 2-D shapes • Pythagorean theorem

Main Points to be Raised

• A pyramid has a polygon base and triangular lateral faces that meet at the top vertex, or *apex*.

• The height of a pyramid or cone is the perpendicular distance from the apex to the base.

• The slant height of a cone is the length of a line that starts at the vertex and goes down the cone to meet the edge of the base at a right angle. The slant height of a pyramid is the height of one of its lateral triangle faces; it connects the apex to the middle of one of the edges of the base.

• Cones are similar to pyramids in that the relationship of a cone to a cylinder is the same as the relationship of a pyramid to a prism. They are different in that the base of the cone is a circle whereas the base of the pyramid is a polygon.

• The volume of a pyramid (or cone) is $\frac{1}{3}$ the volume of the prism (or cylinder) with the same base and the same height: $V = \frac{1}{3}Ah$. If students have done **Lesson 6.1.2**, they will be aware of this relationship.

• The height, the radius, and the slant height of a cone represent three sides of a right triangle. The Pythagorean theorem can be used to calculate one of those lengths when the other two are known.

• In a pyramid, the Pythagorean theorem can be used if two of the height, the distance from the centre of the base to a vertex on the base, and the slant edge are known (the slant edge is the length of one of the lateral edges), or if two of the height, the slant height and the distance from the centre of the base to the middle of an edge of the base (the apothem) are known.

Try This—Introducing the Lesson

A. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *How do you know the triangular prism has half the volume of the rectangular prism?* (The height is the same, but the area of the triangular base is half the area of the rectangular base.)

• *Why do you think the volume of the pyramid in **part iii)** is less than the volume of the triangular prism in **part ii)**?* (The base is the same, but it is not as wide.)

The Exposition—Presenting the Main Ideas

• Before students read through the exposition, hold up a pyramid and a cone. In each case, ask students where the base is and where the height is. Make sure they refer to the perpendicular height of the shape for the height. Also point out and name the slant height and slant edge on the pyramid and the slant height of the cone.

• Work through the exposition with the students. Make sure they understand that the height is used directly in the first and second examples, but that the slant height is used to calculate the height of the cone in the third example.

• Point out that, in the case of the first cone, the volume could have been reported as 100π cm^2 or approximated as a decimal, as shown in the text.

Revisiting the Try This

B. This question allows students to make a connection between the students' volume estimates and the result given by the formula.

Using the Examples

• Copy **example 1** on the board. Allow students to try the problem with their books closed and compare their answers to the information provided in the text. Make sure they understand that because the triangle is equilateral, the height bisects the base so that two triangles with sides of 30 cm and 60 cm are formed.

• Work through **example 2** with the students. Help them understand why it was necessary to refer to similar triangles to calculate the height of the small cone that was cut off. Again, reinforce that the volume of the truncated cone could instead have been reported as the exact value $133\frac{2}{3}\pi\ \text{cm}^2$.

Practising and Applying

Teaching points and tips

Q 2: You may need to help students interpret the hatch marks to understand that each edge of the tetrahedron is the same length.

Q 3: Encourage students to draw a diagram to help them solve the problem. It will be necessary to revisit the **Try This**.

Q 5: Encourage students to write out the formula as an equation to be solved.

Q 7 and Q 8: You might ask students to explain why the ratio is what it is. Their responses may help you understand their ways of thinking about the formulas.

Q 8: The ratio of the volumes is exactly 12:10 or 6:5, but because of rounding, students' results may be slightly different. If students leave their volume results in terms of π, the relationship is more obvious.

Q 10: This question is similar to **question 1**, except that students need to use the Pythagorean theorem to find the height.

Q 11: This question is a good one to talk about in class after students have completed it. They will have a variety of responses, and it will help them understand the volume formula if they hear each other's strategies and ways of thinking about the formula.

Common errors

Students may mistake a slant height for the height of a cone or pyramid. You can use the pyramids you constructed in **lesson 6.1.2** to show that the height of the lateral face (the slant height) must be longer than the height of the pyramid. Generalize this to include cones.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the volume of a clearly-marked pyramid or cone
Question 5	to see if students can use the volume formula to infer other measurements
Question 9	to see if students can apply volume formulas to composite shapes
Question 10	to see if students can solve a more complex problem requiring the volume formula for a cone or a pyramid

Answers

See the note at the beginning of the answers to Getting Started.

A. i) 230 × 230 × 139 = 7,353,100 blocks **ii)** (230 × 139 ÷ 2) × 230 ≈ 3,676,550 blocks **iii)** *Sample response*: between 2 and 3 million	**B.** $\frac{1}{3} \times (230 \times 230) \times 139 \approx 2{,}451{,}033\ \text{m}^3$
1. a) $V = \frac{1}{3} \times 25.0 \times 9.0 = \mathbf{75\ m^3}$ **b)** $V = \frac{1}{3} \times 1\pi \times 2 = \frac{2\pi}{3} \approx \mathbf{2.09\ m^3}$ **c)** $V = \frac{1}{3} \times (3.2)^2\,\pi \times 4.2 \approx \mathbf{14.34\pi}$ or $\mathbf{45.04\ m^3}$	**d)** $V = \frac{1}{3}A \times 2.9 = \frac{1}{3} \times [5 \times (1.2 \times 1.7 \div 2)] \times 2.9 =$ $\mathbf{4.93\ cm^3}$ **2. a)** 62.35 cm^2 **b)** 203.69 cm^3 **3.** Length of rope = $\sqrt{139^2 + 115^2} \approx \mathbf{180.41\ m}$

4. 3.6 m^3

5. a) $V = \frac{1}{3}\pi r^2 h$, $450 = \frac{1}{3} \times \pi(5.5)^2 h$, $h =$ **14.21 cm**.

b) 18 L = 18,000 cm^3; $V = \frac{1}{3}\pi r^2 h$, $18{,}000 = \frac{1}{3} \times \pi r^2(54)$; $r =$ **17.84 cm**.

6. a) 12 cm **b)** 6 cm

c) 144π or 452.39 cm^3 **d)** 18π or 56.55 cm^3

e) 126π or 395.84 cm^3

7. a) A is 6 m^3, B is 4 m^3.

b) To find the volume of a square pyramid, you square the base dimension but not the height, so it makes a difference which dimension applies to the base and which applies to the height.

c) 3:2

8. a) A is 120π or 376.99 m^3; B is 100π or 314.16 m^3.

b) To find the volume of a cone, you square the radius but not the height, so it makes a difference which dimension applies to the base and which applies to the height.

c) 12:10 or 6:5

9. Cylinder: $V = 20\pi$ or 62.83 m^3; Cone: $V = 2\pi$ or 6.28 m^3; Whole structure: $V =$ $\mathbf{22\pi}$ or **69.11 $\mathbf{m^3}$**.

10. a) Height: $10^2 = h^2 + 6^2$, $h = 8$ m;
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi(6)^2(8) =$ $\mathbf{96\pi}$ or **301.59 $\mathbf{m^3}$**.

b) Height: $18^2 = h^2 + 4^2$; $h = 17.55$ mm;
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi(4)^2(17.55) =$ $\mathbf{93.60\pi}$ or **294.05 $\mathbf{mm^3}$**.

c) Height: $7^2 = h^2 + 5.2^2$; $h = 4.69$ cm;
Area of the base: $A = 6 \times (6 \times 5.2 \div 2) = 93.6$ cm^2;
$V = \frac{1}{3} \times Ah = \frac{1}{3} \times 93.6 \times 4.69 =$ **146.21 $\mathbf{mm^3}$**.

d) Height: $5.8^2 = h^2 + 4.6^2$; $h = 3.53$ cm;
Apothem (height of the triangle that is one eighth of the base): $4.6^2 = a^2 + (6.5 \div 2)^2$; $a = 3.26$ cm;
Area of the base: $A = 8 \times (6.5 \times 3.26 \div 2) =$ 84.64 cm^2;
$V = \frac{1}{3} \times Ah = \frac{1}{3} \times 84.64 \times 3.53 =$ **99.67 $\mathbf{cm^3}$.**

11. *Sample response*:
Two cones with diameter 6 cm. One has a height of 3 cm and the other a height of 5 cm.

12. *Sample response*:
Visualize tapering the prism from bottom to top so it becomes a wedge with half the original volume. A pyramid with the same base and height is even smaller than this, so it must be less than half.

Supporting Students

Struggling students

• You might ensure that initially students only work with problems where the height is directly given and does not have to be calculated. You might also provide additional experiences where the base of the pyramid is a rectangle or triangle. Only later would problems be included where the height had to be calculated or where the base had more than four sides.

• Students who are struggling might have difficulty communicating to respond to **questions 8b or 12**. Although it would be ideal if students would attempt these problems, they are not essential for the struggling student.

Enrichment

• You might challenge students to design nets of a cylinder and a cone with the same base and height so they can verify the 3:1 relationship in volumes (as they did for prisms and pyramids in **lesson 6.1.2**).

• You might challenge students to design a truncated pyramid and to decide what measurements would be necessary for calculating its volume.

• You might ask students to work in groups to find the largest square-based pyramid that could fit into your classroom. Make sure they realize they could, theoretically, put the base of the pyramid either along a wall or on the floor.

6.1.4 Volume of Spheres and Composite Shapes

Curriculum Outcomes	Outcome relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate and calculate the volume of spheres • solve problems that involve finding the dimensions of a shape when the volume is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the volume of a variety of composite shapes	Spheres are common shapes, so it is useful to have strategies for determining the space they take up.

Pacing	Materials	Prerequisites
1 h	• Measuring cup • A ball that fits in the measuring cup • Enough water to fill the measuring cup • Calculators	• cubes and cube roots

Main Points to be Raised

• The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

• It is not surprising that the variable is raised to the third power. For a cylinder, the volume involves the product of the second power of r and the first power of h. In a sphere, the height is related to the radius, so you need three r's multiplied together, with the appropriate coefficient.

• Students have already been calculating the volume of composite shapes in **lessons 6.1.1 and 6.1.3**, but in this lesson, **lesson 6.1.4**, these types of shapes are formally defined as shapes created by combining or separating known shapes.

• Subscripts (e.g., A_{cone}) are sometimes useful for referring to measurements of many separate shapes, to make it easier to keep track of which measurements refer to which parts.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *Explain how you know the volume of the third shape is smaller than the volume of the first shape.* (It has the same height at the top but not anywhere else, so it must have a smaller volume.)

• *Explain how you know the volume of the third shape is larger than the volume of the second shape.* (It has the same height as the second one, but it is fuller.)

Write the volumes of the first and second shapes in large numerals on two pieces of paper. Affix these two papers to opposite walls in the classroom or at either end of a wall. Say: *Stand between these two measurements to show where you think the volume of the third shape would be*.

The Exposition—Presenting the Main Ideas

• Have students read through the exposition. There is no easy way of explaining why the formula for the volume of a sphere is true. However, using the **Try This** questions, students can get a sense that the volume should be close to the value given by the formula.

• When students have finished reading the exposition, tell them that you are thinking of a sphere with a particular radius, e.g., a radius of 8 cm. Have them estimate, showing with their hands, how big the sphere is. Then have them calculate the volume. Repeat, but this time give the diameter value for a sphere and not the radius.

Revisiting the Try This

C. This question allows students to see that the formula gives a result that is reasonable compared with other shapes they have already worked with. Remind the students of their answers to **part B**, and ask them why they think they might have predicted incorrectly (for those who predicted incorrectly).

Using the Examples

• Have the students look at the presented problem. Ask how the model is different from the stupa, to draw attention to the inevitable estimation that occurs when representing and estimating the volume of real shapes.
• Have the students read through the example and discuss any questions they have in small groups. Offer to respond to any unresolved questions.

Practising and Applying

Teaching points and tips

Q 3: You might provide an example of n such as $n = 2$ and point out the algebra involved in replacing the r in the formula with $2r$. Since the formula involves r^3, substituting $(2r)^3$ results in $2^3 \times r^3 = 8r^3$.

Q 4: Students might be interested to know that many of the planets are not perfect spheres. The earth's polar diameter is 12,720 km and its equatorial diameter is 12,760 km.

Q 6: If you have a measuring cup and a small ball or another shape, you could demonstrate the process of measuring using displacement of water.

Q 7: Many students will have difficulty figuring out how to determine the size of the hemispheres at the ends. It might be useful to sketch a diagram. Students will need to use the relationship 1 m^3 = 1000 L to convert their volume measurement to capacity.

Q 9: Students will need to recognize that the dimensions of the cube are identical to the diameter of the ball.

Common errors

Some students do not recognize the importance of following the order of operations implied by the formula.
It is important that students substitute for the value of r before multiplying by $\frac{4}{3}$ or $\frac{4}{3}\pi$ rather than multiplying the radius by $\frac{4}{3}$ or $\frac{4}{3}\pi$ before cubing the value.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the volume of a sphere given the radius
Question 4	to see if students can compare volumes when given diameters
Question 7	to see if students can apply the volume formula to a composite shape
Question 10	to see how students connect the formula for the volume of a sphere with the formula for the volume of a cylinder

Answers *See the note at the beginning of the answers to Getting Started.*

A. i) 3141.59 cm^3 **ii)** 1047.20 cm^3 **B. i)** It is larger than the cone and smaller than the cylinder. **ii)** *Sample response*: about 2000 cm^3	**C. i)** 2094.40 cm^3 **ii)** It is exactly half way between the other two (though rounding issues may make it hard to see this).

Answers [Continued]

1. and 2. a) and b) Question / Radius / Volume / Factor a) / 1 / $1\frac{1}{3}\pi$ / × 1 b) / 2 / $10\frac{2}{3}\pi$ / × 8 c) / 3 / 36π / × 27 d) / 4 / $85\frac{1}{3}\pi$ / × 64 **c)** Each factor is the radius cubed. **d)** $5^3 = 125$, so $V = 125 \times 1\frac{1}{3}\pi \approx 166\frac{2}{3}\pi$ m^3. **3.** The volume is multiplied by n^3**.** **4.** Jupiter is about 1394.37 times larger. **5.** Volume of sphere = $\frac{4}{3} \times \pi r^3 = \frac{4}{3} \times \pi(2.5)^3 \approx 65.45$ cm^3; Substitute into cylinder formula: $65.45 = \pi(2.5)^2 h$; h = **3.33 cm**. **6. a)** $\frac{4}{3} \times \pi r^3 = \frac{4}{3} \times \pi(5.5)^3 \approx 696.91$ cm^3 = **696.91 mL** **b)** Substitute into cylinder formula: $696.91 = \pi(6)^2 h$; $h \approx$ **6.16 cm**.	**7.** The radius of the cylinder and of each hemispheric end is 0.45 m. The length of the cylindrical centre is 3.4 – 0.45 – 0.45 = 2.5 m. The volume of the cylinder is $\pi r^2 h = \pi(0.45)^2(2.5) = 1.59$ m^3. The two hemispheres combined make one sphere of radius 0.45 m and volume $\frac{4}{3}\pi r^3 = 0.38$ m^3. The total volume is 1.59 + 0.38 ≈ **1.97 m^3**. **8.** The volume of the sphere is about 8181.23 cm^3, so its capacity is about 8181.23 mL, which is 8181.23 g of water. Add the mass of the water to the mass of the container: 8181.23 + 780 g = 8961.23 g = **8.96 kg**. **9. a)** 1843.29π or 5790.87 cm^3 **b)** 11,059.76 cm^3 **c)** The cube is about 2 times larger. **10.** *Sample response*: - Radius of the sphere does not matter. It is $\frac{2}{3}$ the volume of the cylinder because the height of the cylinder is equal to the diameter of the sphere (or 2 times the radius). - Volume of the cylinder is $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$. - Volume of the sphere is $\frac{4}{3}\pi r^3$. You have to multiply the $2\pi r^3$ by $\frac{2}{3}$ to get to $\frac{4}{3}\pi r^3$.

Supporting Students

Struggling students

Questions 3 and 7 may be too abstract for struggling students. You may choose to omit them for these students. Or, for **question 3**, you might replace n with a specific value, such as 5, to make it less abstract, or hypothetical.

Enrichment

• You might challenge students to design a composite shape that includes a sphere or part of a sphere. They can exchange their designs and calculate the volume of each other's shapes.

• You might calculate the volume of different balls that are available to you in class.

CONNECTIONS: Perspective

Another way you can direct students to explore perspective is to give them photographs of simple 3-D shapes. They can trace the outlines and edges of these shapes from the photographs. For example, though a circular object appears to be a circle in a photograph, the shape you actually see in the photograph is rarely circular. It is actually elliptical, but because of the perspective our eyes trick us into seeing the object as circular. You could use diagrams and photographs from this textbook as examples.

Perspective was commonly used in paintings and drawings starting in medieval times in Europe.

Chapter 2 Surface Area

6.2.1 Surface Area of Prisms

Curriculum Outcomes	Outcome relevance
9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders • estimate and calculate the surface area of prisms [memorization of formulas is not intended at this level **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	Determining the surface area of a prism is a particularly useful skill, for example, when considering packaging or heat conservation.

Pacing	Materials	Prerequisites
1 h	• Calculators	• area of basic 2-D shapes • ability to interpret and create nets • perimeter of basic 2-D shapes • Pythagorean theorem

Main Points to be Raised

• The surface area of a 3-D shape is the sum of the areas of all exposed faces. Although the term surface area means the total of all the surfaces, the term total surface area has been used in the student book to emphasize this.

• One way to determine total surface area is to create a net and examine each face of the net to determine its area.

• When you calculate the total surface area of a prism, it makes sense to take advantage of the relationships between faces. For example, you calculate the area of one base and double it to account for the two bases.

• The total surface area of a prism is calculated by adding double the area of one base to the combined lateral surface areas.

• The lateral surface area of a prism can be calculating by multiplying the height of the prism by the perimeter of the base. This is because the lateral faces, when "flattened," form a rectangle. This is evident when you work with nets.

Try This—Introducing the Lesson

A and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *How did you know that you sketched all the faces of the prism?* (I know there are five faces: two triangles and three rectangles. I have accounted for all of them.)

• *How can you find the dimensions of the largest lateral face?* (I can use the Pythagorean theorem to get the height of the rectangle. I know the base is 13 cm.)

The Exposition—Presenting the Main Ideas

• Draw the two shapes shown in the exposition on the board. Discuss what total surface area is and ask students to suggest how to calculate each total surface area. Follow each suggestion through to actually determine the total surface area. Once you have recorded some of the students' suggestions, ask them to read the exposition to check their answers.

• You could ask students to make a more specific formula for the surface area of a trapezoid by replacing the area of the base portion with the formula for the area of a trapezoid, $A = \frac{1}{2}h(b_1 + b_2)$. If students are not familiar with this formula, you might take a few moments to explain its derivation (based on dissecting a trapezoid into two triangles, one with base 1, and the other with base 2).

• Students may not be sure whether to include the base for any surface area calculation. Tell students to always include the base unless there is something in the question that tells you it should not be included.

• Ensure students realize that a right prism (where the polygon bases are directly opposite each other) has two congruent polygon bases and multiple rectangular lateral faces. If the base is a regular polygon, the lateral faces are congruent.

• Students should observe that the units used for surface area will always be square units.

Revisiting the Try This

C. This question affords an opportunity to apply the formula for surface area to a shape students have already thought about.

Using the Examples

• Have students read the examples in pairs. One person in the pair works through **solution 1** and the other does **solution 2**. They then share the approaches with each other.

•For **example 1**, make sure that students understand why square centimetres (cm^2) were used as the area units.

Practising and Applying

Teaching points and tips

Q 2a: Observe whether students multiply the area of one rectangle by 8 or calculate each area separately.

Q 4: This question is designed to elicit students' understanding about the relationship between the surface areas of similar shapes. They should notice that the small prism has linear dimensions that are half those of the larger prism, but that the area dimensions have a different relationship.

Q 5: To calculate the area of the base, students will need to divide the pentagon into a rectangle and a triangle. To find the dimensions of the roof (triangle) they will need to use the Pythagorean theorem.

Q 7: Observe whether students notice that dimensions are given in different units and that some values must be converted before they can be used.

Q 8: Many students will not realize they have to do two steps—first determine the surface area of one face of the cube, and then use it to calculate the length of an edge.

Q 9: Encourage students to estimate before they calculate area.

Common errors

• A number of students confuse area and volume formulas. To keep the distinction between them clear, you may want to display the formulas along with visuals to support them.

• Students often forget that conditions of the problem affect which faces' areas are required. For example, in **questions 7 and 9**, they need to realize that one lateral face should not be included.

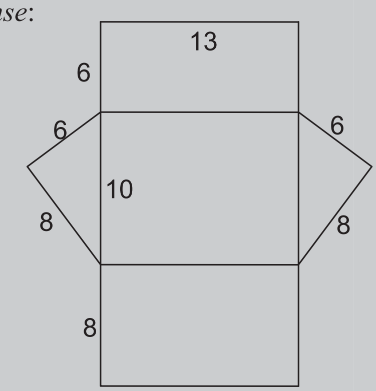
Suggested assessment questions from Practising and Applying

Question 2	to see if students can determine the total surface area of a regular prism given all required measurements
Question 3	to see if students can explain the formula for the surface area of a prism
Question 5	to see whether students can calculate the surface area and volume of a composite shape
Question 9	to see how the students use the surface area formula to infer other measurements

Answers *See the note at the beginning of the answers to Getting Started*.

A. 10 cm using Pythagorean theorem

B. *Sample response*:



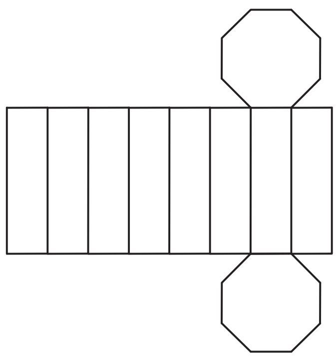
C. 360 cm^2

1. 524 cm^2

2. a) $SA = 2A_{\text{base}} + P_{\text{base}} \times h =$
$2 \times 8 \times (2.5 \times 3.0 \div 2) + (2.5 \times 8) \times 9 =$ **240 cm^2**

b) $SA = 2A_{\text{base}} + P_{\text{base}} \times h =$
$2 \times 5 \times (29 \times 20 \div 2) + (29 \times 5) \times 22 =$ **6090 mm^2**

3. The eight rectangles together make one large rectangle with these dimensions: the perimeter of the octagon × the height of the prism. The two regular octagons are the bases.



4. a) The total surface area of the large prism is 2520 m^2 and the total surface area of the small prism is 630 m^2.

b) 2:1 **c)** 4:1

d) Each face has both dimensions doubled, so it is × 2 × 2, which is the same as × 4.

5. a) Pentagon Faces: $A = 3 \times 4 + (4 \times 0.5 \div 2) =$
$12 + 1 = 13$ m^2; Hypotenuse (angled roof edge):
$x^2 = 2^2 + 0.5^2$; $x \approx 2.06$ m;
Roof rectangles: $A = 2.06 \times 4 \approx 8.25$ cm^2;
Side Rectangles: $A = 3 \times 4 = 12$ m^2;
Floor: $A = 4 \times 4 = 16$ m^2;
Total $SA = 13 + 13 + 8.25 + 8.25 + 12 + 12 + 16 =$
82.49 m^2.

b) $V = Ah = 13 \times 4 =$ **52 m^3**

6. Hypotenuse (angled roof edge): $x^2 = 6^2 + 1.5^2$,
$x \approx 6.18$ m;
Surface area is two rectangles, each with
$A = 6.18 \times 12 \approx 74.22$ m^2;
Total surface area = 74.22 + 74.22 = **148.43 m^2**.

7. $A_{\text{triangle}} = 25 \times 25 \div 2 = 312.5$ cm^2;
$A_{\text{back rectangle}} = 25 \times 200 = 5000$ cm^2;
Hypotenuse: $x^2 = 25^2 + 25^2$, $x \approx 35.36$ cm;
$A_{\text{front rectangle}} = 35.36 \times 200 \approx 7071.07$ cm^2;
Total area = 312.5 + 312.5 + 5000 + 7071.07 ≈
12,696.07 cm^2.

8. a) $A_{\text{square}} = 52 \div 6 \approx 8.67$ cm^2;
Side = $\sqrt{8.7} \approx$ **2.94 cm**.

b) *Sample response*:
2 cm × 2 cm × 5.5 cm

c) $A_{\text{base}} = 6 \times (2.2 \times 1.9 \div 2) = 12.54$ cm^2;
$A_{\text{lateral faces}} = 52 - 12.54 - 12.54 = 26.92$ cm^2;
$A_{\text{rectangle face}} = 26.92 \div 6 = 4.49$ cm^2;
Length of rectangle (height of prism) = 4.47 ÷ 2.2 =
2.04 cm.

9. $1.1 \times 1.4 \times 5 =$ **7.7 m^2**

10. a) Find the cube root of the volume to find the edge, or side length. Square the side length to find the area of each square face. Multiply by 6 because there are 6 square faces.

b) *Sample response*:
No. For example, if a prism has a base area of 20 cm^2 and a height of 5 cm, its volume is the same as a different prism with a base area of 5 cm^2 and a height of 20 cm, but the total surface areas of the two prisms would be very different.

Supporting Students

Struggling students

If students seem uncertain of their answers, ask them to sketch all the faces or a net of the shape, labelling the dimensions of each face.

Enrichment

- You might challenge students by having them estimate the total surface area of prisms in the classroom. (For example, you could ask how much surface area would need to be painted in the room if the walls were painted.)
- You might have students design surface area problems for each other.

6.2.2 Surface Area of Pyramids

Curriculum Outcomes	Outcome relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate and calculate the surface area of pyramids • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	Students extend their knowledge about surface area of prisms to surface area of other shapes, in this case, pyramids.

Pacing	Materials	Prerequisites
1 h	• Paper • Scissors • Calculators	• area of a triangle • Pythagorean theorem

Main Points to be Raised

• The total surface area of a pyramid includes the area of the base and the total area of the triangular lateral faces.

• In the case of right pyramids (the only ones considered in this unit), all the lateral faces will have the same area and all will be isosceles, which includes equilateral triangles. This will simplify calculations.

• In many cases, required measurements are not given directly, but must be calculated using the Pythagorean theorem.

• Calculating the area of the base in order to determine total surface area requires the same approaches students have already practiced when calculating the volume of a pyramid.

• As with other 3-D shapes, visualization skills are needed to ensure that all faces are counted and that the correct measurements apply to the correct lengths.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. It will help if they follow Tshewang's instructions to make the pyramids.

Observe while students work. You might ask:

• *Why did it help to know how the pyramid was constructed to determine the area of each lateral face?* (I knew each face was originally one quarter of a square with area 144 cm^2.)

• *How could you make similar instructions for constructing a pentagon-based pyramid?* (Start with a pentagon, divide it into fifths and cut out one fifth.)

• *How would you find the height of the base triangle?* (Construct a line through one vertex perpendicular to the base and then use the Pythagorean theorem.)

The Exposition—Presenting the Main Ideas

• Work through the exposition with the students. Discuss why the value of *x* was needed to calculate the areas of the lateral faces and how the Pythagorean theorem was used to accomplish this. If necessary, point out to students that the slant height, *x*, is the height of the lateral triangular face. Make sure students understand why the area of only one triangle needed to be calculated since all triangular faces are congruent.

• Make sure students understand how the area of the octagonal base was calculated.

Revisiting the Try This

C and D. Refer students to the pyramids they made (following Tshewang's instructions) and ask them to point out the different parts that might be used to calculate the total surface area of a pyramid: the apothem, the height of the pyramid, and the slant height. These questions encourage visualization, which is an important skill in measurement situations.

Using the Examples

• Pose the two problems presented in the examples on the board. Ask students to work in pairs. Each partner solves one of the problems and then they compare their solutions with those in the text. The students then lead their partners through their own solved examples.
• Ask students how they adapted the formula for the volume of the pyramid to solve **example 2** (by not including the base of the pyramid).

Practising and Applying

Teaching points and tips

Q 2: You may need to remind students that a simple way to construct the net is to start with the base and attach one triangle to each edge of the base.

Q 3: Encourage students to estimate the value of x before they calculate. In particular, ask how they know it must be greater than four.

Q 5: The drawing may confuse students into thinking the base is not a regular polygon. Assure them that it is regular.

Common errors

• Many students continue to use the volume formula even though it is the total surface area that is required. Keep reminding students to consider which measurements they actually need.
• There is often confusion about the h that might be used to represent the height of the pyramid and the h that might be used to represent the height of one of the triangular faces of the pyramid. Students should either use different variable names or make sure they clearly label what part of the shape they are working on at a particular time.
• Many students forget to multiply by a half when they calculate the areas of triangles. It may be necessary to remind them.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the total surface area of a pyramid given all required dimensions
Question 5	to see if students can apply their knowledge to a composite shape
Question 7	to see how students can use visualization to estimate the total surface area of a pyramid

Answers *See the note at the beginning of the answers to Getting Started*.

A. i) 144 cm^2

ii) $\frac{1}{2} \times 12 \times 6 = 36$ cm^2

B. i) equilateral triangle
ii) each side is 12 cm long

C. $3 \times 36 = 108$ cm^2

D. The base is an equilateral triangle with sides 12 cm long. The other faces are isosceles triangles with sides 12 cm, and about 8.5 cm and 8.5 cm (or base 12 cm and height 6 cm). To find the height of the base equilateral triangle:

$x^2 + 6^2 = 12^2$, $x \approx 10.39$ cm; Area of base = $\frac{1}{2} \times 12 \times 10.39 = 62.35$ cm^2; Total $SA = 108 + 62.35 =$ **170.35 cm^2**.

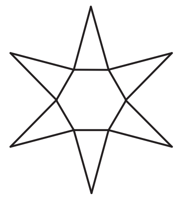
Answers [Continued]

1. a) $SA = 6 \times (8.7 \times 7.5 \div 2) + 6 \times (9.3 \times 8.7 \div 2) =$ **438.48 cm^2**

b) $SA = 8 \times (6 \times 6 \div 2) + 8 \times (11.7 \times 5 \div 2) =$ **378 cm^2**

c) $SA = 6 \times 6 + 4 \times (6 \times 15.5 \div 2) =$ **222 cm^2**

2. *For the area of the six triangles that form the lateral faces*:



Each triangle has a base of 8.7 cm (the side length of the base hexagon) and a height of 9.3 cm (the slant height of pyramid). Use $A = \frac{1}{2}bh$ to find each area and then multiply by 6 to find the area of all six triangles ($\frac{1}{2} \times 8.7 \times 9.3 \times 6 = 242.73$ cm^2).

For the area of the base:
Divide it into six congruent triangles, find the dimensions of one triangle (8.7 cm by 7.5 cm, the apothem), use $A = \frac{1}{2}bh$ to find its area and then multiply by 6 to find the base area ($\frac{1}{2} \times 8.7 \times 7.5 \times 6 = 195.75$ cm^2).

For the total SA:
Add the two areas to get a total SA of 438.48 cm^2.

3. a) 5 m

b) 96 m^2

4. a) Area of base pentagon: 220.35 cm^2;
Height of lateral triangles: 14.31 cm;
Area of each lateral triangle: 80.86 cm^2;
Total $SA = 220.35 + 5 \times 80.86 \approx$ **624.67 cm^2**.

4. b) Base square: 64 cm^2;
Height of lateral triangles: 12.65 cm;
Area of each lateral triangle: 50.60 cm^2;
Total $SA = 64 + 4 \times 50.60 \approx$ **266.39 m^2**.

5. Each rectangle: $25 \times 20 = 500$ cm^2;
Each triangle: $25 \times 30 \div 2 = 375$ cm^2;
Total $SA = 6 \times (500 + 375) =$ **5250 cm^2**.

6. Two cans of wall paint and two cans of roof paint (although there will a lot of roof paint left over):
Each rectangle: $4.8 \times 2.1 = 10.08$ m^2;
Height of each triangle: $1^2 + 2.4^2 = x^2$; $x = 2.6$ m;
Each triangle: $A = \frac{1}{2} \times 4.8 \times 2.6 = 6.24$ cm^2;
SA of walls $= 10.08 \times 4 = 40.32$ m^2 so two cans of wall paint will be needed.
SA of roof $= 6.24 \times 4 \approx 24.96$ m^2 so two cans of roof paint will be needed, although there will be quite a bit left over.

7. a) No; *sample response*: For a square pyramid with a 6 m base and height 4 m, the SA is 96 m^2. When the height is doubled, the SA is 139 m^2, which is less than 192 m^2 (double 96 m^2).

b) Yes; $V = \frac{1}{3}Ah$, so if you double the height you double the volume: $2 \times V = \frac{1}{3} \times A \times 2h$

8. *Sample response*:
The height of the triangles that form the lateral faces is the hypotenuse of the triangle formed by the apothem and the height of the pyramid. Thus the hypotenuse is larger than the apothem, and the triangular lateral faces are larger than the triangles that can be drawn joining the vertices to the centre on the base polygon.

Supporting Students

Struggling students

When using the Pythagorean theorem to find the slant height, students may be unsure which is the hypotenuse of the triangle they are supposed to be visualizing. Ask them to sketch the triangle they are using.

Enrichment

• You might challenge students to design surface area problems that involve pyramids for their classmates.
• You might have students estimate the total surface area of any pyramid-like objects in their environment.

6.2.3 Surface Area of Cylinders

Curriculum Outcomes	Outcome relevance
9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders • estimate and calculate the surface area of cylinders [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	Students extend their knowledge about surface area to surface area of other shapes, in this case the cylinder, which is a very common shape.

Pacing	Materials	Prerequisites
1 h	• Paper (A4 or larger) • Rulers (mm) • Scissors • Calculators	• area of basic 2-D shapes, including circle and rectangle • circumference of a circle

Main Points to be Raised

• The total surface area of a right cylinder is the combined area of the two bases (in situations where both bases are present) and the lateral surface. (Note: The surface area formula only applies to right cylinders because the base of an oblique cylinder is not circular.)

• The total area of the bases is calculated by doubling the area of one base, since the two bases are congruent.

• The lateral surface, when "flattened," can be viewed as a rectangle. The width of the rectangle is the circumference of the circular base of the cylinder. The length of the rectangle is the height of the cylinder.

• The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *Why is the cylinder shorter if you fold it that way?* (because the height of the shape is the shorter length)

• *Why is the cylinder wider if you fold it that way?* (because the long edge forms the diameter instead of the short edge)

The Exposition—Presenting the Main Ideas

• Hold up a cylinder. Ask students how they would determine the area of the bases. Then point out the curved lateral surface. Ask students what shape that surface would roll out to if you cut a slit straight up the height of the cylinder (some students may make the immediate connection to the paper rolling exercise in the **Try This**). It may help to show students a paper cylinder and actually make the cut.

• Once students see that the shape is a rectangle, ask how the length of the rectangle relates to the height of the cylinder (they are the same). Also ask how the width of the rectangle relates to the cylinder (it is the circumference of the base). It may help to run your finger around the curved edge of the base of the cylinder to point out the circumference that turns into the width of the rectangle. Alternatively, you might actually colour the circumference of one base with a crayon. After you cut the cylinder, students will see that the coloured edge is the width of the rectangle formed by the lateral surface.

• Ask students for input and record a student-suggested formula for the surface area of the cylinder. Then have students read the exposition and compare their formula to the one in the text.

• Point out once more that the area can be reported either as the exact value of 42π cm^2 or approximated using the rounded value 131.95 cm^2.

Revisiting the Try This

C. Ask individuals to explain the formula using the curled paper from **parts A and B**. You might ask: - *Which part of the formula for the surface area of a cylinder relates to finding the total surface area of the curled up paper?* ($2\pi rh$) - *Which part of the formula is not used? Why?* (the part for the area of the two circle bases, $2\pi r^2$)

Using the Examples

• Have students read **example 1** independently. Then ask: *Why would the formula $2\pi r^2 + \pi dh$ also work?*
• Draw students' attention to the picture at the top of **example 2**. Work through both solutions with the students. Ask which solution to **example 2** the students prefer and why. (Different students will have different preferences for various reasons.)

Practising and Applying

Teaching points and tips

Q 2: Make sure students realize the three drawings are not drawn to the same scale.

Q 3: Make sure students see the connection between this problem and the **Try This**.

Q 6: Students need to exclude the bottom base of the cylinder and the circle missing from the top face of the prism.

Q 7: Students may forget to include the lateral surface on the inside of the ring.

Q 9: Some students will solve this problem algebraically. They will equate 2 × A with circumference 2 × height and A with circumference 1 × height. They will solve the equation and realize that the second circumference is double the first, so $2\pi r_2 = 2 \times 2\pi r_1$, and $r_2 = 2r_1$. Other students will be able to determine this relationship visually by imagining the rectangles formed by cutting the lateral surface.

Common errors

• Many students, if given the diameter, use it directly rather than taking half to calculate the radius. It might be useful for those students to make a list where they carefully record the value of each variable to be used in the formula. This will draw their attention to the fact that they have not yet calculated r.
• In situations involving composite shapes, such as **question 6**, students often add the separate surface areas rather than remembering that some faces are not exposed because the shapes are attached. Before the students begin the problem, ask them to consider which faces of which shapes should not be included in the final calculation.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can calculate the total surface area of a cylinder given its height and the radius or diameter of the base
Question 5	to see how students apply the formula to find missing dimensions
Question 6	to see if students understand which parts of the formula apply to a composite shape
Question 9	to determine how students understand the formula, whether algebraically or visually

Answers *See the note at the beginning of the answers to Getting Started*.

A. i) short edge	**B. i)** long edge	**C.** 500 cm^2 each
ii) long edge	**ii)** short edge	

1. a) 9 cm

b) 19 cm

c) $9\pi \approx$ **28.27 cm²**

2.a) $SA = 2\pi r^2 + 2\pi rh = 2\pi(2)^2 + 2\pi(2)(5) = 8\pi + 20\pi = 28\pi \approx$ **87.96 m²**

b) $SA = 2\pi r^2 + 2\pi rh = 2\pi(3.4)^2 + 2\pi(3.4)(2.8) = 72.63 + 59.82 \approx$ **132.45 cm²**

c) $SA = 2\pi r^2 + 2\pi rh = 2\pi(6)^2 + 2\pi(6)(1.5) = 72\pi + 18\pi = 90\pi \approx$ **282.74 m²**

3. a) Circumference = 25 cm, so $C = 2\pi r$; $25 = 2\pi r$; $r = 3.98$ cm. $A = \pi r^2 = \pi(3.98)^2 = 16\pi \approx$ **49.74 cm²**.

b) Circumference = 20 cm, so $C = 2\pi r$, $20 = 2\pi r$, $r = 3.18$ cm; $A = \pi r^2 = \pi(3.18)^2 \approx$ **31.83 cm²**.

4. $SA = 2\pi r^2 + 2\pi rh = 2\pi(10)^2 + 2\pi(10)(40) = 200\pi + 800\pi = 1000\pi \approx$ **3141.59 cm²**

5. $A_{\text{lateral surface}} = 2\pi rh$, $9500 = 2\pi(23)h$, $h \approx$ **65.74 cm**.

6. $A_{\text{circular top}} = \pi r^2 = \pi(1)^2 = 3.14\ \text{cm}^2$;
$A_{\text{cylinder's lateral surface}} = 2\pi rh = 2\pi(1)(5) = 10\pi = 31.42\ \text{cm}^2$;
$A_{\text{top of prism}} = 4 \times 5 - A_{\text{circle}} = 20 - 3.14 = 16.86\ \text{cm}^2$;
$A_{\text{bottom prism}} = 4 \times 5 = 20\ \text{cm}^2$;
$A_{\text{lateral surface of prism}} = P \times h = (4 + 5 + 4 + 5) \times 3 = 54\ \text{cm}^2$;
Total $SA = 3.14 + 31.42 + 16.86 + 20 + 54 \approx$
125.42 cm².

7. $A_{\text{top}} = A_{\text{large circle}} - A_{\text{small circle}} = \pi R^2 - \pi r^2 = \pi(11)^2 - \pi(5.5)^2 = 285.49\ \text{mm}^2$;
$A_{\text{outside lateral surface}} = 2\pi rh = 2\pi(11)(2) = 138.23$ mm;
$A_{\text{inside lateral surface}} = 2\pi rh = 2\pi(5.5)(2) = 69.12$ mm;
Total $SA = 285.49 \times 2 + 138.23 + 69.11 \approx$
778.33 mm²;
$V = Ah = 285.49 \times 2 \approx$ **570.99 mm³**.

8. Apothem of hexagon = 13 ÷ 2 = 6.5 mm;
$A_{\text{prism base}} = A_{\text{hexagon}} - A_{\text{circle}} = 6(\frac{1}{2}bh) - \pi r^2 =$
$6 \times \frac{1}{2} \times 7.5 \times 6.5 - \pi(3.5)^2 \approx 146.25 - 38.48 = 107.77\ \text{mm}^2$;
$A_{\text{outside lateral surface}} = 6(lw) = 6(7.5)(6) = 270\ \text{mm}^2$;
$A_{\text{inside lateral surface}} = 2\pi rh = 2\pi(3.5)(6) \approx 131.95\ \text{mm}^2$;
Total $SA = 107.77 + 107.77 + 270 + 131.95 \approx$
617.48 mm².

9. The area of each base is quadrupled (× 4) if the curved surface is doubled and the height is maintained. For example, take a cylinder with height 10 cm and radius 5 cm. The circumference is $2\pi r = 10\pi$. Doubling the lateral surface while maintaining a height of 10 cm would make the circumference 20π. Substituting into the circumference formula, $C = 2\pi r = 20\pi$ and $r = 10$ cm. When the radius is doubled, the area is quadrupled because you square the radius to calculate the area of a circle.

Supporting Students

Struggling students

If students have difficulty with composite shapes (**questions 6, 7, and 8**), ask them to describe with words the 3-D shapes they see in the question. Also, make sure they have additional experience with the individual shapes before they work with combined shapes.

Enrichment

• You might challenge students to estimate the total surface area of cylinder-like objects in the classroom or within the school. They could draw the object and show surface area calculations to display in the classroom.

• You might challenge your students to determine the dimensions of a cylinder that requires the smallest amount of material (the least total surface area) to hold 1 L of water.

6.2.4 Surface Area of Cones

Curriculum Outcomes	Outcome relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate and calculate the surface area of cones • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	Students extend their knowledge about surface area to surface area of other shapes, in this case, cones.

Pacing	Materials	Prerequisites
1 h	• Paper • Compasses • Rulers (mm) • Scissors • Calculators	• area of a circle • circumference of a circle • multiplication of fractions (to follow the formula development)

Main Points to be Raised

• The lateral surface of a cone is a sector of a circle with radius equal to the slant height of the cone.

• The formula for the surface area of a right cone involves two variables: the radius of the base and the slant height. It is made up of two terms, one representing the area of the base of the cone and one representing the area of the lateral surface. (Note: the surface area formula only applies to right cones because the base of an oblique cone is not circular.) The formula is $SA = \pi r^2 + \pi rs$, or $SA = \pi r(r + s)$

• To determine the area of the lateral surface, it is necessary to determine what fraction of a full circle is the lateral surface. This can be done because the base and the lateral surface have identical circumferences.

• The lateral surface is the fraction $\frac{r}{s}$ of a full circle, where r is the radius of the base circle and s is the slant height.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

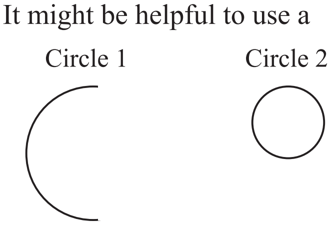
• *How would it affect the cylinder if you cut out less than a quarter of the circle?* (The base circle would be larger and the height of the cone would be less.)

• *How would it affect the cylinder if you cut out more than a quarter of the circle?* (The base circle would be smaller and the height of the cone would be greater.)

The Exposition—Presenting the Main Ideas

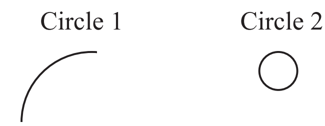
• Work through the exposition with students. Many students will struggle with the portion of the exposition that describes how to determine what fraction of the large circle is the lateral surface. It might be helpful to use a demonstration:

Cut out two circles, one with a radius twice the other. Show students how half of the circumference of the large circle is the same length as the circumference of the small circle. In other words, if the radius is doubled, so is the circumference.



$\frac{1}{2}C_1 = C_2$, where $r_1 = 2r_2$

If students are still confused, follow up by showing how, if one circle has $\frac{1}{4}$ the radius of another, it also has $\frac{1}{4}$ the circumference and so $\frac{1}{4}$ of its circumference of the large circle is the same length as the circumference of the small circle.



$\frac{1}{4}C_1 = C_2$, where $r_1 = 4r_2$

In general, the fraction of the large circle (with radius s) occupied by the portion forming the lateral surface is the same fraction that r is of s.

• Display the formula for the surface area of the cone to help students focus on it.

• Many students will struggle to follow the development of the formula, but will have less trouble applying it.

Revisiting the Try This

C. This question reinforces for students that the total surface area needs to consider both the base and the lateral curved surface. You might ask:

• *What part of the formula for the surface area of a cone relates to the curled up sector?* (πrs)

Using the Examples

Point out to the students that they might want to read through the example before beginning their assigned **practising and applying** questions.

Practising and Applying

Teaching points and tips

Q 2: This question is designed to emphasize the different roles the slant height and the radius play in calculating the total surface area.

Q 2 and Q 6: After students have completed **question 1**, which might help them make predictions, you might ask students to read these two questions and write predictions in their workbooks. After they have recorded their predictions, have a full class discussion asking what predictions students made and why.

Q 4: You may wish to take this question up orally. It is likely that students will have a variety of responses that would be interesting to share.

Q 6: This problem encourages students to use visualization to interpret shapes.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the total surface area of a cone when some dimensions are given but other dimensions must be calculated
Question 2	to see if students can determine the effects of slant height and radius on the surface area
Question 5	to see if students understand which parts of the formula apply to a composite shape
Question 6	to see if students can visually compare the total surface areas of two shapes

Answers *See the note at the beginning of the answers to Getting Started.*

A. i) about 314.15 cm^2

ii) $\frac{3}{4}$ of 314.15 ≈ 235.61 cm^2

B. i) circle

ii) Its circumference is $\frac{3}{4}$ the circumference of the original circle. $\frac{3}{4} \times 62.83 \approx 47.12$ cm.

B. iii) The radius of the base is smaller than the radius of the original circle. The radius of the base can be determined from the circumference $C = 2\pi r$, so $47.12 = 2\pi r$; $r = 7.5$ cm, which is $\frac{3}{4}$ of the radius of the original circle.

C. i) The lateral face is 235.61 cm^2, and the base is $\pi r^2 = \pi(7.5)^2 \approx 176.71$ cm^2, so the total surface area is 235.61 + 176.71 ≈ **412.32 cm^2**.

Answers [Continued]

1. **a)** $SA = \pi r^2 + \pi rs = \pi(12)^2 + \pi(12)(13) = 144\pi + 156\pi = 300\pi \approx$ **942.48 cm²**

b) Slant height: $s^2 = 11.2^2 + 15.5^2$, $s = 19.12$ cm;
$SA = \pi(15.5)^2 + \pi(15.5)(19.12) =$ **1685.96 cm²**.

2. A: Slant height: $s^2 = 3^2 + 4^2$, $s = 5$ cm;
$SA = \pi r^2 + \pi rs = \pi(3)^2 + \pi(3)(5) = 9\pi + 15\pi = 24\pi \approx 75.40$ cm².
B: Slant height: $s^2 = 2^2 + 6^2$, $s = 6.32$ cm;
$SA = \pi r^2 + \pi rs = \pi(2)^2 + \pi(2)(6.3) \approx 52.30$ cm².
Cone A has a greater area.

3. a) Slant height: $s^2 = 3.5^2 + 12^2$, $s = 12.5$ cm;
$SA = \pi rs = \pi(3.5)(12.5) \approx$ **137.44 cm²**.

b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3.5)^2(12) \approx 153.94$ cm³, which is **153.94 mL**.

4. a) Slant height: $s^2 = 3^2 + 4^2$; $s = 5$ cm.
Lateral surface: $A = \pi rs = \pi(3)(5) = 15\pi \approx$ **47.12 cm²**.

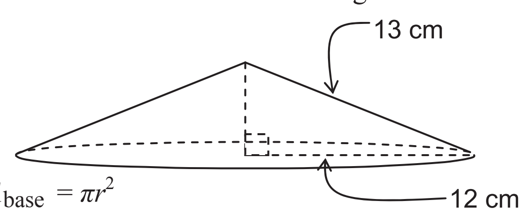
b) Slant height: $s^2 = 1.5^2 + 4^2$, $s \approx 4.27$ cm;
Lateral surface: $A = \pi rs = \pi(1.5)(4.27) \approx 20.13$ cm².
The two cups together require $20.13 \times 2 = 40.26$ cm² of paper, which is less than the amount needed for the original cup. For each of the small cones, the radius is halved, but s is also reduced since s is based on the radius, so the total surface area of each small cone is less than half the total surface area of the original cone.

c) Slant height: $s^2 = 3^2 + 2^2$, $s \approx 3.61$ cm;
Lateral surface: $A = \pi rs = \pi(3)(3.61) \approx 33.98$ cm².
The two cups together require about 67.96 cm² ($33.98 \times 2 = 67.96$) of paper, which is a greater total surface area than that of the original. For each of the small cones, the radius remains the same and only the slant height is reduced, but not to half of the original, so the total surface area of each small cone is more than half the total surface area of the original cone.

5. Slant height of the cone: $s^2 = 2^2 + 1.5^2$, $s = 2.5$ cm;
Area of the lateral surface of the cone $= \pi rs = \pi(2)(2.5) = 5\pi$ m²;
Area of the lateral surface of the cylinder $= 2\pi rh = 2\pi(2)(5) = 20\pi$ m²;
Total $SA = 5\pi + 20\pi = 25\pi \approx$ **78.54 m²**.

6. a) *Sample response*: the pyramid
b) Pyramid: Height of the triangle on each lateral face of the pyramid: $x^2 = 3^2 + 4^2$, $x = 5$ cm;
Area of each triangle $= \frac{1}{2}bh = \frac{1}{2}(6)(5) = 15$ cm²;
Area of base $= 6 \times 6 = 36$ cm²;
$SA_{\text{pyramid}} = 4 \times 15 + 36 = 96$ cm².
Cone: Slant height: $s^2 = 3^2 + 4^2$, $s = 5$ cm;
$SA_{\text{cone}} = \pi r^2 + \pi rs = \pi(3)^2 + \pi(3)(5) = 9\pi + 15\pi = 24\pi \approx 75.40$ cm².
The total surface area of the pyramid is greater:
$SA_{\text{pyramid}} = 96$ cm²; $SA_{\text{cone}} = 75.40$ cm²

7. The slant height must be longer than the radius for the cone to slope upwards. In the formulas for the area of the circle and the lateral surface, the circle area is $A = \pi r^2$ or $\pi \times r \times r$, and the lateral surface area is $A = \pi rs$ or $A = \pi \times r \times s$. s is greater than r, so the area is larger. For example, in the cone below, the radius is 12 cm and the slant height is 13 cm.



$$A_{\text{base}} = \pi r^2 = \pi \times r \times r = \pi \times 12 \times 12$$

$$A_{\text{lateral surface}} = \pi rs = \pi \times r \times s = \pi \times 12 \times 13$$

The lateral surface must be larger.

Supporting Students

Struggling students

If students are struggling with this topic, prioritise **questions 1, 2, 5, and 4** in that order. If necessary, point out where the Pythagorean theorem is needed in **questions 2, 4, and 5**.

Enrichment

• You might challenge students by asking them to design two cones so that one has double the total surface area of the other. (Do this twice, once with an open cone that has no base, and once where the base is included in the total surface area.)

• You might also let students compare the volume and total surface area of the cones formed by cutting different numbers of sectors out of a circle that has been divided into eighths.

6.2.5 Surface Area of Spheres

Curriculum Outcomes	Outcome relevance
9-D2 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres • estimate and calculate the surface area of spheres • solve problems that involve finding the dimensions of a shape when the surface area is given [memorization of formulas is not intended at this level] **9-D3 Volume and Surface Area: measure and calculate for composite 3-D shapes** • estimate and calculate the surface area of a variety of composite shapes	Students extend their knowledge about surface area to surface area of other shapes, in this case, spheres and hemispheres, which are very common shapes.

Pacing	Materials	Prerequisites
1 h	• A soccer ball and a large rectangular piece of paper to wrap around it (22 cm × 69 cm) • Calculators	• squares and square roots

Main Points to be Raised

• Students need to know that the formula for the surface area of a sphere is $4\pi r^2$, which is four times the value of the area of the circle that is the cross-section at the "equator" of the sphere. Students might visualize the cross-section circle at the equator covering each quarter of the surface area of the sphere.

• Although it is difficult to explain the derivation of the formula, students can gain some insight into it by wrapping the ball, as explained in the exposition. The covering will not be exact since the cut out triangles do not fit together perfectly.

Try This—Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *Explain how you know the total surface area of the third shape is smaller than the first.* (It slopes down so you lose surface.)

• *Explain how you know the total surface area of the third shape is larger than the second.* (It is fuller than the cone, so it seems like there is extra surface.)

Write the total surface areas of the first and second shape in large numerals on two pieces of paper. Affix these two papers to opposite walls in the classroom or at either end of a wall. Ask:

• *Where would you stand to show the total surface area of the third shape? Would it be closer to the total surface area of the cylinder or closer to the total surface area of the cone?* (It is likely that students will stand in the middle because the exact middle was the correct place to stand in the similar exercise in **lesson 6.1.4.** However, the middle is incorrect in this case.)

The Exposition—Presenting the Main Ideas

• Have students read the exposition and allow them to ask any questions they have.

• Have one student model for the rest of the class the relationship between the soccer ball and the rectangular paper, using the materials you provide. Invite students to ask questions for clarification.

Revisiting the Try This

C. This question allows students to make a connection between their predictions about the total surface area of the hemisphere and the formula. Invite students to explain why this value might have been difficult to predict.

Using the Examples

Suggest to students that they read through the example before beginning their assigned questions.

Practising and Applying

Teaching points and tips

Q 1: This question requires students to translate the information about the diameter and the circumference into information about the radius in order to calculate the total surface area.

Q 2: Although the Class IX curriculum does not include measurement optimisation, it is certainly reasonable to expose them to it in the context of a unit on volume and surface area. The sphere is usually the most efficient shape as it has the minimum surface area for a given volume. This is an issue in heating spaces and heat loss.

Q 4: Students will have to remember to halve the surface area of the sphere to describe the total surface area of the hemisphere at the top of the balloon.

Q 5: Students will have to remember not to include the portion of the surface of the box that is covered by the hemisphere.

Common errors

Students must be sure to apply the correct order of operations. If they square after multiplying by 4π the answer will be incorrect. Remind students of the order of operations as you discuss the formula—first square the radius and then multiply by 4 and π in either order.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine the surface area of a sphere when given the radius measurement directly or indirectly
Question 5	to see if students can apply the surface area formula to a composite shape
Question 7	to see if students recognize the role of the variable r in the formula for surface area

Answers *See the note at the beginning of the answers to Getting Started.*

A. Cylinder: $SA = 2\pi r^2 + 2\pi rh = 2\pi(10)^2 + 2\pi(10)(10) = 200\pi + 200\pi = 400\pi \approx$ **1256.64 cm²**.

Cone: Slant height: $s^2 = 10^2 + 10^2$, $s \approx 14.14$ cm; $SA = \pi(10)^2 + \pi(10)(14.14) \approx$ **758.45 cm²**.

B. i) It is larger than the cone because you would have to carve away part of the hemisphere to make it the same as the cone. It is smaller than the cylinder because you would have to carve away part of the cylinder to make it the same as the hemisphere.

ii) about 1000 cm^2

C. i) Area of the base circle = $\pi r^2 = \pi(10)^2 = 100\pi$; Area of hemisphere top = $\frac{1}{2}(4\pi r^2) = \frac{1}{2} \times 4\pi(10)^2 = 200\pi$; Total $SA = 100\pi + 200\pi = 300\pi \approx$ **942.48 cm²**.

ii) The surface area of the sphere is larger that that of the cone and smaller than that of the cylinder.

1. a) $SA = 4\pi r^2 = 4\pi(6)^2 = 144\pi \approx$ **452.39 cm²** **b)** $SA = 4\pi r^2 = 4\pi(9)^2 = 324\pi \approx$ **1017.88 m²** **c)** $SA = 4\pi r^2 = 4\pi(7.2)^2 \approx$ **651.44 mm²** **d)** Find the radius: $C = 2\pi r$; $1.0 = 2\pi r$, $r = 1.0 \div 2\pi = 0.16$ m; $SA = 4\pi r^2 = 4\pi(0.16)^2 \approx$ **0.32 m²**. **2.** $SA = \frac{1}{2} \times 4\pi r^2 = \frac{1}{2} \times 4\pi(2.25)^2 \approx$ **31.81 m²** **3. a)** Radius = 3.8 ÷ 2 = 1.9 m; Height of cylinder portion = 9.3 – 1.9 = 7.4 m; Area of the hemisphere = $\frac{1}{2} \times 4\pi r^2 = \frac{1}{2} \times 4\pi(1.9)^2 \approx 22.68$ m²; Area of the lateral surface of the cylinder = $2\pi rh = 2\pi(1.9)(7.4) \approx 88.34$ m²; Total $SA = 22.68 + 88.34 \approx$ **111.02 m²**. **b)** Area of base = $\pi r^2 = \pi(1.9)^2 \approx 11.34$ m²; Volume of cylinder = $\pi r^2 h = \pi(1.9)^2(7.4) \approx 83.92$ m³; Volume of hemisphere = $\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{1}{2} \times \frac{4}{3}\pi(1.9)^3 \approx 14.37$ m³; Total $V = 83.92 + 14.37 =$ **98.29 m³**.	**4.** Radius = 21 ÷ 2 = 10.5 m; Height of cone portion = 35 – 10.5 = 24.5 m; Slant height of cone portion: $x^2 = 10.5^2 + 24.5^2$, $x = 26.66$ m; Surface area of hemisphere top = $\frac{1}{2} \times 4\pi r^2 = \frac{1}{2} \times 4\pi(10.5)^2 \approx 692.72$ m²; Lateral surface of cone = $\pi rs = \pi(10.5)(26.66) \approx 879.27$ m²; Total $SA = 879.27 + 692.72 =$ **1571.99 m²**. **5.** Hemisphere top = $\frac{1}{2} \times 4\pi r^2 = \frac{1}{2} \times 4\pi(5)^2 = 50\pi \approx 157.08$ m²; Top of rectangular prism = $lw - \pi r^2 = (12)(12) - \pi(5)^2 = 65.46$ m²; Lateral faces of prism = $4 \times lw = 4 \times 12 \times 6 = 288$ m²; Total $SA = 157.08 + 65.46 + 288 \approx$ **510.54 m²**. **6.** Radius = 0.9 ÷ 2 = 0.45 m; Length of cylinder centre = 3.4 – 2 × 0.45 = 2.5 m; Surface of two hemisphere ends = $4\pi r^2 = 4\pi(0.45)^2 \approx 2.54$ m²; Lateral surface of cylinder = $2\pi rh = 2\pi(0.45)(2.5) \approx 7.07$ m²; Total $SA = 2.54 + 7.07 \approx$ **9.61 m²**. **7.** The surface area is multiplied by 9, if you triple the diameter, because tripling the diameter triples the radius and the surface area formula squares the radius, $3^2 = 9$. For example: SA of sphere with diameter 2 m = $4\pi r^2 = 4\pi(1)^2 = 4\pi$; SA of sphere with diameter 6 m = $4\pi r^2 = 4\pi(3)^2 = 36\pi$; $36\pi \div 4\pi = 9$

Supporting Students

Enrichment

• You might challenge students to design a composite shape that includes a portion of a sphere. They can exchange sketches of their shapes and calculate the total surface area of each other's shapes.

• To extend **question 7**, you might ask students to generalize the effect on the surface area of multiplying the diameter by a scale factor of n. This can be repeated for volume.

UNIT 6 Revision

Pacing	Materials
1 h	Calculators

Question(s)	Related Lesson(s)
1, 2	Lesson 6.1.1
3, 4	Lesson 6.1.3
5, 6	Lesson 6.1.4
7ab	Lesson 6.2.1
7cd	Lesson 6.2.3
8	Lesson 6.2.2
9	Lesson 6.2.4
10	Lesson 6.2.5
11	Lessons 6.1.1 and 6.2.1
12	Lessons 6.1.3 and 6.2.3
13	Lessons 6.1.1 and 6.2.1
14	Lesson 6.2.4
15, 16, 17	Lessons 6.1.4 and 6.2.5

Revision Tips

Q 2 & 6: Remind students of the relationship between cubic centimetres, millilitres, and grams.

Q 11: There is no single correct answer, but the students' answers will give insight into their mathematical thinking.

Q 13: Students need to reason that they have to add in the total surface area of the triangular prism (less the bases), but in the calculation of volume, the triangle-based prism's volume needs to be subtracted.

Q 17: For this problem, students will have to create their own mathematical model. This may be a challenge for struggling students. Students may notice that you can't measure the diameters but you can measure slant height of cone, circumference of cone, circumference of cylinder (and hence of hemisphere) which is enough.

Answers *See the note at the beginning of the answers to Getting Started.*

1.a) $V = Ah = 5 \times (\frac{1}{2}bh) \times 4 = 5 \times (\frac{1}{2} \times 1.45 \times 1) \times 4 =$ $\mathbf{14.5\ cm^3}$

b) Side of triangle: $14^2 = x^2 + 7^2$; $x \approx 12.12$ cm; $V = Ah = (\frac{1}{2}bh) \times 11 = (\frac{1}{2} \times 7 \times 12.12) \times 11 \approx$ $\mathbf{466.79\ cm^3}$.

c) $V = \pi r^2 h = \pi(3^2)(2) = 18\pi \approx$ $\mathbf{56.55\ cm^3}$

d) $V = \pi r^2 h = \pi(6)^2(9) = 324\pi \approx$ $\mathbf{1017.88\ m^3}$

2. $V = \pi r^2 h = \pi(11.5)^2(14) \approx 5816.66$ cm^3, or 5816.66 mL, or 5816.66 g; 5816.66 g + 1300 g = **7116.66 g** or **7.12 kg**.

3. a) $V = \frac{1}{3}Ah = \frac{1}{3} \times (16 \times 16) \times 15 =$ $\mathbf{1280\ cm^3}$

3. b) Base octagon is 8 triangles with $A = \frac{1}{2}bh = \frac{1}{2}(4.1)(5) = 10.25$; $V = \frac{1}{3}Ah = \frac{1}{3}(10.25 \times 8)(12) =$ $\mathbf{328\ m^3}$.

4. a) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6)^2(8) = 96\pi \approx$ $\mathbf{301.59\ cm^3}$

b) Height of cone: $h^2 + 3^2 = 7^2$; $h \approx 6.32$ cm; $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3)^2(6.32) \approx$ $\mathbf{59.61\ cm^3}$.

5. 11494.04 mm^3 = 11.49 cm^3

6. Volume of hemisphere top = $\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{1}{2} \times \frac{4}{3} \times \pi(3.5)^3 \approx 89.80$; Volume of cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3.5)^2(15) \approx 192.42$ cm^3; Total volume = 192.42 + 89.80 = 282.22 cm^3 or **282.22 mL**.

7. a) $SA = 2A + Ph = 2 \times 5 \times (\frac{1}{2} \times 1.45 \times 1) + (1.45 \times 5)(4) =$ **36.25 cm²**

b) $SA = 2A + Ph = 2(\frac{1}{2} \times 7 \times 12.1) + (7 + 12.1 + 14)(11) =$ **448.8 m²**

c) $SA = 2\pi r^2 + 2\pi rh = 2\pi(3)^2 + 2\pi(3)(2) = 18\pi + 12\pi = 30\pi \approx$ **94.25 cm²**

d) $SA = 2\pi r^2 + 2\pi rh = 2\pi(6)^2 + 2\pi(6)(9) = 72\pi + 108.0\pi = 180\pi \approx$ **565.49 m²**

8. a) Base area = 16 × 16 = 256 cm^2 Each lateral triangle = $\frac{1}{2}bh = \frac{1}{2}(16 \times 17) = 136$ cm^2. Total SA = 256 + 4 × 136 = **800 cm²**.

b) Base area = $8 \times \frac{1}{2}bh = 8 \times \frac{1}{2}(4.1)(5) = 82$ m^2; Each lateral triangle = $\frac{1}{2}bh = \frac{1}{2}(4.1 \times 13) = 26.65$ m^2; Total SA = 82 + 8 × 26.65 = **295.2 m²**

9. a) $SA = \pi r^2 + \pi rs = \pi(6)^2 + \pi(6)(10) = 36\pi + 60\pi = 96\pi \approx$ **301.59 cm²**

b) $SA = \pi r^2 + \pi rs = \pi(3)^2 + \pi(3)(7) = 9\pi + 21\pi = 30\pi \approx$ **94.25 cm²**

10. $SA = 4\pi r^2 = 4\pi(12.8)^2 =$ **2058.87 cm³**

11. *Sample response*:
volume is easier since you just do two multiplications instead of two multiplications and some adding

12. a) Base area = 12 × 12 = 144 m^2; $V = \frac{1}{3}Ah$, $1440 = \frac{1}{3} \times 144h$, $h =$ **30 m**.

b) $SA = 2\pi r^2 + 2\pi rh$; $140 = 2\pi(2.5)^2 + 2\pi(2.5)h$; $140 = 39.27 + 15.71h$; $100.73 = 15.71h$; $h \approx$ **6.41 cm**.

13. Base area = rectangle – triangle = $lw - \frac{1}{2}bh = 23.7 \times 13 - \frac{1}{2}(7.2 \times 6.5) = 284.7$ cm^2; Outside lateral surface = (23.7 + 13 + 23.7 + 13) × 1.2 = 88.08 cm^2; Hypotenuse of triangle: $7.2^2 + 6.5^2 = x^2$, $x = 9.7$ cm; Inside lateral surface = (7.2 + 6.5 + 9.7) × 1.2 = 28.08 cm^2; $V = Ah = 284.7 \times 1.2 \approx 341.64$ cm^3; $SA = 2 \times A$ + outside lateral surface + inside lateral surface = 2 × 284.7 + 88.08 + 28.08 ≈ **685.56 cm²**.

14. a) Lateral surface of cone A: $A = \pi rs = \pi(6)(13) = 78\pi \approx$ **245.04 cm²**; Lateral surface of cone B: $A = \pi rs = \pi(6)(15) = 90\pi \approx$ **282.74 cm²**.
b) 13 ÷ 15 = **0.87** **c)** $78\pi \div 90\pi =$ **0.87**
d) Only one thing is different when you calculate the lateral surface of each cone, $s = 13$ vs. $s = 15$. Because s in both cases is multiplied by the same value (6π), the ratio must be the same, 13 ÷ 15.

15. Volume of prism = 6 × 31 × 16 = 2976 cm^3; Volume of two hemispheres = $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(7)^3 =$ 1436.76 cm^3; Total volume = 2976 + 1436.76 = **4412.76 cm³**. Surface area of two hemispheres = $4\pi r^2 = 4\pi(7)^2 = 196\pi = 615.75$ cm^2.; Top surface = rectangle – two circles = $31 \times 16 - 2 \times \pi(7)^2 = 188.12$ cm^2; Lateral surfaces of prism = 6 × (31 + 16 + 31 + 16) = 564 cm^2; Bottom = 31 × 16 = 496 cm^2; Total SA = 615.75 + 188.12 + 564 + 496 = **1863.88 cm²**.

16. a) Volume of hemisphere = $\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{1}{2} \times \frac{4}{3}\pi(4)^3 \approx 134.04$ m^3; Volume of cylinder = $\pi r^2 h = \pi(4)^2(6) \approx 301.59$ m^3; Volume of cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(4)^2(3) \approx 50.27$ m^3; Total volume = 134.04 + 301.59 + 50.27 = **485.90 m³**.
b) Slant height of cone: $4^2 + 3^2 = s^2$; $s = 5$ m; SA of hemisphere = $\frac{1}{2} \times 4\pi r^2 = \frac{1}{2} \times 4\pi(4)^2 = 100.53$ m^2; Lateral surface of cylinder = $2\pi(4)(6) = 48\pi \approx 150.80$ m^2; Lateral surface of cone = $\pi rs = \pi(4)(5) \approx 62.83$ m^2; Total SA = 100.53 + 150.80 + 62.8 ≈ **314.16 m²**.

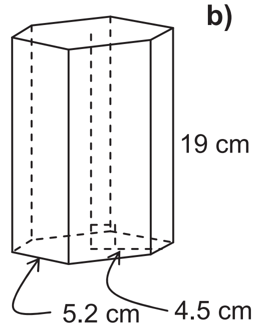
17. a) *Sample response*:
Cone on top of a square prism, hemisphere in the middle, cylinder at the bottom
b) Measure the diameter and slant height of the cone, the height and side length of the prism, the diameter of the hemisphere (and cylinder), and the height and diameter of the cylinder.

UNIT 6 Measurement Test

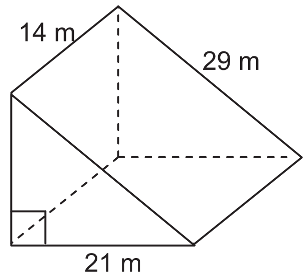
For each question, show your work.

1. What is the volume of each?

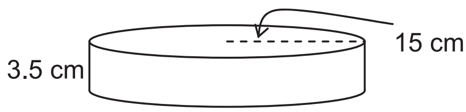
a)



b)



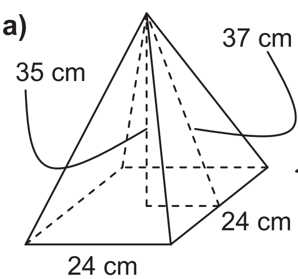
c)



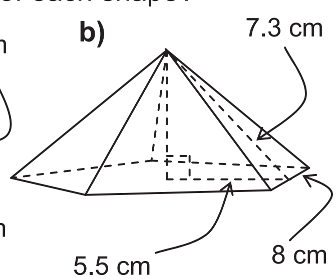
2. A cylindrical container with diameter 14 cm and height 23 cm is filled with water. If the container's mass is 950 g empty, what will be its mass when it is full?

3. What is the volume of each shape?

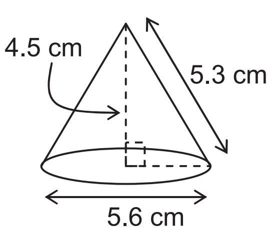
a)



b)



4. Determine the volume of this cone.



5. Determine the total surface area of each shape in **question 1**.

6. Determine the total surface area of each pyramid in **question 3**.

7. Determine the total surface area of the cone in **question 4**.

8. The diameter of a ball bearing (in the shape of a sphere) is 5 cm.

a) Find the volume of the ball bearing.

b) Find its total surface area.

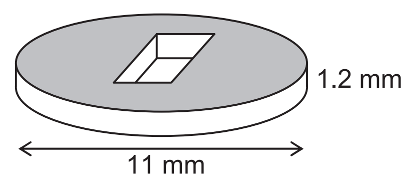
9. Which do you think is easier and why?

- Calculating the volume of a cone
- Calculating the total surface area of a cone

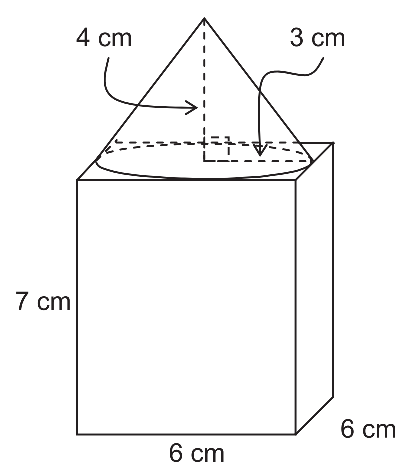
10. a) Determine the height of a triangle-based prism with base area 30 cm^2 and volume 450 cm^3.

b) Determine the diameter of a sphere with total surface area 212 cm^2.

11. This small metal piece is a cylinder with a rectangular hole cut out of it. The rectangle measures 2.3 mm by 6.1 mm. Determine the volume and total surface area of the piece.



12. This model of a building has the shape of a rectangular prism with a cone on top. Find its volume and total surface area.



UNIT 6 Test

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1, 2	Lesson 6.1.1
3, 4	Lesson 6.1.3
5a, b	Lesson 6.2.1
5c	Lesson 6.2.3
6	Lesson 6.2.2
7	Lesson 6.2.4
8	Lessons 6.1.4 and 6.2.5
9	Lessons 6.1.3 and 6.2.4
10	Lessons 6.1.1 and 6.2.5
11	Lessons 6.1.1 and 6.2.3
12	Lessons 6.1.4, 6.2.1, and 6.2.4

Select questions to assign according to the time available.

Answers

1. a) $V = Ah = 6 \times (\frac{1}{2}bh) \times 19 = 6 \times (\frac{1}{2} \times 5.2 \times 4.5) \times 19 =$ **1333.8 cm³**

b) Side of triangle: $29^2 = x^2 + 21^2$, $x = 20$ cm; $V = Ah = (\frac{1}{2}bh) \times 14 = (\frac{1}{2} \times 20 \times 21) \times 14 =$ **2940 cm³**.

c) $V = \pi r^2 h = \pi(15)^2(3.5) \approx$ **2474.00 cm³**

2. $V = \pi r^2 h = \pi(7)^2(23) \approx 3540.57$ cm³, that is about 3540.57 mL, or 3540.57 g; 3540.57 g + 950 g ≈ **4490.57 g** or **4.49 kg**.

3. a) $V = \frac{1}{3}Ah = \frac{1}{3} \times (24 \times 24) \times 35 \approx$ **6720 cm³**

b) Height of pyramid: $x^2 + 5.5^2 = 7.3^2$, $x = 4.8$ m; Base pentagon is five triangles with $A = \frac{1}{2}bh = \frac{1}{2}(8)(5.5) = 22$ m²; $V = \frac{1}{3}Ah = \frac{1}{3}(22 \times 5)(4.8) =$ **176 m³**.

4. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2.8)^2(4.5) \approx$ **36.95 cm³**

5. a) $SA = 2A + Ph = 2 \times 6 \times (\frac{1}{2} \times 5.2 \times 4.5) + (5.2 \times 6)(19) =$ **733.2 cm²**

5. b) $SA = 2A + Ph = 2(\frac{1}{2} \times 20 \times 21) + (20 + 21 + 29)(14) =$ **1400 m²**

c) $SA = 2\pi r^2 + 2\pi rh = 2\pi(15)^2 + 2\pi(3.5)(15) \approx$ **1743.58 cm²**

6. a) Base area = 24 × 24 = 576 cm²; Each lateral triangle = $\frac{1}{2}bh = \frac{1}{2}(24 \times 37) = 444$ cm²; Total SA = 576 + 4 × 444 = **2352 cm²**.

b) Base area = $5 \times \frac{1}{2}bh = 5 \times \frac{1}{2}(8)(5.5) = 110$ m²;

Each lateral triangle = $\frac{1}{2}bh = \frac{1}{2}(8 \times 7.3) = 29.2$ m²;
Total SA = 110 + 5 × 29.2 = **256 m²**.

7. $SA = \pi r^2 + \pi rs = \pi(2.8)^2 + \pi(2.8)(5.3) \approx$ **71.25 cm²**

8. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(2.5)^3 \approx$ **65.45 cm³**.
$SA = 4\pi r^2 = 4\pi(2.5)^2 \approx$ **79.54 cm²**.

9. *Sample response*: If I am given the height, it is easier to calculate volume, and if I am given the slant height it is easier to calculate surface area. This way I would not have to use the Pythagorean theorem to find the missing measurement.

10. a) $V = Ah$; $450 = 30h$; $h =$ **15 cm**.

b) $SA = 4\pi r^2$, $212 = 4\pi r^2$, $r^2 \approx 16.87$; $r \approx 4.11$ cm; $d \approx$ **8.21 m**.

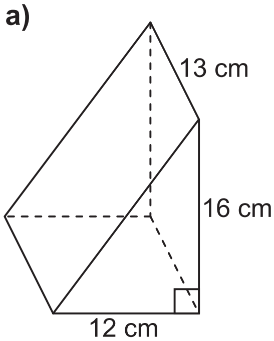
Answers [Continued]

11. Base area = circle – rectangle = $\pi r^2 - lw = \pi(5.5)^2 - (6.1)(2.3) \approx 81.00$ mm^2; Volume = $Ah \approx 81.00 \times 1.2 = 97.20$ mm^3; Lateral surface of rectangle = $1.2 \times (6.1 + 2.3 + 6.1 + 2.3) = 20.16$ mm^2; Lateral surface of cylinder = $2\pi rh = 2\pi(5.5)(1.2) \approx 41.47$ mm^2; Total $SA = 81.00 \times 2 + 20.16 + 41.47 \approx$ **223.64 mm^2**.	**12.** Base of Prism = $6 \times 6 = 36$ cm^2; Volume of Prism = $Ah = 36 \times 7 = 252$ cm^3; Volume of cone= $-\ \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3^2) \times 4 \approx 37.70$ cm^3; Total volume = 252 +37.70 = **289.70 cm^3**. SA of base of prism = $6 \times 6 = 36$ cm^2; SA of lateral surfaces = $4 \times 7 \times 6 = 168$ cm^2; SA of top of prism = square – circle = $36 - 9\pi \approx$ 7.73 cm^2; SA of lateral part of cone = $\pi rs = 15\pi \approx 47.12$ cm^2; Total $SA \approx 36 + 168 + 7.73 + 47.12 =$ **258.85 cm^2**.

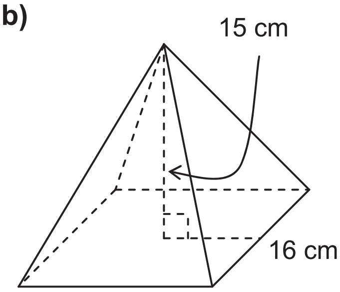
UNIT 6 Performance Task—Changing Volume and Surface Area

1. Determine the volume and total surface area of each figure.

a)

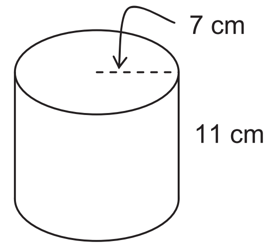


b)

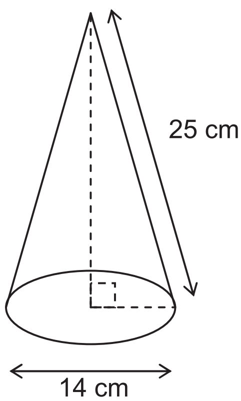


Square-based pyramid

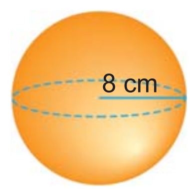
c)



d)



e)



Choose three shapes from question 1 for question 2 and use the remaining two shapes for question 3.

2. For each of the three shapes chosen from **question 1**, change the length of only one dimension so that the new volume will be between 400 cm^3 and 600 cm^3. Explain the reasoning you used to determine which length to change and by how much. Show that the new volume is in the 400 cm^3 to 600 cm^3 range. (For example, for **part 1b)**, you could keep the 16 cm × 16 cm square base and change the height, or you could keep the height at 15 cm and change the side length of the square base.)

3. For the two remaining shapes from **question 1**, change the length of only one dimension so that the new total surface area will be between 1300 cm^2 and 1700 cm^2. Explain the reasoning you used to determine which length to change and by how much. Show that new total surface area is in the 1300 cm^2 to 1700 cm^2 range.

UNIT 6 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-D1 Volume and Surface Area: estimate and calculate for right prisms and cylinders **9-D2** Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres	1 h	• Calculators

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used as enrichment material for some students.
You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Inform them that for some of their revised shapes they will be able to think of strategies for achieving the design specifications, but for others they will probably need to use trial and error. It is preferable to use strategies where possible, but trial and error is a useful approach to solve a problem.

Sample solution

1. a) The length of the hypotenuse:
$12^2 + 16^2 = x^2$, $x = 20$ cm
The area of the base triangle:
$A = \frac{1}{2}bh = \frac{1}{2}(12)(16) = 96 \text{ cm}^2$
The volume of the prism:
$V = Ah = 96 \times 13 = \mathbf{1248\ cm^3}$
The total surface area of the prism:
$$\begin{aligned} SA &= 2A + \text{lateral surface} \\ &= 2 \times 96 + 13 \times (12 + 16 + 20) \\ &= \mathbf{816\ cm^2} \end{aligned}$$

b) The height of each triangle in the lateral surface:
$8^2 + 15^2 = x^2$, $x = 17$ cm
The area of the base square:
$A = s^2 = 16^2 = 256 \text{ cm}^2$
The volume of the pyramid:
$V = \frac{1}{3}Ah = \frac{1}{3} \times 256 \times 15 = \mathbf{1280\ cm^3}$
The total surface area of the prism:
$$\begin{aligned} SA &= A + \text{lateral surface} \\ &= 256 + 4 \times (\tfrac{1}{2} \times 16 \times 17) \\ &= \mathbf{800\ cm^2} \end{aligned}$$

c) The area of the base circle:
$A = \pi r^2 = \pi(7)^2 \approx 153.94 \text{ cm}^2$
The volume of the cylinder:
$V = Ah \approx 153.94 \times 11 \approx \mathbf{1693.32\ cm^3}$
The total surface area of the cylinder:
$$\begin{aligned} SA &= 2A + 2\pi rh \\ &= 2 \times 153.94 + 2\pi(7)(11) \\ &\approx \mathbf{791.68\ cm^2} \end{aligned}$$

1. d) The height of the cone:
$7^2 + h^2 = 25^2$, $h = 24$ cm
The area of the base circle:
$A = \pi r^2 = \pi(7)^2 \approx 153.94 \text{ cm}^2$
The volume of the cone:
$V = \frac{1}{3}Ah = \frac{1}{3} \times 153.94 \times 24 \approx \mathbf{1231.50\ cm^3}$
The total surface area of the cone:
$$\begin{aligned} SA &= A + \pi rs \\ &= 153.94 + \pi(7)(25) \\ &\approx \mathbf{703.72\ cm^2} \end{aligned}$$

e) The volume of the sphere:
$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(8)^3 \approx \mathbf{2144.66\ cm^3}$
The total surface area of the sphere:
$SA = 4\pi r^2 = 4\pi(8)^2 \approx \mathbf{804.25\ cm^2}$

2. a) To make the volume between 400 and 600 cm^3, I can make the prism one third the original volume by multiplying the height by about $\frac{1}{3}$ to make it 5 cm: $V = Ah = 96 \times 5 = 480$ cm^3, which is between 400 cm^3 and 600 cm^3. **b)** To make the volume between 400 and 600 cm^3, I can make the pyramid one third the original volume by multiplying the height by about $\frac{1}{3}$ to make it 5 cm: $V = \frac{1}{3}Ah = \frac{1}{3} \times 256 \times 5 \approx 426.67$ cm^3, which is between 400 cm^3 and 600 cm^3. **c)** To make the volume between 400 and 600 cm^3, I can make the prism about one quarter the original volume by multiplying the height by about $\frac{1}{4}$ to make it 3 cm: $V = Ah = 153.94 \times 3 \approx 461.82$ cm^3, which is between 400 cm^3 and 600 cm^3. **d)** To make the volume between 400 and 600 cm^3, I can make the prism between one half and one third the original volume by multiplying the height by about $\frac{1}{3}$ and by $\frac{1}{2}$, and finding a number in between to make it 10 cm: $V = \frac{1}{3}Ah = \frac{1}{3} \times 153.94 \times 10 = 513.13$ cm^3, which is between 400 cm^2 and 600 cm^3. The slant height is for a height of 10 cm: $7^2 + 10^2 = s^2$; $s \approx 12.21$ cm. **e)** To make the volume between 400 and 600 cm^3, I will use trial and error. If I make the radius 4 cm: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 \approx$ 268.08 cm^3, which is too small. If I make the radius 6 cm: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(6)^3 \approx$ 904.78 cm^3, which is too large. If I make the radius 5 cm: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5)^3 \approx$ 523.60 cm^3, which is between 400 cm^2 and 600 cm^3.	**3. a)** To make the total surface area between 1300 cm^2 and 1700 cm^2, I could add about 600 cm^2. If I add 13 cm to the height, i.e., $600 \div (12 + 16 + 20)$, the bases stay the same but the lateral surface increases. $SA = 2A$ + lateral surface = $2 \times 96 + 26 \times (12 + 16 + 20) = 1440$ cm^2, which is between 1300 cm^2 and 1700 cm^2. **b)** To make the total surface area between 1300 cm^2 and 1700 cm^2, I have to add about 600 cm^2. If I change the base, that changes the lateral surface area too, so I will change the lateral surface by increasing the slant height by trial and error. If the slant height is 40 cm, the total surface area is: $SA = A$ + lateral surface $= 256 + 4 \times (\frac{1}{2} \times 16 \times 40) =$ 1536 cm^2, which is between 1300 cm^2 and 1700 cm^2. If the slant height is 40 cm, then the height needs to be calculated: $8^2 + h^2 = 40^2$, $h \approx 39.19$ cm. **c)** To make the total surface area between 1300 cm^2 and 1700 cm^2, I could add about 700 cm^2. If I add 16 cm to the height, the bases stay the same but the lateral surface increases. The new height would be 27 cm. $SA = 2A + 2\pi rh = 2 \times 153.94 + 2\pi(7)(27) \approx$ 1495.40 cm^2, which is between 1300 and 1700 cm^2. **d)** To make the total surface area between 1300 and 1700 cm^2, I have to add about 700 cm^2 or 800 cm^2. If I change the base, that changes the lateral surface area too, so I will change the lateral surface by increasing the slant height by trial and error. If the slant height is doubled to be 50 cm, then the total surface area is: $SA = A + \pi rs = 153.94 + \pi(7)(50) \approx 1253.50$ cm^2 I will make it longer, a slant height of 60 cm: $SA = A + \pi rs = 153.94 + \pi(7)(60) \approx 1473.41$ cm^2, which is between 1300 cm^2 and 1700 cm^2. **e)** To make the total surface area between 1300 cm^2 and 1700 cm^2, I will use trial and error. If I make the radius 12 cm, a little larger than it was: $SA = 4\pi r^2 = 4\pi(12)^2 \approx 1810$ cm^2, which is too large. If I make the radius a little smaller, 11 cm: $SA = 4\pi r^2 = 4\pi(11)^2 \approx 1521$ cm^2, which is between 1300 cm^2 and 1700 cm^2.

UNIT 6 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Calculating Volume and Surface Area	Completely accurate calculations of volume and surface area with all work shown.	Reasonably accurate calculations of volume and surface area with enough work shown to indicate method.	Calculations of volume and surface area are mostly correct, and indications of method are given.	Major errors in the calculation of volume and surface area and no indication of a method.
Strategies for Revising Shapes for Design Specifications	Creative strategies for achieving design specifications, showing understanding of volume and surface area formulas.	Appropriate strategies for achieving design specifications, showing understanding of volume and surface area, but perhaps with some minor calculation errors.	Some indication of understanding of volume and surface area revealed in design strategies, but not consistently.	Major flaws in understanding.

NOTE: The following assessment interview is for UNIT 5 Geometry.

Unit 5 Assessment Interview

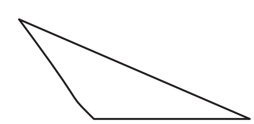
You may want to take the opportunity to interview selected students to assess their understanding of the work in the first chapter in this unit. The results can be used as formative assessment or, if you wish, as a piece of summative assessment data. As the student works, ask him or her to explain their thinking.

T: *I'm thinking of two triangles. They both have a 90° angle and both have a side length of 5 cm. Do they have to be congruent?* (The answer is no since we only know SA and not SSA.)

T: *What if they both also had a 42° angle? Then would they be congruent? How do you know?* (Not necessarily since the 5 could be a hypotenuse of one triangle and a leg of the other.)

T: *What else would I have to tell you for you to be sure they are congruent?* (That the 5 is between the 42° angle and the 90° angle or that the 5 is the hypotenuse.)

T: *Here is a triangle. Draw me a triangle that you know can't be similar to it?*
(e.g., an acute or right triangle)



T: *How would you go about constructing a triangle similar to mine? Can you show me?* (e.g., measure the angles and make another triangle with the same angles) [Observe for an appropriate construction.]

Or you may want to take the opportunity to interview selected students to assess their understanding of the work in the second chapter of this unit. The results can be used as formative assessment or, if you wish, as a piece of summative assessment data. As the student works, ask him or her to explain their thinking.

Draw a triangle with coordinates (0, 0), (4, 2) and (2, 8) on a coordinate grid.

T: *What would the vertices be if I reflected this triangle in the y-axis?* [(0, 0), (–4, 2), (–2, 8)]

T: *What would they be if I reflected the triangle in the x-axis?* [(0, 0), (4, –2), (2, –8)]

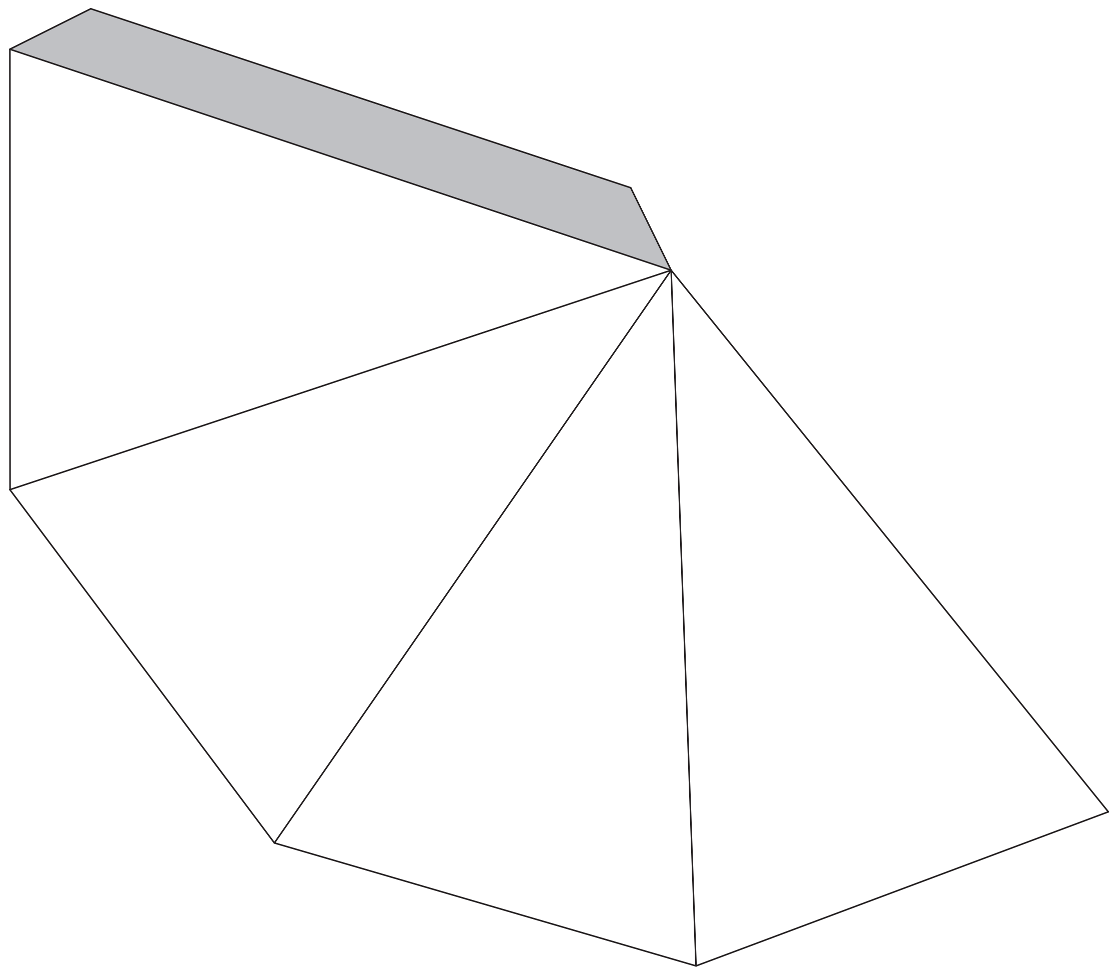
T: *What if I rotated it 180° about the origin counterclockwise?* [(0, 0), (–4, –2), (–2, –8)]

T: *What if I translated it so that (2, 8) moves to (10, 10)?* [(8, 2), (10, 10), (12, 4)]

T: *What if the triangle ended up with the coordinates (4, 3), (6, 7), and (5, 12)? How could I perform transformations to get there?* (*Sample response*: translate [3, 4], then reflect in the *y*-axis, and then rotate 90° counterclockwise around the origin.)

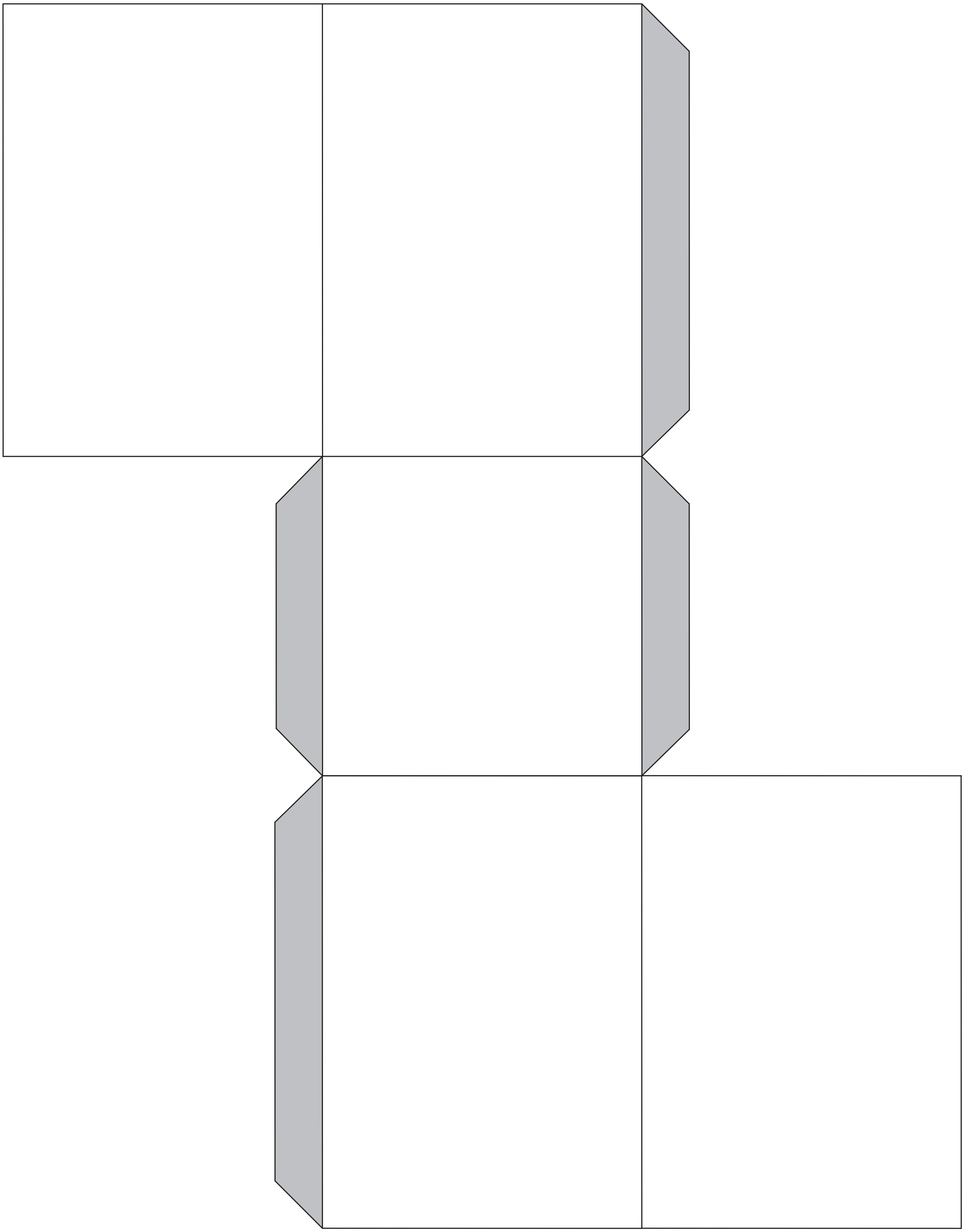
UNIT 6 Blackline Master 1

Net for Square-Based Pyramid (Lesson 6.1.2)



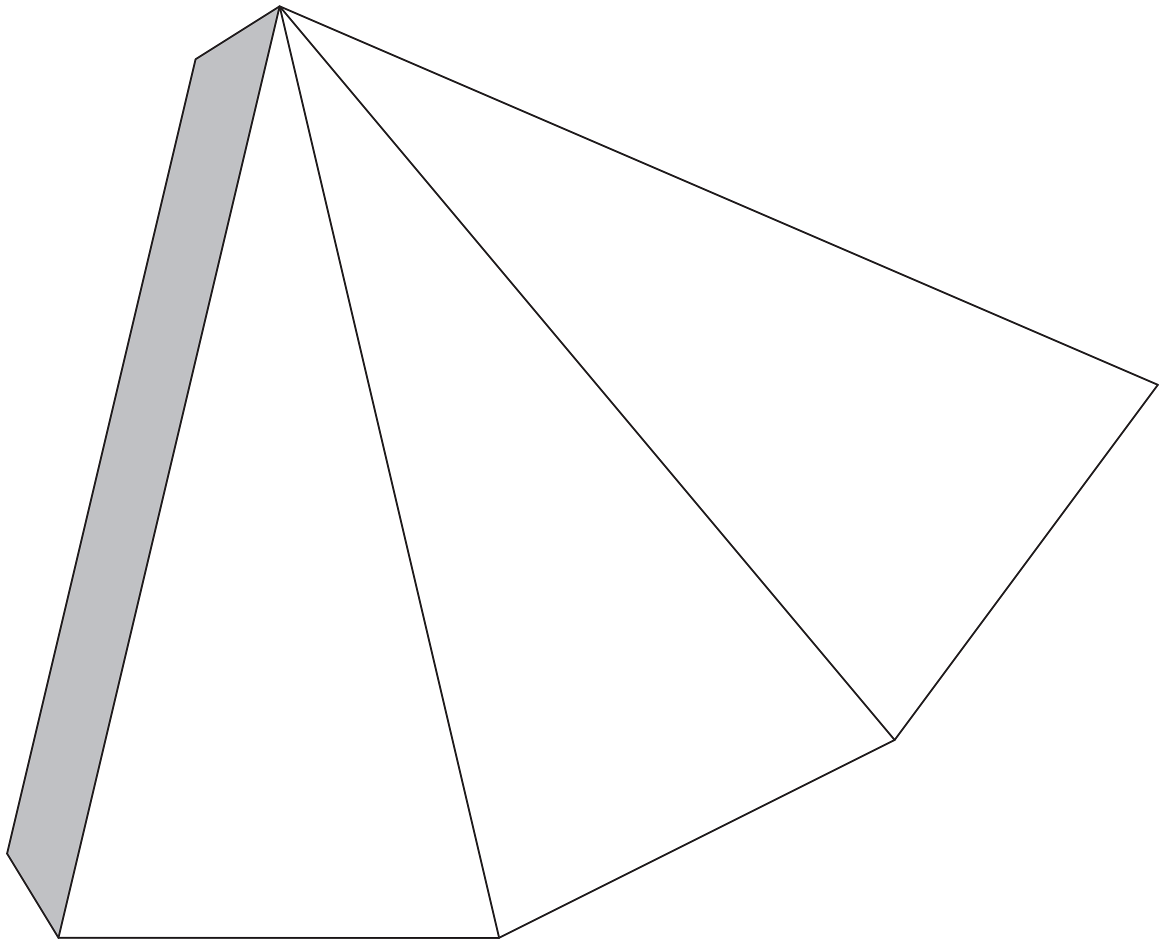
UNIT 6 Blackline Master 2

Net for Square-Based Prism (Lesson 6.1.2)



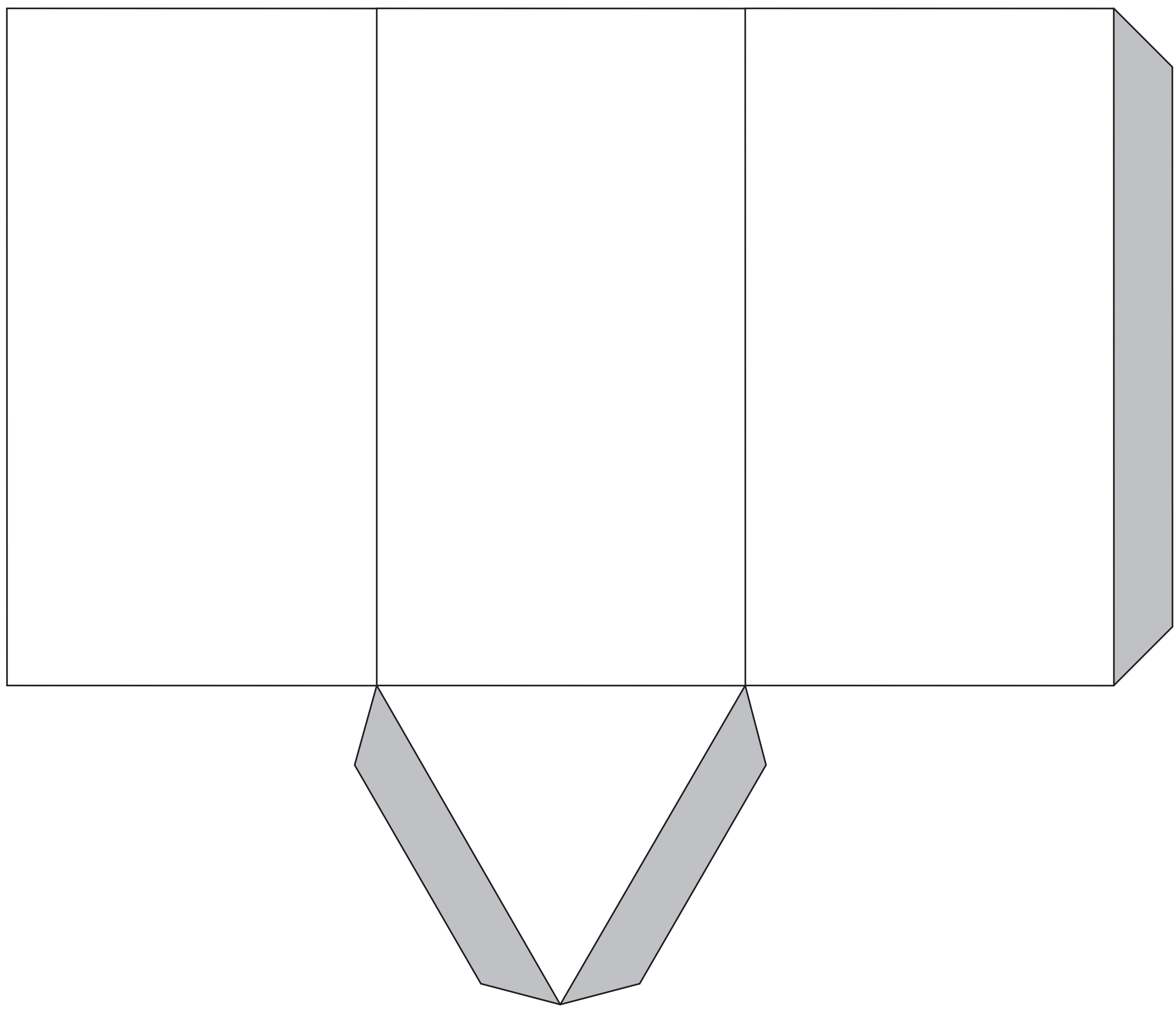
UNIT 6 Blackline Master 3

Net for Triangle-Based Pyramid (Lesson 6.1.2)



UNIT 6 Blackline Master 4

Net for Triangle-Based Prism (Lesson 6.1.2)



UNIT 7 COMMERCIAL MATHEMATICS

UNIT 7 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started	Review prerequisite concepts, skills, and terminology, and pre-assessment	1 h	None	All questions
Chapter 1 Household Finances				
7.1.1 Income and Expenditures	**9-B11: Income, Taxes, and Deductions: estimate and calculate** • estimate and calculate various types of income	1 h	• Calculators	Q1, 4, 6, 7, and 8
7.1.2 Budgets	**9-B12: Budgets: solve problems** • solve problems relating to personal budgets	1 h	• Calculators	Q1, 3, 4, and 5
GAME: Lucky Shopper	Practise calculating percent discount in a game situation	20–30 min	• Dice	N/A
Chapter 2 Taxes				
7.2.1 Reporting Income and Taxes	**9-B11: Income, Taxes, and Deductions: estimate and calculate** • estimate and calculate taxes on income	1 h	• Calculators	Q1, 2, 4, and 7
7.2.2 Income Deductions	**9-B11: Income, Taxes, and Deductions: estimate and calculate** • estimate and calculate income deductions	1 h	• Calculators	Q1, 2, 5, and 6
7.2.3 EXPLORE: Income Tax Rates (Optional)	**9-B11: Income, Taxes, and Deductions: estimate and calculate** • estimate and calculate taxes on income	1 h	• Calculators	Observe and Assess questions
CONNECTIONS: Taxation Around the World	**9-B11: Income, Taxes, and Deductions: estimate and calculate** • estimate and calculate taxes on income	20–30 min	• Calculators	N/A
UNIT 7 Revision	Review the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 7 Test	Assess the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 7 Performance Task	Assess the concepts and skills in the unit	1 h	• Paper or light cardboard for cards • Calculators	Rubric provided

Math Background

• This commercial math unit can be taught at any point in the year; there are no prerequisite skills needed from other Class IX topics. It might be used when a short amount of time is available but you do not want to start a longer unit.

• The focus of the unit is on household budgeting and income tax regulations. It follows work in Class VIII on simple interest, and calculations of cost, selling, and discount prices. It precedes work in Class X on compound interest, stocks, and dividends.

• The mathematical skills students need are relatively simple calculations involving decimal numbers and percentages.

• As students proceed through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in **question 8** of **lesson 7.1.1**, where they combine a variety of pieces of information to solve a problem, in **question 3** of **lesson 7.2.2**, where they might work backwards to determine an investment knowing the interest rate and the applicable taxes and deductions, and in **lesson 7.2.3**, where they explore and synthesize the effect of various tax rates on the amount of tax owing.

- They use communication in **questions 5** and **6** of **lesson 7.1.2**, where they consider the reasons for budgeting and appropriate categories to use, as well as in **question 7** of **lesson 7.2.1**, where they consider the relationship between income and tax paid.

- They use reasoning in answering questions such as **part B** of the **Try This** and **question 4b** of **lesson 7.2.1**, where they analyse the effects of the different tax rates in different slabs.

- They consider representation in **lesson 7.2.1**, where they explore the equivalence of two charts (the one in the **exposition** and the other in **question 1**) describing tax rates.

- They make connections throughout the unit as they learn about financial information that will be relevant in their lives as citizens of Bhutan. There is a specific **Connections** feature highlighting how tax rates in Bhutan compare with rates in other countries and a connection is made to the data strand (interpreting a circle graph) in the **Try This** of **lesson 7.1.2**.

Rationale for Teaching Approach

• This unit is divided into two chapters. **Chapter 1** focuses on personal budgeting. **Chapter 2** focuses on tax regulations.

• The focus on budget precedes the work on taxes because it involves simpler information. It builds on prior understanding of how to use percents.

• The tax scheme for the 2005 year is used since this was the most recent information available at the time of printing of this book. You may have to revise these values if the tax laws change. This is the responsibility of the teacher.

• The **Explore** lesson allows students to see the effect of changing tax rates. It is only through such an investigation that students truly understand the idea of graduated income tax where people who earn more pay a higher rate of tax than those who earn less. This can lead to a valuable discussion about social issues and values, and of why this might be perceived as fair, even though on the surface it might seem unfair.

• The **Game** provides an opportunity to practice work with percents in a pleasant way.

Technology in This Unit

There is substantial use of calculators in this unit. An ordinary calculator is all that is required, but all students should have access to one.

Students will need to be reminded that, to calculate percentages of certain amounts, they will need to convert the percent to a decimal. For example, to find 1.7% of Nu 20,000: 0.017 [×] 20000 [=] 340

Getting Started

Curriculum Outcomes	Outcome relevance
8 Percent: solving and creating real problems in context (including estimation) **8** Percent: increase and decrease **8** Commercial math: discounts, cost price, marked price, and selling price **8** Commercial math: simple interest and commissions	Students require some fundamental skills with percents to accomplish the work in this unit. This introductory activity will allow you to make sure students are comfortable with calculations with percents in the context of price, commissions, and simple interest. If they struggle, some review of Class VIII material may be required.

Pacing	Materials	Prerequisites
1 h	• Calculators	• vocabulary: cost price, selling price, discount, commission, interest • calculation with percents up to 100%

Main Points to be Raised

- A percent can only be determined relative to a whole.
- Knowing any two of the three values—part, whole, and percent—allows you to calculate the missing value.
- There is more than one way to complete a percent calculation.
- Calculations with percents are often simplified by using decimal multiplication or division.

Use What You Know—Introducing the Unit

• Students can work alone or in pairs to complete the activity. If they wish, students can create a chart including the values for both the gho and the shoes. This might make the total more obvious for completing **part A.**

Item	Price	Percent
Gho	Nu 1000	
Shoes	Nu 1500	
Total	Nu 2500	100%

You might point out that if the shoes or the gho had been a different price, the percent of the total purchase related to purchasing the gho would be different. For example, if the costs had been Nu 1000 for the gho but also Nu 1000 for the shoes, the percent for the gho would 50% instead of 40%. In other words, a percent can only be determined relative to a whole.

• **Part B** is designed to focus on alternative calculation methods. They should realize that the middle pair of calculations makes no sense since you cannot mix numbers with percents to perform a calculation. The question is also set up to help them see that calculating 95% of the cost of an item is different from determining 100% when you know 105%. This is because the 100% base is different in the two situations.

• Parts C, **D**, and **E** are designed to help students recall how to solve different types of problems involving percents.

Observe students as they work. You might ask:

• *How do you know that the values will increase if you multiply by 1.05, but decrease if you divide by 1.05?* (When you multiply by 1.05, you get the original amount and an extra 5%, so it is more than you started with. When you divide by 1.05, you are finding out how many times 1.05 fits into an amount and that has to be less than how many times 1 fits in because 1.05 > 1.)

• *Why do you want the values to decrease when finding the cost price?* (The cost price should be less than the selling price or the store owner would lose money.)

• *Why might you multiply by 5 to calculate Deki's father's income?* (20% = $\frac{1}{5}$, so to find 100%, I multiply by 5.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually.

Answers

A. i) 60%	**ii)** 40%	**C.** 2500 ÷ 0.20 = Nu 12,500	
B. i) *Sample response:* about Nu 1425 for shoes and Nu 950 for the gho. **ii)** the third pair: 1500 ÷ 1.05 and 1000 ÷ 1.05 **iii)** No; they are close but not identical: 1 ÷ 1.05 = 0.952 and 1 × 0.95 = 0.95. **iv)** Nu 1428.57 and Nu 952.38		**D.** shoes: (1500 ÷ 1.05) × 1.12 = Nu 1600 gho: (1000 ÷ 1.05) × 1.12 ≈ Nu 1066.67 **E.** more than $33\frac{1}{3}\%$	
1. a) Nu 213	**b)** Nu 299	**3. a)** Nu 251	**b)** Nu 697
2. a) Nu 450	**b)** Nu 494	**4. a)** Nu 400	**b)** Nu 165

Supporting Students

Struggling students

If a student is struggling with percent calculations, you may need to take time to review the calculations that are used, but with simpler values than those in the lesson. For example, present each of these situations:

- You know that a purchase costs Nu 300 altogether, and one item costs Nu 80. What percent of the total cost was the cost of that item? [Show how you could either divide 80 by 300 and rename the decimal as a percent, or you could solve the proportion $\frac{80}{300} = \frac{n}{100}$.]

- You know that the cost of one item represented 40% of the cost of a purchase of Nu 400. What was the cost of that item? [Show how you could either multiply 400 by 0.40 or you could solve the proportion $\frac{40}{100} = \frac{n}{400}$.]

- You know that you spent Nu 300 and it represented 60% of the money you had available to spend. How much money did you have available? [Show how you could either divide 300 by 0.60 or you could solve the proportion $\frac{300}{n} = \frac{60}{100}$.]

After students have tried the suggestions above, change the values to use less rounded numbers, such as 321 for 300 and 42% for 40%.

Enrichment

Students might create three different true sentences involving the numbers 30, 60, and a percent sign somewhere in the sentence. (For example, 30 is 60% of 50 and 60% of 30 is 18.)

Chapter 1 Household Finances

7.1.1 Income and Expenditures

Curriculum Outcomes	Outcome relevance
9-B11: Income, Taxes, and Deductions: estimate and calculate • estimate and calculate various types of income	In their everyday lives, students benefit from understanding how to manage their personal income and expenses. Students need to recognize that income can come from various sources and that sometimes calculations are required to translate percentage descriptions of income to ngultrum descriptions. They also need to consider the various ways income might be expended.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculations with percentages • fraction simplification

Main Points to be Raised

• Income can come from many sources and can be described in terms of an annual, monthly, weekly, or daily amount. It might be paid in different time intervals, such as monthly or weekly.

• It is desirable for expenditures to be less than income.

• Expenditures are sometimes necessary, for example, for food, but can also be discretionary, for example, for a car.

• By saving money in a bank account where interest is offered, a person receives additional money.

Try This—Introducing the Lesson

A. Allow students to work alone or with a partner. Observe while students work. You might ask:

• *How do you know that 25% of the cost is more than Nu 60,000?* (25% is $\frac{1}{4}$, so I think about dividing the price by 4. If the price were Nu 240,000 and I divided by 4, the amount would be Nu 60,000; but the price is even higher, so $\frac{1}{4}$ of that amount would be more than Nu 60,000.)

• *How did you calculate how much extra Tshering has available to save each month?* (10,300 – 9500)

• *Why did you divide to calculate the length of time it would take him to save 25% of the price?* (I needed to know how many Nu 800s were in 25% of the price. To find out how many are in something, you divide.)

The Exposition—Presenting the Main Ideas

• Begin by finding out what students already know about sources of income. For example, you might ask the students to share what they know about how members of their family or friends of the family earn money. Add to the list any items from the exposition that are not suggested by the students.

• Then ask students if they know what the term *expenditures* means. If not, explain it to them. Again, have them suggest different items on which a person might spend money.

• Follow this up by talking to the students about savings. Ask if they know what interest is, why people would choose to save money in a bank to gain interest, and why the bank might offer interest. (to have use of additional money that they can offer as loans to make money for the bank).

• Ask students if they have ever seen a cheque. Make sure they realize how cheques are used. It would be good to show them a real cheque, pointing out its features—the name of the bank, the number of the account so that the bank knows from where the money is to come, the date, the name of the payee, the amount of payment in both written and numerical form, and the cheque number so that the account owner and bank can more easily keep track of cheque payments. Make sure students understand that people use cheques so that they do not have to personally go to the bank to put money in or take money out each time they need to.

• Encourage students to read through the exposition. Make sure they understand why the monthly income for Kinley was multiplied by 12, why the allowance was calculated as 0.35 of Kinley's salary and then multiplied by 12, and why the Nu 500 interest income was not multiplied by 12.

Revisiting the Try This

B. This question provides an opportunity to reinforce the terms *income* and *expenditures*.

Using the Examples

Put students in groups of three. In each group, each student is responsible for working through one example and explaining the example to the other two students. Provide enough time so that each student has time to work through his or her example and then share his or her knowledge with the other two. Remind students that they can use these examples for reference as they work through the **Practising and Applying** questions. If necessary, clarify what a dividend is for students who do not know.

Practising and Applying

Teaching points and tips

Q 1: Students who calculate the weekly income by dividing the monthly income by 4 will have a good estimate but not an exact calculation of the weekly income. Even division by 52 is not exact, but it is close enough to be acceptable. Some students may choose to divide by $52\frac{1}{7}$ to recognize that there are 52 weeks and 1 day in the year. Some students may even want to take leap years into account by dividing by $365\frac{1}{4} \div 7$.

Q 2: Observe whether students estimate to make sure their answers make sense. Estimation should be encouraged. For example, a student might estimate the answer to **part b)** by calculating 10% mentally (instead of 8%) and then reducing the estimated value.

Q 3: Students are asked to predict in order to practise their estimating skills. For example, a student might realize that 4.75% of Nu 22,100 must be greater than 4.5% of Nu 12,200 since it is a greater percentage of a greater amount. It is more difficult to predict whether 4.75% of Nu 22,100 is greater than 4.95% of Nu 19,750.

Q 5: Some students may need support with this question. For example, in **part b)** students need to realize that Nu 14,500 represents 88% of the necessary income and so they must divide 14,500 by 0.88 or set up a proportion to solve the problem.

Q 6: Here students must think to divide by *x*, where *x* represents the decimal equivalent to the percentage, rather than multiplying, for example, dividing by 0.27 in **part a)**.

Common errors

Students will often solve the problems in **question 5** without thinking through the question. They will typically multiply the expenditure amount by the indicated percentage. It might be helpful to use a chart. For example, in **part c)**:

Total Income

Item	Percent	Amount
Expenditures		Nu 9200
Savings	6%	

They would then realize the missing percentage must be 94%. They could calculate the amount of savings (6% of the total) as $\frac{6}{94}$ of 9200 and add it to 9200 to get the total. They could also simply divide 9200 by 0.94 to calculate the necessary income.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can describe income in equivalent forms
Question 4	to see if students can calculate a percentage of an amount
Question 6	to see if students can calculate a whole given a percent
Question 7	to see if students can perform the skill of writing a cheque
Question 8	to see if students can solve a more complex problem involving income and percents

Answers

A. i) about 7.8%
ii) Since he puts away Nu 800 each month and he needs Nu 66,250, it will take 82.8 months or almost 7 years.

B. an expenditure

1. ***Note that weekly incomes were found by dividing by 52. Other answers close to these may also be acceptable.***
a) monthly Nu 9166.67; weekly Nu 2115.38
b) monthly Nu 6666.67; weekly Nu 1538.46
c) monthly Nu 11,833.33; weekly Nu 2730.77
d) monthly Nu 13,000; weekly Nu 3000

2. a) Nu 344 **b)** Nu 960
c) Nu 1450 **d)** Nu 1237.50

3. a) *Sample response:*
I predict **part iii)**; since the amount deposited is greatest and 4.75% interest is almost 4.95%, which is the greatest percentage.

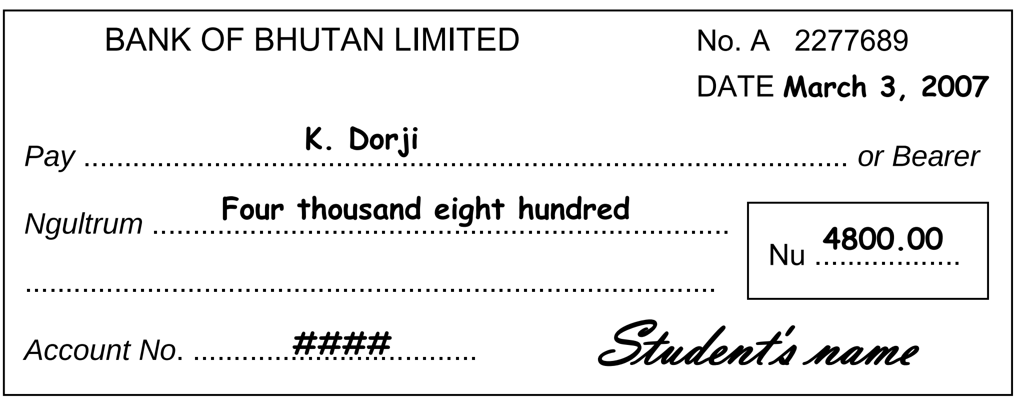
3. b) i) Nu 549 **ii)** Nu 551
iii) Nu 1049.75 **iv)** Nu 977.63

4. a) Nu 2097
b) Nu 3789.50
c) Nu 3336

5. a) Nu 14,666.67 **b)** Nu 16,477.27
c) Nu 9787.23 **d)** Nu 9062.50

6. a) Nu 9259 **b)** Nu 6364
c) Nu 7759 **d)** Nu 7903

7.


BANK OF BHUTAN LIMITED No. A 2277689
DATE March 3, 2007
Pay K. Dorji or Bearer
Ngultrum Four thousand eight hundred
Nu 4800.00
Account No. ####
Student's name

8. The family's interest is Nu 36,000, insurance is Nu 15,000, and income is 12 × (2 × Nu 4500) = Nu 108,000. Therefore, their profit is Nu 108,000 – Nu 36,000 – Nu 15, 000 = Nu 57,000.

9. They would keep track of their income so that they know how much they can afford to spend on rent, food, etc., and they would also keep track of expenses to make sure they do not spend more than they can afford.

Supporting Students

Struggling students

• Before they perform a calculation, encourage students to predict whether the value they will get will be greater or less than the value they are working with. For example, in **question 4**, they should predict the values will always be less than the ngultrum values in the question, but in **questions 5** and **6**, they should predict greater values.

• You might scaffold **question 8** for students by helping them solve the problem in parts. Suggest that they first calculate the income, then the total expenses, and then the difference.

Enrichment

For some students, this work will be very simple. You might have them investigate what typical incomes are for various professions and/or typical interest rates for bank deposits.

7.1.2 Budgets

Curriculum Outcomes	Outcome relevance
9-B12: Budgets: solve problems • solve problems relating to personal budgets	In their everyday lives, students benefit from understanding how to manage their personal income and expenses. Although some students may learn to budget from their families, others may need school intervention to help them with budgets.

Pacing	Materials	Prerequisites
1 h	• Calculators	• relationship between fractions and percents • calculations with percentages

Main Points to be Raised

• A budget is a plan; it is not what actually happens, but indicates how you plan to use available income.
• Budgets can be created for different periods of time, depending on what is most suitable for an individual. Typically, budgets are monthly, but not always.
• Budgets are often organized in terms of charts or, if available, computer spreadsheets. Spreadsheets allow for easy analysis of possible changes in decisions about how income is to be used.
• Bank records are useful for keeping track of income and expenditures.

Try This—Introducing the Lesson

A. and B. Allow students to work alone or in pairs. This particular **Try This** makes a connection between what students have learned about circle graphs and what they will be learning about budgets. Observe while students work. You might ask:

• *How much is 15% as a fraction? Why is it useful to think of 15% as about one sixth?* (If I think of it as one sixth, I would just be looking for a combination of sections that make up one sixth or half of a third of the circle.)

• *How might answering **part iii)** help you check your answer for **part i)**?* (If a section is $\frac{1}{3}$, it should be twice as big as the 15% section, so I could make sure the section for **part i)** is half the size of the section for **part iii)**.)

The Exposition—Presenting the Main Ideas

• Allow students to read through the Exposition on their own. Clarify their understanding by asking them to explain what they understand a budget to be and why a budget might be useful. If you have a passbook available from a bank, you might want to allow students to look at its format.

Revisiting the Try This

C. Make sure that students recognize that a circle graph is a way to show parts of a whole just like a budget is a way to show how parts of the whole income are used.

Using the Examples

• Display the problems shown in **examples 1 and 2** on the board. Have students work through the examples in pairs. When they have finished, they can compare their work with the work shown in the **Thinking**.

• Make sure students understand why it is easier to first calculate the 10% and then to use that to calculate the 15%. Point out that another way to calculate 29% would be to triple the 10% and subtract 1% ($3 \times 1050 - 105$). Similarly, the two values, 30% and 1%, could be added to calculate 31% ($3 \times 1050 + 105$).

Practising and Applying

Teaching points and tips

Q 1: Remind students that although they are completing a monthly budget, most of the information in **part a)** describes amounts needed for two months.

Q 3: Observe how students check their calculated amounts based on the given percentages. For example, see if a student realizes that the 8% and 12% amounts have to average to a 10% amount.

Q 4: This is a very open-ended question, but a very important question. Focus students on whether the amounts they propose in each category make sense.

Q 6: You might discuss with students why budgets for people of different ages might use different categories.

Common errors

Students will sometimes not account for the fact that some income comes more frequently (e.g., weekly) or less frequently (e.g., monthly or yearly) than other income. Remind them to attend to this as they work through the exercises.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can input values into a budget template
Question 3	to see if students can relate assigned percentages to ngultrum amounts
Question 4	to determine whether students allocate percentages to various categories in ways that make sense
Question 5	to see if students can communicate about the usefulness of budgets

Answers

A. more than 50%; more than half the circle.

B. i) *Sample response*: savings and other

ii) *Sample response:* clothes, savings

iii) *Sample response:* clothes, TV/phone, and car loan

C. *Sample response:*
Advantage: You can see at a glance what proportion of your money you are spending in different categories.
Disadvantage: To create the circle graph, you would need to calculate the percentages if the income and expenses are expressed in ngultrums.

1. a) Sonam's monthly budget

Income source	From parents			…..			Total
Amount	900						900
Expense	For clothes	For school supplies	For recreation	For snacks	For gift for mother	Other	Total
Amount	250	100	50	300	150	50	900

b) Dechen's father's monthly budget

Income source	From Salary	From allowance	From interest	…..			Total
Amount	10,200	3500	200				13,900
Expense	For rent	For food	For TV/phone	For clothing, toys	For money to Dechen	Other	Total
Amount	5800	3100	700	200	500	3600	13,900

2. Sonam: Nu 50; Dechen's father: Nu 2600

3. Rent: Nu 2850; Food: Nu 2850:
Household: Nu 1425; Recreation: Nu 475;
Savings: Nu 760; Other: Nu 1140

Answers [Continued]

4. *Sample response:*

Ugyen's budget

Income source	From all sources						Total
Amount	12,200						12,200
Expense	For rent	For food	For TV, phone	For clothing	For savings	Other	Total
Amount	5000	3500	1000	600	1500	600	12,200

5. a) *Sample response:*
If you do not have a lot of money, you need to make sure that you have enough for the necessities.

b) Even a person with more money could end up spending too much on unnecessary things and not have enough left for necessities.

6. a) *Sample response:*
Income would be in one category. I would separate expenditures into these categories: for school, for food, for recreation, for savings, and other expenses, so I could see how I spend my money and find out if I could put more into savings.

b) *Sample response:*
A budget could help me keep track of my expenses so I could explain to my parents why I need a little extra for something like recreation.

Supporting Students

Struggling students

You may begin by providing students a completed budget and having them answer a number of questions about it before asking them to fill in a budget chart, as is required for **question 4.**

Enrichment

Some students might wish to interview different adults they know to find out what kinds of budgets they keep and what information they include in their budgets.

GAME: Lucky Shopper

Some variations of the game are suggested below.

- Students could roll 5 times and be allowed to choose any two digits for the percentage discount and the other three digits for the price.
- The person whose sale price is closer to a particular value, for example, closer to Nu 200, could win instead of the person with the lower sale price. Then students would have to estimate to decide which price and discount would be better to use.

Chapter 2 Taxes

7.2.1 Reporting Income and Taxes

Curriculum Outcomes	Outcome relevance
9-B11: Income, Taxes, and Deductions: estimate and calculate • estimate and calculate taxes on income	As students become adults, they will be required to complete a personal income tax form each year. This lesson will help prepare them for this responsibility and make them aware of social decisions.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculations with percentages • estimating percentages • reading charts

Main Points to be Raised

• Students need to understand that when the PIT (personal income tax) rates are applied, different percentages are used for different income levels. Therefore, to determine the tax owing, it is essential to first determine the appropriate income category to use.

• Tax is based on many sources of income and all of the sources must be considered when calculating tax owing. However, the way tax is calculated may vary depending on the income source.

• A certain percentage of tax may be deducted at source (i.e., before the person receives his or her income) in order to ensure that the government has a regular flow of funding and to ensure that individuals better manage their money and are not caught owing a big amount of tax at the end of the year.

• The rates of tax deducted at source (TDS) vary in terms of the type of income it is.

• When tax owed is calculated, the amount of income is recalculated twice after different types of deductions (or exclusions) are allowed. One of these amounts is called adjusted gross income and the other is called net taxable income.

Try This—Introducing the Lesson

A. Allow students to work in pairs to compare answers. Observe while students work. You might ask:

• *How do you know that the percentage is less than 10%?* (10% of 282,000 is about 28,000 and so 10,080 must be a lot less.)

• *How would you estimate what percentage it is?* (I think about 4% since 10% is about 28,000. One third of that is about 9000. 10,000 is slightly higher than 9000, so the percentage is slightly more than one third of 10%.)

The Exposition—Presenting the Main Ideas

• Draw students' attention to the chart on **page 274** in the textbook that shows the current PIT percentage rates. Make sure they understand that an individual's total income affects what percentage rate is applied. Provide an example or two. For example, show how if the taxable income is Nu 300,000, the tax is applied only to
Nu 100,000 and then the tax is 10% on Nu 50,000, but 15% on the last Nu 50,000. Note that not all income is actually taxed. This will be explained further in the next lesson.

• Suggest a couple of other total net taxable incomes to see if students can apply the chart. For example, have them apply the PIT rates to net taxable incomes of Nu 200,000 and Nu 780,000.

• Explain to students what TDS (tax deducted at source) means. Make sure they understand, it means if a person's basic salary per month is Nu 12,000, he/she might receive less than Nu 12,000 monthly.

• You may wish to show a Bhutan income tax form so that students can see what it looks like. They should be led to understand that the government allows certain deductions from income (to be discussed more explicitly in the next lesson) to recognize that a certain amount of income is essential and should not be taxed. Tell students that there are two sets of deductions applied, one to reduce the total (or gross) income to something called gross adjusted income and another set of deductions applied to reduce the total even further to an amount called a net taxable income. Assure students that these types of deductions will be addressed later.

• Encourage students to read through the exposition on their own and ask any questions they might have.

Revisiting the Try This

B. This question allows students to better interpret the tax form. They begin to see that if 0% is applied to part of the income, 10% to another part, and 15% to yet another part, the actual tax rate may not be any of these values.

Using the Examples

• Write the questions in **example 1** on the board. Encourage students to try them and then to check their work against the thinking in the text.

• Then display the questions in **example 2**, again encouraging students to try them and then checking their work against the thinking in the text.

Practising and Applying

Teaching points and tips

Q 1: This question focuses on the notion that a mathematical calculation or concept can be shown in more than one way. It is very important that students understand how this representation of the tax information is equivalent to the representation in the main ideas.

Q 4: This question involves reasoning. Students need to recognize that because the tax slab for Bishnu's income is the second one, the 10% is applied only to part, and not all, of the income. Consequently, the real percentage rate has to be less than 10%.

Q 6: This question is more about social values than about mathematics, but is important for students to consider.

Q 7: Students might handle this question as a small investigation. They must reason to figure out how to answer it.

Common errors

When calculating the tax, some students may forget to include all parts of the multi-part calculations that might be required. This could be modelled by leaving a calculation you have performed on the board for them to refer to. This would supplement the examples in the book.

Suggested assessment questions from Practising and Applying

Question 1	to see if students recognize the equivalence of representations in chart form
Question 2	to see if students can directly apply the skill of calculating tax using the tax chart
Question 4	to see if students can use mathematical reasoning about percentages to draw a conclusion
Question 7	to see if students can make a decision about what values to consider to answer a problem that is posed in the abstract

Answers

A. about 3.8%

B. You are not paying any of those percentages on the whole income; you are paying those percentages on different parts of the same income.

1. a)
- The first two rows in each chart are the same.
- The third row in one chart says 5000 in place of 10% of the next 50,000 in the other chart, but 10% of 50,000 = 5,000, so these rows are also the same in each chart.
- The fourth row in one chart says 42,500 in place of 10% of 50,000 and 15% on the next 50,000 in the other chart, but 10% of 50,000 + 15% of 250,000 = 5000 + 37,500 = 42,500, so these rows are also the same in each chart.
- The last row in one chart says 142,500 in place of 10% of the 50,000, 15% on the next 250,000, and 20% on the next 500,000 in the other chart, but
10% of 50,000 + 15% of 250,000 + 20% of 500,000
= 5,000 + 37,500 + 100,000 = 142,500
so these rows are also the same in each chart.

1. b) It saves time since some of the calculations are done for you.

c) It shows the rate you are paying on each part of your income. It is also easy to see that as your income increases, so does the tax rate.

2. a) Nu 72,500 **b)** Nu 12,500

c) Nu 0 **d)** Nu 41,000

3. a) Nu 76 **b)** Nu 4,350

c) Nu 1250 **d)** Nu 377

4. a) He is in the second tax slab which means he only pays 10% of Nu 47,000 (247,000 – 200,000), which is a lot less than 10% of Nu 247,000. **b)** He is paying 10% of less than half his total income, so the rate has to be less than half of 10%, which is less than 5%. **5.** Nu 124,000 **6.** People who have less income use a larger percentage of their income for basic necessities like food and rent than those with a greater income, so they have less that they can afford for taxes. People who have more should share.	**7.** Usually, but not always. Even if they pay the same tax rate, the person who has the higher income will pay more because that percentage is applied to a greater amount. For example, if one person has an income of Nu 210,000 and another has an income of Nu 250,000, they both pay 10% but the first person pays 10% on 10,000 (which is Nu 1,000) and the other person pays 10% on 50,000 (which is Nu 5,000). The exception is if the income is Nu 200,000 or less.

Supporting Students

Struggling students

As students are using the tax chart, you might help them realize there are multiple calculations to complete. You could model a format like this:

A	Net taxable Income	Nu
B	Subtract Nu ______	Nu
C	Take __ % of (B)	Nu
D	Add fixed Nu amount of ______ to (C)	Nu

Enrichment

Some students might be interested in the effect of various salary increases in various tax slabs. For example, they could compare the effects of a raise of Nu 1000 a month in each of the tax slabs to see what percentage of the salary increase is actually retained.

7.2.2 Income Deductions

Curriculum Outcomes	Outcome relevance
9-B1: Income, Taxes, and Deductions: estimate and calculate • estimate and calculate income deductions	Students will begin to see how, and to a certain extent why, deductions on certain types of income can have differential effects on the tax that is due.

Pacing	Materials	Prerequisites
1 h	• Calculators	• solving problems with percentages • adding and subtracting whole numbers and decimals

Main Points to be Raised

- Deductions are amounts that are deducted, or subtracted, from income to reduce the amount of income that is actually taxed.
- Allowable deductions vary for different types of income sources.
- Deductions are designed to recognize expenses that are beyond basic necessities but are valued. For example, a deduction for rental income is meant to recognize the value of people building properties to rent and to recognize that there are associated expenses that should not be counted as income. Education deductions are designed to recognize the value the government places on parents spending money to educate the children of Bhutan.
- An allowable deduction can never exceed the amount of income to which it is applied.

Try This—Introducing the Lesson

A. Allow students to work in pairs. Observe while students work. You might ask:

- *How do you know that the orchard owner would keep more of his income?* (The orchard owner gets a 30% deduction on income since his income comes from the sale of a cash crop. The government worker does not get the same deduction.)
- *About how much of the income of the orchard owner is not eligible for deduction?* (70% of 18,000 is about two thirds of 18,000 ≈ Nu 12,000, which will be taxable.)

The Exposition—Presenting the Main Ideas

- Talk with the students about the word *deduction* (to mean that something is deducted or subtracted). Tell the students that the government allows parents to reduce their taxable income by a certain amount for each child they are educating. Ask why the government might do this.
- Then ask why the government might choose to allow some interest or dividend income to be excluded from taxation (to encourage people to invest and grow the economy).
- Make a list on the board of some of the other types of deductions that are allowed: for rental income, for dividends, for interest, for cash crop income, for life insurance premiums, and for donations. Ask students to talk in small groups about why the government might allow these deductions. They can then share their ideas.
- After discussing some of the students' ideas, mention that the deduction for interest income is Nu 10,000. Ask why a person whose interest income is Nu 6000 only takes off Nu 6000 to calculate his taxable income, but the person who earns Nu 12,000 interest income takes off Nu 10,000 to calculate taxable income.
- Encourage students to copy a chart into their notebooks indicating the value (whether a percentage or ngultrum amount) of the possible deduction for each possible type of deduction.
- Make sure students realize they can refer back to the information in the exposition if they wish.

Revisiting the Try This

B. Students will need to consult the tax chart to answer this question. They need to realize that the orchard owner will get a 30% deduction on his income.

Using the Examples

Display the information from the question in the **example** on the board and encourage students to complete the question individually. They can check their work against the work shown in the textbook.

Practising and Applying

Teaching points and tips

Q 2: Some students might round to the nearest ngultrum and others might not. Either approach should be acceptable.

Q 4a: Most students will simply calculate 30% of 41,000 and subtract 10,000. An alternative you might show students is to calculate what share value would have to be associated with a 30% rate to receive an income of Nu 10,000 (33,333.33), subtract that from 41,000, and then calculate 30% of the result. Although this is more complicated, it emphasizes the fact that problems can always be solved in many ways.

Q 7: This question is designed to help students see that part of budgeting is hypothesizing about different options and making choices about how to use money.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can calculate the appropriate deduction
Question 2	to see if students can apply the appropriate deduction
Question 5	to see if students can solve a more complex percentage question involving dividend deductions
Question 6	to see if students can solve a more complex percentage question taking several factors into account

Answers

A. *Sample response:*
Both people might have the same expenses for rent and food and so on, but the person with the orchard might need to pay other people to help him gather his crop and might need to buy trees to keep the orchard going.

B. *Sample response:*
Nu 3888 since both incomes are in the second slab and a rate of 6% is applied to anything over Nu 100,000[6% of (12 × 30% of 18,000)].

1. a) Nu 241,450 **b)** Nu 49,020

2. a) Nu 810 **b)** Nu 1278
c) Nu 2945 **d)** Nu 10,000

3. a) Nu 222,222.22 **b)** Nu 200,000
c) Nu 250,000 **d)** Nu 166,666.67

4. a) Nu 10,000 **b)** Nu 4770 **c)** Nu 4290

5. a) Nu 40,000 **b)** Nu 47,619.05
c) Nu 33,333.33 **d)** Nu 5,555.56

6. Nu 17,916.67
7. the dividend

6. They receive a total education allowance of Nu 150,000. Income up to Nu 100,000 has no tax. Therefore, the family can earn up to Nu 250,000 per year and pay no tax. This is a monthly income of Nu 20,833.33.

7. The dividend; with the dividend you earn more and pay less tax. With the dividend you earn Nu 20,000 in income and pay tax on Nu 10,000 (Nu 20,000 – Nu 10,000). With the rental property, you earn Nu 15,000 and pay tax on Nu 12,000 (20% of 15,000 = 3000 and 15,000 – 3000 = 12,000).

Supporting Students

Struggling students

• You may encourage students to create a well-organized chart that clearly lists the deductions in various categories.

• You may need to help struggling students with **question 5**, where they must divide 10,000 by the decimal equivalent of each percentage rather than multiplying. It might help to ask questions such as:

- *How do you know that the income must be greater than Nu 10,000?*
- *How do you know it must be greater than Nu 20,000?*

Enrichment

Some students might enjoy problems where they are given a total allowable deduction for a particular income and figure out some of the possible scenarios that would lead to that situation.

7.2.3 EXPLORE: Income Tax Rates

Curriculum Outcomes	Lesson relevance
9-B11: Income, Taxes, and Deductions: estimate and calculate • estimate and calculate taxes on income	This optional exploration of the effect of rate changes on tax payable gives students a deeper understanding of the Bhutan tax system.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculations with percentages

Exploration

• Ask each student to consult one of the tax charts shown in **lesson 7.2.1**. One is presented in the main ideas box and the other is presented in Practising and Applying **question 1.**

• Students need to understand that all questions assume that the first Nu 100,000 of income is exempt from tax.

Observe while students work. You might ask:

• *Why did the increase not matter if the income was only Nu 100,000?* (Because there is no tax on the first Nu 100,000 income no matter what the rate is.)

• *Why would a 2% increase have a smaller effect than doubling?* (Because the rate change is not as much. 9% increases to 11% and 6% to 8% instead of increasing to 18% and 12%.)

• *How could you have predicted that the ngultrum increase would be greatest for the highest tax slab? Does that mean the effect was greatest?* (People in the highest slab pay a bigger percentage so the increase would be greater. But if they make a lot of money, the increase might still be a small percentage of their total income.)

Observe and Assess

As students are working, notice:

• Do they use one calculation to help them calculate or estimate another?

• Do they make reasonable predictions of what might happen?

• Are their conclusions explained well?

Share and Reflect

• After students have had sufficient time to work through the exploration, first discuss **parts A to E**, where they consider the mathematics related to changing tax rates. Make sure that students explain how they arrived at their conclusions.

• Afterwards, discuss with them their thinking on **part F.** Encourage different students to participate in the discussion by asking at least several students for their points of view.

Answers

A. i) 0 **ii)** Nu 8000 – Nu 6000 = Nu 2000
iii) Nu 28,500 – Nu 22,500 = Nu 6000
iv) Nu 81,500 – Nu 67,500 = Nu 14,000
v) Nu 211,500 – Nu 181,500 = Nu 30,000

B. i) 100% **ii)** 8000 ÷ 6000 = 1.33 = 133%
iii) 28,500 ÷ 22,500 = 1.27 = 127%
iv) 81,500 ÷ 67,500 = 1.21 = 121%
v) 211,500 ÷ 181,500 = 1.17 = 117%

C. The percentage increase is greatest for the second slab; the ngultrum increase is greatest for the fifth slab.

D. i) 0 **ii)** Nu 12,000 – Nu 6000 = Nu 6000
iii) Nu 45,000 – Nu 22, 500 = Nu 22,500
iv) Nu 135,000 – Nu 67,500 = Nu 67,500
v) Nu 363,00 – Nu 181,500 = Nu 181,500

E. Doubling the rate doubled the tax owing in each slab. *Sample response:* It worked for incomes in each slab that I tried. The increased tax rate is 200% of the old one for everyone.

F. *Sample response:*
The government should take into account the typical cost of rent and food to make sure citizens have enough to live on. But they should also think about how much they need to provide services.

Supporting Students

Struggling students

Some students might benefit from an organized chart to keep track of the calculations. For example, it might look like this:

Income Example for each slab	Current rate	Current Tax	New rate	New tax

Enrichment

Require students to mathematically explain why the doubling would hold for all tax rates in **part E**, rather than just observing that it happened.

CONNECTIONS: Taxation around the World

1. a) Some recent income tax rates are listed below:

Country	Low–High	Range
Australia	15–46%	31
Austria	21–50%	29
Belgium	25–50%	25
Bhutan	0–15%	15
Bulgaria	10–24%	14
Canada	15–29%	14
China	5–45%	40
Czech Rep.	12–32%	20
Denmark	38–59%	21
France	10–48.09%	38
Germany	15–42%	27
Greece	0–40%	40
Hungary	18–36%	18
India	10–30%	20
Ireland	20–42%	22
Israel	10–49%	39
Italy	23–43%	20
Japan	10–37%	27
Mexico	3–29%	26
Morocco	0–41.5%	41.5
Netherlands	0–52%	52
Norway	28–51.3%	23.3
Pakistan	7.5%–35%	27.5
Philippines	5–32%	27
Poland	19–40%	21
Portugal	10.5–40%	29.5
Russia	13%	0
South Africa	18–40%	22
Spain	15–45%	30
Taiwan	6–40%	34
Thailand	5–37%	32
Turkey	15–35%	20
United Kingdom	0–40%	40
United States of America	0–35%	35

Source: http://en.wikipedia.org/wiki/Tax_rates_around_the_world

The greatest range is 52 in the Netherlands and the smallest ranges are Russia at 0, followed by Canada and Bulgaria both at 14 and then Bhutan at 15.

2. *Sample response*:
Most ranges are about 20 and Bhutan's is 15, so Bhutan's range is smaller than most. Bhutan is like a lot of other countries by having a 0% rate, but its high end rate is much lower than most.

Tax Ranges

0	0
1	4 4 5 8
2	0 0 0 0 1 1 2 2 3 5 6 7 7 7 8 9
3	0 0 1 2 4 5 8 9
4	0 0 0 2
5	2

Tax Rates (low end)

0	0 0 0 0 0 0 3 5 5 5 6 8
1	0 0 0 0 0 1 2 3 5 5 5 5 5 8 8 9
2	0 1 3 5 8
3	8

Tax Rates (high end)

1	3 5
2	4 9 9
3	0 2 2 5 5 5 6 7 7
4	0 0 0 0 0 0 2 2 2 3 5 5 6 8 9
5	0 0 1 2 9

UNIT 7 Revision

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1, 2, 3, 4, 5	Lesson 7.1.1
6, 7, 8, 9, 10	Lesson 7.1.2
11, 12, 13, 14	Lesson 7.2.1
15, 16,– 17	Lesson 7.2.2

Revision Tips

Q 4: Some students will need help recognizing that, for example, if 8% of income is saved, the ngultrum amount given as spent is 92% of the income.

Q 6: Students will have to recognize that most of the values are for four months.

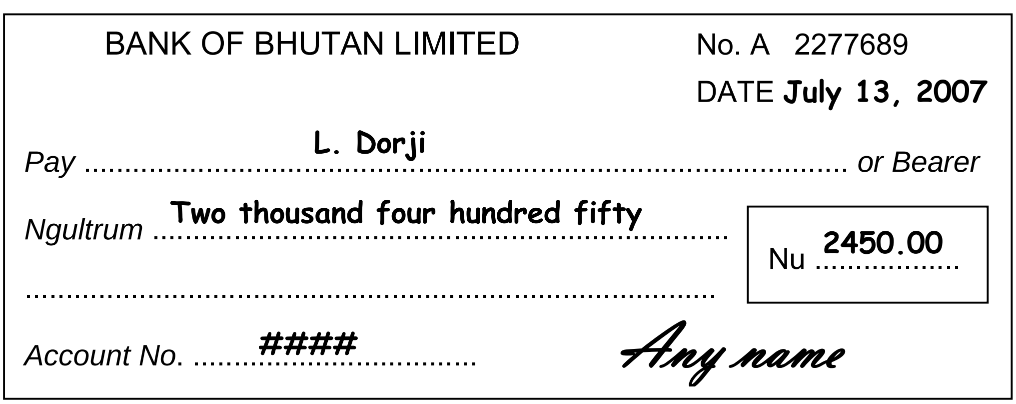
Q 11: Some students might prefer to use the chart in the exposition in **question 1** of **lesson 7.2.1.**

Q 16: Students might need to be reminded to consider each piece of information provided in the problem.

Answers

1. a) Nu 126,000	**b)** Nu 103,480	**3. a)** Nu 6000	**b)** Nu 5850
c) Nu 117,520	**d)** Nu 105,600	**c)** Nu 10,608	**d)** Nu 2633.75
2. a) Nu 935	**b)** Nu 883.50	**4. a)** Nu 9435	**b)** Nu 10,495
c) Nu 1485.75	**d)** Nu 803	**c)** Nu 15,989	**d)** Nu 13,989

5. a)



5. b)

BANK OF BHUTAN LIMITED No. A 2277689

DATE **Aug. 14, 2007**

Pay **G. Tshering** *or Bearer*

Ngultrum **Thirty-one thousand two hundred** Nu **31,200.00**

Account No. **####** *Any name*

6. a) *Sample response:*
For four months

Income source	From parents					Total
Amount	3000					3000
Expense	For clothes	For school supplies	For recreation	For snacks	Other	Total
Amount	1000	250	300	800	650	3000

b) For one month

Income source	From salary						Total
Amount	12,800						12,800
Expense	For rent	For food	For household expenses	For clothing	For loan payment	Other	Total
Amount	4800	4200	800	300	1500	1200	12,800

7. Sonam: 21.7%; U. Pem: 9.4%

8. To make sure you do not spend more than you have, to see how much you spend in different categories like rent and food, and to make sure you are not overspending in one category.

9. a) Nu 2410 **b)** Nu 2602.80 **c)** Nu 1446

d) Nu 482 **e)** Nu 2699.20

10. a) Nu 18,929 **b)** Nu 20,357 **c)** Nu 17,667

11. a) Nu 8130 **b)** Nu 24,750

c) Nu 76,250 **d)** Nu 260,000

12. a) Nu 1230 **b)** Nu 39

c) Nu 2400 **d)** Nu 180.80

13. *Sample response:*
People with less money spend a greater percentage of their money on necessities. The tax rates take this into account.

14. To make sure money comes in regularly to the government to pay for services and to make sure taxpayers do not run out of money to pay their taxes when they are due.

15. a) 24.24% **b)** Nu 48,888.89

16. B; rental income deduction is
$0.2 \times 220{,}000 + 32{,}000 + 1200 =$ Nu 77,200;
education deduction: Nu 100,000

17. a) Nu 710 **b)** Nu 9150

UNIT 7 Commercial Mathematics Test

1. Calculate the annual income for each.

a) monthly income of Nu 12,200

b) monthly income of Nu 6250

c) weekly income of Nu 1440

d) weekly income of Nu 2530

2. Calculate the annual dividend or interest income on each amount.

a) dividend rate: 21.3% per annum

amount: Nu 13,400

b) interest rate: 4.5% per annum

amount: Nu 22,100

3. Describe some of the expenses a family with two children might have.

4. You are writing a cheque to K. Thinley for Nu 3450. What information will be found on the cheque?

5. Show what a budget chart for Pema's mother might look like if her monthly income and expenses are as follows:

Monthly Salary: Nu 11,500

Monthly Allowance: 25% of salary

Monthly Rent: Nu 5500

Monthly Food: Nu 3500

Monthly Loan payment: Nu 1200

Phone, TV, etc: Nu 800

6. Sonam's family income is Nu 19,000 a month.

How much would they spend each month in each category of expenditures if these are the percentages?

a) Rent: 30%

b) Food: 22%

c) Car: 9%

d) Donations: 5%

7. Manju spends Nu 5000 on food each month. This represents 32% of her income. What is her monthly income?

8. Consult the tax chart your teacher has displayed. For each net taxable income below, calculate the following:

A net taxable income: Nu 252,000

B net taxable income: Nu 495,000

a) tax owed

b) the percentage of income going to taxes

c) the percentage of income going to taxes if there were a salary increase of 2% (keeping in mind the taxes to be paid). Round to the nearest hundredth of a percent.

9. Explain why a 2% raise in gross income is not always a 2% raise in take-home pay.

10. How would the tax deducted at source be different for Nu 24,000 in tourist income than for Nu 24,000 in dividend income?

11. How might two people receive different amounts of interest income but have the same allowable deduction for the purpose of personal income tax?

12. a) An individual's dividend income deduction was more than his interest income deduction.
Is it possible that he earned more interest than dividends? Explain.

b) Would your answer be the same if one were rental income and one were interest income? Explain.

13. Why does the government allow deductions for the purpose of computing net taxable income?

UNIT 7 Test

Pacing	Materials
1 day	• Calculators

Question(s)	Related Lesson(s)
1, 2, 3, 4	Lesson 7.1.1
5, 6, 7	Lesson 7.1.2
8, 9	Lesson 7.2.1
10, 11	Lesson 7.2.2

Select questions to assign according to the time available.

Answers

1. a) Nu 146,400 **b)** Nu 75,000 **c)** Nu 74,880 **d)** Nu 131,560

2. a) Nu 2854.20 **b)** Nu 994.50

3. *Sample response:*
rent, food, clothing, toys, school supplies, phone/TV

4. bank name, bank account number, cheque number, amount of Nu 3450 in written and numerical form, K. Thinley's name, date, and signature

5. For one month

Income source	From salary	From allowance					Total
Amount	11,500	2875					14,375
Expense	For rent	For food	For loan	For phone, TV, etc.	Other		Total
Amount	5500	3500	1200	800	3375		14,375

6. a) Nu 5700 **b)** Nu 4100

c) Nu 1710 **d)** Nu 950

7. Nu 15,625

8. a) **A:** Nu 5300 **B:** Nu 41,750

b) A: 2.05% **B:** 6.27%

c) A: 2.13% **B:** 6.36%

9. You have to pay tax on the new money for most tax slabs, so you do not keep the entire 2%.

10. Only 2% is deducted on tourist income, but 10% is deducted for dividend income

11. Each might have received more than Nu 10,000 in interest income, but they could each deduct only Nu 10,000.

12. a) No, since the maximum deduction amounts are the same for both.

b) Yes, since the deduction for interest income is a fixed value but the deduction for rental income is a percentage. If, for example, interest income was Nu 60,000 and rental income was Nu 55,000, the interest deduction would be Nu 10,000, which is less than the rental income deduction of Nu 11,000, even though more was earned.

13. *Sample response:*
Partly to encourage certain kinds of investments in the economy, such as in education, but also to recognize that there are costs associated with earning certain income, for example, through rental properties or farming.

UNIT 7 Performance Task — Financial Figuring

Passang's mother is planning to be out of the country for a number of months. She wants to make sure Passang and his father understand how to budget the family income during those months and to pay the annual personal income tax for herself and her husband.

Here are their financial details:

- Passang's father earns Nu 21,500 each month with an allowance of 15% of his salary.
- Passang's mother earns Nu 14,000 per month with no allowance.
- Passang and his sister study in private school.
- They own a car and pay the bank Nu 4000 per month on a car loan.
- The family owns one rental property for which they receive Nu 7000 each month; they pay no insurance or local taxes on this property, but there are some maintenance costs.
- They receive no dividend income, but they do receive Nu 520 in interest income for the year.

A. Prepare a reasonable monthly budget for Passang's family.

- Show all sources of income and all expense categories.
- Describe the expenses both in ngultrum values and in percentages. Round ngultrum values to the nearest whole ngultrum and percentages to the nearest tenth of a percent.

B. Calculate the amount of net taxable income for the family, if there are no deductions other than those based on the information provided.

C. What percentage, rounded to the nearest tenth of a percent, of the family income goes to taxes?

D. How much more salary money must the family earn in order for them to have five percent more to spend or save after taxes have been paid?

UNIT 7 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-B11 Income, Taxes, and Deductions: estimate and calculate **9-B12** Budgets: solve problems	1 h	• Calculators

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of the outcomes in this unit.
It could replace or supplement the unit test. It could also be used as enrichment material for some students.
You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution

A. *Sample response:*
For one month

Income source	father's salary	father's allowance	mother's salary			Rental	Interest		Total
Amount	21,500	3,225	10,000			5,000	520		40,245
Expense	rent	food	loan	fuel	clothing	phone/TV	Rental maintenance	Savings /others	Total
Amount	7,000	11,000	4,000	3,000	2,000	2,000	5,000	6,245	40,245
Percentage	17.39%	27.33%	9.94%	7.45%	4.97%	4.97%	12.42%	15.52%	100.00%

B. Income is Nu 482,940 annually.
Deductions are Nu 150,000 for education, Nu 12,000 on rental income, Nu 520 on interest income
Net taxable income is Nu 320,420
Tax due is Nu 5000 + 15% of (320,420 – 250,000) = 5000 + 10,563 = Nu 15,563/-
C. About 2.2%

D. $1.05 \times (482,940 - 15,563) \approx$ Nu 490,746/-
Deductions = 150,000 + 12,000 + 520 = 162,520/-
Net taxable income would be 490,746 – 162,520 = 328,226/-
Required income, *I*, is the solution to the equation
$i - (5000 + 0.15 \times (328,226 - 250,000)$ = Nu 490,746/-
$i \approx$ Nu 507,480/- annually or Nu 42,290/- monthly. This is an increase of about 5.1%.

UNIT 7 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Correctness of calculations	Almost completely correct calculations	Mostly correct calculations	Many correct calculations	Errors in most calculations
Thoroughness of budget plan	Comprehensive consideration of possible budget categories	Correct use of provided data with a few other suggestions provided	Correct use of most of the provided data	Omission of many pieces of provided data
Problem solving and reasoning in calculating the needed income	Takes into account money available after taxes and develops an appropriate method for calculating the new required salary	Takes into account money available after taxes and develops a method for calculating the new required salary with only minor flaws	Recognizes that the salary increase must be more, but not much more than 5%, but does not develop an appropriate method for calculating the amount	Does not recognize that the salary increase is not just 5% of current income

PHOTO CREDITS

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