Teacher's Guide to

Understanding Mathematics

Textbook for Class X

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Ministry of Education and Skills Development
Royal Government of Bhutan

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ROYAL GOVERNMENT OF BHUTAN
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THIMPHU : BHUTAN

MINISTER

FOREWORD

Provision of quality education for our children is a cornerstone policy of the Royal Government of Bhutan. Quality education in mathematics includes attention to many aspects of educating our children. One is providing opportunities and believing in our children's ability to understand and contribute to the advancement of science and technology within our culture, history and tradition. To accomplish this, we need to cater to children's mental, emotional and psychological phases of development, enabling, encouraging and supporting them in exploring, discovering and realizing their own potential. We also must promote and further our values of compassion, hard work, honesty, helpfulness, perseverance, responsibility, *thadamtsi* (for instance being grateful to what I would like to call '*Pham Kha Nga'*, consisting of parents, teachers, His Majesty the King, the country and the Bhutanese people, for all the goodness received from them and the wish to reciprocate these in equal measure) and *ley-ju-drey* — the understanding and appreciation of the natural law of cause and effect. At the same time, we wish to develop positive attitudes, skills, competencies, and values to support our children as they mature and engage in the professions they will ultimately pursue in life, either by choice or necessity.

While education recognizes that certain values for our children as individuals and as citizens of the country and of the world at large, do not change, requirements in the work place advance as a result of scientific, technological, and even political advancement in the world. These include expectations for more advanced interpersonal skills and skills in communications, reasoning, problem solving, and decision-making. Therefore, the type of education we provide to our children must reflect the current trends and requirements, and be relevant and appropriate. Its quality and standard should stem out of collective wisdom, experience, research, and thoughtful deliberations.

Mathematics, without dispute, is a beautiful and profound subject, but it also has immense utility to offer in our lives. The school mathematics curriculum is being changed to reflect research from around the world that shows how to help students better understand the beauty of mathematics as well as its utility.

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The development of this textbook series for our schools, *Understanding Mathematics,* is based on and organized as per the new School Mathematics Curriculum Framework that the Ministry of Education has developed recently, taking into consideration the changing needs of our country and international trends. We are also incorporating within the textbooks appropriate teaching methodologies including assessment practices which are reflective of international best practices. The *Teacher's Guides* provided with the textbooks are a resource for teachers to support them, and will definitely go a long way in assisting our teachers in improving their efficacy, especially during the initial years of teaching the new curriculum, which demands a shift in the approach to teaching and learning of Mathematics. However, the teachers are strongly encouraged to go beyond the initial ideas presented in the Guides to access other relevant resources and, more importantly to try out their own innovations, creativity and resourcefulness based on their experiences, reflections, insights and professional discussions.

The Ministry of Education is committed to providing quality education to our children, which is relevant and adaptive to the changing times and needs as per the policy of the Royal Government of Bhutan and the wish of our beloved King.

I would like to commend and congratulate all those involved in the School Mathematics Reform Project and in the development of these textbooks.

I would like to wish our teachers and students a very enjoyable and worthwhile experience in teaching, learning and understanding mathematics with the support of these books. As the ones actually using these books over a sustained period of time in a systematic manner, we would like to strongly encourage you to scrutinize the contents of these books and send feedback and comments to the Curriculum and Professional Support Division (CAPSD) for improvement with the future editions. On the part of the students, they can and should be enthusiastic, critical, venturesome, and communicative of their views on the contents discussed in the books with their teachers and friends rather than being passive recipients of knowledge.

Trashi Delek!

Thinley Gyamtsho

MINISTER

Ministry of Education

January of 2007

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INTRODUCTION

HOW MATHEMATICS HAS CHANGED

Mathematics is a subject with a long history. Although newer mathematical ideas are always being created, much of what your students will be learning is mathematics that has been known for hundreds of years, if not longer.

The learning of mathematics helps a person to solve problems. While solving problems, skills related to, representation of mathematical ideas, making connections with other topics in mathematics and connections with the real world, providing reasoning and proof, and communicating mathematically would be required. The textbook is designed to promote the development of these process skills.

Worldwide, there is now a greater emphasis on students understanding the mathematics they learn rather than memorizing and applying rote procedures.

Worldwide, there is now a greater emphasis on the need for students to understand the mathematics they learn rather than to memorize rote procedures. There are many reasons for this. • In the long run, it is very unlikely that students will remember the mathematics they learn unless it is meaningful. It is much harder to memorize “nonsense” than something that relates to what they already know.

- Some approaches to mathematics have not been successful; there are many adults who are not comfortable with mathematics even though they were successful in school.

In this new program, you will find many ways to make mathematics more meaningful.

It is important to always talk about why something is true, not simply that it is true.

- We will always talk about why something is true, not simply that it is true. For example, the reason why the number of lines of symmetry of a regular shape is always equal to its order of rotational symmetry is not just stated. It is explained using deductive reasoning.
- Mathematics should be taught using contexts that are meaningful to the students. They can be mathematical contexts or real world contexts. The student textbook uses both Bhutanese and international contexts.

  For example, in Unit 1 (Matrices and Networks), a task with an international context involves multiplying a matrix by a scalar to determine the price in ngultrums of several items priced in Thai baht. One task with a Bhutanese context involves considering the precision of a common balance used at a
  local market (Unit 4 Measurement). Meaningful contexts will help students see and appreciate the value of mathematics.


Working with Thai baht and a common balance from the Paro market

• When discussing mathematical ideas, we expect students to use the processes of problem solving, communication, reasoning, making connections (connecting mathematics to the everyday world and connecting mathematical topics to each other) and representation (representing mathematical ideas in different ways, such as manipulatives, graphs, and tables). For example, in Unit 2 (Commercial Math and Number) students connect radicals (both rational and irrational numbers) to the hypotenuse of right triangles, use reasoning to see why different representations of radical expressions are equivalent, and communicate their thinking while solving problems involving radicals.

We expect students to use the processes of problem solving, communication, reasoning, making connections, and representation.

• There is an increased emphasis on problem solving because the reason we learn mathematics is to solve problems. Once students are adults, they are not told when to factor or when to multiply; they need to know how to apply those skills to solve certain problems. Students will be given opportunities to make decisions about which concepts and skills they need in certain situations.

A significant amount of research evidence has shown that these more meaningful approaches work. Scores on international tests are higher when the emphasis is on higher level thinking and not only on the application of skills.

Scores on international tests are higher when the emphasis is on higher level thinking and not only on the application of skills.

THE DESIGN OF THE STUDENT TEXTBOOK

Each unit of the textbook has the following features:

• a *Getting Started* to review prerequisite content

• two or three chapters, which cluster the content of the unit into sections that contain related content

• regular lessons and at least one *Explore* lesson

• a *Game* (usually)

• at least one *Connections* feature

• a *Unit Revision*

Getting Started

There are two parts to the *Getting Started*. They are designed to help you know whether the students are missing critical prerequisites and to remind students of knowledge and terminology they should have already learned that will be useful in the unit.

The Getting Started is designed to help you know whether the students are missing critical prerequisites and to remind students of knowledge and terminology they will need for the unit.

• The *Use What You Know* section is an activity that takes 20 to 30 minutes. Students are expected to work in pairs or in small groups. Its purpose is consistent with the rest of the text's approach, that is, to engage students in learning by working through a problem or task rather than being told what to do and then just carrying it through.

• The *Skills You Will Need* section is a more straightforward review of required prerequisite skills for the unit. This should usually take about 30 minutes.

Regular Lesson

• Each lesson might be completed in one or two hours (i.e., one or two class periods). The time is suggested in this *Teacher's Guide*, but it is ultimately at your discretion.

• Lessons are numbered #.#.#. The first number tells the unit, the second number the chapter, and the third number the lesson within the chapter. For example, Lesson 4.2.1 is Unit 4, Chapter 2, Lesson 1.

Lessons are numbered #.#.#. The first number tells the unit, the second number the chapter, and the third number the lesson within the chapter.

• Each lesson is divided into five parts:

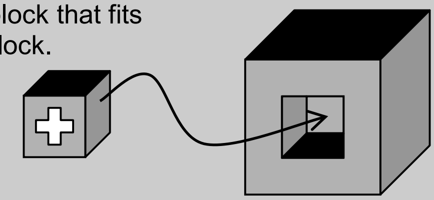
- A *Try This* task or problem
- The exposition (the main points of the lesson)
- A question that revisits the *Try This* task, called *Revisiting the Try This* in this guide
- *Examples*
- *Practising and Applying* questions

Try This

• The *Try This* task is in a shaded box, like the one below from **lesson 8.1.2**.

A baby is playing with a small cube block that fits into a square hole in a bigger cube block.

A. How many ways can the baby fit the small cube block into the hole?



• The *Try This* is a brief task or problem that students might often complete in pairs or small groups. It serves to motivate new learning. Students can do the *Try This* without the new concepts or skills that are the focus of the lesson, but the problem is related to the new learning. It should be completed in 5 to 10 minutes. The reason to start with a *Try This* is that we believe students should do some mathematics independently before you intervene.

The Try This is a brief task or problem that students might often complete in pairs or small groups to motivate new learning.

• Some lessons have *Try This* activities that take longer (e.g., in Unit 5, where students might be graphing). In these cases, the lesson itself is likely to be a two-period lesson.

• The answers to the *Try This* questions are not found in the back of the student book (but they are in this *Teacher's Guide*).

The Exposition

• The exposition presents the main concepts and skills of the lesson. Examples are often included to clarify the points being made.

The exposition presents the main concepts and skills of the lesson and is taught in different ways, as suggested in this guide.

• You will help the students through the exposition in different ways (as suggested in this *Teacher's Guide*). Sometimes you will present the ideas first, using the exposition as a reference. Other times students will work through the exposition independently, in pairs, or in small groups.

• Key mathematical terms are introduced and described in the exposition. When a key term first appears in a unit, it is highlighted in **bold type** to indicate that it is found in the glossary (at the back of the book).

• Students are not expected to copy the exposition into their notebooks either directly from the book or from your recitation.

Revisiting the Try This

• The *Revisiting the Try This* question follows the exposition and appears in a shaded area, like this example from **lesson 8.1.2**, which follows from the above *Try This* about the small cube block.

B. i) How does the question in **part A** relate to rotational symmetry?
ii) What is the order of turn symmetry of the small cube? Explain.

• The *Revisiting the Try This* question links the *Try This* task or problem to the new ideas presented in the exposition. This is designed to build a stronger connection between the new learning and what students already understand.

The Revisiting the Try This question links the Try This task to the new ideas presented in the exposition.

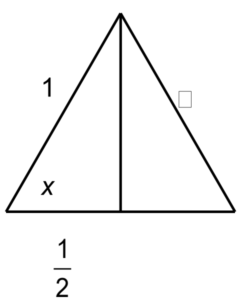
Examples

• The *Examples* are designed to provide additional instruction by modelling how to approach some of the questions students will meet in *Practising and Applying*. Each example is a bit different from the others so that students have multiple models from which to work.

The Examples model how to approach some of the questions students will meet in Practising and Applying

• The *Examples* show not only the formal mathematical work (in the left hand *Solution* column), but also student reasoning (in the right hand *Thinking* column). This model should help students learn to think and communicate mathematically. Photographs of students are used to further reinforce this notion.

• Many of the *Examples* present two or even three different solutions. The example below, from **lesson 7.1.4** shows two possible ways to approach the task, *Solution 1* and *Solution 2*.

Example 2 Using the Reciprocal Ratio	
In a right triangle, sec x = 2. What is the value of x?	
Solution 1 If sec x = 2, then cos x = $\frac{1}{2}$.  The angle must be 60°.	**Thinking**  • I knew that sec x = $\frac{1}{\cos x}$, so I used the secant ratio to find cosine. • I knew that cosine was $\frac{1}{2}$ for one of the angles in a 30°-60°-90° triangle. I drew that triangle as half of an equilateral triangle to help me figure out whether x was 30° or 60°. • Cosine is based on the adjacent side, so the angle with cosine of $\frac{1}{2}$ must be the 60° angle.
Solution 2 If sec x = 2, then cos x = $\frac{1}{2}$. $\cos^{-1} \frac{1}{2} = 60°$	**Thinking**  • My calculator doesn't have a button for $\sec^{-1}$ that gives the angle if you enter the secant, but it does have a button for $\cos^{-1}$. That's why I used the relationship between secant and cosine to find the value of the cosine ratio.

The Examples show the formal mathematical work in Solution column and model student reasoning in the Thinking column. This model should help students learn to think and communicate mathematically.

• The treatment of *Examples* varies and is discussed in the *Teacher's Guide*. Sometimes the students work through these independently, sometimes they work in pairs or small groups, and sometimes you are asked to lead them through some or all of the examples.

• A number of the questions in the *Practising and Applying* section are modelled in the *Examples* to make it more likely that students will be successful.

Practising and Applying

• Students work on the *Practising and Applying* questions independently, with a partner, or in a group, using the exposition and *Examples* as references.

• The questions usually start like the work in the *Examples* and get progressively more conceptual, with more explanations and more problem solving required later in the exercise set.

• The last question in the section always brings closure to the lesson by asking students to summarize the main learning points. This question could be done as a whole class.

Students work on the Practising and Applying questions independently, with a partner, or in a group, using the exposition and Examples as references.

Explore Lessons

• *Explore* lessons provide an opportunity for students to work in pairs or small groups to investigate some mathematics in a less directed way. Often, but not always, the content is revisited more formally in a regular lesson immediately before or after the *Explore* lesson. The *Teacher's Guide* indicates whether the *Explore* lesson is optional or essential.

• There is no teacher lecture in an *Explore* lesson, so the parts of the regular lesson are not there. Instead, a problem or task is posed and students work through it by following a sequence of questions or instructions that direct their investigation.

• The answers for these lessons are not found in the back of the book, but are found in this *Teacher's Guide*.

Explore lessons provide an opportunity for students to work with a partner or in small groups to investigate some mathematics in a less directed way.

Connections

• The *Connections* is an optional feature that takes many forms. Sometimes it is
a relevant and interesting historical note. Sometimes it relates the mathematical content of a unit to the content of a different unit. Other times it relates the mathematical content to a real world application.

• There are always one or more *Connections* features in a unit. The placement of
a *Connections* feature is not fixed; it depends on the content knowledge required.

• The *Connections* feature always gives students something to do beyond simply reading it.

• Students usually work in pairs or small groups to complete these activities.

The Connections feature takes many forms. Sometimes it is a relevant and interesting historical note. Sometimes it relates the mathematical content of a unit to the content of a different unit. Other times it relates the mathematical content to a real world application.

Game

• There is usually one *Game* per unit, and sometimes there are two. If there is no *Game*, there is an extra *Connections* feature.

• The *Game* provides an enjoyable way to practise skills and concepts introduced in the unit.

• Its placement in the unit is based on where it makes most sense in terms of the content required to play the *Game*.

• In most *Games* students work in pairs or small groups, as indicated in the instructions.

• The required materials and rules are listed in the student book. Usually, there is a sample shown to make sure that students understand the rules.

• Most *Games* require 15 to 20 minutes, but students can often benefit from playing them more than once.

The Game provides an enjoyable way to practise skills and concepts introduced in the unit.

Unit Revision

• The *Unit Revision* provides an opportunity for review for students and for you to gather informal assessment data. *Unit Revisions* review all lesson content except the *Getting Started* feature, which is based on previous class content or optional explore lessons. There is always a mixture of skill, concept, and problem solving questions.

• The order of the questions in the *Unit Revision* generally follows the order of the lessons in the unit. Sometimes, if a question reflects more than one lesson, it is placed where questions from the later lesson would appear.

• Students can work in pairs or on their own, as you prefer.

• The *Unit Revision*, if done in one sitting, requires more than one hour. If you wish, you might break it up and assign some questions earlier in the unit and some questions later in the unit.

The Unit Revision provides an opportunity for review for students and allows you to gather informal assessment data.

Glossary

• At the end of the student book, there is a glossary of key mathematical terminology introduced in the units. When new terms are introduced in the units, they are written in **bold type**. All of these terms are found in the glossary.

• The glossary also contains important mathematical terms from previous classes that students might need to refer to.

• In addition, there is a set of instructional terms commonly used in the *Practising and Applying* questions (for example, justify, explain, predict, …) along with descriptions of what those terms require the student to do.

The Glossary contains key mathematical terminology introduced in the units, important terms from previous classes, and a set of instructional terms.

Answers

• Answers to most numbered questions are provided in the back of the student book. In many cases, the full solutions are not shown, only the final answers.

• There is often more than one possible answer. This is indicated by the phrase *Sample Response*.

• Full solutions to many of the questions are provided in the *Teacher's Guide*,
as are the answers to the lettered questions (such as A or B) in the *Try This* and the *Explore* lessons.

The Answers to most of the numbered questions are found in the back of the student book. This Teacher's Guide contains a full set of answers.

THE DESIGN OF THE TEACHER'S GUIDE

The *Teacher's Guide* is designed to complement and support the use of the student textbook.

• The sequencing of material in the guide is identical to the sequencing in the student textbook.

• The elements in the *Teacher's Guide* for each unit include:

- a *Unit Planning Chart*
- *Math Background* for the unit
- a *Rationale for Teaching Approach*
- a brief overview of *Technology in This Unit* (as required)
- support for each lesson
- a *Unit Test*
- a *Performance Task*
- an *Assessment Interview* (Units 2 and 5 only)

The Teacher's Guide is designed to complement and support the use of the student textbook.

The support for each lesson includes:

- *Curriculum Outcomes* covered in that lesson
- *Outcome relevance* (*Lesson relevance* in the case of *Explore* lessons)
- *Pacing* in terms of hours
- *Materials* required to teach the lesson
- *Prerequisites* that the lesson assumes students possess
- *Main Points to be Raised* explicitly in the lesson
- suggestions for working through the parts of the lesson
- *Suggested assessment* for the lesson
- *Common errors* to be alert for
- *Answers*, often with more complete solutions than are found in the student text
- suggestions for *Supporting Students* who are struggling and/or for enrichment

Unit Planning Chart

The Unit Planning Chart provides an overview of the unit.

This chart provides an overview of the unit and indicates, for each lesson, which curriculum outcomes are being covered, the pacing, the materials required, and suggestions for which questions to use for formative assessment.

Math Background and Rationale for Teaching Approach

Information about the critical mathematics behind the unit, an explanation of why the math is approached the way it is in the unit, and an overview of how technology is used or could be used in the unit is provided at the beginning of each unit.

This explains the organization of the unit or provides some background for you that you might not have, particularly in the case of less familiar content. In addition, there is an indication of why the material is approached the way it is.

Technology in This Unit

Most units include a brief overview of the role of technology in the unit. It is assumed that students have access to scientific calculators. Any reference to the use of the Internet, graphing software, or spreadsheets, for example, is considered optional.

Regular Lesson Support

Regular lesson support includes grouping and instructional strategies, alerts for common errors, suggestions for assessment, and teaching tips.

• Suggestions for grouping and instructional strategies are offered under the headings *Try This, Revisiting the Try This, The Exposition — Presenting the Main Ideas, Using the Examples,* and *Practising and Applying — Teaching Tips.*

• *Common errors* are sometimes included. If you are alert for these and apply some of the suggested remediation, it is less likely that students will leave the lessons with mathematical misunderstandings.

• A number of *Suggested assessment questions* are listed for each lesson. This is to emphasize the need to collect data about different aspects of the student's performance — sometimes the ability to apply skills, sometimes the ability to solve problems or communicate mathematical understanding, and sometimes the ability to show conceptual understanding.

• It is not necessary to assign every *Practising and Applying* question to each student, but they are all useful. You should go through the questions and decide where your emphasis will be. It is important to include a balance of skills, concepts, and problem solving. You might use the *Suggested assessment questions* as a guide for choosing some of the questions to assign.

• You may decide to use the last *Practising and Applying* question to focus a class discussion that revisits the main ideas of the lesson and to bring closure to the lesson.

Explore Lesson Support

• As with regular lessons, for *Explore* lessons there is an indication of the curriculum outcomes being covered, the relevance of the lesson, and whether the lesson is optional or essential.

• Because the style of the lesson is different, the support provided is different than for the regular lessons. There are suggestions for grouping the students for the exploration, a list of *Observe and assess* questions to guide your informal formative assessment, and *Share and reflect* ideas on how to consolidate and bring closure to the exploration.

Because the style of an Explore lesson is different than a regular lesson, the support provided in the Teacher's Guide is also different.

Unit Test

A pencil-and-paper unit test is provided for each unit. It is similar to the unit revision, but not as long. If it seems that the test might be too long (for example,
if students would require more than one class period to complete it) some questions may be omitted. It is important to balance the items selected for the
test to include questions involving skills, concepts, and problem solving, and to include at least one question requiring mathematical communication.

If the Unit Test seems too long, some questions may be omitted but is important to ensure a balance of questions involving skills, concepts, problem solving, and communication.

Performance Task

• The *Performance Task* is designed as a summative assessment task, although
it can be also used as an additional assignment. Performance on the task can be combined with performance on a *Unit Test* to give a mark for a student on a particular unit.

• The task requires students to use both problem solving and communication skills to complete it. Students have to make mathematical decisions to complete the task.

• It is not appropriate to mark a performance task using percentages or numerical grades. For that reason, a rubric is provided to guide assessment. There are four levels of performance that can be used to describe each student’s work on the task. A level is assigned for each different aspect of the task and, if desired, an overall profile can be assigned. For example, if a student’s performance is level 2 on most of the aspects of the task, but level 3 on one aspect, an overall profile of level 2 might be assigned.

• A sample solution is provided for each task.

The Performance Task is designed as a summative assessment task that can be combined with a student’s performance on the Unit Test to give an overall mark.

Unit Assessment Interviews

• Selected units (Units 2 and 5) provide a structure for an interview that can be used with one or several students to determine their understanding of the outcomes.

• Interviews are a good way to collect information about students since they allow you to interact with the students and to follow up if necessary. Interviews are particularly appropriate for students whose performance is inconsistent or who do not perform well on written tests, but who you feel really do understand the content.

• You may use the data you collect in combination with class work or even a unit test mark.

Interviews are a good way to collect information about students, since interviews allow you to interact with the students and to follow up if necessary.

ASSESSING MATHEMATICAL PERFORMANCE

Forms of Assessment

It is important to consider both formative (continuous) and summative assessment.

Formative

Formative assessment is observation to guide further instruction.

• Formative assessment is observation to guide further instruction. For example, if you observe that a student does not understand an idea, you may choose to re-teach that idea to that student using a different approach.

• Formative assessment opportunities are provided through
  - prerequisite assessment in the *Getting Started*
  - suggestions for assessment questions in each regular lesson
  - questions that might be asked while students work on the *Try This* or during an *Explore* lesson
  - the *Unit Revision*
  - the unit *Assessment Interview* (Units 2 and 5 only)

• Formative assessment can be supplemented by
  - everyday observation of students' mathematical performance
  - formal or informal interviews to reveal students' understanding
  - journals in which students comment on their mathematical learning
  - short quizzes
  - projects
  - a portfolio of work so students can see their progress over time, for example, in problem solving or mathematical communication (see *Portfolios* below)

Summative

Summative assessment is used to see what students have learned and is often used to determine a mark.

• Summative assessment is used to see what students have learned and is often used to determine a mark or grade.

• Summative assessment opportunities are provided through
  - the *Unit Test*
  - the *Performance Task*
  - the *Assessment Interview*

• Summative assessment can be supplemented with
  - short quizzes
  - projects
  - a portfolio that is assessed with respect to progress in, for example, problem solving or communication

Portfolios

One of the advantages of using portfolios for assessment is that students can observe their own growth over time, leading to greater confidence.

One of the advantages of using portfolios for assessment is that students can observe their own growth over time, leading to greater confidence. In each unit, items are identified in the section on math background as pertaining to the mathematical processes. Student work items related to one of the mathematical processes: problem solving, communication, reasoning, or representation, could form the basis of the portfolio.

Assessment Criteria

It is right and fair to inform students about what will be assessed and how it will be assessed.

• It is right and fair to inform students about what will be assessed and how it will be assessed. For example, students should know whether the intent of a particular assessment is to focus on application or on problem solving.

• A student's mark and all assessments should reflect the curriculum outcomes for Class X. The proportions of the mark assigned for each unit should reflect both the time spent on the unit and the importance of the unit. The modes of assessment used for a particular unit should be appropriate for the content. For example, if the unit focuses mostly on skills, the main assessment might be a paper-and-pencil test or quizzes. If the unit focuses on concepts and application, more of the student's mark should come from activities like performance tasks.

The modes of assessment used for a particular unit should be appropriate for the content.

• The focus of this curriculum is not on procedures for their own sake but on a conceptual understanding of mathematics so that it can be applied to solve problems. Procedures are important too, but only in the context of solving problems. Assessment should balance procedural, conceptual, and problem solving items, although the proportions will vary in different situations.

Assessment should balance procedural, conceptual, and problem solving items, although the proportions may vary in different situations.

• Students should be informed whether a test is being marked numerically, using letter grades, or with a rubric. If a rubric is being used, then it should be shared with students before they begin the task to which it is being applied.

Determining a Mark

It is important to remember that the mark should reflect student competence with the mathematical outcomes of the course and not behaviour, neatness, participation, and so on.

• In determining a student's mark, you can use the tools described above along with other information such as work on a project or poster. It is important to remember that the mark should, as closely as possible, reflect student competence with the mathematical outcomes of the course. The mark should not reflect behaviour, neatness, participation, and other non-mathematical aspects of the student's learning. These are important to assess as well, but not as part of the mathematics mark.

• In looking at a student's mathematics performance, the most recent data might be weighted more heavily. For example, suppose a student does poorly on some of the quizzes given early in Unit 1, but later you observe that he or she has a better understanding of the material in the unit. You might choose to give the early quizzes less weight in determining a student's mark for the unit.

• At present, you are required to produce a numerical mark for a student, but that should not preclude your use of rubrics to assess some mathematical performance. One of the values of rubrics is their reliability. For example, if a student performs at level 2 on a particular task one day, he or she is very likely to perform at the same level on that task another day. On the other hand, a student who receives a test mark of 45 one day might have received a mark of 60 on a different day if one question had changed on the test or if he or she had read an item more carefully.

• You can combine numerical and rubric data using your own judgment. For example, if a student's marks on tests average 50%, but the rubric performance is higher, for example, level 3, it is fair and appropriate to use a higher average for that student's class mark.

THE CLASSROOM ENVIRONMENT

This new curriculum requires a change in the classroom environment to include more pair and group work and an increased emphasis on communication.

This new curriculum requires a change in the classroom environment to include more pair and group work and an increased emphasis on communication. It is only in this way that students will really become engaged in mathematical thinking instead of being spectators.

• In every lesson, students should be engaged in some pair or small group work (for the *Try This*, selected *Practising and Applying* questions, or during an *Explore* lesson).

• Students should be encouraged to communicate with each other to share responses that are different from those offered by other students or from the responses you might expect. Communication involves not only talking and writing but also listening and reading.

Pair and Group Work

• There are many reasons why students should be working in pairs or groups, including

- to ensure that students have more opportunities to communicate mathematically (instead of competing with the whole class for a turn to talk)
- to make it easier for them to take the risk of giving an answer they are not sure of (rather than being embarrassed in front of so many other people if they are incorrect)
- to see the different mathematical viewpoints of other students
- to share materials more easily

• Sometimes students can work with the students who sit near them, but other times you might want to form the groups so that students who are struggling are working together. Then you can help them while the other students move forward. Students who need enrichment can also work together so that you can provide an extra challenge for them all at once.

• For students who are not used to working in pairs or groups, you need to set down rules of behaviour that require them to attend to the task and to participate fully. You need to avoid a situation where four students are working together, but only one of them is really doing the work. You might display Rules for Group Work, as shown here.

• Once students are used to working in groups, you might sometimes be able to base assessment on group performance rather than on individual performance.

Communication

It is through communication that students clarify their own thinking as well as show you and their classmates what they do or do not understand.

Students will be reluctant to communicate unless the environment is risk-free.

Students should be communicating regularly about their mathematical thinking.
It is through communication that they clarify their own thinking as well as show you and their classmates what they do or do not understand. When they give an answer to a question, you can always be asking questions like, *How did you get that? How do you know? Why did you do that next?*

• Communication is practised in small group settings, but is also appropriate when the whole class is working together.

• Students will be reluctant to communicate unless the environment is risk-free.
In other words, if students believe that they will be reprimanded or made to feel badly if they say the wrong thing, they will be reluctant to communicate. Instead, show your students that good thinking grows out of clarifying muddled thinking. It is reasonable for students to have some errors in their thinking and it is your job to help shape that thinking. If a student answers incorrectly, it is your job to ask follow-up questions that will help the student clarify his or her own thinking. For example, suppose a problem requires a student to determine the x-intercepts of a quadratic function from its equation but the student instead gives the y-intercept. Rather than saying the student is wrong, you could ask these questions:

- *How did you figure out your intercept?*
- *Where would the intercept be on the graph of the function? What are the coordinates at that point? How do you know?*
- *Is the point on the x-axis or the y-axis? Which intercept is it?*

• Many of the questions in the textbook require students to explain their thinking. The sample *Thinking* in the *Examples* is designed to provide a model for mathematical communication.

• One of the ways students communicate mathematically is by describing how they know an answer is right. Even when a question does not ask students to check their work, you should encourage them to think about whether their answer makes sense. When they check their work, they should usually check using a different way than the way they used to find their answer so that they do not make the same reasoning error twice. This will enhance their mathematical flexibility.

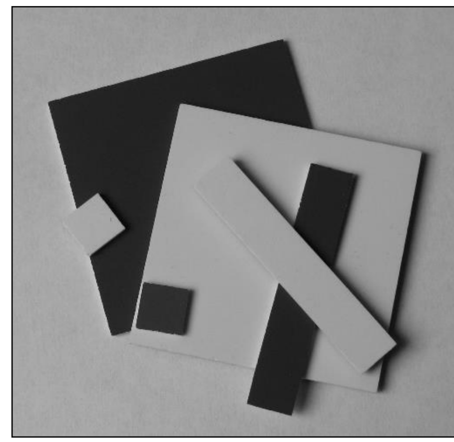
The Thinking beside each Solution in the Examples is designed to provide a model for mathematical communication.

MATHEMATICAL TOOLS

Manipulatives

• As the students move up the classes, there will be less use of manipulative (hands-on) materials than in earlier classes. Nevertheless, there is value in using manipulative materials even for older students who are good at mathematics. For example,
Unit 5 makes use of algebra tiles for factoring polynomials. Although some students can be successful without these materials, all students benefit from their use. Students start to see not only how to perform algebraic manipulations, but why they are done the way they are.

• Manipulative materials are important in Class X, particularly, but not exclusively, in the units on quadratics, probability, geometry, and measurement.



Algebra tiles for factoring polynomials

Appropriate Calculator Use

• In Class X, the calculator should be used as a regular tool. Just like a pencil, a calculator is a mathematical tool. At this point in their mathematical education, students are no longer asked simply to perform routine calculations. Calculations are now part of more sophisticated mathematical tasks that are the real focus of their learning.

• Because calculators can change from year to year and because not all students have the same calculator, it is your responsibility to teach students how to use their calculators correctly. There may be some mention of general calculator skills in the text or guide, but the specifics vary from calculator to calculator.

Calculators should be used as a regular tool.

THE STUDENT NOTEBOOK

It is valuable for students to have a well-organized, neat notebook to look back at to review the main mathematical ideas they have learned. However, it is also important for students to feel comfortable doing rough work in that notebook or doing rough work elsewhere without having to spend time copying all of it neatly into their notebooks. In addition to the things you tell students to include in their notebooks, they should be allowed to make some of their own decisions about what to include in their notebooks.

Students should be allowed to make some of their own decisions about what to include in their notebooks.

CLASS X CURRICULUM

STRAND A: NUMBER

KSO Number *By the end of **Class 10** students should*
♦ *demonstrate an understanding of the real number system through appropriate application of concepts and procedures related to real numbers.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-A1 Irrational Numbers: understand role
• develop awareness of errors in decimal approximations and rounding off
• understand when to approximate and when to continue with radical expressions
• convert between entire and mixed radicals (e.g., within the application of the Pythagorean theorem)

10-A2 Inequalities: relate sets of numbers to solutions
• solve problems with a restricted solution set
• include choice of number systems

10-A3 Matrices: represent problems
• understand that matrices are used as a means of storing data
• understand how the rows and columns of a matrix are identified

10-A4 Network Problems: represent networks using matrices and vice versa
• represent a network as a matrix and interpret a matrix in terms of a corresponding network situation

STRAND B: OPERATIONS

KSO Operations *By the end of **Class 10** students should*
♦ *derive, analyse, and apply computational procedures in situations involving all representations of real numbers,*
♦ *derive, analyse, and apply algebraic procedures in problem situation, and*
♦ *recognize and use the relationship between algebraic and arithmetic operations to solve problems.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-B1 Roots: apply properties to operations
• develop and apply properties for operations involving roots

10-B2 Purchasing Decisions: solve problems
• use percentage to solve problems involving purchases

10-B3 Simple and Compound Interest: demonstrate an understanding
• understand the long term difference between simple and compound interest
• investigate both investments and financing situations

10-B4 Matrices: adding, subtracting, and scalar multiplication — model, solve, and create problems
• understand that to add or subtract two matrices the dimensions of the two matrices must be the same
• understand that to multiply a matrix by a scalar multiply each entry in the matrix by the scalar
• create and solve matrix problems

10-B5 Matrix Multiplication: develop and apply procedures
• develop and apply the algorithm for multiplication
• understand that matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix

10-B6 Matrices: solve network problems
• represent and solve network problems using matrices

STRAND C: PATTERNS AND RELATIONSHIPS

KSO Patterns and Relationships *By the end of **Class 10** students should*
♦ *model real-world problems using functions, equations, inequalities, and discrete structure,*
♦ *represent functional relationships in multiple ways and describe connections among those representations,*
♦ *perform operations on and between functions,*
♦ *analyse and explain the behaviors, transformations, and general properties of types of equations and relations, and*
♦ *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-C1 Transformations: express algebraically or using mapping rules
• express transformations from a graph using algebraic expressions or using mapping rules
• express mapping rules algebraically and vice versa

10-C2 Patterns and Real-World Relationships: describe, identify, and apply
• use graphs, tables of values, and written descriptions to describe patterns and relationships
• identify patterns in graphs and/or tables of values

10-C3 Linear Inequalities: write and describe graphs
• describe a given graph using inequalities

10-C4 Graphs and Tables: construct and analyse
• analyse graphs and tables to determine mathematical characteristics
• interpret characteristics in relation to given contexts

10-C5 Graphs and Tables: explore dynamics of change
• determine how changes in one variable affect another through the analysis of tables or graphs

10-C6 Graphs: sketch
• create graphs given information in a variety of formats
• sketch the graph of a quadratic function in vertex or factored form
• translate among tabular, written, symbolic, and graphical representations of functions

10-C7 Graphs: create by constructing a table of values and graphing
• construct a graph from a table of values
• understand when to choose to graph by the y-intercept slope method

10-C8 Systems of Linear Equations: solve
• realize that the graphing method will not always give exact solutions
• solve linear equations by substitution method, including comparison of equations
• determine the solution to an equation by graphing one side of the equation against the other and identifying the intersection point
• solve linear equations by elimination method

10-C9 Non-linear Equations: evaluate and interpret
• determine the roots of quadratic equations from the corresponding graph

10-C10 Equations: solve linear and simple radical, exponential, and absolute value equations and linear inequalities
• encourage proficiency with algebraic manipulation
• use strategies to check answers for reasonableness within the problem context

10-C11 Equations and Inequalities: graph and analyse graphs
• graph equations and inequalities by constructing a table of values, using knowledge of transformations, and by identifying characteristics

10-C12 Problems: express in terms of equations
• analyse and interpret a variety of situations and model algebraically as equations

10-C13 Equations: rearrange
• transform equations from one form to another

10-C14 Equations: solve using graphs
• use the x-intercept to determine the solution of quadratic equations

10-C15 Equations: apply properties of numbers upon expressions
• solve equations by applying associative, distributive, identity, and inverse properties

10-C16 Quadratic Equations: solve by factoring
• understand the zero product rule: if $ab = 0$, then either $a = 0$ or $b = 0$
• apply the zero product rule to solve quadratic equations by factoring
• convert a quadratic equation to two linear equations by the factoring method
• develop factoring strategies for polynomials in one variable that are products of degree one binomials
• solve equations including those which involve common factors, regular equations, perfect square trinomials, and difference of squares

10-C17 Functional Relationships and Notation: explore formally and informally
• understand the relationship between a relation and a function
• start with functional relationships then apply the mathematical concept of function
• use mathematical notation and vocabulary

10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions
• apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function
• apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the absolute value function

10-C19 Data: gather, plot, and demonstrate understanding of independent and dependent variables, and domain and range
• make decisions regarding independent and dependent variables

STRAND D: MEASUREMENT

KSO Measurement *By the end of* ***Class 10*** *students should*
♦ *measure quantities indirectly, using techniques of algebra, geometry, and trigonometry,*
♦ *determine measurements in a wide variety of problem situations, and consider accuracy and precision, and*
♦ *apply measurement formulas and procedures in a wide variety of contexts.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-D1 Measurement: accuracy and precision
• understand that accuracy depends upon the usage of the measurement instrument
• understand that precision depends on how finely an instrument is graduated
• address precision issues when performing calculations on measurement data
• express answers to problems with significant figures

10-D2 Perimeter and Area: explore and apply properties
• examine maximizing area while restricting perimeter
• examine minimizing perimeter while restricting area

10-D3 Volume, Capacity, and Surface Area: demonstrate understanding
• understand that surface area, capacity, and volume apply to 3-D shapes
• understand the connection between volume and surface area
• compare prism measurements

10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine
• apply formulas for area, perimeter, surface area, and volume in a variety of contexts
• determine capacity using volume
• develop non-routine formulas to determine area
• understand that areas of regular polygons can be determined by dividing the area into familiar shapes

10-D5 Similar Triangles: apply properties
• apply side and angle relationships when developing the primary trig ratios

10-D6 Similar Triangles and Right Triangles solve problems
• solve problems using the proportionality relationship among sides in similar triangles
• apply the Pythagorean theorem in appropriate situations

10-D7 Trigonometric Functions: relate to ratios in similar right triangles
• understand that primary trig ratios are equivalent for the equal angles in similar right triangles
• investigate the three primary ratios between the lengths of pairs of sides in right angle triangles
• relate reciprocal ratios to primary trig ratios

10-D9 Trigonometric Identities:
• understand what identities are
• test statements to see if they are identities
• understand why $\sin x = \cos (90 - x)$
• understand why $\sin^2 x + \cos^2 x = 1$

10-D10 Trigonometric Values: special angles
• use the Pythagorean theorem to determine exact values for the sine, cosine, and tangent of 30°, 60° and 45° angles

10-D11 Trigonometric Values: use calculators
• use calculators to determine the trig ratios sin θ, cos θ, and tan θ

10-D12 Trigonometric Values: right triangles — apply to solve problems
• find areas of polygons using right triangle trigonometry
• explore angles of elevation (measured from the horizon up) and angles of depression (measured from the horizon down) in real world settings

10-D13 Trigonometric Ratios: solve problems
• calculate side lengths and angles using trig ratios (use of calculators is required)

10-D14 Vectors and Bearings: solve problems
• solve bearing and vector problems using the Pythagorean theorem and/or trigonometric ratios

10-D15 Networks: traversability
• recognize that a network with more than two odd vertices is not traversable

STRAND E: GEOMETRY

KSO Geometry *By the end of **Class 10** students should*
♦ *make and test conjectures about, and deduce properties of and relationships between, 2- and 3-dimensional shapes in multiple contexts,*
♦ *analyse and apply Euclidean transformations, including representing and applying translations,*
♦ *represent problem situations with geometric models and apply properties of shapes, and*
♦ *demonstrate an understanding of reasoning, justification, and proof.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-E1 Congruent Triangles and Angle Properties: informal deductions
• distinguish between inductive and deductive reasoning using both mathematical and non-mathematical reasoning

10-E2 Geometric Reasoning: inductive and deductive
• use inductive and deductive reasoning in situations such as generalizing relationships and proving theorems

10-E3 2-D and 3-D Shapes: explore properties and test conjectures
• compare 2-D and 3-D mirror symmetry
• compare 2-D and 3-D rotational symmetry
• compare mirror and rotational symmetry

10-E4 Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors)
• consider the concepts of perpendicular and angle bisectors, medians, and altitudes of triangles
• locate incentres and circumcentres and construct incircles and circumcircles using perpendicular and angle bisector constructions
• locate centroids (centres of gravity) and orthocentres using median and altitude constructions

STRAND F: DATA MANAGEMENT

KSO Data Management *By the end of **Class 10** students should*
♦ *determine, interpret and apply as appropriate a wide variety of statistical measures and distribution and*
♦ *use curve fitting to determine the relationship between, and make predictions from, sets of data and be aware of bias in the interpretation of results.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-F1 Correlations: develop an intuitive understanding
• understand that a correlation coefficient is a description of how well data fits a linear pattern
• identify the difference between a strong and weak correlation and between a negative and positive correlation based on the scatter plot and the value of the correlation coefficient

10-F2 Curves of Best Fit: non-linear data
• explore curve fitting for non-linear data
• understand that non-linear models often show a better relationship than linear models

10-F3 Normal Curves: explore measurement issues
• understand that a frequency polygon is created by joining the midpoints of the top of each bar in a histogram
• identify situations that give rise to common distributions (e.g., U-shaped, skewed, and normal)
• demonstrate an understanding of the properties of the normal distribution (e.g., the mean, median, and mode are equal; the curve (and data) is symmetric about the mean)
• understand that a normal curve is based upon a certain type of histogram with infinitely small bins

10-F4 Data Analysis: distribution of data
• understand that box and whisker plots are useful when comparing data

10-F5 Displaying Data: construct and interpret
• compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: stem and leaf plots, box plots, and histograms

STRAND G: PROBABILITY

KSO Probability *By the end of **Class 10** students should*
♦ *represent and solve problems involving uncertainty,*
♦ *make predictions and carry out simulations to answer real world issues of interest, and*
♦ *determine theoretical probability for dependent and independent events and apply to real life issues.*

Toward this, students in **Class 10** will be expected to master the following **SO** (Specific Outcomes):

10-G1 Theoretical Probability: independent and dependent events
• distinguish between two events that are dependent or independent using reasoning and calculations

UNIT 1 MATRICES AND NETWORKS

UNIT 1 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 1** **TG p. 4**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	None	All questions
Chapter 1 Matrices				
1.1.1 Introducing Matrices **SB p. 2** **TG p. 6**	**10-A3 Matrices: represent problems** • understand that matrices are used as a means of storing data • understand how the rows and columns of a matrix are identified	1 h	• Grid paper	Q5 and 9
1.1.2 Adding and Subtracting Matrices **SB p. 5** **TG p. 8**	**10-A3 Matrices: represent problems** • understand that matrices are used as a means of storing data **9-B4 Matrices: adding, subtracting, and scalar multiplication — model, solve, and create problems** • understand that to add or subtract two matrices the dimensions of the two matrices must be the same	1 h	None	Q2, 9, 10, and 11
1.1.3 Multiplying a Matrix by a Scalar **SB p. 8** **TG p. 11**	**9-B4 Matrices: adding, subtracting, and scalar multiplication — model, solve and create problems** • understand that to multiply a matrix by a scalar multiply each entry in the matrix by the scalar • create and solve matrix problems	1 h	• Calculators	Q1, 6, and 7
1.1.4 Multiplying Matrices **SB p. 11** **TG p. 14**	**10-B5 Matrix Multiplication: develop and apply procedures** • develop and apply the algorithm for multiplication • understand that matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix	2 h	• Grid paper	Q1, 7, 8, and 9

UNIT 1 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Networks				
1.2.1 EXPLORE: Travelling Networks (Essential) **SB p. 15** **TG p. 18**	**10-D15 Networks: traversability** • recognize that a network with more than two odd vertices is not traversable	1 h	None	Observe and Assess questions
CONNECTIONS: The Seven Bridges of Konigsberg **SB p. 16** **TG p. 19**	To explore an application of networks presented as a puzzle	20 min	None	N/A
GAME: Sprouts **SB p. 16** **TG p. 19**	To explore networks in a game situation	20–30 min	None	N/A
1.2.2 Describing a Network With a Matrix **SB p. 17** **TG p. 20**	**10-A4 Network Problems: represent networks using matrices and vice versa** • represent a network as a matrix and interpret a matrix in terms of a corresponding network situation	1 to 2 h	None	Q1, 2, and 8
1.2.3 Solving Network Problems **SB p. 22** **TG p. 24**	**10-B6 Matrices: solve network problems** • represent and solve network problems using matrices	2 h	None	Q2, 3, and 8
UNIT 1 Revision **SB p. 28** **TG p. 27**	Review the concepts and skills in the unit	1 h	None	All questions
UNIT 1 Test **TG p. 29**	Assess the concepts and skills in the unit	1 h	None	All questions
UNIT 1 Performance Task **TG p. 31**	Assess concepts and skills in the unit	1 h	• Map of Bhutan	Rubric provided

Math Background

• Work with matrices is a natural extension of previous work with charts and tables. The matrix itself is no different from a chart or table (although the column and row headers are not part of the matrix).

• Addition and subtraction of matrices and multiplication by a scalar are likely to make sense to students as they are a logical extension of work with tables and charts. Multiplication of matrices is the operation that will be most unfamiliar and will require more time for students to make sense of it.

• One of the most important applications of matrices is the solution of a system of linear equations, but this unit does not get to that level of sophistication. However, another useful application of matrices is explored — describing directed graphs (networks made of paths) and solving problems involving those digraphs using matrix multiplication.

• As students work with networks, they are exposed to Euler's discovery about traversability of networks — a network is traversable without repeating an edge and starting and ending at the same point only if there are no more than 2 *odd* vertices (that is, there are 0, 1, or 2 odd vertices). (A vertex is odd if an odd number of edges meet there.)

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in **lesson 1.2.1**, where they explore the traversability of networks, and in **lesson 1.2.3** where they determine the number of two-stopover trips in **questions 2 and 4**, and create digraphs in **question 6** to satisfy given conditions.

- They use communication in answering questions such as **question 9** of **lesson 1.1.1** and **question 8** of **lesson 1.1.3**. You might notice that the last **Practising and Applying** question in each lesson often involves and element of communication.

- They use reasoning in answering questions such as **question 11** in **lesson 1.1.2**, where they think about how the procedure for adding matrices has implications on what size matrices can be added, or in **questions 6 and 7** of **lesson 1.1.4**, where they reason about the ordering and (non)commutative nature of matrix multiplication.

- They use representation and visualization and make connections as they represent a digraph as a matrix or vice versa in **lessons 1.2.2 and 1.2.3**. They also make connections as they relate the square of an adjacency matrix to solving network problems in **lesson 1.2.3**.

Rationale for Teaching Approach

• The unit is divided into two chapters. **Chapter 1** introduces the concept of matrices and the operations of simple addition, subtraction, and scalar multiplication of matrices. **Chapter 2** introduces networks and connects the work with matrices to networks and digraphs.

• Matrix multiplication is introduced after matrix addition and subtraction and after scalar multiplication because of its complexity.

• There is one **Explore** lesson, **lesson 1.2.1**, about the "traversability" of networks as an introduction to networks. This is followed by a chance to learn about a famous mathematical problem in the **Connections** feature — The Seven Bridges of Konigsberg — and to play a **Game**, both related to networks.

Getting Started

Curriculum Outcomes	Outcome relevance
7 Add, subtract, multiply, and divide: whole numbers and decimals **7** Add and subtract integers and decimals mentally **7** Multiply and divide integers and decimals mentally	Students will need to review some calculation skills from previous classes to help them work with matrices as they add, subtract, and multiply them.

Pacing	Materials	Prerequisites
1 h	None	• three-addend addition of single digit numbers • operations with integers and decimals

Main Points to be Raised

• The sums of all rows, all columns, and diagonals of a magic square add to the same value.

• Adding two magic squares or multiplying every value in a magic square by the same amount results in another magic square. Note that, for a square to be officially declared a magic square, all the elements must be different numbers. For the purposes of this activity, it is quite acceptable for students to refer to their resulting matrices as magic squares, even if there are repeated numbers, as long as there is one "magic" sum.

NOTE: The magic square activity should be re-examined in the context of matrices after matrix terminology has been introduced in **lessons 1.1.2 and 1.1.3**.

Use What You Know — Introducing the Unit

• Once you are certain students understand what a magic square is, allow them to work on their own, or with a partner or small group.

• Use this activity as a way to gently introduce the topic of matrices in an engaging way.

• Be aware that a new magic square can be created from an existing one by adding, subtracting, or multiplying all of the entries by the same value. Rotating a magic square a quarter turn or reflecting its rows or columns will also produce new magic squares**.**

Observe students as they work. You might ask:
• *How do you know that the missing number in the top row is one more than the missing number in the bottom left corner?* (4 + 8 is one less than 4 + 9 and the sum is the same when you add the missing numbers.)

• *How do you know that the sum of all the numbers in the square must be* 1 + 2 + 3 + … + 9 = 45*?* (There are 9 numbers in the 3-by-3 magic square and all the numbers from 1 to 9 are included.)

• *If you know that the sum of all the numbers is 45, how do you know that the sum of each row, column, and diagonal (the magic sum) must be* $\frac{1}{3}$ *of this, or 15?* (There are 3 rows and they all add to the same amount.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign all of these questions.

• Students can work individually.

Answers

A.

4	3	8
9	5	1
2	7	6

B. 15

C. *Sample response*:

2	9	4
7	5	3
6	1	8

D. i) *Sample response*:

6	12	12
16	10	4
8	8	14

ii) It is magic because the numbers in the rows, columns, and diagonals all add to the same value.

E. i) *Sample response*:

1	4.5	2
3.5	2.5	1.5
3	0.5	4

ii) It is magic because the numbers in the rows, columns, and diagonals all add to the same value.

iii) Yes; since each term is half as much, the sum of the terms in all the rows, columns, and diagonals would still be the same, but half the previous value.

1. a) 2 **b)** –6 **c)** –7 **d)** –24

2. a) –4 **b)** –2 **c)** 13 **d)** –12

3. a) –4 **b)** –12 **c)** 40 **d)** 72

4. a) 2.1 **b)** 0.5 **c)** 2 **d)** 40

5. a) 0.21 **b)** 36 **c)** 10.08 **d)** 18

Supporting Students

Struggling students

If some students are struggling with decimal or integer work, you may need to re-teach those skills to those individuals. For certain students you might choose to replace the decimals and/or integers in the matrices in this unit with whole numbers to allow for more immediate success. Students will still learn the underlying concepts and skills related to matrices.

Enrichment

Students might enjoy constructing magic squares using the following method:

1. Write the number 1 in the centre cell on the top row.
2. Move one cell up and one cell to the right. (To do this, you have to assume that the top row "wraps around" to the bottom row, and the right column "wraps around" to the left column.)
3. If this cell is empty, write in the next highest number in the sequence.
4. If this cell is not empty, move down one cell within the same column, "wrapping around" from the bottom row to the top row if necessary.
5. Repeat steps 2 to 4 until you have completed all the cells. The largest number in the sequence should be in the middle of the bottom row. If this is not the case, then you have made a mistake somewhere.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Chapter 1 Matrices

1.1.1 Introducing Matrices

Curriculum Outcomes	Outcome relevance
10A3 Matrices: represent problems • understand that matrices are used as a means of storing data • understand how the rows and columns of a matrix are identified	Students become acquainted with matrix terminology and are introduced to a number of mathematical situations and real life contexts where matrices are used.

Pacing	Materials	Prerequisites
1 h	Grid paper (BLM in Unit 6)	• prime factorisation of whole numbers • familiarity with a multiplication table • coordinate plotting

Main Points to be Raised

• A matrix is a set of items in a rectangular table or chart without row and column labels.

• Matrices are often named using capital letters.

• Matrices are usually shown with square brackets on the left and right.

• *Matrices* is the plural of *matrix*.

• A description of a matrix always includes the dimensions: number of rows by number of columns.

• The term *element*, which describes an item in a matrix, can sometimes be called an *entry*.

• There are some special matrices — *square matrices*, which have as many columns as rows; *row matrices*, consisting of one row with multiple columns; and column matrices, consisting of one column with multiple rows.

• There are many applications of matrices, including mathematical uses (e.g., displaying prime factors of a number or describing the vertices of a shape) and everyday uses (e.g., describing personal relationships or describing inventory or a set of scores).

Try This — Introducing the Lesson

A. Allow students to try these alone or with a partner. These questions provide an opportunity to use a matrix in an informal way, before formal matrix language is introduced in the lesson.

Observe while students work. You might ask:
• *Who are Thinley's friends?* (Dema and Karma)
• *What do Dema and Sonam have in common?* (neither is a friend of Yuden and both have one friend)
• *What does the number of 1s in a row tell you?* (how many friends that person has)

The Exposition — Presenting the Main Ideas

• You may have students read along with you as you introduce the terminology in the exposition. Or you may simply write a matrix on the board to introduce the language required and leave the exposition as another example to which students can refer later.

• For some students, it might be useful for you to write a matrix on the board and divide it up using horizontal and vertical chalk lines to see the cells.

• Students who have encountered spreadsheets on computers may be familiar with the term *cell*.

• You may wish to display a poster with all of the parts of a matrix labelled.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** and the new, more formal matrix language after the exposition has been presented. You might approach **part B** as a whole class.

Using the Examples

• Students can read through the examples, which introduce them to two different applications of matrices. You might ask students to restate the student thinking on the right side of each example in their own words for the whole class.
• For **example 2**, if necessary provide samples of prime factorisations for some of the multiples of 3 (e.g., 6 = 2 × 3; 24 = 2 × 2 × 2 × 3) and review plotting points on a coordinate grid.

Practising and Applying

Teaching points and tips

Q 2: Students might use the same matrix for **part a) and part c)**. You might ask students how they know that the matrix for **c)** will have multiple rows and columns.

Q 3: Some students might notice that the dates broke the pattern between 1981 and 1987.

Q 5: Students should be aware that the same information could be shown in a column matrix.

Q 7: Students might describe the shape in different ways. They might say it is a quadrilateral, a parallelogram, or, more specifically, a parallelogram with a base of 6 and height of 4.

Suggested assessment questions from Practising and Applying

Question 5	to see if students recognize that a matrix stores data and understand the role of the rows and columns of a matrix
Question 9	to see if students can connect the new topic of matrices to more familiar knowledge

Answers

A. i) Karma

ii) Karma has 3 friends and all of the others have either 1 or 2.

B. i) The part with the 1s and 0s without the names.

ii) It is a rectangular array of information, in this case, numbers.

1. A

2. a) *Sample response*: $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$

b) *Sample response*: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

c) *Sample response*: $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

3. a) 9-by-2
b) The only information needed is the plan number (column 1) and the year (column 2).

4. *Sample response*:

$$\begin{array}{c c} & \begin{array}{cccc} F1 & F2 & C1 & C2 \end{array} \\ \begin{array}{c} F1 \\ F2 \\ C1 \\ C2 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

5. a) 1-by-5
b) A 5-by-1 matrix could show the same information in 1 column with 5 rows instead of 1 row with 5 columns.

6. a) the number of items (ghos and kiras) each woman made
b) the total number of ghos or kiras that were made by all three women

7. It is a parallelogram.

8.

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 2 & 3 & 2 & 5 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

9. The numbers in a multiplication table are in a 10-by-10 square array.

Supporting Students

Struggling students

Some students might be more successful if the first matrices they see describe real life situations only, rather than the kind of mathematical situations described in **questions 7 and 8**.

1.1.2 Adding and Subtracting Matrices

Curriculum Outcomes	Outcome relevance
10-A3 Matrices: represent problems • understand that matrices are used as a means of storing data **10-B4 Matrices: adding, subtracting, and scalar multiplication — model, solve, and create problems** • understand that to add or subtract two matrices the dimensions of the two matrices must be the same	Because matrix addition and subtraction are defined conventions, students need to learn the rules for using these operations. They need to be aware that the dimensions of the matrices are critical.

Pacing	Materials	Prerequisites
1 h	None	• addition and subtraction of whole numbers and decimals

Main Points to be Raised

• Matrices are added or subtracted by adding or subtracting corresponding elements. For that reason, two matrices can only be added or subtracted if they have the same dimensions.

• Matrices can be added in any order (commutative property).

• The elements in the matrix resulting from the subtraction Matrix A – Matrix B have opposite signs to those resulting from Matrix B – Matrix A.

Try This — Introducing the Lesson

A. Some students might solve the problem initially without using matrix notation. In **part ii)** they are required to translate what they did into a matrix. Other students might work with a matrix from the start, thus working on **parts i) and ii)** at the same time.

Observe while students work. You might ask:

• *Will you subtract the 2001 figures from the 2003 figures or the 2003 figures from the 2001 figures? Why?* (e.g., I'll subtract 2001 from 2003 to look for growth.)

• *Which segment of the population seemed to decrease?* (ages 15 to 64)

• *Sometimes reported data cannot be trusted. Why might you mistrust this data?* (It does not make sense that the population decrease for age 15 to 64 was so large.)

The Exposition — Presenting the Main Ideas

• Have students read the exposition that shows how to add and subtract matrices.

• Check their understanding by recording two matrices with the same dimensions on the board. Ask students to explain how to add and how to subtract them.

• Students may choose to use coordinate addresses to describe the elements of a matrix, but it should not be required.

• You might ask students to relate matrix addition to the Magic Square activity, **part D**, in the **Getting Started**.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** within a real life context and the new, more formal matrix language of, and procedures for, the subtraction of matrices, after the exposition has been formally presented. You might handle **part B** as a whole class.

Using the Examples

Allow students time to read the example:
• Ask them why the student (in the **Thinking**) first checked the sizes of the matrices.
• Point out that the student calculated A – B for **part b)** and so subtracted elements in a particular order. Then point out the student could have subtracted B – A instead, and the result would have been different. Relate that to calculating the difference between 3 and 4. It could be 1 or –1, depending on how one chooses to look at the situation. Note that the interpretation of the resulting matrix would change.

Practising and Applying

Teaching points and tips

Q 3 and Q 5: Together, these questions should solidify for students the relationship between the elements of A – B and B – A.

Q 5: You might suggest that students compare their answers with classmates' to see if they got the same results.

Q 7: A matrix with all zeros is often called a *null* matrix.

Q 8: This question refers to what is known as the associative principle for addition.

Q 9: Asking students to work backwards by providing the resulting matrix not only provides an opportunity to assess their understanding, but it allows for many possible responses and more participation.

Q 10: Students might refer back to the previous lesson for ideas.

Common errors

Often students will lose track of which elements must be added to or subtracted from which. They might benefit from making small check marks on the elements they have already added or subtracted. Working on grid paper might help students keep track of the rows and columns.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can perform the required operations
Question 9	to see if students can add or subtract two matrices to achieve a certain result
Question 10	to see if students recognize the type of data that might be stored in matrices
Question 11	to see if students realize that only certain matrices can be added or subtracted

Answers

A. i) under 15: increase by 8600; 15 to 64: decrease by 44,671; 65 and over: increase by 681

ii) $\begin{bmatrix} 8600 \\ -44{,}671 \\ 681 \end{bmatrix}$ or $\begin{bmatrix} -8600 \\ 44{,}671 \\ -681 \end{bmatrix}$

B. I subtracted because I wanted to show the difference between the values to show the change.

1. A

2. a) $\begin{bmatrix} 5 & 1.2 & 5.9 \\ 0 & 2 & 1.1 \end{bmatrix}$ **b)** $\begin{bmatrix} -7.7 & -12 \\ 6.5 & 2 \end{bmatrix}$ **c)** $\begin{bmatrix} 13 & -13.3 \\ 3.9 & 1.4 \\ -8.8 & 8.5 \end{bmatrix}$

3. a) $\begin{bmatrix} 15.2 & 12.9 & 10.6 \\ 11.9 & 12.4 & 11.6 \\ 9.9 & 9.3 & 7.5 \end{bmatrix}$

The numbers tell how much higher the higher average temperature was in each place for each month.

b) $\begin{bmatrix} -15.2 & -12.9 & -10.6 \\ -11.9 & -12.4 & -11.6 \\ -9.9 & -9.3 & -7.5 \end{bmatrix}$

The numbers tell how much the temperature went down in each place for each month.

4. $\begin{bmatrix} 3 & 3 & 2 & 2 \\ 1 & 5 & 4 & 1 \end{bmatrix}$

5. a) *Sample response*:

A = $\begin{bmatrix} 3 & 1 & 0 & 2 & 4 \\ 3 & 1 & 2 & 5 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & 1 & 0 \end{bmatrix}$

b) *Sample response*: A – B = $\begin{bmatrix} 3 & 0 & -2 & 3 & 1 \\ 3 & -1 & -2 & 4 & 1 \end{bmatrix}$

c) *Sample response*:

B – A = $\begin{bmatrix} -3 & 0 & 2 & -3 & -1 \\ -3 & 1 & 2 & -4 & -1 \end{bmatrix}$

d) The signs for the corresponding elements are opposite; the opposite of *a* – *b* is *b* – *a*, e.g., 6 – 4 = 2 but 4 – 6 = –2.

6. $\begin{bmatrix} 3 & 10 & 6 & 7 \\ 1 & 20 & 3 & 7 \end{bmatrix}$

7. A = $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Answers [Continued]

8. a) *Sample response*:

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 10 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -6 & 0 \end{bmatrix}$

b) *Sample response*: $\begin{bmatrix} 11 & 3 \\ 2 & 3 \\ 5 & 2 \end{bmatrix}$

c) *Sample response*: $\begin{bmatrix} 12 & 3 \\ 4 & 8 \\ -1 & 2 \end{bmatrix}$

d) *Sample response*: $\begin{bmatrix} 11 & 3 \\ 2 & 7 \\ -4 & 1 \end{bmatrix}$

e) *Sample response*: $\begin{bmatrix} 12 & 3 \\ 4 & 8 \\ -1 & 2 \end{bmatrix}$

f) You get the same final matrix because the order does not matter when you add numbers or matrices.

9. a) *Sample response*:

$A = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -9 & -1 & -10 & 5 \\ 14 & -4 & 28 & -4 \end{bmatrix}$;

$A = \begin{bmatrix} -8 & 0 & -12 & 1 \\ 8 & -8 & 15 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 4 & 4 \\ 9 & 5 & 13 & -2 \end{bmatrix}$

9. b) *Sample response*:

$A = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 1 & 10 & -5 \\ -14 & 4 & -28 & 4 \end{bmatrix}$;

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 8 & -5 \\ -15 & 4 & -27 & 4 \end{bmatrix}$

10. *Sample response*:
If you have two matrices showing the number of tourists from five different countries in two different years, you could subtract them to find the change in the number of visitors from each country over the two years.

11. To add or subtract matrices, you add or subtract elements that are in the same position in both matrices, so you need the matrices to be the same size.

Supporting Students

Struggling students

If students are uncomfortable working with negative integers, stick to matrices with only positive integers when adding and subtracting matrices.

Enrichment

Have students make up their own matrices, A, B, and C, with the same dimensions. Students will then work out A + B and A + B – C for their own matrices. Students can then pair up. Each student gives the values of A and A + B to their partner who must then determine B. When the partner successfully finds B, then the result of A + B – C is provided and the partner must then determine C.

1.1.3 Multiplying a Matrix by a Scalar

Curriculum Outcomes	Outcome relevance
10-4 Matrices: adding, subtracting, and scalar multiplication — model, solve and create problems • understand that to multiply a matrix by a scalar multiply each entry in the matrix by the scalar • create and solve matrix problems	Students are introduced to a variety of numerical and geometric situations where multiplying a matrix by a scalar is meaningful.

Pacing	Materials	Prerequisites
1 h	Calculators	• multiplication of whole numbers and decimals

Main Points to be Raised

• When a matrix is multiplied by a single value, usually called a *scalar*, each element is multiplied by that value.

• The size, or dimensions, of a matrix does not change when it is multiplied by a scalar.

• The dimensions of the matrix do not matter when multiplying by a scalar.

Try This — Introducing the Lesson

A. Students can work alone or with a partner. To prepare students for working on **part A**, you might ask:

• *If 1 baht is worth Nu 1.16, how many Nu can I buy with 10 baht?* (11.6) *What about 100 baht?* (116)

Students might use these values to make sense of the answers they calculate in **A i)**.

NOTE: You might point out that the rates of exchange between Bhutanese ngultrums and most other currencies, including Thai baht, may change daily (the rates float). By contrast, the rate of exchange between Bhutanese ngultrums and Indian rupees is 1:1 (it is at par) and does not change (the rate is pegged).

Observe while students work. You might ask:

• *Why will the values in the new ngultrum matrix be greater than in the original baht matrix?* (You multiply each baht value by 1.16 to get the equivalent ngultrum value because Nu 1.16 is worth 1 baht.)

• *What will be the dimensions of the new matrix? Why?* (4-by-1 as there are still 4 things to buy and 1 price for each.)

• *Which element will be greatest? Which will be least? Why?* (The haircut will be greatest and the water least; changing the currency does not change the relative values when you multiply by a positive value.)

The Exposition — Presenting the Main Ideas

• Before students read the exposition, record a matrix such as Matrix A below on the board. In order to connect the new learning about multiplying by a scalar to the already familiar matrix addition, ask students to add A to itself. Then have them multiply Matrix A by the scalar 2. Talk about why the sum matrix might be written as A + A or 2 × A (it is just like 3 + 3 = 2 × 3). Have students observe that each element of the original Matrix A is doubled.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 0 & -2 & 4 \\ -6 & 0 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} \qquad 2 \times \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 0 & -2 & 4 \\ -6 & 0 & 4 \end{bmatrix}$$

• Make sure students understand that all scalar multiplication works the same way, that is, elements in the resulting matrix are calculated by multiplying each element in the original matrix by the same scalar value.

• Ask students to relate matrix multiplication to the Magic Square activity, **part E**, in the **Getting Started**.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** within a real life context and the new, more formal matrix language of, and procedures for, multiplying a matrix by a scalar, after the exposition has been formally presented. You might handle **part B** as a whole class.

Using the Examples

• Some students might benefit from solving the problem in **Example 2** on their own. They could sketch or draw on grid paper.
• For **Example 2**, make sure that students notice that although the linear dimensions were doubled, the area was quadrupled.

Practising and Applying

Teaching points and tips

Q 1 c): Students who are uncomfortable multiplying fractions might replace $\frac{2}{3}$ with a value like 0.6. The results will be comparable, but not equivalent.

Q 5: Students should have access to calculators for this question. Some students might multiply 0.021 by the population and then add it to the population, while others will directly multiply the population by 1.021.

Q 6: Students who are not comfortable with fraction multiplication and division may struggle with this. You might need to point out, for example, how $7 \times \frac{1}{2} = \frac{7}{2}$ and $14 \times \frac{1}{4} = \frac{14}{4} = \frac{7}{2}$ to get them started. Most students will not realize that they can factor $\frac{1}{4}$ or $\frac{1}{2}$ unless they think of $7 \times \frac{1}{2}$ as $\frac{7}{2}$ and $\frac{7}{2}$ as $\frac{14}{4}$ in order to factor out $\frac{1}{4}$.

Q 7: Students might benefit from completing **question 6** first.

Common errors

• The computation required to multiply a matrix by a scalar is fairly simple, so students are not likely to miscalculate if they multiply directly. Errors are more likely in situations where they combine operations, for example, 2A + B in **question 2 a)**. They might first add A + B and then multiply by 2.
• Students might lack skills in percent and decimal work for **question 5**.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can perform the required operation
Question 6	to see if students can solve a problem by thinking backwards
Question 7	to see if students recognize potential applications for the outcome

Answers

A. i) and ii) $\begin{bmatrix} 58 & 10.44 & 31.32 & 69.60 \end{bmatrix}$

B. I multiplied each value in the matrix by the same value, 1.16.

1. a) $\begin{bmatrix} 6 & 1.2 \\ 2.4 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 16 & 10\frac{2}{3} \\ 6\frac{2}{3} & 10 \\ 6 & 20\frac{2}{3} \\ 12 & 14 \end{bmatrix}$

c) $\begin{bmatrix} 4.8 & 0.72 & 2.4 \\ 2.1 & 0.36 & 3 \end{bmatrix}$

d) $\begin{bmatrix} 6 & 15 \\ -1.2 & -2.7 \end{bmatrix}$

2. a) $\begin{bmatrix} 4 & 1 & -4 \\ 20 & 1 & -4 \\ -5 & 1 & 21 \end{bmatrix}$

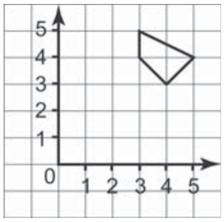
b) $\begin{bmatrix} 2 & -2 & -7 \\ 5 & 3 & -2 \\ -5 & 8 & -2 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 2 & -2 \\ 22 & 0 & -4 \\ -4 & -2 & 26 \end{bmatrix}$

d) $\begin{bmatrix} -4 & 1 & 8 \\ -16 & -3 & 4 \\ 7 & -7 & -11 \end{bmatrix}$

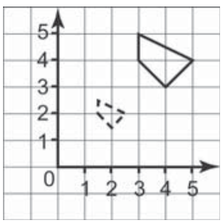
3. a) There are 4 columns in the matrix of coordinates so I know there are 4 vertices, and any shape with 4 vertices is a quadrilateral.

3. b)



c) $\begin{bmatrix} 2 & 1.5 & 1.5 & 2.5 \\ 1.5 & 2 & 2.5 & 2 \end{bmatrix}$

d) The new shape is smaller and similar.



4. $\begin{bmatrix} 1.81 & 6.32 & 0.56 & 0.36 \end{bmatrix}$

5. $\begin{bmatrix} 16{,}454 & 100{,}748 & 75{,}949 & 37{,}847 \end{bmatrix}$

$\begin{bmatrix} 16{,}800 & 10{,}2864 & 77{,}544 & 38{,}642 \end{bmatrix}$

6. a) 5 **b)** $\begin{bmatrix} -4 & 6 & 8 \\ 20 & -18 & 16 \\ 38 & 16 & -14 \end{bmatrix}$

c) *Sample response*: $\frac{1}{4} \times \begin{bmatrix} 3 & 14 \\ -5 & 8 \end{bmatrix}$

7. *Sample response*:
To find the total cost of phone use each day for each person, you would multiply each number of minutes by the rate per minute.

8. You multiply every element in the matrix by the scalar, so it does not matter how many elements there are or how they are arranged.

Supporting Students

Struggling students

Some students might be more comfortable using matrices with fewer elements and/or with matrices with whole numbers instead of fractions, decimals, and integers. If this is the case, you may want to replace some of the matrices in the questions.

Enrichment

Students who are very comfortable with the content might be provided with more situations where they factor matrices, as is done in **question 6**. They should observe that there are an infinite number of possible answers, even though some are more likely than others. For example:

$\begin{bmatrix} 4 & 2 & 6 \\ 0 & -2 & 4 \\ -6 & 0 & 4 \end{bmatrix}$ equals $2 \times \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix}$, but also equals $\frac{1}{3} \times \begin{bmatrix} 12 & 6 & 18 \\ 0 & -6 & 12 \\ -18 & 0 & 12 \end{bmatrix}$ or $0.8 \times \begin{bmatrix} 5 & 2.5 & 7.5 \\ 0 & -2.5 & 5 \\ -7.5 & 0 & 5 \end{bmatrix}$

1.1.4 Multiplying Matrices

Curriculum Outcomes	Outcome relevance
10-B5 Matrix Multiplication: develop and apply procedures • develop and apply the algorithm for multiplication • understand that matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix	Matrix multiplication will be used in this unit as a vehicle for solving network problems (in **lesson 1.2.3**). Matrix multiplication is an extremely valuable tool to solve many types of mathematical problems, particularly systems of linear equations.

Pacing	Materials	Prerequisites
2 h	Grid paper (BLM in Unit 6)	• multiplication of integers and decimals

Main Points to be Raised

• Matrices can only be multiplied if the number of columns in the first matrix matches the number of rows in the second matrix.

• The dimensions of the resulting product matrix are based on the dimensions of the matrices that are multiplied. The product matrix has the number of rows of the first matrix and the number of columns of the second matrix.

• Matrix multiplication is not commutative; the order matters.

• To calculate the value of the element at (j, k) in the product matrix, the elements in row j of the first matrix are matched with the elements in column k of the second matrix and multiplied. All the products are then added together.

• Matrix multiplication usually involves the collection into one value of many pieces of separate, but related, data.

• Matrix multiplication can be applied in a variety of situations, including numerical and geometric ones.

Try This — Introducing the Lesson

A. Students can solve the problem alone or in pairs. Observe while students work. You might ask:

• *Which multiplications did you have to complete to solve the problem?* (the number of tourists by the tourist rate and the number of government employees by the government rate, for each hotel)

• *Why do you have to add some of the values you calculate? Which ones?* (because the income for the hotel comes from two different sources — tourists and government employees — that have to be totalled)

The Exposition — Presenting the Main Ideas

• Rather than reading through the exposition with the students, record Matrices A and B on the board and demonstrate to students how to multiply the two matrices.

• Point out how each entry, or element, is calculated by using values from one row of the first matrix and one column of the second matrix. For example, for position (2, 3) in the product matrix, values from row 2 in the first matrix are multiplied by the corresponding values in column 3 of the second matrix and then added. Help students see that this is why the number of columns in the first matrix must match the number of rows in the second matrix for multiplication of matrices to be possible.

• Students might read through the example on their own to get further clarification.

• You may wish to let students practise one multiplication on their own before assigning **part B** of the **Try This**.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** within a real life context to the new, more formal matrix language of, and procedures for, multiplying matrices, after the exposition has been formally presented. You might handle **part B** as a whole class.

Using the Examples

• Allow the students to read through both examples. Answer any questions they might have.

• Point out to students how **Example 1** shows a typical application of matrix multiplication: a variety of values need to be multiplied, one by one, by appropriate weightings (%) and then totals calculated. Point out how each row of the product matrix describes the grade of one student, which was calculated by starting with each student's raw scores and using the same weightings (%) to multiply the raw scores for all students.

• Tell students that matrix multiplication is also used in other contexts. **Example 2** shows how matrix multiplication is used to describe a geometric transformation. Ask why it is the *y*-coordinate and not the *x*-coordinate that changes sign (each *y*-coordinate, in the second row of Matrix C, is multiplied by –1). Some students might speculate about what matrix might reflect the original shape in the *y*- axis. Make sure students know to always position the transformation matrix on the left to multiply. (The matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ would reflect a shape in the *y*-axis.) Students should be aware that the order of vertices in the matrix is arbitrary.

Practising and Applying

Teaching points and tips

Q 2: Students might benefit from writing the multiplications in a format like this:

[ ]-by-[ ] × [ ]-by-[ ] results in [ ]-by-[ ], such as 4-by-5 × 5-by-2 results in 4-by-2.

Q 3: Students might relate this back to **Example 2** to figure out why both coordinates changed.

Q 4 and Q 7: Both of these questions require students to create a matrix model. Some students might need support — you may need to show them what matrices to use and ask them why your suggestions make sense.

Q 5: Square matrices with 1s on the main diagonal and 0s elsewhere are called identity matrices. If you multiply a matrix by the appropriate identity matrix, the product is the original matrix.

Q 6: You might have students compare their results to see if they reach the same conclusion for **6 b)**.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can perform matrix multiplication
Question 7	to see if students can model a problem related to matrix multiplication
Question 8	to see if students understand the distinction between situations requiring scalar multiplication and those requiring matrix multiplication
Question 9	to see if students can explain the procedure for matrix multiplication

Answers

A. $\begin{bmatrix} 38{,}900 \\ 37{,}650 \end{bmatrix}$

B. $\begin{bmatrix} 1100 & 950 \\ 1050 & 950 \end{bmatrix} \times \begin{bmatrix} 25 \\ 12 \end{bmatrix} = \begin{bmatrix} 38{,}900 \\ 37{,}650 \end{bmatrix}$ You would get the total amount each hotel would earn, since you are multiplying the number of tourists by the tourist rate and the number of government employees by the government rate and adding these for each hotel.

1. a) A × B, B × C, B × D, C × D, D × A, D × B

b) A × B = $\begin{bmatrix} 4 & 2 & 6 \\ 6 & 7 & 10 \end{bmatrix}$; B × C = $\begin{bmatrix} 10 & 1 & 6 \\ 2 & 4 & 5 \end{bmatrix}$;

B × D = $\begin{bmatrix} 5 & 6 \\ 12 & 2 \end{bmatrix}$; C × D = $\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 2 & 2 \end{bmatrix}$; D × A = $\begin{bmatrix} 2 & 0 \\ 6 & 0 \\ 6 & 2 \end{bmatrix}$

D × B = $\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 0 & 8 & 2 \end{bmatrix}$

2. a) *Sample response*:

4-by-1 × 1-by-2, or

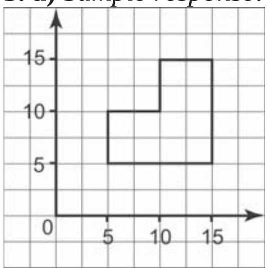
4-by-3 × 3-by-2, or

4-by-anything × anything-by-2 as long as both "anythings" are the same.

b) The number of columns in the first matrix has to match the number of rows in the second, and there must be 4 columns in the first matrix and 2 rows in the second matrix for the product matrix to be 4-by-2.

Answers [Continued]

3. a) *Sample response*:

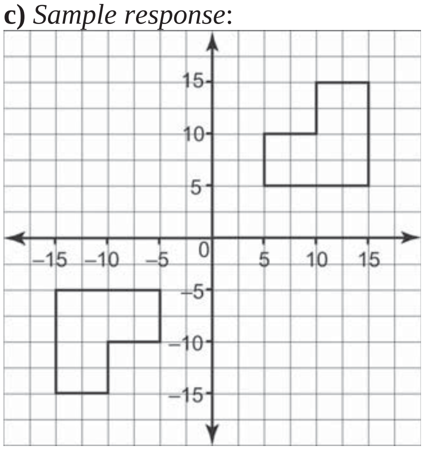


$$B = \begin{bmatrix} 5 & 15 & 15 & 10 & 10 & 5 \\ 5 & 5 & 15 & 15 & 10 & 10 \end{bmatrix}$$

b) *Sample response*:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 5 & 15 & 15 & 10 & 10 & 5 \\ 5 & 5 & 15 & 15 & 10 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} -5 & -15 & -15 & -10 & -10 & -5 \\ -5 & -5 & -15 & -15 & -10 & -10 \end{bmatrix}$$

c) *Sample response*:



d) It turned, or rotated, a half turn about the origin.

4. $\begin{bmatrix} 50 & 100 & 500 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 6 \end{bmatrix} = \begin{bmatrix} 5500 \end{bmatrix}$; Nu 5500 in notes

5. a) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

b) You get the same matrix you started with.

6. a) You cannot multiply a 3-by-2 matrix by a 1-by-3 matrix since the number of columns in the first matrix does not match the number of rows in the second one, but you can multiply a 1-by-3 by a 3-by-2 matrix.

b) *Sample response*:

No; $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 16 \\ 2 & 10 \end{bmatrix}$

but $\begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 6 \end{bmatrix}$

Normally the results are different, but there are exceptions, i.e., when it involves the identity matrix:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$

and

$\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$.

7. a) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

b) A × B; A would be first so the number of columns in the first matrix, 3, matches the number of rows in the second matrix, 3.

8. a) *Sample response*:
If you had a number of prices in ngultrums to exchange for U.S. dollars, you would multiply a matrix with the prices by a scalar, the exchange rate.

b) *Sample response*:
If you were trying to figure out several students' final grades, you could multiply a matrix with their marks for each part of their final grade by a matrix with the percent weighting for each part.

9. a) 3-by-4

b) Multiply the numbers in the 2nd row of the first matrix by the matching numbers in the 3rd column of the second matrix and add the products.

Supporting Students

Struggling students

- You might create a colour-coded visual showing students which numbers are multiplied together when matrices are multiplied. You would colour the element in a particular row of the first matrix the same as a matching element in a particular column in the second matrix, and then show where it will end up in the product matrix.
- You might want to spend more time with 2-by-2 matrices before working with larger matrices.

Enrichment

• Advanced students might explore "identity" matrices (matrices you can multiply by without changing the value of the matrix being multiplied) for 3-by-3 or 4-by-4 matrices to parallel what they discover in **question 4**.

These square matrices always have 1s on the main diagonal and 0s elsewhere (e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$).

• Students might also explore "division," calculating what matrix to multiply by to result in a particular product matrix. For example, the question below is asking what to divide the product matrix by to result in the second matrix. $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 2 & 8 \end{bmatrix}$ can be thought of as $\begin{bmatrix} 7 & 14 \\ 2 & 8 \end{bmatrix} \div \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

($\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$ is $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$)

Chapter 2 Networks

1.2.1 EXPLORE: Travelling Networks

Curriculum Outcomes	Lesson relevance
10-D15 Networks: traversability • recognize that a network with more than two odd vertices is not traversable	One of the interesting and useful applications of matrices is to investigate networks for planning flight paths or driving routes. This core open lesson sets the stage for this connection to be made in the next lesson. Note that the material is not covered directly in any other lesson.

Pacing	Materials	Prerequisites
1 h	None	None

Main Points to be Raised

• If the vertex of a network has an even number of edges, it means that you can get both into and out of the network from that vertex. But if the vertex has an odd number of edges, it means that, at least once, you can get into and not out of the network (or out and not into the network) at that vertex.

• A network is traversable without repeating an edge only if there are 0 or 2 odd vertices.

• If there are two odd vertices, to travel the network completely without repeating an edge, one odd vertex must be the starting point and the other the end point.

Exploration

• Ask students to work on **parts A to C** with a partner or in a small group.

• Make sure they know that they need to check each network, making sure they can travel every part of the network without lifting their pencils and without repeating any part of it.

Observe while students work. You might ask:
• *That route did not work. Are you sure there is not another one that will?*
• *Why did you start there? Where else could you have started?*
• *How are you keeping track of each part of the network you have travelled?* (see answer to **part A** below.)

Observe and Assess

As students are working, notice:
• Do they persevere? If one approach fails, do they realize they could try to travel the network a different way?
• Do they learn from experience? As they are successful with one network, do they use a similar strategy on another one?
• Do they generalize appropriately? Do they look for commonalities in their various attempts to help them answer **part B** appropriately? Do they use what they have learned in creating a network for **part C**?

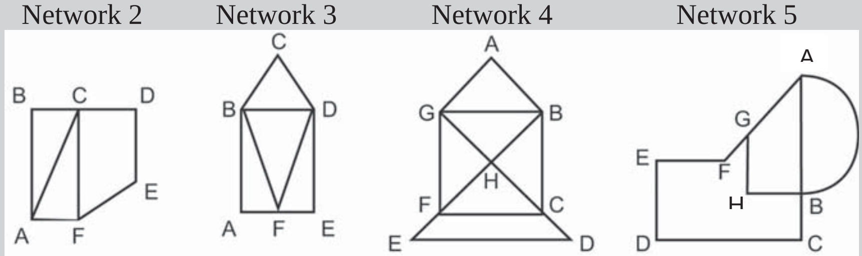
Share and Reflect

After students have had enough time to answer **parts A, B and C**, encourage different groups to come forward and describe how they approached the problem and what they learned. They should notice these main points:
• If all vertices are even, the network can be traversed starting anywhere and ending where you started.
• If exactly two vertices are odd, the network can be traversed starting at one odd vertex and ending at the other.

Answers

A. *Note that network vertices have been labelled so routes can be described. Students may label their networks differently.*

Network 2 Network 3 Network 4 Network 5



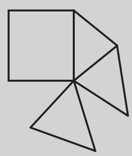
Route: ABCDEFCAF ABCDEFDBFA ABHFEDCHGBCFGA ABAGFEDCBHG

B. Traversable Networks — Number of Edges at Each Vertex

Vertex	Network 2	Network 3	Network 4	Network 5
A	3	2	2	3
B	2	4	4	4
C	4	2	4	2
D	2	4	2	2
E	2	2	2	2
F	3	4	4	2
G			4	3
H			4	2

Networks 3 and 4 have all even vertices and networks 2 and 5 have exactly two odd vertices.

C. *Sample response*:



Supporting Students

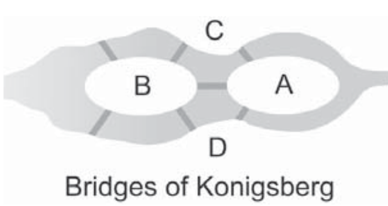
Enrichment

Interested students might explore other networks that can and cannot be travelled. There is a very famous problem called *The Travelling Salesman Problem* where the objective is to visit each city in a network (represented by vertices) exactly once, returning to the home city in the most cost-effective way.

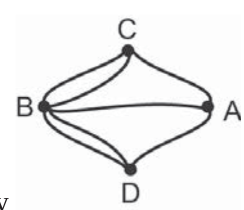
CONNECTIONS: The Seven Bridges of Konigsberg

1. It is not possible to travel all seven bridges exactly once, starting and ending at the same place, because all four vertices (that represent land areas) have an odd number of edges.

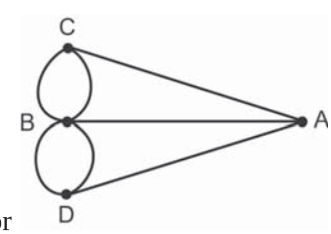
The network can be drawn in several ways. For example:



can be represented by



or



2. No; because there are 4 odd vertices and you need to have two odd vertices so you enter on one of them and exit on the other

Students may be interested in exploring the problem if another bridge is added or a bridge is removed.

GAME: Sprouts

John Conway, an American mathematician, invented this game in 1967. Although the game is presented here as an interesting way to practise network ideas, mathematicians continue to study the game and have developed hypotheses and proved theorems about it. For example, it as been shown that if a game starts with n vertices,
- the maximum number of regions is $2n + 1$
- the minimum number of regions is $n + 2$ for 3 or fewer vertices and $n + 3$ for 4 or more vertices

A region is a defined area inside the network.

It would be useful if students use different colours to make their edges to make it easier to analyze the game. It is probably a good idea to start with 3 or more dots. Students should be encouraged to keep their edges simple.

1.2.2 Describing a Network With a Matrix

Curriculum Outcomes	Outcome relevance
10-A4 Network Problems: represent networks using matrices and vice versa • represent a network as a matrix and interpret a matrix in terms of a corresponding network situation	In preparation for using matrices to solve network problems, students need to learn how to represent a network with a matrix. Because some paths are uni-directional, the concept of a digraph (directed graph) is introduced.

Pacing	Materials	Prerequisites
1 to 2 h	None	• ability to interpret a matrix

Main Points to be Raised

• A digraph is a directed network; the arrows on the edges indicate the direction in which they must be followed. If there is no direction indicated on a particular edge, it is assumed to be bi-directional.

• Digraphs can be simplified by replacing two one-way edges between two vertices that go in opposite directions with one bi-directional edge that has no arrow. As well, if there are multiple edges going in the same direction from one vertex to another, they can be replaced with one edge that has the number of edges recorded on it.

• Matrices used to describe digraphs are called adjacency matrices. The matrix is a way to record information about the number of edges from each vertex to each other vertex.

• One can create an adjacency matrix to describe a digraph or a digraph to describe an adjacency matrix.

• Adjacency matrices are always square since they show all possible connections between (from and to) each vertex and every other vertex, including itself.

• It is always possible to draw two digraphs that are not congruent but that share the same adjacency matrix. Mathematicians would say that the digraphs are *topologically equivalent*, that is, they are the same if the only issue that is being looked at is connections.

• Note that arrows on lops are optional because an edge from a vertex to itself is the same in either direction.

Try This — Introducing the Lesson

A. Students can work on **part A** alone or with a partner. Observe while students work. You might ask questions such as the following to have them explain their rationale for how they travel each digraph:

• *Why do you have to go in that direction?*

• *After you go from A to B, why do you have to travel around the loop before you go on to C?*

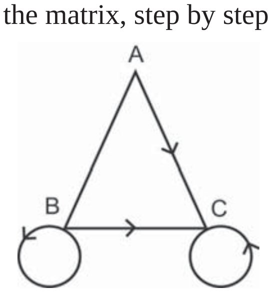
• *I notice you went from A to D. Could you travel the network if you went from A to B first?*

The Exposition — Presenting the Main Ideas

• Introduce the term **adjacency matrix** as a matrix describing a digraph.

• Have the students read the part of the exposition that describes how the first row of the matrix was created. To check their understanding, ask different students to explain how the elements in the 2nd, 3rd, and 4th rows were determined.

• If another example is necessary, record the digraph below on the board and have students tell you how to create the matrix, step by step:



$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \end{array}$$

Create a 3-by-3 matrix

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \end{array}$$

Complete Row 1
Edges from A

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ ? & ? & ? \end{bmatrix} \end{array}$$

Complete Row 2
Edges from B

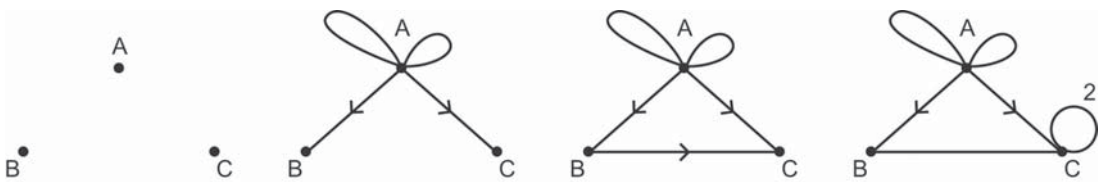
$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Complete Row 3
Edges from C

• Indicate that you can also create digraphs from an adjacency matrix. Have students read the part of the exposition that describes how the first row of Matrix R was used to create the digraph. To check their understanding, ask different students to explain how the elements in the 2nd row and 3rd row were used to create the digraph. Help students see that there is often more than one digraph that could be created from any particular adjacency matrix. The positions of points can vary, as can the number of paths if some are bi-directional and some are not.

• If another example is necessary, record the matrix below on the board and have students tell you how to create the digraph step by step, as shown below:

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$



Draw 3 vertices Draw edges for Row 1 Draw edges for Row B Draw edges for Row C

Draw 3 vertices to represent A, B and C.

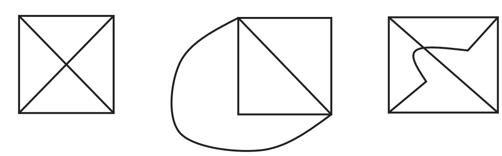
Draw edges for Row 1: draw **2** edges (loops) from A to A; **1** edge from A to B, and **1** edge from A to C

Draw edges for Row 2: draw **1** edge from B to C

Draw edges for Row 3: instead of drawing **1** edge from C to B, remove the arrow on the edge from B to C (two edges going in opposite directions can be replaced by one bi-directional edge with no arrow); instead of drawing **2** edges (loops) from C to C, draw 1 edge and record a 2 on it.

• You may want to draw student's attention to the value in avoiding creating unintentional vertices. For example, the network shown at the left below might lead you to think there is a fifth vertex when this is not the intent. This can be avoided by drawing the network in either of the two ways shown at the right.

Avoid this Use one of these instead



Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** and the new, more formal matrix language of, and procedures for the creation, of adjacency matrices, after the exposition has been formally presented. Students can work with a partner.

Using the Examples

• Suggest that students examine the digraph in **example 1** and create their own adjacency matrix. They can check their thinking against the solution.
• Similarly, they might try to draw a digraph for the matrix in **example 2** before examining the solution. Reinforce that neither the digraph with bi-directional paths nor the one with uni-directional paths is preferred; either is acceptable.

Practising and Applying

Teaching points and tips

Q 4: Suggest that students choose vertex letters that reflect their meaning such as I for insects and C for caterpillars. An arrow from I to P could mean I (insects) eat P (plants) or I are eaten by P, as long as it is done one way consistently throughout the digraph.

Q 5 and Q 7: These questions provide examples of real-world applications of matrix multiplication.

Q 6: Students could draw several digraphs for the same matrix to support their explanation. An exception you may introduce is the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Q 8: Students have the opportunity to synthesize what they have observed in many situations in this unit to come up with generalizations about all the information an adjacency matrix holds.

Q 9: Students will interpret "complicated" in personal ways. There is no one interpretation. What matters is whether the student's reasoning makes sense.

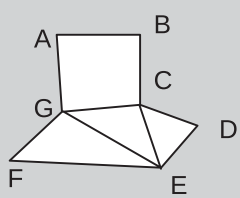
Suggested assessment questions from Practising and Applying

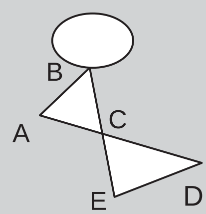
Question 1	to see if students can create a matrix from a digraph
Question 2	to see if students can create a digraph from a matrix
Question 6	to see if students recognize which aspects of a digraph matter and which do not in describing it and how this suggests that there might be different digraphs described by the same matrix
Question 8	to see if students can interpret an adjacency matrix

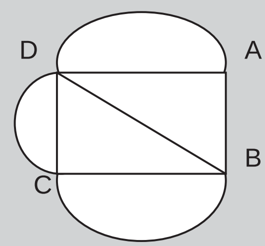
Answers

A. ii) Digraphs 2 and 3

B. *Sample responses*: 1. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$ 2. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$





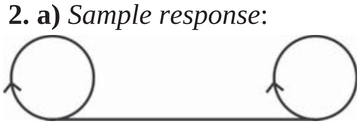


Note that these diagrams have been included to show the labelling of the networks used in the matrices above.

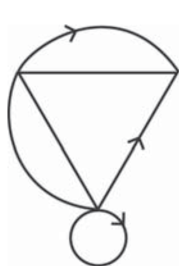
1. a) $\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$ **b)** $\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$

c) $\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

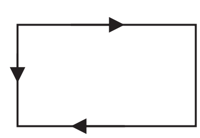
2. a) *Sample response*:



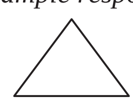
b) *Sample response*:



c) *Sample response*:

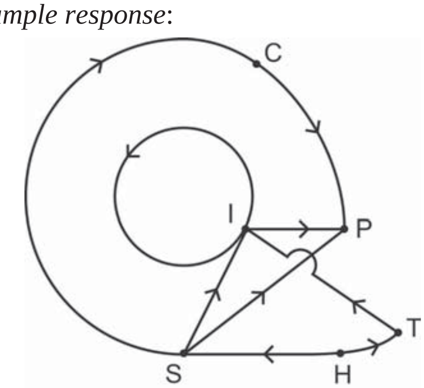


2. d) *Sample response*:



3. There are no loops connecting any vertex to itself.

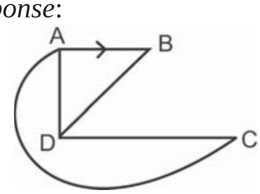
4. a) *Sample response*:



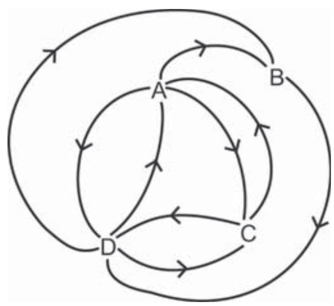
b)

$$\begin{array}{c c} & \begin{matrix} C & I & P & S & H & T \end{matrix} \\ \begin{matrix} C \\ I \\ P \\ S \\ H \\ T \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

5. *Sample response*:

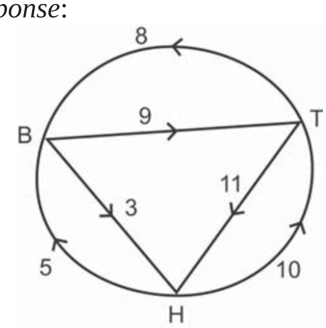


and



6. *Sample responses*:
- You can put your vertices in different locations.
- Wherever you have a curved line, you can make a straight line or vice versa.
- You can use two uni-directional arrows instead of one bi-directional path.

7. *Sample response*:



8. a) by the number of rows or columns

b) by numbers other than 0 where the row and column names are the same (the principal diagonal)

c) wherever there is a 0 in both entries, from A to B and from B to A, where the row for one vertex intersects the column for the other

d) the total of the numbers in the row for that vertex

e) the total of the numbers in the column for that vertex

9. *Sample response*:
If there are many 0s, the graph is not as complicated as when there are many 1s and 2s. But if it was all 1s, it wouldn't be too complicated.

Supporting Students

Struggling students

Some students may have much more difficulty going from the matrix to the digraph than from the digraph to the matrix. For those students, provide additional experience with simple matrices where they are successful before moving to the more difficult problems.

1.2.3 Solving Network Problems

Curriculum Outcomes	Outcome relevance
10-B6 Matrices: solve network problems • represent and solve network problems using matrices	Using matrix multiplication to solve problems involving network travel helps students see an important application of matrix work.

Pacing	Materials	Prerequisites
2 h	None	• ability to interpret digraphs and matrices • matrix multiplication

Main Points to be Raised

• One of the types of problems students might solve with digraphs is determining how many routes there are that are not direct (for example, involving one stopover or two stopovers) for airline or bus routes.
• These problems can be solved either directly by exploring the graph or indirectly using matrices.
• To indirectly determine the number of one-stopover routes between vertices of a digraph, the adjacency matrix can be squared.
• To indirectly determine the number of two-stopover routes between vertices of a digraph, the adjacency matrix can be cubed.

Try This — Introducing the Lesson

A. Students can work on **part A** with a partner. Observe while students work. You might ask:
• *How do you know there is more than one way to get from Paro to Jakar with one stopover?* (I know I could take three different helicopters from Paro to Thimphu and then another to Jakar. That's already three ways.)

• *How do you know there is not a one-stopover route between Paro and Trashigang?* (The only way from Paro is through Thimphu and the only way to Trashigang is through Jakar, so there has to be more than one stopover.)

The Exposition — Presenting the Main Ideas

• Work with the students as they follow along with the explanation in the student book. Make sure they understand that the problem is solved first directly by analyzing the digraph and then indirectly by squaring the adjacency matrix and interpreting it.

• The explanation after the **Try This** focuses on a single entry, 1st row, 5th column, and how it is created. It may be necessary to give a few more examples before students fully understand how the elements in the squared matrix are calculated. For example, the element of the squared matrix in the 4th row, 3rd column, 1, describes the number of one-stopover trips from D to C. This element is the result of the following:

D to A and then A to C is not possible since you cannot get from D to A or from A to C (0 × 0)
D to B and then B to C is possible 1 way since there is a path from D to B and a path from B to C (1 × 1)
D to C and then C to C is not possible since there is no way to get from D to C or from C to C (0 × 0)
D to D and then D to C is not possible since there is no loop from D to D or path from D to C (0 × 0)
D to E and then E to C is not possible since there is a path from D to E but no way to get from E to C (1 × 0)

• Help students see that cubing the adjacency matrix describes two-stopover trips. The squared matrix describes one-stopover trips. If, for example, you multiply the elements of row 2 of the squared matrix and column 4 of the adjacency matrix and add them, you are calculating the number of one-stopover trips from B to each other vertex followed by an edge from each of those vertices to D. This results in two-stopover trips.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between what was done in **part A** to the new, more formal matrix language of, and procedures for, working with adjacency matrices to determine one-stopover routes, after the exposition has been presented. Students can work with a partner.

Using the Examples

Ask students to read through **Example 1**. In pairs, have one student explain to a partner what the student in the example was thinking in Solution 1 and then switch roles for Solution 2. A volunteer can share his or her re-statement of the student thinking with the class. Lead students through **Example 2**.

Practising and Applying

Teaching points and tips

Each problem is quite complex, so you may wish to assign fewer problems for this lesson.

Q 1: Note that students are not required to use a matrix to answer **part a)** of this question.

Q 2: Some students will struggle with the idea of calling a path ABB a one-stopover path. You might liken this to the idea of taking a trip from one city (A) to another (B) and then doing another leg of the trip within that city. That is essentially what ABB means.

Q 5: Students can refer to **Example 2** for help with this question. Encourage them to consider different possibilities, that is, networks with only one one-directional path as well as ones with two one-directional paths.

Q 6: Students need to do some analysis to answer **part c**. Most will go directly from the digraph, but they should verify their prediction with matrix multiplication. Some will realize that any polygon with all its diagonals is a possibility.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can solve a route problem using a digraph
Question 3	To see if students can use a matrix to solve network problems
Question 5	to see if students can create a hypothesis about the squares of adjacency matrices and test it
Question 8	to see how well students can communicate their understanding

Answers

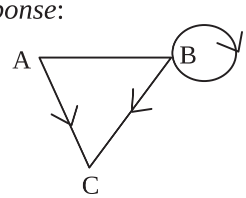
A. 3; you can get from Paro to Thimphu 3 ways and then, for each of those ways, there is 1 way to get to Jakar and 3 × 1 = 3.

B. $\begin{array}{c} \\ P \\ T \\ J \\ Tr \end{array}\begin{array}{c} \begin{array}{cccc} P & T & J & Tr \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array} \times \begin{array}{c} \begin{array}{cccc} P & T & J & Tr \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} P & T & J & Tr \end{array} \\ \begin{bmatrix} 9 & 0 & 3 & 0 \\ 0 & 10 & 0 & 2 \\ 3 & 0 & 5 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \end{array}$; from Paro to Jakar is in row 1, column 3 (1, 3)

1. a)
- 1 from A to A (A-C-A); 1 from A to B (A-C-B); 1 from A to C (A-B-C)
- 1 from B to A (B-C-A); 1 from B to B (B-C-B); 0 from B to C
- 0 from C to A; 1 from C to B (C-A-B); 2 from C to C (C-A-C or C-B-C)

b) You would create an adjacency matrix for the digraph $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and square it to get $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

2. a) *Sample response*:



b)
- 1 from A to A (ABA); 1 from A to B (ABB); 1 from A to C (ABC)
- 1 from B to A (BBA); 2 from B to B (BBB and BAB); 2 from B to C (BBC and BAC)
- 0 from C to anywhere

c)
- 1 from A to A (ABBA); 2 from A to B (ABBB and ABAB); 2 from A to C (ABBC and ABAC)
- 2 from B to A (BBBA and BABA); 3 from B to B (BBBB, BABB, BBAB); 3 from B to C (BBBC, BBAC, BABC)
- 0 from C to anywhere

3.

one-stopover

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- A to A: 1; A to B: 1; A to C: 1
- B to A: 1; B to B: 2; B to C: 2
- C to anywhere: 0

two-stopover

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- A to A: 1; A to B: 2; A to C: 2
- B to A: 2; B to B: 3; B to C: 3
- C to anywhere: 0

Answers [Continued]

4. a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

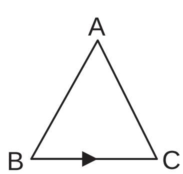
b) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$

There are always 3 one-stopover trips from a vertex to itself and 2 one-stopover trips from a vertex to another vertex.

c) $\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$

There are always 6 two-stopover trips from a vertex to itself and 7 two-stopover trips from a vertex to another vertex.

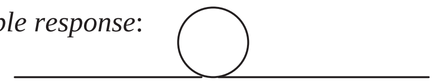
5. There can only be 0, 1, or 2 one-stopover edges. It's usually 0 or 1, but it can be 2 if there is only 1 edge that is one-directional. The two edges are C-A-C and C-B-C.



6. a) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$;

There are no zeroes since there is always a way to get from a vertex to itself or to another vertex with one stopover: from A to A through B, from B to B through A, from A to B through B, from B to A through B (ABA, BAB, ABB, or BBA).

c) *Sample response*:



The adjacency matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and if you square it,

you get $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and it has no zeroes.

7. a) When a ball is passed amongst the players, it is like a network — the people are the vertices and the edges are the paths of the ball.

b) It would tell about passes from each player to each other player (or passes from a player to him or herself) that went through another person.

8. The element in the 2nd row, 4th column tells the number of one-stopover trips from B to D. It is 0 because there cannot be any one-stopover trips from B to D — the only place that B can get to is D and that happens with no stopovers.

Supporting Students

Struggling students

• Some students will need to improve their facility with multiplying matrices to be successful with this lesson. You may refer them back to the exposition and examples in **lesson 1.1.4**.

• Other students will be comfortable multiplying matrices and will understand the rule for squaring or cubing an adjacency matrix to count one-stopover or two-stopover trips, but will not really understand why the method works. Those students should continue to work with fairly simple networks and one-stopover trips. They should look at the digraph and the matrix side by side, so they can see how the elements in the squared matrix relate to the adjacency matrix and the digraph.

Enrichment

You might pose a problem where some, but not all, elements in two matrices and some, but not all, elements in their product are given and the students must determine the missing elements. For example, you might ask them to find the missing elements in these matrices: $\begin{bmatrix} 4 & ? \\ ? & 2 \end{bmatrix} \times \begin{bmatrix} 1 & ? \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} ? & 17 \\ 6 & 18 \end{bmatrix}$.

UNIT 1 Revision

Pacing	Materials
1 h	None

Question(s)	Related Lesson(s)
1	Lesson 1.1.1
2	Lessons 1.1.1 to 1.2.3
3, 4	Lesson 1.1.2
5, 6	Lesson 1.1.3
7 – 10	Lesson 1.1.4
11	Lesson 1.2.1
12, 13	Lesson 1.2.2
14, 15	Lesson 1.2.3

Revision Tips

Q 4: Students might focus on what happened to the linear dimensions (they increased by a factor of 1.5) or to the area (which increased by a factor of 2.25).

Q 10: Encourage students to write down all the dimensions in a list and then to think about how they can be combined:
A: 2-by-4 B: 2-by-1 C: 4-by-1

Q 12 & 13: Make sure students recognize there is only one adjacency matrix to describe a particular network, but many networks may be described by the same adjacency matrix.

Q 15: Students have to recognize that the intersecting paths do not create new vertices.

Answers

1. *Sample response*:

$$\begin{bmatrix} -2 & -1 & -5 & -6 \\ 0 & 0 & -2 & -3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

2. *Sample responses*:
to describe networks or paths, prices of many different items, an ecosystem, a set of student grades, coordinates of a shape on a grid

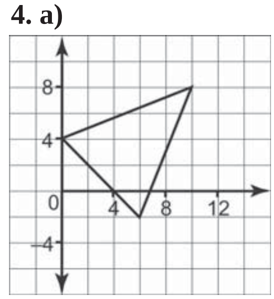
3. a) A and B; because they have the same dimensions

b) $\begin{bmatrix} 3 & -2 & 0 & 3 \\ -2 & 3 & -2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

c) A and B

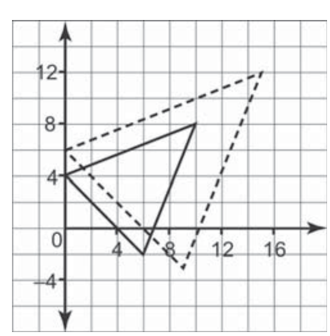
d) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & -1 & -8 & 2 \\ -1 & 2 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 & 0 & -3 \\ -2 & 1 & 8 & -2 \\ 1 & -2 & 0 & 0 \end{bmatrix}$;
you can subtract A – B or B – A

4. a)



b) $\begin{bmatrix} 0 & 15 & 9 \\ 6 & 12 & -3 \end{bmatrix}$

c)



d) The new triangle became bigger but it's similar.

Answers [Continued]

5. 3 × 4

6. a) $\begin{bmatrix} 10 & -2 \\ -2 & 4 \\ -7 & 4 \end{bmatrix}$

b) They describe how much the numbers changed during the year; a negative element means the numbers went up and a positive element means they went down.

c) Yes, because the matrices are the same size.

d) There is no real meaning to the total number of students in both months.

7. a) A × B = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 6 \\ 4 & -2 & 0 \end{bmatrix}$; B × A = $\begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}$

b) The product matrices have different dimensions.

8. 2 (first matrix), 1 (second matrix),
5 (product matrix)

9. a) The number of rows in B is the same as the number of columns in A, so you can multiply A × B. The number of rows in A is the same as the number of columns in B, so you can multiply B × A.

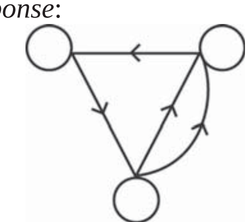
b) 3-by-anything × the same anything-by-2, e.g., a 3-by-5 matrix × 5-by-2 matrix

10. a) B **b)** A × C

11. Start at left and end at right, or start at right and end at left.

12. a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

13. *Sample response*:



14. a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

b) There is 1 one-stopover path (ABC) and
1 two-stopover path (ABDC).

15. Some vertices are not connected, e.g., A to C, and A to F, so those teams have not played each other yet.

UNIT 1 Matrices and Networks Test

1. List three situations for which it might be useful to use a matrix.

2. The matrices below describe the number of craft items two stores have sold in the last three months. Each matrix shows where in Bhutan the crafts were from.

Store A

$$\begin{array}{cc} & \begin{array}{cc}\text{East} & \text{West}\end{array} \\ \begin{array}{c}\text{March}\\ \text{April}\\ \text{May}\end{array} & \begin{bmatrix}15 & 31\\ 28 & 40\\ 22 & 28\end{bmatrix}\end{array}$$

Store B

$$\begin{array}{cc} & \begin{array}{cc}\text{East} & \text{West}\end{array} \\ \begin{array}{c}\text{March}\\ \text{April}\\ \text{May}\end{array} & \begin{bmatrix}35 & 17\\ 50 & 28\\ 41 & 15\end{bmatrix}\end{array}$$

a) Add the two matrices. What do the elements in the sum matrix represent?

b) Subtract the two matrices. What do the elements in the difference matrix represent?

c) What does a negative element in your answer to **part b)** mean?

3. The numbers of students who signed up for football in four different schools in one year are listed below.

$$\begin{bmatrix}42 & 65 & 21 & 38\end{bmatrix}$$

Suppose the numbers grew by 10% the following year. What would the new matrix look like?

4. The coordinates of the three vertices of a triangle are listed in this matrix.

$$T = \begin{bmatrix}6 & 12 & -2\\ 0 & 5 & 3\end{bmatrix}$$

a) Plot the points on a grid.

b) Multiply the matrix by 4.

c) Plot the new coordinates.

d) What happened to the shape?

5. Matrix A is a 4-by-2 matrix. Suppose you multiplied Matrix A by Matrix B.

a) Can A $\times$ B be a 2-by-2 matrix? Explain.

b) Can A $\times$ B be a 4-by-1 matrix? Explain.

6. a) Matrix A shows four students' scores on class work and an exam. Multiply Matrix A × Matrix B.

$$A = \begin{array}{cc} & \begin{array}{cc}\text{C} & \text{E}\end{array} \\ \begin{array}{c}\text{Dechen}\\ \text{Karma}\\ \text{Lobzang}\\ \text{Pema}\end{array} & \begin{bmatrix}69 & 78\\ 72 & 60\\ 75 & 55\\ 53 & 48\end{bmatrix}\end{array} \qquad B = \begin{bmatrix}0.4\\ 0.6\end{bmatrix}$$

b) What does the product matrix tell you?

7. What numbers are missing?

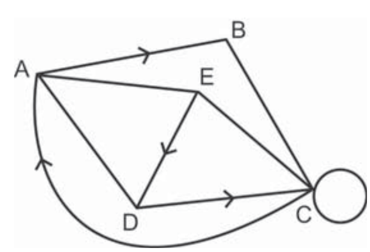
$$\begin{bmatrix}5 & -2 & 3\\ 6 & ? & 0\end{bmatrix} \times \begin{bmatrix}? & 1\\ 2 & -2\\ 0 & ?\end{bmatrix} = \begin{bmatrix}16 & 18\\ 26 & 4\end{bmatrix}$$

8. Create two different digraphs for this adjacency matrix.

$$\begin{bmatrix}0 & 3 & 2\\ 0 & 1 & 0\\ 2 & 2 & 1\end{bmatrix}$$

9. Why is an adjacency matrix always square?

10. a) Create an adjacency matrix for this digraph.



b) What does the number in the position (4, 2) in the matrix tell you? Explain.

c) Why are there more 1s in the C column than any other?

d) Why are there more 1s in the B column than the B row?

e) Square the adjacency matrix. List five facts you know from that matrix.

f) How many two-stopover paths are there from B to E?

UNIT 1 Test

Pacing	Materials
1 h	None

Question(s)	Related Lesson(s)
1	Lesson 1.1.1
2	Lesson 1.1.2
3, 4	Lesson 1.1.3
5 – 7	Lesson 1.1.4
8, 9	Lesson 1.2.2
10	Lesson 1.2.3

Select questions to assign according to the time available.

Answers

1. *Sample responses*:
to keep track of the scores for each player on an archery team; to list the amount of money earned by different people for different months; to list the cost of different items in different stores

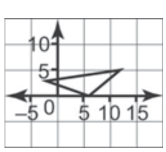
2. a) $\begin{bmatrix} 50 & 48 \\ 78 & 68 \\ 63 & 43 \end{bmatrix}$; the elements represent the total number of items sold by the two stores for each month from each part of the country.

b) $\begin{bmatrix} -20 & 14 \\ -22 & 12 \\ -19 & 13 \end{bmatrix}$; the elements represent how many more items in each month and from each part of the country were sold by store A compared to store B.

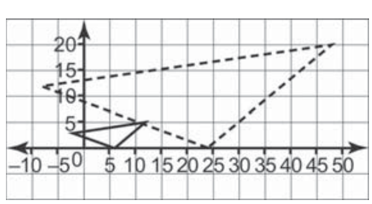
c) A negative element means that store A sold fewer items from that area of the country than store B for that month.

3. $[46 \quad 72 \quad 23 \quad 42]$

4. a)



b) $\begin{bmatrix} 24 & 48 & -8 \\ 0 & 20 & 12 \end{bmatrix}$ **c)**



d) The shape is similar and bigger.

5. a) No; if multiplication is possible, A must have the same number of rows.

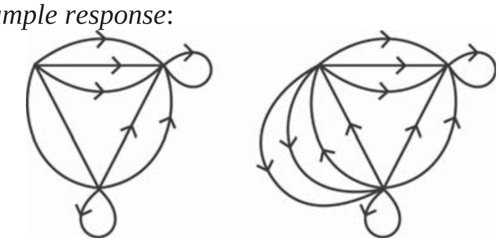
5. b) Yes, if B is 2-by-1

6. a) $\begin{bmatrix} 74.4 \\ 64.8 \\ 63.0 \\ 50.0 \end{bmatrix}$ (also accept rounded values)

b) the final course mark for each student

7. 1 in the first matrix, 4 in the top row of the second matrix, and 3 in the bottom row of the second matrix

8. *Sample response*:



9. You are always showing the number of edges from any vertex to any other vertex. Since you have edges coming to or going from each vertex, there have be the same number of rows (to show edges going from) as columns (to show edges coming to) in the matrix.

10. a) $\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$

b) the number of paths from D to B (which is 0)
c) because there are more ways to get to C than to any other vertex
d) because there are more ways to get into B than out
e) *Sample response*: There are 2 one-stopover paths from A to itself, 1 from B to itself, 3 from C to itself, 1 from D to itself, and 2 from E to itself.
f) 2

UNIT 1 Performance Task — Travelling in Bhutan

A. i) Select four communities (cities, towns, or villages) in Bhutan. Sketch the outline of Bhutan below and then place the communities in their approximate locations.

ii) Decide on four communities you would like to connect by roads. Create a network of four roads to connect the communities. The communities do not have to be connected by roads in reality.

B. Build a network to display the roads between those pairs of adjacent communities you chose in **part A ii)**.

C. Create an adjacency matrix to describe your network in **part B**.

D. Which places CANNOT be connected by two-stopover trips? Explain.

E. Create two other problems you could solve with your matrix or network. Provide a solution for each problem.

UNIT 1 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-B4 Matrices: adding, subtracting, and scalar multiplication — model, solve, and create problems **10-B5** Matrix Multiplication: develop and apply procedures **10-A4** Network Problems: represent networks using matrices and vice versa **10-B6** Matrices: solve network problems	1 h	• Map of Bhutan

How to Use This Performance Task

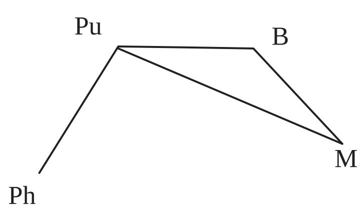
You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used, if you wish, as enrichment material for some students. You can assess performance on the task using the rubric below. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution

A. i) My communities will be Punakha, Bhumtang, Mongar, and Phunsholing.

ii) I will create these four roads:
from Punakha to Bhumtang
from Bhumtang to Mongar
from Mongar to Punakha
from Punakha to Phunsholing

B.



C.

$$\begin{array}{c} \\ Pu \\ B \\ M \\ Ph \end{array}\begin{array}{c} \begin{array}{cccc} Pu & B & M & Ph \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

D. The only one that doesn't work for a two-stopover trip is from Phunsholing to itself. This matrix describes two-stopover trips and there is only one 0 in the matrix and it's for a two-stopover trip from Phunsholing to itself.

$$\begin{array}{c} \\ Pu \\ B \\ M \\ Ph \end{array}\begin{array}{c} \begin{array}{cccc} Pu & B & M & Ph \end{array} \\ \begin{bmatrix} 2 & 4 & 4 & 3 \\ 4 & 2 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

E.
- How many more one-stopover trips would there be from Punakha to itself compared with from Bhumtang to itself? (There are 3 trips from Punakha to itself with one stopover but only 2 from Bhumtang to itself. That is one more path.)
- If I multiplied the adjacency matrix by 2, what would that tell me? (It means that wherever I had one road or edge, now I have two.)

UNIT 1 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Description of network	Completely accurate network and adjacency matrix to describe the 4 roads	Reasonably accurate network and adjacency matrix to describe the 4 roads	Reasonably accurate network or adjacency matrix to describe the 4 roads	Major errors in either the network or the adjacency matrix
Creation of problems	Original and appropriate problems created	Two appropriate problems created modelled on ones solved in the unit	Creation of at least one appropriate problem	No appropriate problems created
Solution of problems	Insightful and correct solutions to both problems	Correct or almost correct solutions to the problems	Correct or almost correct solution to one problem	Incomplete or incorrect solutions to created problems

UNIT 2 COMMERCIAL MATH AND NUMBER

UNIT 2 PLANNING CHART

	Outcomes or Purpose	**Suggested Pacing**	**Materials**	**Suggested Assessment**
Getting Started **SB p. 31** **TG p. 36**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Calculators	All questions
Chapter 1 Commercial Math				
2.1.1 Purchasing Decisions **SB p. 32** **TG p. 38**	**10-B2 Purchasing Decisions: solve problems** • use percentage to solve problems involving purchases	1 h	• Calculators	Q2, 5, and 8
2.1.2 Compound Interest **SB p. 36** **TG p. 41**	**10-B3 Simple and Compound Interest: demonstrate an understanding** • understand the long term difference between simple and compound interest • investigate both investments and financing situations	2 h	• Calculators	Q2, 5, and 9
CONNECTIONS: The Rule of 72 **SB p. 43** **TG p. 45**	Connect what students have learned about compound interest to a real-world application	20 – 30 min	• Calculators	N/A
GAME: Target 200 **SB p. 43** **TG p. 45**	**10-B3 Simple and Compound Interest: demonstrate an understanding** • investigate both investments and financing situations	30 min	• Spinners • Dice • Calculators	N/A
2.1.3 Dividends and Stocks **SB p. 44** **TG p. 46**	**10-B3 Simple and Compound Interest: demonstrate an understanding** • investigate both investments and financing situations	1 h	• Calculators • Newspaper stock reports (optional)	Q1, 3, 5, and 9
2.1.4 Using Commercial Math **SB p. 47** **TG p. 48**	**10-B3 Simple and Compound Interest: demonstrate an understanding** • understand the long term difference between simple and compound interest • investigate both investments and financing situations	2 h	• Calculators	Q1, 3, 7, and 9

UNIT 2 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Radicals				
2.2.1 EXPLORE: Representing Square Roots (Essential) **SB p. 52** **TG p. 51**	**10-A1 Irrational Numbers: understand role** • understand when to approximate and when to continue with radical expressions • convert between entire and mixed radicals	1 h	• Grid paper or 10-by-10 grid	Observe and Assess Questions
2.2.2 Simplifying Radicals **SB p. 53** **TG p. 54**	**10-A1 Irrational Numbers: understand role** • develop awareness of errors in decimal approximations and rounding off • understand when to approximate and when to continue with radical expressions • convert between entire and mixed radicals **10-B1 Roots: apply properties to operations** • develop and apply properties for operations involving roots	1 h	• Calculators • Rulers (optional)	Q1, 4, 8, and 9
2.2.3 Operations with Radicals **SB p. 56** **TG p. 57**	**10-A1 Irrational Numbers: understand role** • convert between entire and mixed radicals **10-B1 Roots: apply properties to operations** • develop and apply properties for operations involving roots **10-C15 Equations: apply properties of numbers upon expressions** • solve equations by applying associative, distributive, identity and inverse properties	1.5 h	• Calculators (optional)	Q1, 3, 6, and 9
GAME: Five Radicals **SB p. 61** **TG p. 59**	Practise and apply operations with radicals in a game situation	25 min	• Constructed cards	N/A
UNIT 2 Revision **SB p. 62** **TG p. 60**	Review the concepts and skills in the unit	2 h	• Calculators	All questions
UNIT 2 Test **TG p. 62**	Assess the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 2 Performance Task **TG p. 64**	Assess concepts and skills in the unit	1 h	• Calculators	Rubric provided
UNIT 2 Assessment Interview **TG p. 67**	Assess concepts and skills in the unit	20 min	• Calculators	All questions
UNIT 2 Blackline Master **TG p. 68**	BLM 10-by-10 grid			

Math Background

• In this unit, students continue their study of commercial math. The emphasis is on compound interest, which involves complex mathematical thinking and was therefore not developed in Class IX. Some earlier work is revisited in relation to percentage calculations in the context of pricing (markups and discounts), commission, stocks, and dividends.

• The other new mathematical thinking in this unit relates to working with radicals. Students have met square roots in earlier classes. In this unit the concept of radical becomes a bit more abstract. Students are introduced to calculations involving operations with radicals and work with other sorts of roots, such as cube roots or fourth roots.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in **question 5** of **lesson 2.1.1** where a salesperson has a problem predicting income that is based on commissions. Other examples of problem solving are found in the **Try This** and **question 7** of **lesson 2.1.2**, where students solve problems about compound interest, and in **question 7** of **lesson 2.1.4**, where they solve a problem related to determining a price to meet certain conditions.

- Students use communication to explain their thinking, for example, in **question 9**, **part c)** of **lesson 2.1.2**, where they describe how different graphs represent the difference between various interest rates, and in **question 9** of **lesson 2.2.3**, where they explain the steps for operating with radicals.

- Students use reasoning in answering questions such as **Revisiting the Try This** in **lesson 2.1.1**, where they must realize there is more than one case to consider, in **question 3**, **part b)** of **lesson 2.1.4** where they develop a conjecture about the effect of an interest rate, and in **question 9** of **lesson 2.1.4**, where they reason about a financial choice. They also use reasoning in **question 5** of **lesson 2.2.2** and in **question 7** of **lesson 2.2.3**, where they form conjectures about radicals.

- They use visualization in **question 9**, **part c)** of **lesson 2.1.2**, where they use a graph to visualize the effect of an interest rate, and in the Explore **lesson 2.2.1**, where they consider a visual model for simplification of radicals, e.g., why $\sqrt{8} = 2\sqrt{2}$. They also use that same visual interpretation of a radical in **question 8** of **lesson 2.2.2.**

- Students make connections, for example, in **question 9** of **lesson 2.1.1** where they relate the concepts of discount, markup and commission to everyday life. They also use connections in **question 10** of **lesson 2.1.2**, where they relate the concept of compound interest to exponential growth, in **lesson 2.1.3**, where they think about why people invest in stocks rather than in a bank account, and in **question 5** of **lesson 2.1.4**, where they consider a choice in winning a lottery. A connection is made between geometric and numerical expressions of radicals both in **lesson 2.2.1** and in **question 8** of **lesson 2.2.2.** The **Connections** feature focuses on a quick way to consider the effect of an interest rate.

Rationale for Teaching Approach

• This unit is divided into two chapters.

- **Chapter 1** deals with commercial math. Before compound interest is introduced, students are given a chance to reactivate their knowledge about simpler interest calculations. The material on stocks and dividends is not mathematically complex, but it is useful for students as citizens. The last lesson in the chapter brings together many of the commercial math ideas learned in Class IX and Class X.

- **Chapter 2** deals with radicals. The chapter allows students to extend their knowledge of how number properties operate with rational numbers to using them with real numbers in the form of radicals. Many students will find the ideas intuitively obvious, but others may need more support, particularly with simplifying radicals. Prior skills with factoring, algebra, and manipulation of expressions will be reinforced here.

• The **Connections** feature in **Chapter 1** is designed to help students see the power of investing when compound interest is applied. They learn to mentally estimate effects of changes in rates.

• The **Explore** feature in **Chapter 2** provides a physical model of simplifying radicals for students to refer back to. Although the feature is optional, it is particularly beneficial for those seeking a visual basis for understanding irrational numbers and radicals. The Pythagorean theorem figures prominently in this exploration.

Technology in This Unit

Students will want to use calculators, or spreadsheets if they are available, to perform the compound interest calculations. Calculators may also be useful to support them both in making conjectures and in confirming their thinking when working with radicals.

Getting Started

Curriculum Outcomes	Outcome relevance
8 Percent: solving and creating real problems in context **9** Irrational numbers: demonstrate and understand meaning **9** Exponent laws: integral exponents	Students will experience more success in this unit if they review what they already know about working with exponents, square roots, and percentages.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculating percentages

Main Points to be Raised

• If the first number is greater than the second number and both are greater than one, then the first number raised to a power is greater than the second number raised to the same power.

• If a number is greater than one, then a greater power of that number is greater than a lesser power of that same number.

Use What You Know — Introducing the Unit

• Students can try this activity alone or with a partner.

• Observe students as they work. You might ask:

- *Once you knew 1.01^5, how did you calculate 1.01^6? Why did that work?* (I would multiply the 1.01^5 by 1.01 because $1.01^6 = 1.01^5 \times 1.011$ because of the exponent laws)

- *Which of these did you expect to be greater?* (I thought 1.02^3 would be greater than 1.02^2 since you are multiplying an extra time.)

- *Why do you think 1.03^3 is greater than 1.02^3 by a different amount than 1.03^2 is greater than to 1.02^2?* (Because you are multiplying by 1.02 for one number and by 1.03 for the other, the difference will increase as the powers get larger.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually, but it will be helpful if each student has a partner with whom he or she can discuss answers or difficulties.

Common Errors

Many students will assume that $\sqrt{40}$ is four times as much as $\sqrt{10}$. Encourage students to estimate each, e.g., $\sqrt{10}$ is just a bit more than 3 and $\sqrt{40}$ is just a bit more than 6. This will help them answer **question 2**.

Answers

A.

$1.01^2 = 1.0201$	$1.02^2 = 1.0404$	$1.03^2 = 1.0609$
$1.01^3 = 1.0303$	$1.02^3 = 1.0612$	$1.03^3 = 1.0927$
$1.01^4 = 1.0406$	$1.02^4 = 1.0824$	$1.03^4 = 1.1255$
$1.01^5 = 1.0510$	$1.02^5 = 1.1041$	$1.03^5 = 1.1593$
$1.01^6 = 1.0615$	$1.02^6 = 1.1262$	$1.03^6 = 1.1941$

B. Increase; each time you are multiplying greater numbers that are more than 1, so the differences have to be greater.

C. Increase; each time you are multiplying again by a number greater than 1, so the differences have to be greater.

1. a) 13 cm **b)** 52 cm; ratio of perimeters = ratio of corresponding sides so $\frac{5+12+13}{120} = \frac{13}{x}$ and x = 52 cm **2.** $\sqrt{40} \times \sqrt{40}$ = 40, but $(2\sqrt{10})(2\sqrt{10}) = 4 \times 10 =$ 40. That means that $2\sqrt{10}$ must also be $\sqrt{40}$.	**3.** Nu 2100; 0.07 × Nu 6000 × 5 years = Nu 2100 **4. a)** 160 **b)** 62.5 **c)** Equal; both are 10. **5.** Because of the power law of exponents, $x^6 = (x^2)^3$, which makes it a perfect cube and $x^6 = (x^3)^2$, which makes it a perfect square.

Supporting Students

Struggling students

For **question 5**, you might encourage students to write the question in this form: $x^6 = x^{3\times2}$. Then you might need to remind them to think about the laws of exponents they already know.

Enrichment

You might challenge students to make up questions like **question 4** for partners to solve. An example might be: 36 is 75% of a number. What is 50% of that number? (You may even try to calculate the answer without finding the number, as in 12 is 25% of the number and hence 24 is 50% of the number.)

Chapter 1 Commercial Math

2.1.1 Purchasing Decisions

Curriculum Outcomes	Outcome relevance
10-B2 Purchasing Decisions: solve problems • use percentage to solve problems involving purchases	Students will make purchasing decisions as a regular part of their everyday life as adults.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculating percentages

Main Points to be Raised

• A *discount* describes a reduction in a marked price.

• A *percent discount* can be calculated using the original marked price and the final selling price. Either the ratio of the two prices can be calculated, converted to a percentage, and subtracted from 100%, or the ratio of the difference between the two prices to the original marked price can be converted to a percentage.

• When a discount is applied, a *selling price* can be calculated by subtracting the discount from the marked price or by describing the selling price as a different percentage (100% minus the percent discount) of the marked price.

• A *markup* describes an increase to the cost price applied by a seller. The seller's purpose is to make a profit.

• A markup can be described as a percentage of the cost price. If the ratio of the marked price to the cost price is written as a percentage, you subtract 100% to get the percent markup.

• A selling price can be calculated by multiplying the cost price by the percent markup (written as a decimal).

• A *commission* is an amount a salesperson receives, usually described as a percentage of the selling price.

Try This — Introducing the Lesson

A. Students can solve the problem alone. Encourage the use of mental math with these simple percentages.

Observe while students work. You might ask:

• *How much did the item cost at store A after the discount?* (Nu 1800)

• *How would knowing the price at store A help you figure out the price at store B?* (I knew that 80% of the marked price at store B was Nu 1800. I solved the equation $0.8x = 1800$ by dividing 1800 by 0.8.)

The Exposition — Presenting the Main Ideas

• Describe a situation where a discount, say 15%, is applied to a simple marked price, for example Nu 1000. Ask students to calculate both the amount of the discount and the amount of the selling price. If students do not suggest several approaches, help students see that not only could you subtract the 15% of 1000 from Nu 1000, but you could also calculate 85% of Nu 1000 directly.

• Describe a situation where you know a marked price and a selling price and ask students to calculate the percent discount. Again, use simple values, but not quite as simple as in the previous example. For example, the marked price could be Nu 900 and the selling price Nu 850. Help students see that they could get the ratio 850:900, write it as a percentage, and subtract it from 100%. Or, they could simply determine the percentage equivalent to the ratio 50:900 (the ratio of the amount of the discount to the original marked price). Ask them which method they prefer.

• Repeat the situations above, but using markups instead.

• Invite students to read through the exposition and ask if there are any questions.

Revisiting the Try This

B. Students have the opportunity to apply what they learned in the exposition to a problem that requires them to do some reasoning. They must realize that there are many possible scenarios and that they cannot draw a conclusion from just one example. The ratio is dependent upon the percent discount being used.

Using the Examples

- Have students close their texts. Write the questions from **examples 1, 2, and 3** on the board for students to work through. They can then compare their responses to those in the text.
- Offer an opportunity for students to ask questions.

Practising and Applying

Teaching points and tips

Q 2: Remind students that they have to figure out what they are calculating the percentage of.

Q 3: You may have to draw students' attention to the fact that one option describes a discount and the other option describes a markup.

Q 4: This question focuses students on the relationship between decimals and percentages. Since 50% = 0.5, then the reverse process is 1 ÷ 0.5 = 2, or 200%. Then if the selling price is 200% of the cost price, the increase (markup) has to be 100%.

Q 5b: Some students will solve the problem using an equation ($5000 = 1000 + 0.03x$), but it is not necessary.

Q 7: Some students may need to be referred back to **example 3** to remember how commissions work.

Q 9: This question might work better as a whole class discussion or with students working in partners.

Common Errors

In solving the **Try This**, many students will realize that Nu 1800 is 80% of the marked price from Store B, but will then add the extra 20% by using 20% of the Nu 1800 instead of recognizing that they instead need to use 20% of the unknown price. Encourage students to check their answers to observe their mistake. Encourage them to write a proportion or equation to solve the problem.

Suggested assessment questions from Practising and Applying

Question 2	to see if students apply percentages to the correct "wholes" to solve a problem
Question 5	to see if students can solve a problem involving commission
Question 8	to see if students can reason about the difference between values described as percentages and values described as absolute amounts

Answers

A. Nu 2250

B. *Sample response*:
It depends on the percent discount. If Store B's discount is 20% and Store A's discount is 10%, then the ratio of the marked prices is 1.125, but if Store B's discount is 50% and Store A's discount is 25%, then the ratio of the marked prices is 1.5.

1. a) Nu 20 **b)** 20% ($100 - \frac{20}{25} \times 100$)

2. No. Jigme's calculation is incorrect. The discount percent is actually 20%, not 25%, because the percent discount is based on the marked price of Nu 140 not on the discounted selling price of Nu 112.

3. Option A;
Option A results in a price of 480 × 1.20 = Nu 576.
Option B results in a price of 700 × 0.85 = Nu 595.

4. 100%; *Sample response*:
A cost price of Nu 30 would increase to a selling price of Nu 60 with a markup of 100%. A 50% discount on a selling price of Nu 60 would take the price back to the original cost price of Nu 30.

5. a) Nu 3100.00

b) Nu 133,333.34

Answers [Continued]

6. If a cost price is increased by more than 100%, the markup is more than the cost price and results in a selling price that is more than double the cost price. This is a reasonable sales practice to allow the seller to make a profit. If a marked price is reduced by a discount of more than 100%, the discount is more than the marked price and results in a selling price of less than 0, which would mean that the seller pays the buyer. This does not make sense.

7. *Sample response*:
A 10% commission on a sale of a Nu 3000 item would result in Nu 300 commission, which would be less than a fixed commission of Nu 500 per item.

8. a) The percentage markup resulting from a markup of Nu 100 cannot be determined unless the cost price is known. Likewise, the amount of markup resulting from a 7% markup cannot be determined unless the cost price is known. Therefore, you cannot compare them.

8. b) *Sample response*:
If the cost price was Nu 100: Nu 100 vs. Nu 7.

c) *Sample response*:
If the cost price was Nu 10,000: Nu 100 vs. Nu 700.

9. *Sample response*:
If the same discount amount, e.g., Nu 100, were offered on an expensive item, e.g., Nu 10,000, and an inexpensive item, e.g., Nu 1000, it would make a significant difference to the selling price of the inexpensive item, making it attractive to buyers, but it would probably not affect the price of the expensive item significantly enough to make it attractive to buyers. If the same percent discount were offered on both items, the selling price of both items would be affected proportionally and there would be an equal incentive to buy both items. The same idea applies to markup and commission.

Supporting Students

Struggling students

• To help them solve **question 2**, some students might benefit if you model a diagram such as the one below.

$$\boxed{\text{Marked Price}} \xrightarrow{\text{Multiply by } x} \boxed{\text{Discounted Price}}$$

$$\boxed{\text{Marked Price}} \xleftarrow{\text{Divide by } x} \boxed{\text{Discounted Price}}$$

If they realize that 112 was divided by x to result in 140, they are more likely to perform the correct calculation. (They would get $x = 0.8$, which means the discounted price is 80% of the marked price and the discount is 20%.)

• Some students may need to partner with another student to help them reason through **question 8.**

Enrichment

• Students might be interested in investigating the values of discounts and markups that are typically used in stores where their families shop.

• Students might create a whole variety of percentage questions involving discounts, markups, and commission, all with the same answer, for example, Nu 25.

2.1.2 Compound Interest

Curriculum Outcomes	Outcome relevance
10-B3 Simple and Compound Interest: demonstrate an understanding • understand the long term difference between simple and compound interest • investigate both investments and financing situations	As adults, students will have to make decisions that involve both earning interest and paying interest when compound interest is involved. Learning about how it works will help them make informed decisions.

Pacing	Materials	Prerequisites
2 h	• Calculators	• calculating percentages

Main Points to be Raised

• Whereas *simple interest* is calculated only on the original principal, *compound interest* is added to the principal before the next instalment of interest is calculated.

• Compound interest can be calculated using different *compounding periods*. The period is often semi-annually or quarterly (two or four times a year respectively), but it can also be monthly, weekly, or daily.

• Compound interest is generally expressed as a yearly (or *per annum*) rate. It can be restated as an equivalent rate based on the compounding period.

• The more frequently interest is compounded for a particular rate, the higher will be the resulting growth.

• Compound interest grows much more quickly than simple interest. That makes it favourable to investors but costly for borrowers.

• The formula for compound interest calculations is

$A = P\left(1+\frac{r}{n}\right)^{nt}$ where

A = amount earned or paid

P = initial amount invested or borrowed, known as the principal

r = interest rate for one year

n = number of interest periods in one year

t = time expressed in years

• A table, or spreadsheet, if available, is a useful tool for keeping track of the steps for calculating compound interest over a number of periods.

Try This — Introducing the Lesson

A. Students can solve the problem alone or with a partner. Allow them to use calculators if they wish.

Observe while students work. You might ask:

• *Which percent will be greatest? Why?* (The markup for the customer, since it has been marked up more times.)

• *Why do you not just add 20% each time?* (Each time the resulting price is 120% of the original price and that means you are multiplying by 1.2. If you keep multiplying by 1.2, you are not adding 0.2. For example, 1.2 × 1.2 = 1.44, which is an increase of 0.24, not 0.2.)

The Exposition — Presenting the Main Ideas

• Lead students through the first two bullets in this section. Help them see the difference between simple and compound interest. Introduce the term *compounding period* and make sure students notice that the difference between simple and compound interest only shows up after the first compounding period.

• Describe how compounding can happen at any regular interval that one might choose, but is typically annual, semi-annual, quarterly, or monthly. It can sometimes be weekly or daily. Mention that the rate is almost always described as an annual rate and discuss the abbreviation p.a. for per annum. Explain that if the annual rate is, say, 5%, you gets only 2.5% (or half of it) each period if interest is compounded semi-annually, 1.25% (or a quarter of it) each period if interest is compounded quarterly, and only 0.4167% (one twelfth of it) each period if interest is compounded monthly.

• Write the formula for interest shown on **page 37**, but help students see how it relates to the table on **page 36**.

In that table, P is 10,000. The value in row 1 relates to $n = t = 1$, $r = 0.04$. The value in row 2 relates to $n = t = 2$, $r = 0.04$. The value in row 3 relates to $n = t = 3$, $r = 0.04$, etc. Then have them look at the table on **page 37** that shows this. Make sure they realize that they are repeatedly multiplying by an extra 1.04 each time, which is why 1.04 is raised to a higher power each time.

• Ask them why they think more frequent compounding might result in a greater total amount.

• Confirm their predictions by applying the formula using an annual rate and a semi-annual rate.

• Suggest to students that they can use the exposition as a reference.

Revisiting the Try This

B. Students have the opportunity to relate what they learned about compounding to the markup problem in the **Try This.**

Using the Examples

• Discuss **example 1** with the whole class. Then assign students to groups of four. Write **example 2** on the board and have the students work on it in their groups. They can then check their answers against the textbook. Then one pair in each group can do **example 3** while the others do **example 4**. One person in each pair can then explain to the others how the problem worked.

• Circulate as students are working to see if students understand the concepts being explored.

• Ask one student from each of three groups to describe to the class his or her approach to the worked examples.

• Make sure that students understand that in **example 4**, the annual rate would always be greater than the rate for the shorter time periods. If you wish, you can tell students that some people call these *effective annual rates.*

Practising and Applying

Teaching points and tips

Q 4: Watch to make sure that students remember to use the adjusted rate for the various compounding periods. For example, they need to use a value of 1% for each monthly compounding period.

Q 5: Students need to remember to subtract their first payment when calculating the interest charge for the second month.

Q 6: Again, students need to remember to subtract the first payment and then the second one when calculating the interest owed. In other words, the value owed at the end of the first year is $1.16 \times 30\ 000$, but the value owed at the end of the second year is $1.16 \times (1.16 \times 30{,}000 - 10{,}000)$.

Q 7: Students need to recognize that 10,950 represents $(1 + \frac{r}{4}) \times 12{,}000 - 1500$.

Q 8: If you introduced the term *effective annual rate* in **example 4**, you might make that connection here.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can calculate an amount resulting from compound interest
Question 5	to see if students can solve a problem involving compound interest
Question 9	to see if students can use graphs to communicate about compound interest

Answers

A. i) 20% **ii)** 44% **iii)** 72.8%

B. i) *Sample response*:
The repeated application of a 20% markup is like compounding at 20% per period:

$A = P(1 + \frac{r}{n})^{nt} \rightarrow A = 30(1 + 0.20)^1$ = Nu 36.00 (cost to business person)
$A = 30(1 + 0.20)^2$ = Nu 43.20 (cost to shopkeeper)
$A = 30(1 + 0.20)^3$ = Nu 51.84 (cost to customer)

ii) $A = 30(1 + 0.20)^3$ = Nu 51.84. The markup is 51.84 – 30.00 = 21.84. Nu 21.84 is 72.8% of the manufacturing cost of Nu 30.

1. 6.25%

2. a) Nu 624.36 **b)** Nu 649.71

3. a) Nu 624.00 **b)** Nu 648.00
The simple interest pays less than the compound interest.

4. a) Nu 1141.43 **b)** Nu 1112.40 **c)** Nu 1080

5. a) Nu 9816.67 **b)** Nu 9631.20

6. a) Nu 15,970.88 **b)** Nu 10,970.88

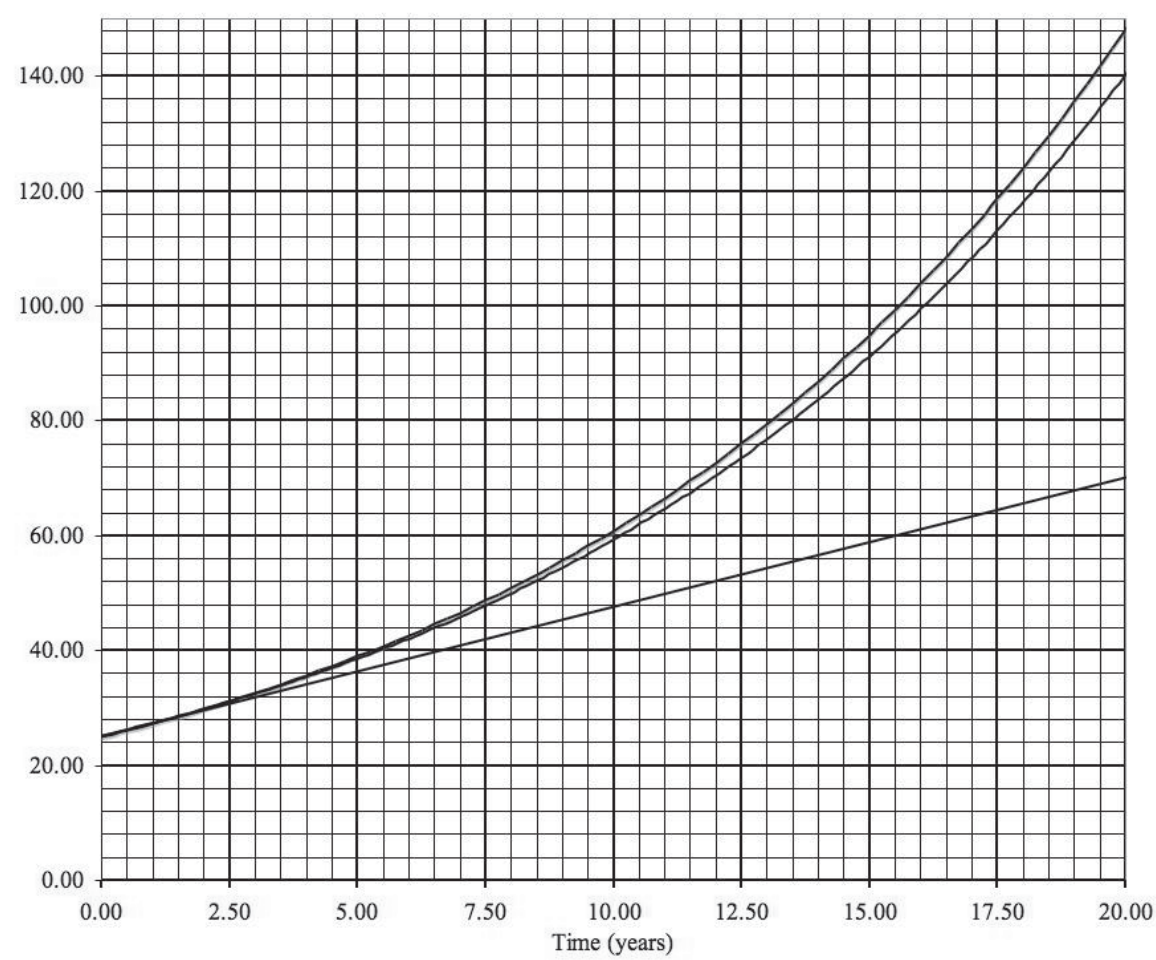
7. 15% (compounded quarterly) since 10,950 = $12{,}000(1 + \frac{r}{4})$ – 1500 (r is the annual rate of interest)

8. 12.68%

9. a) I: Nu 27,327.08; Nu 60,879.73; Nu 148,253.63
II: Nu 27,250.00; Nu 59,184.09; Nu 140,110.27
III: Nu 27,250.00; Nu 47,500.00; Nu 70,000.00

9. b)

Vertical axis is 1000s of ngultrums



Answers [Continued]

9 c) *Sample response*:	**10.** *Sample response*:
The simple interest graph is linear so the investment grows at a constant rate. It starts to be noticeably less than the compounded interest investments after the fifth year; both compound interest graphs are exponential, which means the investments grow faster and faster over time. The graph of interest compounded quarterly curves a bit more sharply than the graph of interest compounded annually. This means the investment with interest compounded quarterly grows faster.	Compound interest calculates interest on interest already earned so it increases faster and faster over time, which is an exponential relationship. Simple interest calculates the same amount of interest each time since the interest is always based on the original principal, so the growth is constant and therefore the relationship is linear.

Supporting Students

Struggling students

• Students who are struggling should be encouraged to use organized charts to keep track of values related to each compounding period. For example:

Amount at start of period	Rate for the period (annual rate divided by number of payments per year)	Amount due at end of period	(Payment)	(Amount still due)

The final two columns are only used if payments are involved. Students should understand that the value at the end of one row becomes the initial value in the next row.

• Some students might find it more meaningful to add the calculated interest to the starting amount separately for each compounding period rather than use the formula.

Enrichment

• Students might do some internet research to compare typical interest rates in Bhutan to rates in other countries.

• Students might investigate the costs of borrowing. For example, they could choose a car, find out its price, and then estimate how much it will actually cost with a particular repayment scheme.

CONNECTIONS: The Rule of 72

This feature is designed to provide students with information that is useful for estimating when compound interest is involved. It also presents a real-world situation involving compound interest.

Answers

1. It appears to work; *Sample response*:

If $r = 9$, then the number of years should be $72 \div 9 = 8$ years.

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + 0.09)^8 = 1992.56$, which is close to double Nu 1000.

If $r = 8$, then the number of years should be $72 \div 8 = 9$ years.

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + 0.08)^9 = 1999.00$, which is close to double Nu 1000.

2. Yes; *Sample response*:
I would expect it to at least double because the number of years to double if it were compounded annually would be about $72 \div 12 = 6$ years, so the number of years to double if it were compounded monthly would have to be less than 6 years, because more frequent compounding results in more interest.

3. About 70 months; *Sample response*:

I used a principal of Nu 1000 and tried different values for *nt* in the exponent of the compound interest formula, starting with 72 (using the Rule of 72 as an estimate, it should take about 6 years, and 6 years with monthly compounding is $6 \times 12 = 72$), until I found a value that was close to double the original investment.

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + \frac{0.12}{12})^{72} = 2047.10$, which is more than double Nu 1000

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + \frac{0.12}{12})^{71} = 2026.83$, which is a bit more than double Nu 1000

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + \frac{0.12}{12})^{70} = 2006.76$, which is just a little bit more than double Nu 1000

$A = P(1 + \frac{r}{n})^{nt} \rightarrow 1000(1 + \frac{0.12}{12})^{69} = 1986.89$, which is a bit less than double Nu 1000

Note that the method used in **example 3** of **lesson 2.1.2** could have been used. The equation would have been $2000 = 1000(1 + \frac{0.12}{12})$, where *m* represents the number of months.

GAME: Target 200

This game provides an opportunity to practice percentage calculations in an enjoyable way.

2.1.3 Dividends and Stocks

Curriculum Outcomes	Outcome relevance
10-B3 Simple and Compound Interest: demonstrate an understanding • investigate both investments and financing situations	Students will read about stocks in the newspaper and may eventually wish to invest in them. It is important that they have some fundamental background.

Pacing	Materials	Prerequisites
1 h	• Calculators • Newspaper stock reports (optional)	• calculation of percentages

Main Points to be Raised

• Shareholders who own stock in a company receive money in the form of dividends from the company.

• The *face value* of a stock can also be called the *nominal value* or the *issue price*. This value is not necessarily the selling or buying price at any particular point in time. The buying or selling price is called the *market value* of the stock.

• If the market value of a stock is higher than its face value, it can be sold *at a premium*. If its market value is lower than its face value, it is sold *at a discount*.

• A *yield percentage*, which is also called a *percentage return*, describes the ratio of money received in dividends to money invested. A good investment has a high yield percentage.

Try This — Introducing the Lesson

A. Students can solve the problem alone. Encourage the use of mental math with these simple percentages.

Observe while students work. You might ask:

• *How do you know that each family member will get more than Nu 1000 in interest?* (10% of Nu 10,000 is Nu 1000 and 12% is more than 10%.)

• *How much money would the company have to earn in the year so that she can pay all her investors without taking a loss?* (It would have to earn 10 × Nu 1200, which is Nu 12,000.)

• *Why might you calculate the amount of interest for her first year investment payments either as 12% of Nu 100,000 or as 10 × 12% of Nu 10,000?* (You can either think about all of it at once or you can total up the ten equal payments to each investor.)

The Exposition — Presenting the Main Ideas

• Ask students to read through the exposition.

• If possible, bring in copies of a stock report from a newspaper so that students can see which companies' shares are sold in Bhutan and so that they can see some of the terms they have met in the lesson.

Revisiting the Try This

B. Students have the opportunity to apply what they learned in the exposition to the problem in the **Try This.**

Using the Examples

Have students close their texts. Write the questions from **examples 1 and 2** on the board for students to work through. They can then compare their responses to those in the text.

Practising and Applying

Teaching points and tips

Q 2: Some students will calculate the total face value and calculate the cost of the stock based on the premium or discount, whereas other students will calculate the value for each share and then multiply.

Q 6: Because this question involves many steps, encourage students to organize their work.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can apply percentages to calculate a market price for stocks
Question 3	to see if students can calculate dividends and yield percentages
Question 5	to see if students can calculate a dividend rate when they know the dividend amount and the investment amount
Question 9	to see if students can make the connection between yield percentages and the attraction of buying stock

Answers

<table>
<tr><td colspan="2">A. Nu 11,200 (100,000 × 1.12 ÷ 10)

B. i) People are investing in a company and getting interest back. The interest is like a dividend.
ii) Preferential shares, because the percent interest was fixed at the start.</td></tr>
<tr><td>1. A. Nu 22,000 B. Nu 40,000 C. Nu 42,500

2. A. 772 shares B. 212 shares C. 100 shares

3. a) A. Nu 3300 B. Nu 16,500 C. Nu 33,000

b) A. 20% B. 27.5% C. 25.88%

c) B; The discount to buy it was greatest, so it resulted in a higher yield percentage because the dividend rate was the same for each stock.

4. a) Nu 2500 b) 58.82%

5. a) 19.8% b) Nu 250 c) Nu 5500

6. Nu 2160

7. Nu 6250</td><td>8. Sample response:
If a company earns more profit, it can afford to pay a higher dividend rate. A company might also pay a higher dividend to try to attract more investors.

9. a) The yield rate for stock investments has the potential to be much higher than the interest you would earn in a bank account. Stock investors are willing to take the risk.

b) Money in a savings account in a bank is more secure than money invested in stocks. There is always a risk involved in stocks because the stock value can drop suddenly and you end up losing money if you have to sell it at a discount. Also, you can assure a fixed amount is available at a particular time with money invested in savings accounts.</td></tr>
</table>

Supporting Students

Struggling students

Question 6 is fairly complex. Partner students who are struggling with other students in order to successfully work through this question.

Enrichment

Students might wish to investigate the history of certain stocks and find out how much individuals might have earned over the past five years if they had held that stock.

2.1.4 Using Commercial Math

Curriculum Outcomes	Outcome relevance
10-B3 Simple and Compound Interest: demonstrate an understanding • understand the long term difference between simple and compound interest • investigate both investments and financing situations	As adults, students will have to make many everyday life decisions that will require them to apply math skills related to investments and financing.

Pacing	Materials	Prerequisites
2 h	• Calculators	• compound interest

Main Points to be Raised

• Students must consider all the factors involved in a situation in order to make the best financial decision. For example, one bank might offer a slightly higher interest rate than another, but the compounding frequency also has to be considered.

• One way to compare investment choices is to describe each choice using equivalent per annum rates, which can easily be compared.

Try This — Introducing the Lesson

A. and B. Students can answer **parts A and B** alone or with a partner.

Observe while students work. You might ask:

• *Why might someone think that the investment at Bank A is better?* (More frequent compounding)

• *Why might someone think that the investment at Bank B is better?* (Higher interest rate)

The Exposition — Presenting the Main Ideas

• On the board, present the problem described in the box on **page 47**. Work through the problem with the class.

• Then work through the second problem, on **page 48**, with the students.

• Have them open their texts to see where the problems have been worked out so they know where to look for later reference.

Revisiting the Try This

C. This question allows students to summarize their understandings about the compound interest formula. It helps them focus on all the factors that are involved in making financial decisions relating to compound interest.

Using the Examples

• Assign half of the class, working in pairs, to read through **example 1**. They can work through the problem until they feel they could explain the problem to another student. Similarly, the other half of the class can in pairs to read through **example 2** in the same way. Before students begin, you may need to explain to them what a down payment means, i.e., that it is an initial amount you must pay in cash, even though the remainder is being financed.

• For each example, call on a pair of volunteers to explain it to the class. Encourage the class to raise questions. The volunteers should be prepared to answer their classmates' questions.

Practising and Applying

Teaching points and tips

Q 1: This problem could be solved using equivalent interest rates, as in **example 2**.

Q 2: Students might use different strategies to solve this. Some might simply try different rates in the compound interest formula using P = 60,000, A = 90,000, n = 12, and t = 10. You may wish to allow students to estimate the rate. The actual rate can be found by solving $90{,}000 = 60{,}000\left(1+\frac{r}{12}\right)^{120}$ for r (dividing by 60,000, raising both sides to the power $\frac{1}{120}$, and isolating r).

Q 3a and Q 8: Students may find it helpful to review **example 3** in **lesson 2.1.2**. Students could solve for the rate, r, in the equation $1.25 = (1 + r)^3$.

Q 5: Some students may be uncomfortable with this problem since the number of years the winner can expect to live is not given. Assure them that they can make their own reasonable assumptions to solve the problem.

Q 7: Students will have to think about whether to add the 1 Lakh before or after they calculate the 10%. Encourage them first to try whichever approach makes sense to them, but then to test it to make sure.

Q 9: This question forces students to show their understanding of the underlying concepts because no actual calculations are possible.

Common Errors

Students may improperly set up calculations with rates or not properly manipulate the numbers in equations such as those in **question 3**, **part a) or question 8**, for example.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can solve a reasonably straightforward problem requiring compound interest calculations
Question 3	to see if students recognize that they can solve problems involving the effects of various rates by choosing their own values to test
Question 7	to see if students can solve a two-step problem about commissions
Question 9	to see if students can reason about the effect of more frequent compounding

Answers

A. *Sample response*: No; the other option might be better because it has more frequent compounding and the interest rate is only slightly lower **B. i)** Bank A: Nu 40.02; Bank B: Nu 40; Bank A is Nu 0.02 more **ii)** Bank A: 480.47; Bank B: 480.24; Bank A is Nu 0.23 more **iii)** Bank A: 1191.78; Bank B: 1191.12; Bank A is Nu 0.66 more	**C. i)** Bank A (3.93% p.a. compounded monthly) is the better choice. **ii)** Interest rate, compounding frequency, and time period of the investment. These three factors play a role in determining the amount an investment will earn: a higher interest rate will earn more, more frequent compounding will earn more, and a longer investment period will earn more. However, higher interest with less frequent compounding may not earn more than a lower rate with more frequent compounding.
1. Option 3 is best and Option 2 is worst. Option 3 has an equivalent rate of 14.93%. Option 2 has an equivalent rate of 15.13%, and Option 1 has a rate of 15% since it is the ratio of 3: 20. A lower rate is better when you are borrowing, rather than earning interest. **2.** 4.1%	**3. a)** 7.73% (rounding would give 7.72% but the guarantee would not be met) **b)** *Sample response*: Yes; since Nu 100 increases by Nu 25 to Nu 125 in three years, it will increase by even more than Nu 25 in the next three years due to compounding. Therefore, the increase over six years would be more than 50%.

Answers [Continued]

4. a) 8% **b)** 16% **5. a)** *Sample response*: Nu 10,000 monthly since I expect to live for more than 8.3 years. **b)** *Sample response*: An elderly person or a very ill person may choose to collect the lump sum prize because they might not live another 8.3 years. Or, someone who needs the money right away might choose the lump sum. **6.** Nu 25,000,000 **7.** Nu 211,111.11 (or 2.11111 Lakhs)	**8. a)** 14.29% **b)** Nu 5625.35 **9.** *Sample response*: The option with a payment of Nu 12,000 at the end of the year has a lower interest rate because both options require payment of Nu 12,000 in total. That means the plan with the later payment is charging the lower interest rate.

Supporting Students

Struggling students

Most of the problems in this lesson are based on work in the previous lessons. Some students might benefit from looking back at the examples in those lessons.

Chapter 2 Radicals

2.2.1 EXPLORE: Representing Square Roots

Curriculum Outcomes	Lesson relevance
10-A1 Irrational Numbers: understand role • understand when to approximate and when to continue with radical expressions • convert between entire and mixed radicals	This essential lesson gives students a visual model to help explain how to simplify square root radicals. They will be able to refer to this model in later work.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6) or 10-by-10 grid (BLM in this unit)	• meaning and calculation of square roots • Pythagorean theorem

Main Points to be Raised

• You can use the Pythagorean theorem to determine the lengths of diagonal lines on a grid.

• By relating the lengths of certain diagonal lines on a grid, you can see how to represent those lengths in more than one way

• The lengths of some diagonals on a grid have a *rational number* of units, but the lengths of other diagonals have an *irrational number* of units.

Exploration

• Before inviting students to work on **parts A to D**, review the Pythagorean theorem using a 1-1-$\sqrt{2}$ triangle.

• Ask students to work with a partner or in a small group. Observe while students work. You might ask:

- *How did you decide which segment represented* $\sqrt{8}$ *?* (I chose the segment that looked the longest since $\sqrt{8}$ had to be the biggest number.)

- *How did you decide that* $\sqrt{8} = 2\sqrt{2}$ *?* ($\sqrt{8}$ was made up of two equal pieces and each piece measured $\sqrt{2}$.)

- *How did you make sure you found all the other possible lengths?* (First I drew diagonals where the points were one line apart, then two lines apart, then three lines apart, and so on, to make sure I did not forget any.)

Observe and Assess

As students are working, notice:

• Do they use the Pythagorean theorem correctly to determine lengths of diagonals?

• Are they using a systematic method to find all the possible line segments?

• Can they distinguish between rational and irrational numbers?

• Do they use visualization to relate different irrational numbers?

Share and Reflect

Ask students to share their responses to **part C**.

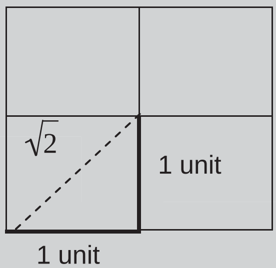
Answers

A. $\sqrt{2}$ is III; $\sqrt{5}$ is I; $\sqrt{8}$ is II

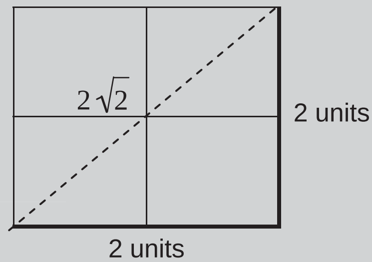
B. The hypotenuse of a right triangle with legs of 1 unit is $\sqrt{2}$ ($\sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$) so the hypotenuse of a right triangle with legs of 2 units is twice that, or $2\sqrt{2}$. When you use the Pythagorean theorem to calculate the length of the hypotenuse of the right triangle with 2 unit legs, you get $\sqrt{8} = \sqrt{2^2 + 2^2}$, so $\sqrt{8}$ must be equal to $2\sqrt{2}$.

Answers [Continued]

Right triangle with 1 unit legs



Right triangle with 2 unit legs



C. All possible lengths are shown below. The first row represents all possible vertical or horizontal lengths. The bottom of the grid is not completed since the values would duplicate the top portion, e.g., a triangle with legs measuring 1 unit and 2 units is the same as a triangle with legs measuring 2 units and 1 unit.

One leg

Other leg

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{17}$	$\sqrt{26}$	$\sqrt{37}$	$\sqrt{50}$ or $5\sqrt{2}$	$\sqrt{65}$	$\sqrt{82}$	$\sqrt{101}$
2		$2\sqrt{2}$	$\sqrt{13}$	$\sqrt{20}$ or $2\sqrt{5}$	$\sqrt{29}$	$\sqrt{40}$ or $2\sqrt{10}$	$\sqrt{53}$	$\sqrt{68}$ or $2\sqrt{17}$	$\sqrt{85}$	$\sqrt{104}$ or $2\sqrt{26}$
3			$3\sqrt{2}$	$\sqrt{25}$ or 5	$\sqrt{34}$	$\sqrt{45}$ or $3\sqrt{5}$	$\sqrt{58}$	$\sqrt{73}$	$\sqrt{90}$ or $3\sqrt{10}$	$\sqrt{109}$
4				$\sqrt{32}$ or $4\sqrt{2}$	$\sqrt{41}$	$\sqrt{52}$ or $2\sqrt{13}$	$\sqrt{65}$	$\sqrt{80}$ or $4\sqrt{5}$	$\sqrt{97}$	$\sqrt{116}$ or $2\sqrt{29}$
5					$\sqrt{50}$ or $5\sqrt{2}$	$\sqrt{61}$	$\sqrt{74}$	$\sqrt{89}$	$\sqrt{106}$	$\sqrt{125}$ or $5\sqrt{5}$
6						$\sqrt{72}$ or $6\sqrt{2}$	$\sqrt{85}$	$\sqrt{100}$ or 10	$\sqrt{117}$ or $3\sqrt{13}$	$\sqrt{136}$ or $2\sqrt{34}$
7							$\sqrt{98}$ or $7\sqrt{2}$	$\sqrt{113}$	$\sqrt{130}$	$\sqrt{149}$
8								$\sqrt{128}$ or $8\sqrt{2}$	$\sqrt{145}$	$\sqrt{164}$ or $2\sqrt{41}$
9									$\sqrt{162}$ or $9\sqrt{2}$	$\sqrt{181}$
10										$\sqrt{200}$ or $10\sqrt{2}$

D. i) Numbers 1 to 10; *Sample response*: All integers are rational numbers.

ii) All of the remaining lengths in the chart; *Sample response*: These lengths can not be expressed exactly without using root symbols.

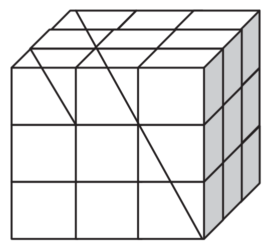
Supporting Students

Struggling students

You could limit the size of the grid to, say, 5 × 5 or 6 × 6 for students who are struggling.

Enrichment

Some students might think about an equivalent situation involving cube roots. They could consider the diagonals of a large cube made up of small cubes and compare the lengths of the diagonals of the various possible cubes they could create.



2.2.2 Simplifying Radicals

Curriculum Outcomes	Outcome relevance
10-A1 Irrational Numbers: understand role • develop awareness of errors in decimal approximations and rounding off • understand when to approximate and when to continue with radical expressions • convert between entire and mixed radicals **10-B1 Roots: apply properties to operations** • develop and apply properties for operations involving roots	Solving equations involving quadratics will become more meaningful once students have a better understanding of the relationships between various expressions involving radicals.

Pacing	Materials	Prerequisites
1 h	• Calculators • Rulers (optional)	• Pythagorean theorem • square roots

Main Points to be Raised

• *Irrational numbers* are decimals that cannot be written as repeating or terminating decimals. When you calculate the measurements of objects, you might be content with a decimal approximation, which may vary depending on the precision required. But if you want to express the exact length, you might need to use a radical expression. The term *radical* is used to describe a root (whether a square root, a cube root, or any other root).

• The square root of n can be written as $n^{\frac{1}{2}}$. Using the exponent principles, $n^{\frac{1}{2}} \times n^{\frac{1}{2}} = n^1 = n$. Similarly, other roots can be expressed using radical notation or fractional exponents. For example, the cube root of n can be written as either $\sqrt[3]{n}$ or $n^{\frac{1}{3}}$ and the fourth root as either $\sqrt[4]{n}$ or $n^{\frac{1}{4}}$.

• Radicals can often be written in another form that is seen to be simpler. It is called simplified form. For example, $\sqrt{90}$, an *entire radical*, can be simplified to $3\sqrt{10}$, called a *mixed radical* (since there is a whole number multiplier).

• Radicals are simplified using exponent principles. Since $(ab)^n = a^n \times b^n$, if a radical includes a perfect square (or cube or other power), the root of that perfect power can be brought out as a factor of the remaining factors in the number under the radical sign.

Try This — Introducing the Lesson

A. Allow students to try these alone or with a partner. Some may measure with a ruler, but others might use the Pythagorean theorem. Either approach is acceptable. Observe while students work. You might ask:

• *What is the length of the first hypotenuse? How do you know?* (It is $\sqrt{2}$ since the triangle has legs of 1 unit and 1 unit and I can use the Pythagorean theorem.)

• *How do you know that each hypotenuse is getting longer as you go up and to the right?* (Even though one leg is 1 unit each time, the other leg keeps getting longer – the first leg is $\sqrt{2}$, but the next leg is $\sqrt{3}$)

The Exposition — Presenting the Main Ideas

• Make sure students understand that $\sqrt{2}$ is irrational because it cannot be written as a fraction. At this point, most students will not be able to do a proof, but they can try squaring, for example, 1, then 1.4, then 1.41, then 1.414, and so on to see that they never get exactly 2.

• Point out that we are often content to estimate a measurement with an approximation, much like we estimate the value of π by using the fraction $\frac{22}{7}$, but that is not really the exact value. If we want to express the exact value, we need to use an expression that might involve an irrational number like π or $\sqrt{2}$. (In Unit 4, they will discuss to a greater extent the amount of precision and the appropriate approximation to describe a measurement.)

• Point out that 2 is the square root of 4 since $2^2 = 4$. Introduce the terms *cube root* and *fourth root* to discuss other kinds of roots. For example, 2 is the cube root of 8 since $2^3 = 8$ and 2 is the fourth root of 16 since $2^4 = 16$. Introduce the radical notation for cube roots and square roots, i.e., $\sqrt[3]{}$ and $\sqrt[4]{}$. Note that the square root does not require a number in front of the root symbol.

• Lead the students through the exposition to help them understand why $\sqrt[n]{x}$ is written as $x^{\frac{1}{n}}$.

• To introduce the terms *entire radical, mixed radical* and *simplification,* ask students to use their calculators to observe the decimal estimates for $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$, and $\sqrt{32}$. They should observe that $\sqrt{8} = 2\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}$, and $\sqrt{32} = 4\sqrt{2}$.

• Next, lead students through the last part of the exposition.

Revisiting the Try This

B. For students who measured in **part A** rather than using the Pythagorean theorem, encourage them to write the lengths first as entire radicals and then as mixed radicals. Those who used the Pythagorean theorem for **part A** only need to change what they did earlier to mixed radicals.

B. and C. These questions are intended to make the connection to simplification, discussed in the exposition. If students completed **lesson 2.2.1**, relate what happens in these questions to what they saw in that lesson.

Using the Examples

Ask students to try to simplify the radicals in the **example** before reading through the solution. They can then check their work against the solution.

Practising and Applying

Teaching points and tips

Q 1: Many students might simplify $\sqrt{1000}$ to 10; suggest that they check their answer by squaring it.

Q 3: Some students might simplify $\frac{\sqrt{45}}{\sqrt{5}}$ first to $\sqrt{\frac{45}{5}}$ and then realize this is just $\sqrt{9}$, which is 3. Others will do it differently, as $\frac{3\sqrt{5}}{\sqrt{5}}$, which is also 3.

Q 4: Some students will "unsimplify" the first expression, whereas others will try to simplify the second expression. The latter makes the problem somewhat more difficult to solve, as 920 does not readily break down into factors that are squares.

Q 5 and Q 6: Both of these questions require reasoning. Students will need to consider what the situation requires. For example, in **question 5**, **part a)** they must realize they need a power that is even but not a multiple of 3. For **question 6**, they need to consider the relationship between square roots and fourth roots.

Q 8: This question relates back to the visual idea in optional **lesson 2.2.1**, where a mixed radical is related geometrically to an entire radical.

Q 9: Students may wish to use a numerical example to help them through the explanation.

Common Errors

Sometimes when students simplify radicals, they use the power rather than the root as the multiple in the mixed radical. For example, to simplify $\sqrt{20}$, they might write $4\sqrt{5}$ instead of $2\sqrt{5}$. Get students to estimate values to check their work. For example, if they know $\sqrt{20}$ is between 4 and 5 and that $\sqrt{5}$ is a bit more than 2, then they will see that $4\sqrt{5}$ would be more than 8, which is too much.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can simplify radicals
Question 4	to see if students can interpret and compare mixed and entire radicals
Question 8	to see if students can interpret radical simplification geometrically
Question 9	to see if students can communicate about manipulating radicals

Answers

A. The seventh is twice the length of the first.

B. i)

1st triangle	2nd triangle	3rd triangle	4th triangle
$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$ or 2	$\sqrt{5}$
5th triangle	**6th triangle**	**7th triangle**	
$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$ or $2\sqrt{2}$	

ii) The first hypotenuse length is $\sqrt{2}$ and the seventh is $2\sqrt{2}$. $2\sqrt{2}$ is twice $\sqrt{2}$.

C. The 31st hypotenuse; the seventh hypotenuse is $2\sqrt{2}$ and the length that is twice that is $4\sqrt{2}$ which is $\sqrt{16} \times \sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}$.
The hypotenuse of the 31st triangle has a length of $\sqrt{32}$ because the length of the hypotenuse is the square root of 1 more than the triangle number.

1. a) $4\sqrt{3}$ **b)** $10\sqrt{10}$ **c)** $2\sqrt[3]{4}$ **d)** $\sqrt{110}$

2. $9\sqrt{2}$, 11, $3\sqrt{13}$, $4\sqrt{7}$, $6\sqrt{3}$;
$3\sqrt{13} = \sqrt{117}$; $4\sqrt{7} = \sqrt{112}$; $9\sqrt{2} = \sqrt{162}$; $11 = \sqrt{121}$;
$6\sqrt{3} = \sqrt{108}$

3. a) $\sqrt{18} = 3\sqrt{2}$; $\sqrt{45} = 3\sqrt{5}$ **b)** 6; 3

4. $\sqrt{920}$ because $13\sqrt{5}$ is the same as $\sqrt{13^2 \times 5}$, which is $\sqrt{169 \times 5}$, or only $\sqrt{845}$. Since 845 < 920, $\sqrt{845}$ is less than $\sqrt{920}$

5. a) *Sample response*: 4
b) This is impossible because for $\sqrt[3]{n}$ to be a whole number, n must be a perfect cube with three identical factors. But $\sqrt{n}$ is an entire radical only if no pair of identical factors can be found.
c) *Sample response*: 27

6. **a)** No; *Sample response*:
$\sqrt[4]{n}$ is a whole number only if n is a fourth power ($a \times a \times a \times a$). However, any fourth power is a perfect square ($a^2 \times a^2$). For example, $\sqrt[4]{16} = \sqrt[4]{2 \times 2 \times 2 \times 2} = 2$ and $\sqrt{16} = \sqrt{2 \times 2 \times 2 \times 2} = \sqrt{4 \times 4} = 4$.
b) Yes; *Sample response*: $\sqrt[4]{9} = 1.732...$ and $\sqrt{9} = 3$.

7. a) None because the 11^3 part is not a perfect square.

7. b) Any non-negative multiple of 3 such as 0, 3, 6, 9, ... since the cube root would be $3^2 \times 11 \times 5^{3m}$ where m is one-third of the multiple of 3. The cube root means raising each power to $\frac{1}{3}$ its value ($\frac{1}{3}$ of the multiple of 3 is an integer).

8. The diagonal in the top left square is $\sqrt{2}$ units long and the full diagonal is 3 times as long, or $3\sqrt{2}$. Using the Pythagorean theorem to find the length of the full diagonal, it is $\sqrt{3^2 + 3^2} = \sqrt{18}$. That means $3\sqrt{2} = \sqrt{18}$.

9. a) *Sample response*:
Factor n to include as many pairs of identical factors as possible. For each identical pair under the square root symbol, create a term outside the square root symbol that is equal to the square root of the product of the identical factors;
$\sqrt{56} = \sqrt{2 \times 2 \times 2 \times 7} = \sqrt{2 \times 2} \times \sqrt{2 \times 7} = \sqrt{4} \times \sqrt{2 \times 7} = 2\sqrt{14}$. If no terms are left inside the root symbol, the square root symbol can be removed, e.g., $\sqrt{324} = \sqrt{9 \times 9 \times 2 \times 2} = \sqrt{9 \times 9} \times \sqrt{2 \times 2} = 9 \times 2 = 18$.
b) *Sample response*:
The term outside the radical must be moved inside the radical. This is done by multiplying the value inside the radical by the square of the term outside;
$m\sqrt{n} = \sqrt{m \times m} \times \sqrt{n} = \sqrt{m \times m \times n}$ or $\sqrt{m^2 \times n}$.

Supporting Students

Struggling students

Students who are struggling might be encouraged to write out each number under the radical sign (in an entire radical) in fully factored form, e.g., $45 = 3^2 \times 5$ or $225 = 3^2 \times 5^2$. In this way it becomes easier to see whether the power is a multiple of the type of root being considered.

Enrichment

Archimedes' spiral is a famous figure. Students who have Internet access might wish to learn more about it.

2.2.3 Operations with Radicals

Curriculum Outcomes	Outcome relevance
10-A1 Irrational Numbers: understand role • convert between entire and mixed radicals **10-B1 Roots: apply properties to operations** • develop and apply properties for operations involving roots **10-C15 Equations: apply properties of numbers upon expressions** • solve equations by applying associative, distributive, identity and inverse properties	In higher mathematics, students will often be required to multiply, divide, add, and subtract entire and mixed radicals.

Pacing	Materials	Prerequisites
1.5 h	• Calculators (optional)	• simplifying radicals

Main Points to be Raised

• The same properties that apply to operations with rational numbers apply to operations with radicals. These include the commutative, distributive, and associative properties, as well as the conventions for order of operations.

• The distributive principle allows one to add or subtract like radicals.

• Unlike radicals, or radicals that cannot be made alike, cannot be added or subtracted, but they can be multiplied or divided.

Try This — Introducing the Lesson

A. Allow students to try these alone or with a partner. Observe while students work. Encourage them not to use a ruler, but instead to use the Pythagorean theorem. You might ask:

• *How do you know that the first hypotenuse is greater than* $4\sqrt{5}$ *?* (The hypotenuse is always longer than either of the other sides.)

• *What is the square of* $5\sqrt{3}$ *? How do you know?* (75. I squared 5 to get 25 to multiply it by the square of $\sqrt{3}$, which is 3.)

• *Does the fact that* $5\sqrt{3} > 4$ *and* $5\sqrt{3} > 4\sqrt{5}$ *tell you which hypotenuse will be longer?* (No. You have to use the lengths of both sides to determine the lengths of the hypotenuse.)

The Exposition — Presenting the Main Ideas

• Pose the question: *What do you think is the sum of* $3\sqrt{2}$ *and* $2\sqrt{2}$ *? Why?*

• See if students come up with $5\sqrt{2}$ without assistance — many will. Talk about how three of anything and two more of that same thing are five of that thing. In this case, the thing is $\sqrt{2}$.

• Then ask about the sum of $\sqrt{2}$ and $\sqrt{8}$. Some students are likely to recall that $\sqrt{8}$ is $2\sqrt{2}$, so if you add it to another $\sqrt{2}$, the sum is $3\sqrt{2}$. Point out how simplifying $\sqrt{8}$ made it possible to add the radicals because the radicals became *like terms*. Make sure students understand the meaning of *like terms* by contrasting like terms with unlike terms. Show how unlike terms, e.g., $\sqrt{2}$ and $\sqrt{3}$, cannot be combined into a single radical.

• Let students read **page 56** and answer any questions they might have.

• Next, write on the board the calculation $\sqrt{3} \times \sqrt{27}$. Ask students if it can be simplified. Some might use a calculator and notice that the product is 9, whereas others might write it as $\sqrt{3} \times 3\sqrt{3} = 3 \times 3 = 9$.

• Ask students to continue reading through the exposition on **page 57**. Ask different students to explain each of the examples within the exposition.

Revisiting the Try This

B. This question allows an opportunity to make a connection between the work on operations with radicals and the work done in **part A.**

Using the Examples

Put students in pairs. One person can work through **parts a) and c)** in **examples 1 and 2** and the partner can work through **parts b) and d)**. When they finish, they can share their thinking with each other. The pair can work together on the remaining parts of **example 2** and all of **example 3**. It may be advisable to do some of **example 3** with the class as a whole before expecting students to complete the questions. To test understanding, ask all students to try these questions: $8\sqrt{15} - 3\sqrt{60}$ and $12\sqrt{32} \div 2\sqrt{18}$.

Practising and Applying

Teaching points and tips

Q 1: Some students may need a reminder to combine like terms after simplifying.

Q 3: To simplify expressions as in **part a)**, students could separately simplify numerator and denominator (or dividend and divisor) and then simplify the whole fraction, or they could first divide 48 by 12 and then take the square root.

Q 4: Students need to apply the distributive principle twice to simplify these expressions. You can model this by showing how

$(30 + 2)(40 + 8) = 30 \times 40 + 2 \times 40 + 8 \times 30 + 2 \times 8.$

Q 7: This question encourages students to form a conjecture and test it. This is an important mathematical activity.

Q 8: Without formal consideration of rationalizing denominators, students get some initial experience with this notion. It will help them understand later why $\sin 45°$ can be written as either $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$.

Common Errors

Many students add radicals by simply adding the numbers below the radical signs, e.g., $\frac{17}{72} \times 100 + \sqrt{5} = \sqrt{10}$.

Encourage students to test some of their results using a calculator or mental estimation.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can add and subtract radicals
Question 3	to see if students can multiply and divide radicals
Question 6	to see if students can solve simple equations involving operations with radicals
Question 9	to see if students can communicate about operating with radicals

Answers

A. Triangle B has the longer hypotenuse;

Triangle A:

$h^2 = 4^2 + (4\sqrt{5})^2 \rightarrow h^2 = 16 + 16 \times 5$

$\rightarrow h^2 = 16 + 80 \rightarrow h^2 = 96 \rightarrow h = 9.8...$

Triangle B:

$h^2 = (5\sqrt{3})^2 + (5\sqrt{3})^2 \rightarrow h^2 = 25 \times 3 + 25 \times 3$

$\rightarrow h^2 = 150 \rightarrow h^2 = 150 \rightarrow h = 12.25\ldots$

The hypotenuse of Triangle B is longer than the hypotenuse of Triangle A by about 2.45 units.

B. Triangle A:

$h^2 = 4^2 + (4\sqrt{5})^2 \rightarrow h^2 = 16 + 80 \rightarrow h^2 = 96 \rightarrow h = 4\sqrt{6}$

Triangle B:

$h^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 \rightarrow h^2 \rightarrow h^2 = 150 \rightarrow h = 5\sqrt{6}$

The hypotenuse of Triangle B is longer than the hypotenuse of Triangle A by exactly $5\sqrt{6} - 4\sqrt{6} = \sqrt{6}$ units.

1. a) $6\sqrt{3}$ **b)** $(x-7)\sqrt{x}$ **c)** $10\sqrt{k}$ **d)** $2\sqrt{11}+\sqrt{55}-11$ **2. a)** 6 **b)** $2x^4\sqrt{5}$ **c)** $2\sqrt{21}$ **3. a)** 2 **b)** $\frac{3}{\sqrt{2}}$ **c)** $3x^2$ **d)** $\frac{2\sqrt{10}}{\sqrt{3}}$ **4. a)** $12\sqrt{5} - 6\sqrt{55} - 4\sqrt{11} - 22$ **b)** $13-9x$ **5.** *Sample response*: **a)** $\sqrt{7}$ **b**) $4\sqrt{5}$ **c)** $3\sqrt{14}$ **d)** 6 or 8 **6. a)** $m=5$ **b)** $p=6$ **c)** $k=19$ **d)** $s=128$	**7. a)** $\sqrt{6}+\sqrt{4}$ is the greatest. **b)** The expression with the numbers that were closest together was the greatest. **c)** $\sqrt{11}+\sqrt{8}$ will be greatest because 11 and 8 are closer together than 15 and 4 or 17 and 2. **d)** $\sqrt{11}+\sqrt{8}$ **8. a)** $\frac{3\sqrt{11}}{11}$ **b)** Multiplying $\frac{3}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}$ is the same as multiplying $\frac{3}{\sqrt{11}} \times 1$. **c)** $\frac{\sqrt{13}}{\sqrt{13}}$ **9.** You should advise them to look for ways to simplify terms and combine like terms, keeping in mind that $\sqrt{20} = 2\sqrt{5}$ and $\sqrt{x^3} = x\sqrt{x}$.

Supporting Students

Struggling students

Many students will have more difficulty working with variables than with numbers. You might provide additional practice with numbers before assigning questions involving variables. The results using variables may be checked by substituting simple numerical values.

Enrichment

Some students might create a variety of expressions involving operations with radicals with a particular result, for example, ten different radical expressions all resulting in $3\sqrt{6}$, but using different radicals each time. Two examples are $\sqrt{6} + 2\sqrt{6}$ and $9\sqrt{2} \times \sqrt{3} \div 3$.

GAME: Five Radicals

This game allows students to practise operations with radicals in a non-traditional way.

UNIT 2 Revision

Pacing	Materials
2 h	• Calculators

Question(s)	Related Lesson(s)
1	Lesson 2.1.1
2 – 4	Lesson 2.1.2
5, 6	Lesson 2.1.3
7 – 11	Lesson 2.1.4
12, 13, 15	Lesson 2.2.2
14, 16	Lesson 2.2.3

Revision Tips

Q 2b: Observe whether students perform a calculation or whether they realize immediately that simple interest and compound interest are the same for the first compounding period.

Q 3: Students might try this by guess and test or they might solve the equation $(1 + \frac{r}{12})^{12} = 1.115$ more formally by taking the twelfth root of both sides and then solving for *r*.

Q 4: You may discuss why the amounts of the investments do not affect the rates (answers).

Q 5: Watch to make sure that students do not think about the stock price as Nu 28, but rather as Nu 72. You may have to remind students that the yield percentage in **part c)** is based on the cost paid, and not the face value of the owned stock.

Q 8: Again, some students might guess and test rates, whereas others might solve the equation: $22{,}000(1 + \frac{r}{4}) - 3000 = 19{,}770$. The latter method of setting up and solving the equation should be reviewed, as it works in all such situations. Note that some students may identify the amount of interest as Nu 770 and divide by Nu 22,000 to get 0.035. Remind them that this is per quarter and that is why the rate is 14% p.a. compounded quarterly.

Q 9: You may need to remind students of what a down payment means.

Q 12: Make sure students attend to which root is being taken. Encourage them to first write down the factored form of the number under the radical sign if they are struggling.

Q 16d: The missing value is for *m*, not *x*.

Answers

1. a) Nu 160 **b)** 25%
c) The item listed at Nu 300 with a 25% discount costs less. (300 × 0.75 = Nu 225 and 280 × 0.85 = Nu 238)

2. a) Nu 1575 **b)** Nu 1575
c) No; the compounded rate is better because the interest from the first year will earn additional interest above the basic Nu 75 in the second year.

3. 10.93% p.a. compounded monthly

4. a) Simple: 4.42% Compounded annually: 4.07%
b) Simple: 4.52% Compounded annually: 3.80%
c) Simple: 5.68% Compounded annually: 4.60%
d) Simple: 4.71% Compounded annually: 3.94%

5. a) 347 shares **b)** Nu 5899
c) 23.61% (which is $\frac{17}{72}$)

6. a) 158.82% **b)** Nu 3400

7. a) 63.64%
b) I would choose Option 3. Option 2, 8% of 70,000, is Nu 5,600, and Option 3, 3.5% of 180,000 is Nu 6,300. Both of these options exceed Option 1.

8. 14% p.a. compounded quarterly

9. Nu 182,560.01
Amount of loan = 0.75 × 250,000 = Nu 187,500
Still owing after first payment: Nu 185,296.88
Still owing after second payment: Nu 183,072.64
Still owing after third payment: Nu 180,827.08
Due at the end of the four month: Nu 182,560.01

10. $\frac{21}{50}$ (Amount of loan = 280,000 × 0.75 = Nu 210,000, Interest in first month = 210,000 × 0.01 = Nu 2100, and $\frac{2100}{5000}=\frac{21}{50}$) **11.** 15% compounded semi-annually; it would be equivalent to an annual interest rate of 15.56% instead of an annual rate of 15.79% in the first year. **12. A** and **B**. • The expression in **A** simplifies to 0. • In **B**, all of the terms in the numerator simplify to something of the form $n\sqrt{2}$, and because the denominator is $\sqrt{2}$ the expression will have a value of n. • I did not choose **C** because although 49 and 64 are perfect squares, 108 is not, so the product will not simplify to an integer. **13. a)** $3\sqrt{3}$ **b)** $10\sqrt{3}$ **c)** $5\sqrt[3]{2}$ **d)** $\sqrt{30}$	**14. a)** $9\sqrt{2}-\sqrt{7}$ **b)** $4\sqrt{3}+10\sqrt{6}-3\sqrt{2}-15$ **c)** 2 **d)** $\frac{4}{3}$ **e)** $3x^2$ **15.** *Sample response*: • 2007^6 is a perfect cube because 2007^6 = (2007 × 2007) × (2007 × 2007) × (2007 × 2007) and a perfect square because 2007^6 = (2007 × 2007 × 2007) × (2007 × 2007 × 2007). • 6^{2007} is a perfect cube because 2007 is a multiple of 3 (using the divisibility rule of a digit sum of 9) which means three identical pairs of factors could be created. However, 6^{2007} is not a perfect square because 2007 is not a multiple of 2, which means it would be impossible to create two identical sets of factors. **16. a)** $k=4$ **b)** $p=3$ **c)** $n=28$ **d)** $x=15$

Supporting Students

Struggling students

Encourage students who are struggling to go back to the expositions and examples for the relevant lesson to review the items they are having difficulty with.

UNIT 2 Commercial Math and Number Test

1. An item at one shop is sold at a discount of 30%, and its selling price is the same as the selling price of another item that has been marked up by 40%.

a) What is the ratio of the marked price of the first item to the cost price of the second item?

b) Why can you not be sure what the two selling prices are?

2. You invest Nu 1000 in each account.

Account A: 6% p.a. simple interest

Account B: 5% p.a. compounded annually

Account C: 4.75% p.a. compounded quarterly

Account D: 4.5% p.a. compounded monthly

a) How much interest will you earn in each account by the end of

i) one year? **ii)** two years?

b) Some of the answers to **part a) ii)** are exactly double the answers to **part a) i)** and some are not. Why is that?

3. You invested Nu 2500 in an account. After four years, the amount of money had grown to Nu 2903.91. What was the annual interest rate compounded monthly?

4. Yuden paid Nu 116 per share to buy some stock with a face value of Nu 100 per share.

a) If she bought as many shares as she could for Nu 2500 and received a dividend of 23%, how much did she earn in dividends?

b) What is her yield percentage?

5. Arjun bought some stock at a premium and sold it at a discount. Bijoy bought the same stock at a discount and sold it at a premium. Why can you not be sure who made more money, in total, from that stock?

6. A salesperson is offered the choice of earning a 4% commission on sales each month or a 2% commission with an additional guaranteed monthly income of Nu 500. How much would he have to sell each month to earn more from the 4% commission choice?

7. You borrow Nu 30,000 to purchase a new computer. The bank charges 13.25% interest, compounded semi-annually. You pay back the loan by making payments of Nu 5000 every six months.

a) How much will you still owe after the first payment?

b) How much interest will you pay in the first year of the loan?

8. Draw a picture that shows $\sqrt{20} = 2\sqrt{5}$. Describe how the picture shows the idea.

9. k and $\sqrt[6]{k}$ are both whole numbers. State whether each claim below is true and explain your thinking.

a) k is a perfect square

b) $\sqrt[3]{k}$ is a rational number

c) $\sqrt[4]{k}$ is a whole number

10. Solve each equation for m.

a) $5\sqrt{2} \times 4\sqrt{2} + \sqrt{50} - 40 = m\sqrt{2}$

b) $\dfrac{7\sqrt{27} - 4\sqrt{12}}{\sqrt{m}} = 3$

c) $\dfrac{\sqrt{20}}{5}\left(\sqrt{45} - \sqrt{m}\right) = 5$

11. Simplify the following expressions.

a) $\dfrac{\sqrt{5x^3} \times \sqrt{9x^5}}{\sqrt{80x}}$ **b)** $\dfrac{\sqrt{5x^3} + 4\sqrt{9x^5}}{3\sqrt{80x}}$

12. The expression $\sqrt{3} \times \sqrt{12} + \dfrac{\sqrt{8}}{\sqrt{2}} = 8$ represents 8 using exactly one of each of these radicals: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{8}$, and $\sqrt{12}$. Create another expression using each radical exactly once to represent an integer of your choice.

UNIT 2 Test

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1	Lesson 2.1.1
2, 3	Lesson 2.1.2
4, 5	Lesson 2.1.3
6, 7	Lesson 2.1.4
8, 9	Lessons 2.2.1 and 2.2.2
9	Lesson 2.2.2
10 – 12	Lesson 2.2.3

Select questions to assign according to the time available.

Answers

1. a) 2:1
b) It can be any pair of values in the ratio of 2:1 because no prices, selling or marked, are given in the question.

2. a) i)
Account A: Nu 60.00
Account B: Nu 50.00
Account C: Nu 48.35
Account D: Nu 45.94
ii)
Account A: Nu 120.00
Account B: Nu 102.50
Account C: Nu 99.04
Account D: Nu 93.99

b) The simple interest amount is exactly twice as much for two years as for one year because simple interest is applied to the same principal, Nu 1000, each year.
The other interests are compounded, so after the first compounding period the interest is applied to a slightly greater principal.

3. 3.75% p.a. compounded monthly

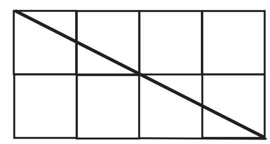
4. a) 21 shares **b)** Nu 483 **c)** 19.83%

5. Nothing is known about the dividend income.
Sample response:
Even though Arjun made less profit from selling his purchased stock, he might have made sufficiently more money in dividends if he held the stock during a period when the dividend rate was higher.

6. Nu 25,000 monthly (or slightly more)

7. a) Nu 26,987.50 **b)** Nu 3775.42

8.



The diagonal is $\sqrt{20}$ units long, since it is the hypotenuse of a right triangle with legs 2 units and 4 units [$2^2 + 4^2 = h^2 \rightarrow 20 = h^2 \rightarrow h = \sqrt{20}$].
Half the diagonal is $\sqrt{5}$ units long, since it is the hypotenuse of a right triangle with legs 1 unit and 2 units [$1^2 + 2^2 = h^2 \rightarrow 5 = h^2 \rightarrow h = \sqrt{5}$].
That means $\sqrt{5}$ is half of $\sqrt{20}$ and $\sqrt{20}$ is twice $\sqrt{20}$, or $2\sqrt{5}$.

9. a) True; k can be written as n^6, which is $(n^3)^2$
b) True; k can be written as n^6, which is $(n^2)^3$.
That means the $\sqrt[3]{k}$ is n^2, which is a whole number and is therefore rational
c) Not necessarily true; if $\sqrt[6]{k} = 2$, then $k = 2^6$, or 64. Since $2^4 = 16$ and $3^4 = 81$, there is no whole number fourth root for 64.

10. a) 5 **b)** 13 **c)** 4

11. a) $\dfrac{3x^2}{4}$ **b)** $\dfrac{5x + 6\sqrt{5}x^2}{12}$

12. *Sample responses*:
$\dfrac{\sqrt{5}\times\sqrt{18}}{\sqrt{10}} + \dfrac{\sqrt{32}}{\sqrt{8}} = 5$, $\dfrac{\sqrt{5}\times\sqrt{32}}{\sqrt{10}} + \sqrt{18}\times\sqrt{8} = 16$,
$\sqrt{18}\times\sqrt{32} + \sqrt{5}\times\sqrt{8}\times\sqrt{10} = 44$,
$\sqrt{18} + \sqrt{8} - \sqrt{32} - \dfrac{\sqrt{10}}{\sqrt{5}} = 0$

UNIT 2 Performance Task — Investing Wisely

Suppose that when you were born, your relatives had been able to invest Nu 10,000 in an interest-earning account or in stocks. The money would have grown in value until you finished school at age 18.

1. a) Would it be possible to invest Nu 10,000 in an interest-bearing account earning 6% p.a. compounded annually so that it would grow to a value of Nu 40,000 in 18 years? Explain.

b) Would more frequent compounding allow the investment to grow to Nu 40,000? How do you know?

c) What interest rate would cause the investment to grow to Nu 40,000? Show your work. Find at least one more answer.

d) Why is there more than one answer to **part c)**?

2. Suppose Nu 10,000 was invested in a stock that yielded an annual dividend of 8.33% for 18 years. The stock had a face value of Nu 100 and was purchased at par. The stock was sold at Nu 150 per share after 18 years. The dividend earnings were not reinvested each year.

a) Would the value of the investment reach Nu 40,000 in 18 years? Explain.

b) If the answer to **part a)** is no, determine a selling price for the stock that would result in a total investment value of Nu 40,000. Show your work.

3. a) Would you choose the investment option described in **question 1 a)** or the option in **question 2 a)**? Why?

b) Why might someone else choose the option you did not choose?

UNIT 2 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-A1 Irrational Numbers: understand role **10-B3** Simple and Compound Interest: demonstrate an understanding	1 h	• Calculators

How to Use This Performance Task

Ask students to read the task and the assessment rubric. Ask if they have any questions before allowing them the time they need to perform the task.

Assign the performance task. If you are using this task for the purposes of recording a mark, students should work alone. Otherwise, they might work with a partner.

Sample Solution

1. a) No; 6% p.a. compounded annually would grow to only Nu 28,543.39.
$10{,}000(1 + 0.06)^{18} = 28{,}543.39$

b) No; even daily compounding results in only Nu 29,444.18.

$10{,}000(1 + \frac{0.06}{365})^{365 \times 18} = 29{,}444.18$

c) *Sample response*:

7.73% p.a. compounded monthly; $10{,}000(1 + \frac{r}{12})^{12 \times 18} = 40{,}000$

$$\begin{aligned} (1 + \tfrac{r}{12})^{12 \times 18} &= 4 \\ 1 + \tfrac{r}{12} &= 4^{\frac{1}{216}} \\ 1 + \tfrac{r}{12} &= 1.00644 \\ \tfrac{r}{12} &= 0.00644 \\ r &= 0.0773\text{, which is } 7.73\% \end{aligned}$$

8.01% p.a. compounded annually would also work; $10{,}000\ (1 + r)^{18} = 40{,}000$

$$\begin{aligned} (1 + r)^{18} &= 4 \\ 1 + r &= 4^{\frac{1}{18}} \\ 1 + r &= 1.08006 \\ r &= 0.08006 \end{aligned}$$

d) There is more than one answer because there are two factors that affect the result — the compounding frequency and the interest rate. More frequent compounding requires a lower interest rate; less frequent compounding requires a higher interest rate.

2. a) No;
- The yield is the ratio of earnings to amount invested. So if the yield is 8.33% annually, that means they receive 8.33% of their investment as extra earnings each year. Assuming they did not reinvest their earnings, over 18 years that would be 18 × 0.0833 × 10,000 = Nu 14,994.
- A total of 100 shares were purchased at Nu 100 each. Since they sold the stock at Nu 150 per share, the sale of the stock would be worth 100 × Nu 150 = Nu 15,000.
- The total result would be Nu 29,994 (14,994 + 15,000 = 29,994).

b) The 100 shares of stocks would have to have sold for Nu 25,006 (40,000 – 14,994 = 25,006). So, each share would have to have sold at a price of Nu 250.06.

3. a) *Sample response*:
I would have taken the stock option because it results in more money even though it is riskier. The dividends could have been reinvested, which would have resulted in even more money.

b) *Sample response*:
An investment in a bank account is less risky and is predictable.

UNIT 2 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Calculations of dividend and interest amounts	Performs all interest and dividend calculations accurately and efficiently, using appropriate values.	Performs most interest and dividend calculations reasonably accurately, using appropriate values.	Performs some interest and dividend calculations reasonably accurately but makes some inappropriate assumptions.	Rarely performs interest and dividend calculations correctly and makes inappropriate assumptions.
Inferring to draw conclusions	Uses calculated values to draw reasonable conclusions, communicates them clearly, acknowledges the uncertainty of the situation, and offers other reasonable possibilities	Uses calculated values to draw reasonable conclusions, communicates reasonably clearly, and either acknowledges the uncertainty of the situation or offers other reasonable possibilities.	Uses calculated values to draw reasonable conclusions and communicates them to a reasonable extent, but neither acknowledges the uncertainty of the situation nor offers any further information.	Does not draw reasonable conclusions from calculated values, has difficulty communicating mathematical thinking, and does not acknowledge the uncertainty of the situation or offer any further information.

Unit 2 Assessment Interview

You may want to take the opportunity to interview selected students to assess their understanding of the work. The results can be used as formative assessment or, if you wish, as a piece of summative assessment data. As each student works, ask him or her to explain his or her thinking.

Ask the student to respond to at least two or three questions in each part.

- Have a calculator available for **Part I**.
- Provide cards with the following numbers written on them for **Part 2**: $\sqrt{2}$, $\sqrt{20}$, $\sqrt{5}$, $\sqrt{8}$

Part 1

- *Why would you choose to invest your money in a bank that compounded interest daily rather than monthly?*
(You get interest on interest much more often.)

- *How would you decide whether or not a rate of 4.25% compounded daily was better than a rate of 4.35% compounded monthly? Show me your steps.*

(I would use the formula $A = P(1 + \frac{r}{n})^{nt}$. twice. I'd use $P = 1$, $\frac{r}{n} = \frac{0.0425}{365}$, and $nt = 365$ the first time and then $P = 1$, $\frac{r}{n} = \frac{0.0435}{12}$, and $nt = 12$ the second time. I'd just see which was greater.)

- *If I had a lot of money to invest, would you advise me to invest in stocks or in a bank account? Tell me your reasons and try to convince me that you are right.*
(If I was fairly certain the stocks would not lose a lot of value, I would invest in stocks since dividends are usually greater than interest on a bank account. But, it's possible I could lose all my money with the stocks, so might pick the bank account to be safer.)

- *Suppose I borrow Nu 35,000 from the bank at a rate of 12.5% compounded semi-annually and I make payments of Nu 5000 every six months. Why will it take me more than seven payments to pay it off? Show me how you would figure out how much I still owe after my first payment. Then explain to me how you would find out how much I would owe after two years.*
(Seven payments would pay back the principal, but it wouldn't pay back the interest. To figure out the balance after the first payment, I would multiply 35,000 by 1.0625 (to also include the interest for six months) and then I'd subtract the Nu 5000 I paid.)

Part II

Display the four cards you made.

- *Which two values could be turned into expressions with like terms? Is there more than one possibility?*
($\sqrt{2}$ and $\sqrt{8}$ as $\sqrt{2}$ and $2\sqrt{2}$; or $\sqrt{20}$ and $\sqrt{5}$ as $2\sqrt{5}$ and $\sqrt{5}$)

- *If you added those two, what value would you get?*
($\sqrt{2} + \sqrt{8} = 3\sqrt{2}$ and $\sqrt{20} + \sqrt{5} = 3\sqrt{5}$).

- *If you multiply one of the values by 25, the result is a little over 100. Which value is it? How do you know?*
(It would be $\sqrt{20}$ since that's the only one that has a value that is a little more than 4.)

- *If you multiply two of the values, you can get 10. Which two values would you use?*
($\sqrt{20}$ and $\sqrt{5}$ since you get the square root of 100 when you multiply them.)

- *How could you combine all of these values using addition, subtraction, multiplication, and/or division to get a result of 2? Each value must be used once and no other values can be used.* ($\sqrt{2} \times \sqrt{8} - \sqrt{20} \div \sqrt{5} = 2$)

- *What is another integer value that you could get by combining all of these values using addition, subtraction, multiplication, and/or division? Explain how you got that integer value.* ($\sqrt{2} \times \sqrt{8} \times \sqrt{20} \times \sqrt{5} = 40$)

UNIT 2 Blackline Master

10-by-10 Grid

UNIT 3 LINEAR FUNCTIONS AND RELATIONS

UNIT 3 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 65; TG p. 73**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Grid paper	All questions
Chapter 1 Linear Functions and Relations				
3.1.1 Linear Functions **SB p. 67** **TG p. 76**	**10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C17 Functional Relationships and Notation: explore formally and informally** • understand the relationship between a relation and a function • start with functional relationships then apply the mathematical concept of function • use mathematical notation and vocabulary	1 h	• Grid paper	Q1, 2, and 4
3.1.2 Applications of Linear Functions **SB p. 72** **TG p. 79**	**10-C2 Patterns and Real-world Relationships: describe, identify, and apply** • use graphs, tables of values, and written descriptions to describe patterns and relationships • identify patterns in graphs and/or tables of values **10-C7 Graphs: create by constructing a table of values and graphing** • understand when to choose to graph by the y-intercept slope method **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C17 Functional Relationships and Notation: explore formally and informally** • understand the relationship between a relation and a function • start with functional relationships then apply the mathematical concept of function • use mathematical notation and vocabulary	2 h	None	Q1, 6, 8, and 9
3.1.3 Graphs of Linear Inequalities **SB p. 78** **TG p. 83**	**10-A2 Inequalities: relate sets of numbers to solutions** • solve problems with a restricted solution set • include choice of number systems **10-C3 Linear Inequalities: write and describe graphs** • describe a given graph using inequalities **10-C6 Graphs: sketch** • create graphs given information in a variety of formats [Continued]	1.5 h	• Grid paper • Rulers	Q4, 5, 8 and 9

UNIT 3 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 1 Linear Functions and Relations [Continued]				
3.1.3 Graphs of Linear Inequalities [Continued]	**10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C11 Equations and Inequalities: graph and analyse graphs** • graph equations and inequalities by constructing a table of values, using knowledge of transformations, and by identifying characteristics **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations			
3.1.4 EXPLORE: Transforming Graphs of Linear Functions (Essential) **SB p. 84** **TG p. 87**	**10-C1 Transformations: express algebraically or using mapping rules** • express transformations from a graph using algebraic expressions or mapping rules **10-C2 Patterns and Real-world Relationships: describe, identify, and apply** • use graphs, tables of values, and written descriptions to describe patterns and relationships • identify patterns in graphs and/or tables of values **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs	1 h	• Grid paper • Rulers	Observe and Assess questions
GAME: True or False **SB p. 85** **TG p. 89**	Practise solving linear relations and functions in a game situation	20–30 min	• Calculators	N/A
Chapter 2 Solving Systems of Linear Equations				
3.2.1 Solving Algebraically — The Comparison Strategy **SB p. 86** **TG p. 90**	**10-C7 Graphs: create by constructing a table of values and graphing** • understand when to choose to graph by the *y*-intercept slope method **10-C8 Systems of Linear Equations: solve** • realize that the graphing method will not always give exact solutions • solve linear equations by substitution method, including comparison of equations • determine the solution to an equation by graphing one side of the equation against the other and identifying the intersection point **10-C10 Equations: solve linear equations and linear inequalities** • use strategies to check answers for reasonableness within the problem context [Cont'd]	1 h	None	Q2, 7, 8 or 11, and 12

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
	10-C12 Problems: express in terms of equations • analyse and interpret a variety of situations and model algebraically as equations			
3.2.2 Solving Algebraically — The Substitution Strategy **SB p. 91** **TG p. 93**	**10-C8 Systems of Linear Equations: solve** • solve linear equations by substitution method, including comparison of equations **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations	1 h	None	Q1, 4, 5, and 7
3.2.3 Solving Algebraically — The Elimination Strategy **SB p. 95** **TG p. 96**	**10-C8 Systems of Linear Equations: solve** • solve linear equations by elimination method **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C13 Equations: rearrange** • transform equations from one form to another	1 h	None	Q1, 6, and 9
CONNECTIONS: Matrix Solution of a Linear System **SB p. 99; TG p. 99**	Make a connection between matrices and linear systems	1 h	None	N/A
3.2.4 EXPLORE: Counting Solutions for Different Systems (Optional) **SB p. 100** **TG p. 100**	**10-C8 Systems of Linear Equations: solve** • solve linear equations by substitution method, including comparison of equations • solve linear equations by elimination method **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation **10-C13 Equations: rearrange** • transform equations from one form to another	1 h	None	Observe and Assess questions
UNIT 3 Revision **SB p. 101** **TG p. 102**	Review the concepts and skills in the unit	2 h	• Grid paper • Rulers	All questions
UNIT 3 Test **TG p. 104**	Assess the concepts and skills in the unit	1 h	• Grid paper • Rulers	All questions
UNIT 3 Performance Task **TG p. 106**	Assess concepts and skills in the unit	1 h	• Grid paper • Rulers	Rubric provided

Math Background

• In this unit, students are introduced to the idea of a function as a special type of relation. They begin to use function notation.

• The focus in this unit is on linear functions and systems of linear equations, but there is also attention to linear inequalities, which are relations, rather than functions.

• Students solve systems of linear equations using a variety of methods — comparison, substitution, and elimination. Through a **Connections** feature, students see how matrices can be used as a tool to solve a system of linear equations.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- They use problem solving in **question 8** of **lesson 3.1.2**, where they use the relationship between distance, rate, and time to solve a problem. Other examples of problem solving are found in **question 10** of **lesson 3.2.1**, where they again use the formula relating distance, rate, and time, and in **question 2** of **lesson 3.2.2** and **question 6** of **lesson 3.2.3**, where they create mathematical models to represent and solve a problem.

- They use communication frequently to explain their thinking, for example, in **question 9** of **lesson 3.1.3**, where they explain the steps for solving an inequality, and in **question 12** of **lesson 3.2.1**, where they explain the conditions necessary for using a particular strategy to solve an equation.

- They use reasoning in answering questions such as **question 2** of **lesson 3.1.1**, where they consider the concept of function applied to a real-world situation and in **question 8** of **lesson 3.1.3**, where they reason about the relationship between the concepts of inequality and function. They also use reasoning in the optional **lesson 3.2.4**, where they consider how characteristics of equations affect the number of solutions.

- They use representation in many situations, for example, in **question 3** of **lesson 3.1.2**, where they use the representation of numbers in a 100 chart to create functions. They use visualization, for example, in **lesson 3.1.3** where they visualize an inequality in terms of a graph, and in Explore **lesson 3.1.4,** where they use graphs to interpret algebraic transformations of functions and in **question 9** of **lesson 3.2.1**, where they use the visual representation of a rectangle to support their algebraic thinking.

- Students see connections, for example, in **lessons 3.2.1 to 3.2.3**, where they explore the connection between different strategies for solving systems of linear equations, in the **Try This** in **lesson 3.2.1**, where they explore the real-world concept of "break even" in terms of a mathematical model, and in **question 5** of **lesson 3.2.2**, where algebra is connected to geometry. The **Connections** feature connects what students have already learned about matrices with what they are learning in this unit about solving systems of linear equations.

Rationale for Teaching Approach

• This unit is divided into two chapters.

- **Chapter 1** focuses on the introduction of the concept of function. It explores linear functions and linear inequalities through graphs, tables of values, and algebraic statements. Students solve a variety of problems about situations that can be described using a linear system of equations.

- **Chapter 2** focuses on the different strategies that can be used to solve a system of linear equations. In this unit, the focus is on two equations in two unknowns, although there is also some exploration of other situations.

• The **Explore** feature in **Chapter 1** focuses on the relationship between the graphs of various linear functions. This will support later work with other types of functions such as quadratic and exponential functions later in Class X and in subsequent classes.

• The **Connections** feature relates the work students did in **Unit 1** on matrices to this new work on systems of equations. Matrices will be an important tool for solving systems of equations in later mathematics work.

Technology in This Unit

Scientific calculators might be useful to solve some equations or create some tables of values, but the need for technology is limited in this unit.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Patterns and Relationships: determine non-algebraic representations **9** Scatter Plots: characteristics of relationships **9** Graphs of Linear Relations: interpret and create **9** Equation of a Line: use graph to determine equation **9** Single Variable Equations: solve algebraically and graphically **9** Inequalities: solve and verify	Students will experience more success in this unit if they review what they already know about the relationships between linear graphs and linear equations and review their skills in creating scatter plots, solving simple linear equations and inequalities, and distinguishing linear situations from other situations.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6) • Rulers	• tables of values and scatter plots • notion of discrete vs. continuous • slope-intercept form of a line • distinction between linear, quadratic, and exponential relations • using an algebraic model to describe a situation • solving simple linear equations • solving simple linear inequalities

Main Points to be Raised

• In a linear function, output values grow by a constant amount for given increases in input values.

• A data set involving figure numbers is discrete, since there is a countable number of figure numbers.

• When an equation is written in slope-intercept form, the slope describes the ratio of the rate of change in *y* for a corresponding change in *x*. The intercept describes the point where the graph crosses the *y*-axis. The line goes up and to the right if the slope is positive. It goes down and to the right if the slope is negative.

• A quadratic relation grows so that second differences are constant; the graph looks like a parabola. An exponential relation grows more quickly than a linear or quadratic relation; the graph is usually quite steep.

• Solving an equation involves finding the value that makes the equation true. There are many strategies for solving a linear equation. Using inverse operations is a particularly useful strategy.

• An inequality can have many solutions. The solution can be expressed as a simpler inequality that describes the possible numerical values for the variable.

Use What You Know — Introducing the Unit

• Some students may recognize from the visual representation that each subsequent figure has one more row of four circles in it. They might use multiplication facts to help them fill in the chart; for example, they would know Figure 5 has four rows of four plus three more, which is 16 + 3 = 19. Other students are more likely to notice the pattern of adding four in the right column in the table of values and to use that pattern to complete the table.

• The reason for asking students to predict the type of relationship without drawing a graph (in **part D)** is to reactivate their knowledge that linear functions grow at a constant rate. If they draw the graph, it is the linearity of the graph, rather than the concept of rate of change, that would lead them to recognize the function as linear.

• Encourage students to work in small groups and to discuss their results as they progress through the activity.

• Observe students as they work. You might ask:
- *What does discrete mean?* (You should not join the points on the graph since there is nothing between them.)
- *How do you know that your equation is in slope-intercept form?* (There is a *y* on one side of the equation. On the other side the *x* is multiplied by a number that is the slope. There is a number added that is the intercept.)
• *Where do you see the slope on the graph? Where do you see the intercept?* (The intercept is where the line crosses the *y*-axis. The slope tells how steep it is; using these two points, I can divide the rise by the run to find the slope.)

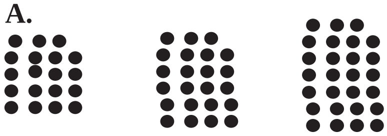
Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually, but it will be helpful if each student has a partner with whom he or she can discuss answers or difficulties.

Answers

A.



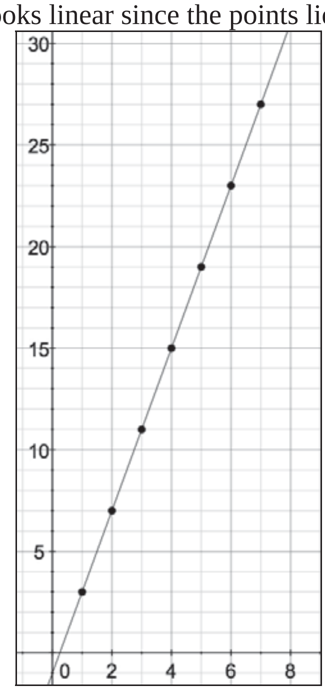
B.

Figure number	Number of circles
1	3
2	7
3	11
4	15
5	**19**
6	**23**
7	**27**

C. It is discrete because there is no figure number between those given.

D. Linear; first differences are all 4 (constant), so the relationship is linear.

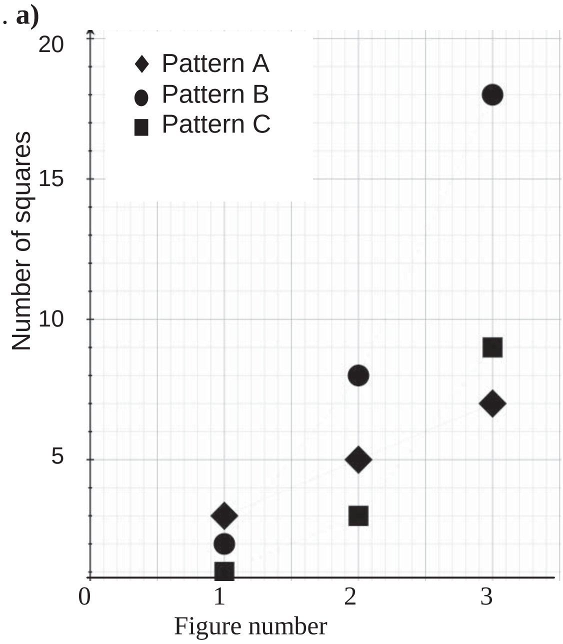
E. It looks linear since the points lie on a line.



F. $C = 4n - 1$ where C is the number of circles and n is the figure number.

G. It is in $y = mx + b$ form. The intercept (b) is –1 and the slope (m) is 4.

1. a)



b) Pattern A is linear because the number of squares increases by two for each figure and since this is a constant increase, the relationship is linear.

Pattern B appears to be quadratic because the plotted points form the right half of a parabola. For the three plotted points, the first differences are not constant but the second differences are constant.

Pattern C is exponential because the plotted points form a curve with a steep increase. The number of squares in each figure is three times the number in the previous figure so neither first nor second differences are constant but the ratios of first differences are constant.

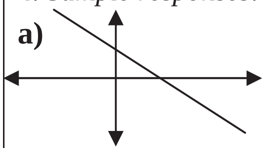
2. $a = 76$, $b = 68$, $c = 64$

3.

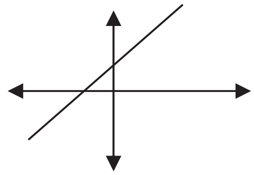
	a)	**b)**	**c)**	**d)**
slope	2	$-\frac{3}{4}$	–3	0.75
y-intercept	3	–2	2	–1.65

4. *Sample responses*:

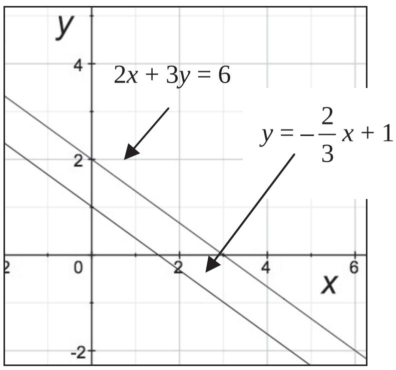
a)



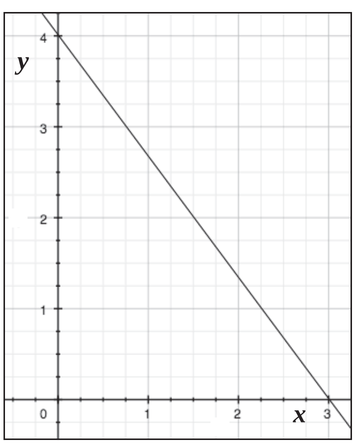
b)



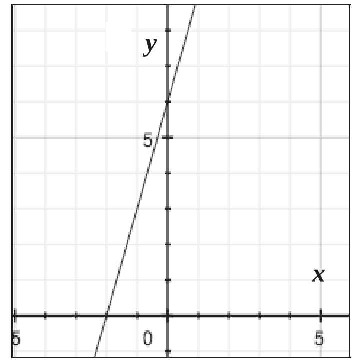
5.



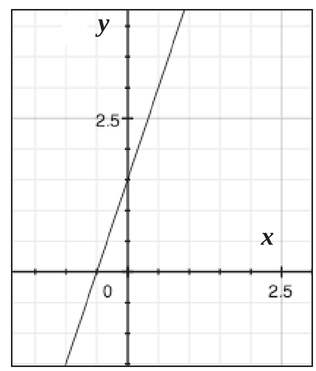
6. a) $4x + 3y = 12$



b) $3x – y = –6$



c) $6x – 2y = –3$



7. a) $x = -\frac{7}{5}$ **b)** $a = 4$

8. a) $a < -3\frac{1}{2}$ **b)** $x < 2$

9. 2

10. a) $200 + 2m$ **b)** $60s$ **c)** $1000n$ **d)** $0.04n$

Supporting Students

Struggling students

• It may be necessary to review the concepts of linear, quadratic, and exponential relations. Focus on the rate of change rather than on the algebraic descriptions, i.e., the linear function changes in a steady way, the quadratic function tends to increase more rapidly, but the exponential function tends to increase even more rapidly.

• Students will have an opportunity to revisit linear relations in this chapter, but it may be necessary to review what the meaning of slope, the ratio of rise to run, and what the slope tells about the rate of change.

• Students who are not comfortable with inequalities might choose values to substitute into the inequality to help them see the pattern of the solution set. For example, for the inequality $5a + 7 < 3a$, they might observe that there is no positive value they can use to make this true, and 0 does not work either. They could start trying negative values and observe that although –1, –2 and –3 do not solve the inequality, lower values do. This helps them see why the solution $a < -3\frac{1}{2}$ makes sense.

Enrichment

Some students might create their own designs like those in **question 1** that contrast linear, quadratic, and exponential functions.

Chapter 1 Linear Functions and Relations

3.1.1 Linear Functions

Curriculum Outcomes	Outcome relevance
10-C4 Graphs and Tables: construct and analyse • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C17 Functional Relationships and Notation: explore formally and informally** • understand the relationship between a relation and a function • start with functional relationships then apply the mathematical concept of function • use mathematical notation and vocabulary	The concept of function is fundamental to higher mathematics, where mathematical models are used to solve real-world problems. Linear functions are one of the simplest for students to understand and have many real-world applications.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• linear relation • creating tables of values

Main Points to be Raised

• A relation describes how pairs of numbers are connected.

• A function is a special type of relation where there is only one output value for any given input value. Functions are particularly useful for creating mathematical models to describe situations. Because there is only one output value for each input value, you can use a function to make predictions.

• One way to determine whether a relation is a function is to apply the "vertical line test" to a graph of the function. There should be only one point of the graph on any vertical line drawn through the function.

• The input value for a function is called the independent variable. The output value is called the dependent variable (since it depends on the input).

•A function is described by a function rule, which is often expressed algebraically. The function itself can be described by a set of ordered pairs, input/output values, a graph, a rule described in words, or a rule described symbolically. The notation $f(x)$ is used to show that the output values are dependent on the input values of x.

Try This — Introducing the Lesson

A. and B. Observe while students work. You might ask:

• *Suppose the equation started y = 5x… .What would have to come next? Why?* (–15. Since 5 tens make 50, I would have to subtract 15 to get back to 35.)

• *Could you use a value involving the square of the input to come up with a rule?* (Yes; if you square 10, you get 100 and if you subtract 65, you get 35. The rule would be $y = x^2 – 65$.)

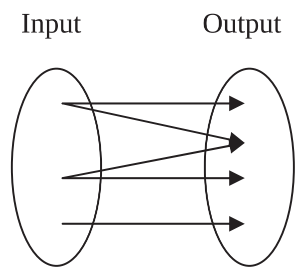
• *Can you create another rule that uses squaring?* (Yes. I could take half of the number, square it, and add 10.)

The Exposition — Presenting the Main Ideas

• Tell the students that you are thinking of a rule. A student says a number and you use the rule to respond with the next number. Repeat this several times and ask the students how you decided what number to say. Talk about how they could tell you were applying a rule. Try the experiment again, but this time when a student gives a number, reply with any number at all that is greater than the given number. Make sure that the number you say is sometimes slightly greater and other times much greater. After four or five responses, ask students to tell you what rule you were using. Usually, this will be much harder to predict. Now tell the students what your rule was: a greater number. Help them understand that it is usually easier to figure out a rule where there is only one possible result than to guess a rule where there are many possible results.

• Re-introduce the term relation (students should know this from Class IX) and introduce the term function. Explain that your first rule was a function — there was only one answer you could have given for any number a student could say. The second rule was not a function.

• Ask students to read through the exposition. Ask a few questions when they have finished to make sure they understand. For example, draw a picture like the one below and ask whether or not it describes a function.



You might also draw a scatter plot on the board and ask students whether it is a function. Observe whether they apply the vertical line test.

• Make sure students understand that a linear function is a function that involves only the first power of the independent variable and that it relates to what they learned in Class IX. Sometimes a function involves only a constant. For example, $f(x) = 3$ is a function since it passes the vertical line test when it is graphed.

Revisiting the Try This

B. Students can work with a partner. This question is a practical application of the notion that once you know two points on a line, the line is well-defined.

Using the Examples

Have students read through the three examples independently. You may wish to revisit e**xample 3** as a whole group to make sure students are comfortable determining slope and intercept, and that they see how the equation of the line, and therefore the linear function, relate to those values. Help students understand that the function rule and the equation of a line are closely related; the only difference is the use of the function notation to emphasize that the value of y is based on the value of x.

Practising and Applying

Teaching points and tips

Q 1: Although students might use a table of values to answer this question, encourage them to consider using the vertical line test.

Q 2: You might answer why **part a)** would not describe a function if the columns were reversed, i.e., if the ages were the input and student names were the output.

Q 6: This is an example of a function that students would not yet be able to describe algebraically.

The function is $h = \frac{10}{\cos a}$.

Common Errors

Some students believe that in order for to have a function, there also has to be one input value for any given output value. Draw their attention to the visual representation of functions shown in the exposition, which makes it clear that this is not the case.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can determine whether a graph represents a function
Question 2	to see if students understand the role of the input and the output in determining whether a relation is a function
Question 4	to see if students can create and use a table of values to determine whether a relation is a function

Answers

A. *Sample response*: $y = 3x + 5$, $y = 10x - 65$, and $y = x^2 - 65$

B. i) $y = 3x + 5$ or $f(x) = 3x + 5$ **ii)** 5

1. a) Not a function; (3, 2) and (3, 4) have the same x-coordinate, but different y-coordinates.
b) Function; no ordered pairs have the same x-coordinate and different y-coordinates.
c) Not a function; except for (0, 0), there are two ordered pairs with different y-coordinates for each x-coordinate.

2. a) Function; each student has only one age.
b) Usually not a function; there would likely be more than one student in the class with identical numbers of siblings.

3. a)

3	4	5
☆	☆	☆
☆	☆	☆
☆	☆	☆
	☆	☆
		☆

b) $f(n) = n$

4. a) *Sample response*:

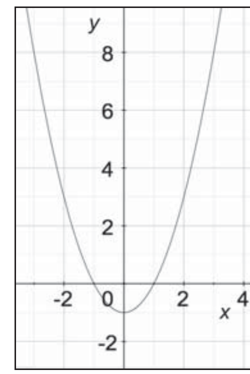
x	$f(x) = 4x - 3$
–1	–7
0	–3
1	1
2	5

b) For each x value, there is only one y value.

c) *Sample responses*:
You could draw a graph and extend it to see that the line has one y-value for each x-value.
OR
You can be sure it continues this way, since each next value of y is exactly 4 greater than the previous one.

5. a) *Sample response*:

x	$f(x) = x^2 - 1$
–1	0
0	–1
1	0
2	3
3	8



b) A vertical line will cross the graph in only one place.

6. Yes; for each value in the first column, there is only one possible value in the second column.

7. a) For each value in the first column, there is only one value in the second column.
b) There are no ordered pairs that have different y-coordinates for the same x-coordinate. This means that if you draw a vertical line, it will hit the graph at only one place no matter where you draw it.

Supporting Students

Struggling students

- Some students might need additional practice creating tables of values. It might be easier to use a number of different linear functions at the start.
- **Question 2** might initially be difficult for struggling students. It is essential to understand, so you should ensure that they grasp this idea.
- Some students might benefit from drawing several triangles for **question 6** to help them see that the hypotenuse really does change with the angle.

Enrichment

Some students might want to use the idea in **question 3** to create non-linear pattern rules that are built on Figures 1 and 2.

3.1.2 Applications of Linear Functions

Curriculum Outcomes	Outcome relevance
10-C2 Patterns and Real-world Relationships: describe, identify, and apply • use graphs, tables of values, and written descriptions to describe patterns and relationships • identify patterns in graphs and/or tables of values **10-C7 Graphs: create by constructing a table of values and graphing** • understand when to choose to graph by the *y*-intercept slope method **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C17 Functional Relationships and Notation: explore formally and informally** • understand the relationship between a relation and a function • start with functional relationships then apply the mathematical concept of function • use mathematical notation and vocabulary	• Students begin to see how a scatter plot can reveal information about a relation. They learn to pay attention to which variable is independent and which is dependent and to consider whether data values are discrete or continuous. This will serve them well in future mathematical work. • Students begin to see the strengths and limitations of interpolating and extrapolating using a graph to make predictions. This prepares them for future experiences with mathematical modelling of natural and social phenomena.

Pacing	Materials	Prerequisites
2 h	• None	• graphing linear relations • determining the equation of a line from a graph • manipulating linear equations • percents • lines of best fit

Main Points to be Raised

• When a linear relation is written in slope-intercept form, the *y* appears alone on one side of the equation. When the relation is written in standard form, both variables appear on one side of the equation, with only a constant on the other side.

• It is very easy to transform the slope-intercept form of a linear relation to an algebraic function rule, but to change to function form from standard form requires some algebraic manipulation.

• In a two-variable situation, sometimes one of the variables must be the independent variable, but other times either variable could meaningfully represent the independent variable.

Try This — Introducing the Lesson

A. Encourage students to work in pairs. Observe while students work. You might ask:

• *Why does a dashed line make more sense?* (You cannot have fractions of balls.)

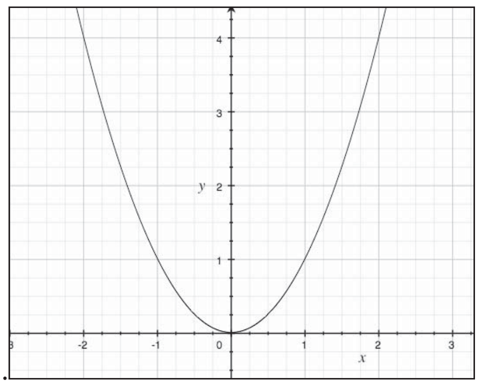
• *Why does the graph slope down?* (If you have more white balls, you have to have fewer black ones.)

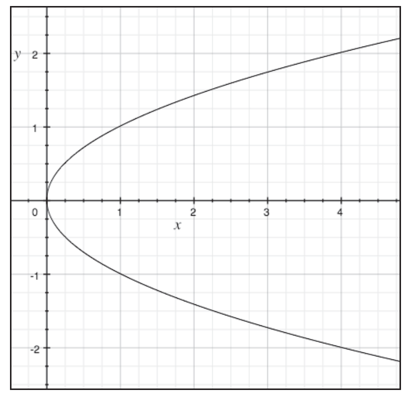
The Exposition — Presenting the Main Ideas

• Write the relationship $y = 2x + 5$ on the board. Ask students to transform it to function notation. Then ask why the relation $2x - y = -5$ is really the same relation. Some students might suggest graphing, others might want to substitute values for x and y, and still others will realize that the y and 5 could be subtracted from both sides of the original equation. Ask students how they would write the second relationship in function notation.

Some students will realize that they could use the original function since it is the same relation that is being described. Some students may choose to write x as a function of y ($f(y) = \frac{y-5}{2}$). Still others might perform algebraic manipulations to write the second function into function notation.

• Have students read through the exposition. Make sure they understand the last section about $y = x^2$ by drawing a graph and showing how one graph ($y = x^2$) displays a function and the other does not ($x = y^2$).





Revisiting the Try This

B. This question allows students to make a formal connection between the graph created in **part A** and the notion of independent and dependent variables as elements of function rules.

Using the Examples

• Lead students through **examples 1 to 3**.

• You may need to spend more time on **example 2**, helping students see where the equation came from. It is built on the representation of 3% as 0.03 and of 4% as 0.04. You may also need to help them see the freedom they have in choosing values for b to get corresponding values for a.

• Although students do not really need to know what a line of best fit is to work through **example 3**, you should remind them that the line should lie close to the data values, but not necessarily exactly on them. The fact that lines of best fit are created to describe data is an excellent example of how mathematicians prefer functions over relations. Whereas the scatter plot might be a relation, with two different y values for some x values, the line of best fit is a function.

• Make sure that students understand that the choice of using $x = 2005$ in **example 3** was arbitrary and that other values of x could have been used instead. Some students might suggest extending the line to find the intercept, but this would be misleading in this situation since the x-axis does not begin at 0. You may wish to point this out if the students do not notice.

Practising and Applying

Teaching points and tips

Q 2: Ask students why this function is discrete and not continuous.

Q 3: You might suggest students sketch a full 100 chart in their notebooks to work on this question. Question students to make sure that they realize that the solution is dependent on the form of the 100 chart. If there were not 10 columns, the results would be different. **Part c)** previews later work on inverse functions.

Q 4: Encourage students to use **example 2** as a model for this question.

Q 5: Students will need to use the relationship Distance = Rate × Time in order to solve this problem. This is a fairly complex question and you may encourage students to work in pairs to solve it.

Q 8: This question, like **question 5**, requires students to be familiar with the relationship
Distance = Rate × Time.

Q 9: Some students will not think of the situation of a horizontal or vertical line. If they do not, you might draw a horizontal line such as $f(x) = 3$ and ask if it is a function. From one perspective, students might think it is not a function since there is no x-variable in the equation, but from another perspective it would be considered a function (based on the vertical line test or because the ordered pairs that describe the line are in the form of a function). You could then draw a vertical line and ask whether it is a function. This is not a function since there are many outputs for a given x-input.

Common Errors

For **question 8,** many students are likely to write the functions $8t + 12$ and $7t$ rather than the equations $8t$ and $7t + 12$. Encourage them to substitute 0 for t to see if their equations make sense in terms of who should be where at the start.

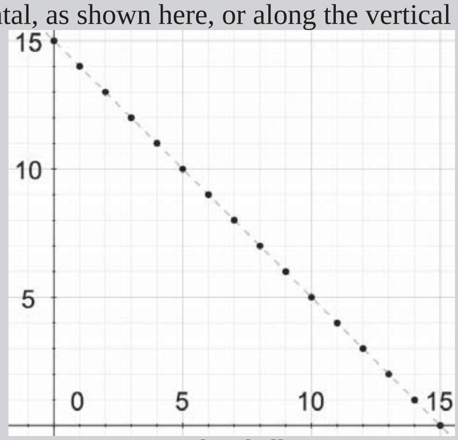
Suggested assessment questions from Practising and Applying

Question 1	to see if students can use standard form to write each variable as a function of the other
Question 6	to see if students are able to use functions to interpolate and extrapolate
Question 8	to see if students can represent and solve a problem involving a linear function
Question 9	to see if students can generalize to recognize why almost every line describes a function and to see if students can communicate their understanding

Answers

A. i) Note that white balls could be plotted along the horizontal, as shown here, or along the vertical axis.

Black balls



White balls

ii) Dashed; since there are only whole balls in the combinations, the relation is discrete.

B. i) $b + w = 15$ (or any variation of the equation)

ii) The number of white balls can be calculated if you know the number of black balls, or the number of black balls can be calculated if you know the number of white balls.

iii) $w = 15 – b$ or $f(b) = 15 – b$

1. a) $f(t) = \dfrac{4-2t}{5}$

b) $f(m) = \dfrac{4-5m}{2}$

2. a) $20t + 50f = 2000$, or any variation of it, e.g., $20t = 2000 – 50f$, where t represents the number of Nu 20 notes and f the number of Nu 50 notes.

b) $f(f) = \dfrac{2000 - 50f}{20} = \dfrac{200 - 5f}{2}$

3. a) $f(l) = 5l + 91$

b) 24 **c)** $f(s) = (s – 91) \div 5$

4. a) $I = 0.042o + 0.045n$, where I represents total interest, o represents the money invested at 4.2% and n represents the money invested at 4.5%.

b) $f(o) = \dfrac{I - 0.042o}{0.045}$

5. a) $30r + 20s = 51$, where r is the number of hours at 30 km/h and s is the number of hours at 20 km/h.

b) $\dfrac{51-20s}{30}$ or $f(s) = \dfrac{51-20s}{30}$

c) 0.9 hours or 54 minutes

6. a) *Sample response*: (1976, 45) and (2002, 60) **b)** $e = \frac{15}{26}a - 1095$, or $f(a) = \frac{15}{26}a - 1095$, where a represents the year and e represents life expectancy. **c)** $f(2010) = 64.6$ **d)** $f(1970) = 41.5$; the actual plotted value is just a bit more than 40 so it is very close. **7. a)** $F = \frac{9}{5}C + 32$ **b)** $C = \frac{5}{9}(F - 32)$	**8. a)** Deki: $f(t) = 8t$; Thinley: $f(t) = 7t + 12$ **b)** Deki: 72 m; Thinley: 75 m **c)** Deki; I solved the equations $8t = 100$ for Deki and $7t + 12 = 100$ for Thinley to get the time it would take each to finish the course. For Deki, t was 12.5 s and for Thinley, t was 12.57 s. **d)** Because where you are on the course depends on how much time has passed. **9.** No; If $y = mx + b$, there is one y-value for each x-value. That's why it's a function. But that also means that $x = \frac{y - b}{m}$. As long as m is not 0, there is only one x-value for each y-value, so this is also a function. If $m = 0$, the original line was horizontal and x did not appear in the equation, so you won't be able to write x in terms of y.

Supporting Students

Struggling students

Some students might benefit from additional practice with writing an equation to describe a situation before they start solving problems involving those relations. For example, you might propose situations like those below where students create a model to describe the situation.
- Someone charges Nu 20 an hour for a service in addition to Nu 50 as a fixed fee
- A bank pays 5% simple interest on a certain account and you know you have received Nu 70 in interest. You want to know what the principal was.
- A parallelogram has a base of 20 cm. You want to know what its area will be for different heights.

Enrichment

Students might enjoy making other shapes in the 100 chart as in **question 3** to create and solve similar problems. They could also describe the sum as a function of the greatest value rather than as a function of the least value.

3.1.3 Graphs of Linear Inequalities

Curriculum Outcomes	Outcome relevance
10-A2 Inequalities: relate sets of numbers to solutions • solve problems with a restricted solution set • include choice of number systems **10-C3 Linear Inequalities: write and describe graphs** • describe a given graph using inequalities **10-C6 Graphs: sketch** • create graphs given information in a variety of formats **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C11 Equations and Inequalities: graph and analyse graphs** • graph equations and inequalities by constructing a table of values, using knowledge of transformations, and by identifying characteristics **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations	Although many relationships that can be modelled mathematically are equalities, inequalities are also useful to model real-world situations. Linear inequalities are among the simplest inequalities that are useful for this purpose.

Pacing	Materials	Prerequisites
1.5 h	• Grid paper (BLM in Unit 6) • Rulers	• graphing linear relations • substituting into an equation

Main Points to be Raised

• Any linear inequality is related to a linear equality; the related equality looks exactly the same except that the equals sign is replaced by an inequality sign.

• Linear inequalities are not functions since there are multiple *y* values for any *x* value.

• When an inequality also allows for equality, the related line is shown as a solid line. Otherwise, the convention is to draw a dashed line.

• You begin to graph a linear inequality by graphing the related linear equation. You then shade the one side of the line that satisfies the inequality. To determine which side of the line to shade, you need to test only one point to see if that point should be on the shaded side or on the unshaded side.

• Any point on the shaded side represents a solution to the linear inequality.

• Points on a solid line also represent solutions.

Try This — Introducing the Lesson

A. Students might work in partners on the **Try This** task.

Observe while students work. You might ask:

• *How do you know that if the number of tennis paddles is 80, the number of badminton racquets is 30 or fewer?* (Because the total has to be 110 or less.)

• *Will you make more profit with 50 table tennis paddles and 40 badminton racquets or with 40 table tennis paddles and 50 badminton racquets?* (You earn more profit with a badminton racquet, so it would be better to have only 40 table tennis paddles and 50 badminton racquets.)

The Exposition — Presenting the Main Ideas

• Remind students of the meaning of the signs $<$, $>$, $\leq$, and $\geq$ by asking them to indicate which of these statements are true: $3 < 4$, $5 > 6$, $3 \geq 3$, and $8 \geq 5$.

• Next, ask the students to suggest five or six different numbers to make each of these true: $y < 4$, $2y > 4$, $3y \leq 3$, and $5y \geq 10$.

• Then ask students to explain why each of these ordered pairs would be a solution to $y < 4$: (2, 0), (2, 1), (2, 2), (2, 3), (4, 0), (4, 1), and (4, 2). Discuss why the expression $y < 4$ is not a function (There are many *y*-values for each *x*-value.)

• Create an inequality involving two variables, e.g., $y < x$, and ask students to suggest ten ordered pairs that would solve the inequality. Plot the points and observe that they are all below the line $y = x$. Talk about why that makes sense (The points above the line would have greater y values and so the y value could exceed the x value.)

Discuss with students why any point below the line $y = x$ solves the inequality.

• Ask students to suggest how they might graph the solutions to the inequality $y < 2x$. If they do not suggest it themselves, help them see that they might begin with the line $y = 2x$. Clearly, any point not on the line has a y-value that is either more or less than $2x$. Ask students to figure out how they would decide which points have values that are less than $2x$. Show the convention of shading one side of the line to show the solutions. Discuss why the line is dashed and why it would be solid for $y \leq 2x$.

• Lead students through the exposition. Show why you only need to test one point to decide which side of a line to shade to display the solutions of the inequality. Then discuss why (0, 0) is often a convenient point to test to make the calculations easier. If (0, 0) is on the line, another point must be selected.

Revisiting the Try This

B. This question allows for a formal connection between the informal inequalities students solved in **part A** and the inequality notation introduced in the exposition. It also provides another opportunity for students to see why linear inequalities are not functions.

Using the Examples

Assign groups of three students to work through the three examples. Each student in the group can be responsible for one example and explain it to the others in the group. After this, you may wish to ask one student to explain **example 3** to the whole class.

Practising and Applying

Teaching points and tips

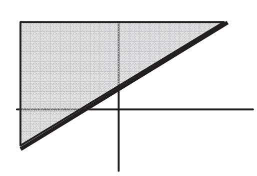
Q 1: You might need to remind students that "in the shaded region" means that the point satisfies the inequality.

Q 3: If students do not read the signs carefully, they might find this question confusing. Ask them to watch for the direction of the sign and whether or not the sign includes equality.

Q 5: Encourage students first to observe the intercept and then to use the coordinates of two points on the graph to determine the slope.

Q 6: Some students may need support in translating the phrase "at least" into an inequality facing the appropriate direction.

Q 8: Some students may be concerned about whether their limited experience is enough to be sure that inequalities can **never** be functions. Encourage them to use reasoning and not just to depend on examples. There are always many values of y for any value of x, even if, for example, y were restricted to positive values.



Common Errors

Some students simply assume that the > side of an inequality is the upper (or right) side and the < side of an inequality is the lower (or left) side. Encourage them to test a point to be sure.

Suggested assessment questions from Practising and Applying

Question 4	to see if students can graph an inequality presented as an algebraic expression
Question 5	to see if students can write the algebraic expression represented by a graphed inequality
Question 8	to see if students can communicate clearly the relationship between the concepts of function and inequality
Question 9	to see if students can communicate the steps for graphing an inequality

Answers

A.

i) *Sample response*:

Paddles	0	80	80	70	50
Racquets	50	0	30	40	50
ii) Profit (Nu)	1500	1600	2500	2600	2500

B. i) If p is the number of paddles and r is the number of racquets, the inequalities describe the maximum number of paddles, the maximum number of racquets, and the maximum total number of pieces of equipment.

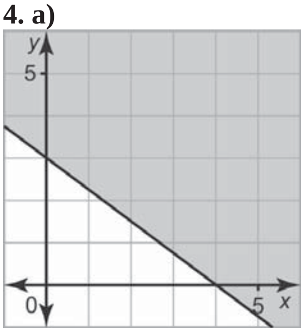
ii) There are many combinations of 50 or fewer racquets and 80 or fewer paddles that will result in fewer than 110 total items.

1. a) No **b)** Yes **c)** No

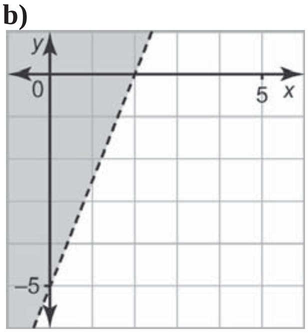
2. a) Below; *Sample response*: I tested the point (0, 0). Since $0 < 0 + 6$, then (0, 0) belongs in the shaded region and it is below the boundary line.

3. a) Both have dashed boundary lines and the lines are the same; the first one has shading on one side of the line and the second has it on the other side.
b) Both have the same boundary line and are shaded below the line; the first one has a solid line and the second has a dashed line.

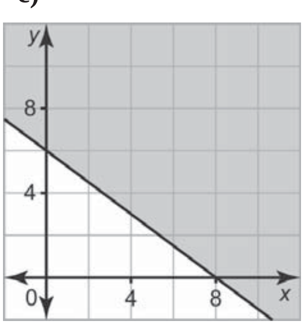
4. a)



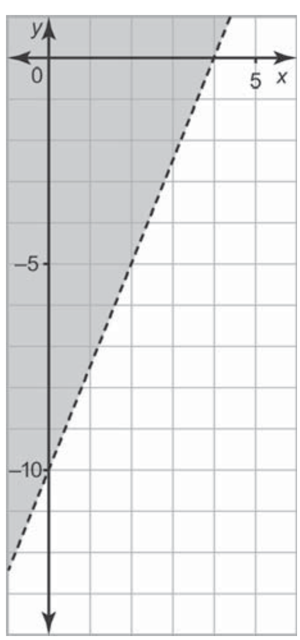
b)



c)



d)

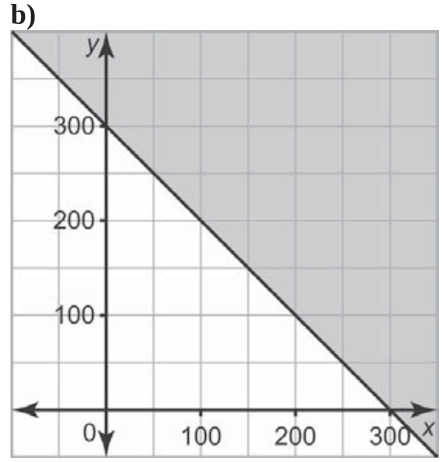


5. a) $y < -\frac{5}{2}x + 2$ **b)** $y \leq \frac{4}{3}x - 4$

c) $< \rightarrow \leq$; $y \leq -\frac{5}{2}x + 2$ **d)** $\leq \rightarrow \geq$; $y \geq \frac{4}{3}x - 4$

6. a) $x + y \geq 300$ (or any variation), where x and y represent the amount he earned at each job.

b)



7. a) A **b)** C

c) *Sample response*:
It could only be A or D since the line had to be solid. I tested the point (0, 0) and found that it should not be in the shaded region, so it had to be A.

d) *Sample response*:
It had to be the region opposite to my answer in **part a)** and the line had to be dashed.

8. If you choose any x-value, there are multiple y-values that go with it.

9. *Sample response*:
- Graph $2x - 4y = 10$ using the two intercepts (5, 0) and (0, –2.5).
- Draw a dashed line because the inequality sign is <.
- Test the point (0, 0) to see whether it belongs in the shaded area. Since $2x - 4y < 10 \rightarrow 0 - 0 < 10$, it does.
- Shade the side of the line that contains (0, 0).

Supporting Students

Struggling Students

• If students are still struggling with graphing linear equations, it might be important to give them some additional experiences with graphing linear equations written in standard form before working with inequalities.

• Struggling students might find a conceptual question like **question 8** difficult. This question is not required for these students.

Enrichment

Some students might explore the similarities and differences between graphing these inequalities:

$2x + 3y < 10$

$-2x - 3y < -10$

$-2x - 3y > 10$

This will allow them to discover what happens when an inequality is multiplied through by –1.

3.1.4 EXPLORE: Transforming Graphs of Linear Functions

Curriculum Outcomes	Lesson relevance
10-C1 Transformations: express algebraically or using mapping rules • express transformations from a graph using algebraic expressions or mapping rules **10-C2 Patterns and Real-world Relationships: describe, identify, and apply** • use graphs, tables of values, and written descriptions to describe patterns and relationships • identify patterns in graphs and/or tables of values **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs	• This core (essential) lesson provides students with experience that will help them make sense of subsequent formal work with different kinds of functions and with composition of functions. • Students see how transforming a function affects the resulting graph. This will be repeated with quadratic functions in a later unit.

Pacing	Materials	Prerequisites
1 day	• Grid paper (BLM in Unit 6) • Rulers	• graphing linear equations • transformation language: translation, reflection • slope language: steep, less steep

Main Points to be Raised

• Substituting an algebraic expression into the input values for a linear function produces a new, related function.

• $f(x + h)$ is always a horizontal translation of $f(x)$.

• $f(kx)$ is steeper than $f(x)$ if $k > 1$.

• $f(–x)$ is a reflection of $f(x)$.

• $f(x) + k$ is a vertical translation of $f(x)$.

Exploration

You might divide the class into groups of two to four students and have each group work on one of **parts A**, **B, or C**. Ask each group to examine the relations listed.

Observe while students work. You might ask:

• *Suppose f(x) were 4x + 2 instead of 3x + 2. Would any of your answers change?* (No, since translations would still be translations, reflections would still be reflections, and lines that were steeper or less steep would still have the same properties.)

• *Why would f(x – 10) be a translation right of 10 units?* (Since $f(x – 1)$ was a translation to the right of 1 unit and $f(x – 3)$ was a translation to the right of 3 units, it makes sense it would be a translation to the right of 10 units.

Observe and Assess

As students are working, notice:

• Are they able to apply order of operations rules and evaluate the expressions correctly?

• Are they able to plot the points and draw the lines with reasonable accuracy?

• Do they use what they learn in one situation to predict what might happen in the next situation?

Share and Reflect

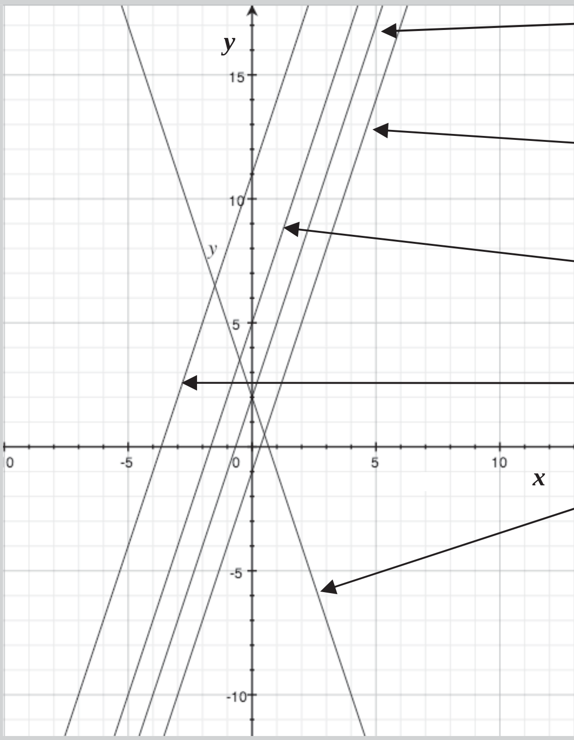
• If you had asked students to work in groups, ask all the students who worked on the same part to get together and report what they observed about the graphs. They could record their observations on a large sheet of poster paper or on the board for the entire class to see.

• You could encourage students to predict what other variations of each type of relation would look like. This would emphasize that the shape of the graph does not change even if the steepness and position change.

• You might ask students to explain how they predicted the transformations that went with various algebraic changes. For example, a student might recognize that if $f(x) = mx + b$ and if $g(x) = –f(x)$, then $g(x)$ would have to be a reflection in the x-axis. Or if $f(x) = mx + b$ and $g(x) = f(x – 1)$, then the y values do not change, but they are associated with x values that are moved 1 unit to the right. This results in a horizontal translation of 1 unit.

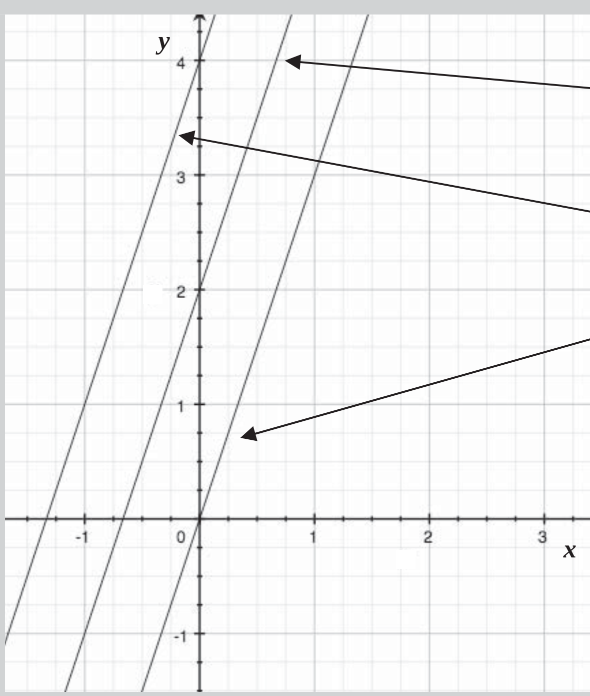
Answers

A.



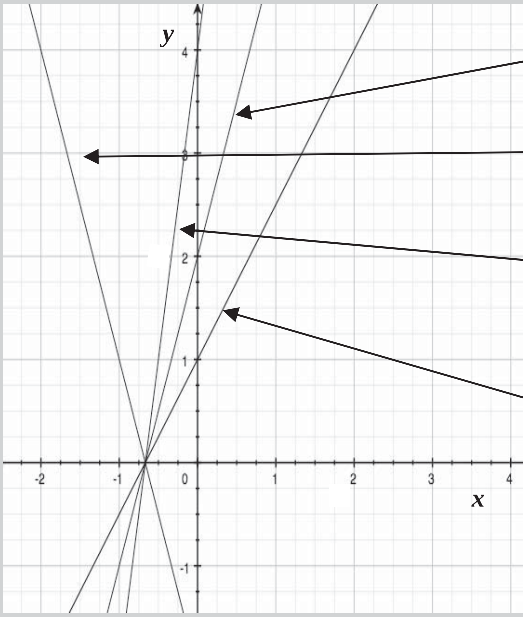
$f(x)$	
	D.
i) $f(x-1)$	Translation 1 to the right
ii) $f(x+1)$	Translation 1 to the left
iii) $f(x+3)$	Translation 3 to the left
iv) $f(-x)$	Reflection in the y-axis

B.



$f(x)$	
	D.
i) $f(x)+2$	Translation up by 2
ii) $f(x)-2$	Translation down by 2

C.



	D.
$f(x)$	
i) $-f(x)$	Reflection in x-axis
iii) $2f(x)$	Steeper graph through the same x-intercept
iv) $0.5f(x)$	Not as steep but through same x-intercept

Supporting Students

Struggling students

Assign struggling students to work with stronger students who you know will be able to assist them with the task.

GAME: True or False

This game provides a pleasant way to review what it means to solve equations and inequalities.

3.2.1 Solving Algebraically — The Comparison Strategy

Curriculum Outcomes	Outcome relevance
10-C7 Graphs: create by constructing a table of values and graphing • understand when to choose to graph by the *y*-intercept slope method **10-C8 Systems of Linear Equations: solve** • realize that the graphing method will not always give exact solutions • solve linear equations by substitution method, including comparison of equations • determine the solution to an equation by graphing one side of the equation against the other and identifying the intersection point **10-C10 Equations: solve linear equations and linear inequalities** • use strategies to check answers for reasonableness within the problem context **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations	• Many real life situations are described by mathematical models involving more than one constraint. Systems of linear equations are often used to describe and make predictions about such situations. • There are a number of strategies for solving systems of equations. It is important for students to meet several of these strategies so that they can choose the most appropriate strategy for a given situation.

Pacing	Materials	Prerequisites
1 h	• None	• solving linear equations with decimal or integer coefficients • percents • creating an equation to describe a line • formula relating distance, speed, and time

Main Points to be Raised

• A linear system is a set of linear equations that are all required to represent a situation. The solution of the system must make each equation true.

• Using a graph, you can look for the intersection point of two lines to determine the solution to a system of equations involving two equations in two unknowns.

• If two lines do not intersect at a grid point, it might be hard to determine graphically the exact values for the solution of the system of equations.

• The comparison strategy for solving a system of equations is useful when both equations are in the form $y = \ldots$ (or $x = \ldots$). In this case, the right hand sides of the equations can be set equal. This determines either the *x*-coordinate or *y*-coordinate of the intersection point. The other value is calculated by substituting the newly-determined value into either of the equations in the system.

Try This — Introducing the Lesson

A. Have students work alone or in pairs. Observe while students work. You might ask:

• *Why is his profit not Nu 15 per biscuit?* (Even though the profit per biscuit is Nu 15, you also have to take into account the equipment costs.)

• *What algebraic expression might you use to describe Passang's profit?* (If x is the number of biscuits, his costs are $4800 + 5x$ and his income is $20x$, so his profit is $20x - (4800 + 5x)$.)

• *How do you know quickly that he would have to sell more than 200 biscuits to make a profit?* (With 200 biscuits, he would earn only Nu 4000 and the equipment costs more than that amount.)

The Exposition — Presenting the Main Ideas

Have students read through the exposition on their own. Check their understanding by asking a question like, *What is the intersection point of the lines described by the equations* $y = 3x + 2$ *and* $y = 2x - 8$?

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between students' informal approaches to solving the problem in **part A** and the concept of systems of linear equations.

Using the Examples

• Work through **example 1** with the students. Make sure they understand why two equations are used to describe the situation and why the equations are those indicated in the solution. You might ask students how they know that when the comparison strategy is employed, the result will be a linear equation in one unknown.

• Encourage students to read through **example 2** on their own. Allow time for them to ask questions.

Practising and Applying

Teaching points and tips

Q 1: The first part of this question lets students see how a model is created. They can then apply the strategy in other problems.

Q 2: Encourage students to predict whether the intersection will be in Quadrant I, II, III, or IV before they solve the equations. Then they can test their predictions.

Q 3: Many students will seek to solve this problem numerically. Although this should not be disallowed, encourage them to also represent the situation with a system of linear equations.

Q 4: Before they solve the problem, encourage students to explain why the first number (the number you take half of) has to be less than the second number.

Q 5: Some students will solve this using a combination of reasoning and guess-and-check strategies. They might realize that if all the balls were practice balls, they could have bought 22. Since they bought 20 balls, they could not have bought too many official balls.

Q 9: Students will have to create the equation of a line that goes through two points (to form a diagonal). They will need to determine the slope using those points, but will also have to calculate the appropriate y-intercept.

Q 10: Some students may need to be reminded of the formula Distance = Rate × Time to solve this problem.

Common Errors

Some students determine the value for one of the two variables and forget that they may need to determine the value of the other in order to have a complete solution. You might ask students to record their solution as an ordered pair to help them remember. Some contexts may require the value of only one variable, as with the number of biscuits in the **Try This**.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can solve a system of simple linear equations with integer coefficients
Question 7	to see if students can create a system of equations to represent a situation and use the system to solve the problem
Question 8 or 11	to see if students can solve a system of simple linear equations with integer (**question 8**) or rational (**question 11**) coefficients when the equations are not in the form where the same variable is isolated in both
Question 12	to see if students can articulate the conditions for applying the comparison strategy

Answers

A. 320 biscuits; 4800 ÷ 15 = 320 (Note that the profit on each biscuit is Nu 15.)	**B. i)** $C = 4800 + 5c$ and $C = 20c$, where C is the cost and c is the number of biscuits. **ii)** $C = 320$; Check: 4800 + 5(320) = 6400 and 20(320) = 6400.
1. a) The number of minutes when both plans cost the same; If Plan A is $C = 1200 + 1.6m$ and Plan B is $C = 1400 + 1.4m$, where m is the number of minutes and C is the total cost, then $1200 + 1.6m = 1400 + 1.4m$ because $C = C$. If you solve $1200 + 1.6m = 1440 + 1.4m$ for m, you get the number of minutes for which both plans cost the same. **b)** 1000 min; $1200 + 1.6m = 1440 + 1.4m \rightarrow$ $0.2m = 200 \rightarrow m = 1000$ **c)** Plan A for 600 min Plan B for 1200 min **2. a)** (–1, –9) **b)** (–2, –7**)** **c)** (1.2, 0.8) **3.** 41 and 7 (equations: $x + y = 48$ and $x - y = 34$) **4.** 38 and 57 (equations: $\frac{1}{2}x = \frac{1}{3}y$ and $x + y = 95$) **5.** 16 practice balls and 4 official balls (equations: $p + o = 20$ and $400p + 600o = 8800$) **6.** 60.2 L (equations: $m = 1295 + 0.737l$ and $m = 1290 + 0.82l$) **7.** 20 at Nu 4 and 40 at Nu 9 (equations: $4x + 9y = 440$ and $x + y = 60$)	**8. a)** (5, 13) **b)** (2, 2) **c)** (–1, 3) **9. a)** AC: $y = 2x - 2$; BD: $y = -2x + 10$ **b)** (3, 4) **10.** Wind speed: 40 km/h Plane's speed if there were no wind: 280 km/h **11. a)** (2, 2) **b)** (–1, 4) **c)** (6, 12) **12.** So you can compare the expressions involving the other variable and then solve for that variable.

Supporting Students

Struggling students

Struggling students will need more support to model situations using linear equations. Students can often solve the system of equations but find it difficult to create the equations to match the situation. If necessary, model this for several situations until students are more comfortable with the process.

Enrichment

• Some students might create their own problems that can be modelled by a system of equations.
Pairs of students can share their problems and solve each other's problems.

• Students may redo **example 2** by isolating a variable involving x rather than y.

3.2.2 Solving Algebraically — The Substitution Strategy

Curriculum Outcomes	Outcome relevance
10-C8 Systems of Linear Equations: solve • solve linear equations by substitution method, including comparison of equations **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations	• Many real-life situations are described by mathematical models involving more than one constraint. Systems of linear equations are often used to describe and make predictions about such situations. • There are a number of strategies for solving systems of equations. It is important for students to meet several of these strategies so that they can choose the most appropriate strategy for a given situation.

Pacing	Materials	Prerequisites
1 h	• None	• solving linear equations with decimal or integer coefficients • angle relationships in a triangle

Main Points to be Raised

• The substitution strategy is like the comparison strategy but it differs in that the variable, or sometimes a multiple of the variable, needs to be isolated in only one equation, not both.

• Once one variable, or a multiple of it, has been expressed in terms of another variable using one linear equation, that expression can be substituted into the other equation. The result is a linear equation in one unknown. This can be solved. The value for the other variable can then be calculated using either of the original equations.

• It is usually preferable to select the variable that is easier to isolate.

Try This — Introducing the Lesson

A. Suggest that students work in pairs. Observe while they work. You might ask:

• *Why can you add the two prices and multiply by 50 to determine the income if she sells 50 of each mask?* (Multiplying two different numbers by 50 and adding them is the same as multiplying the sum by 50; $50x + 50y = 50(x + y)$.)

• *Which operation did you use to solve **part ii)**?* (Division)

• *What strategy did you use to estimate?* (The total sales would be Nu 110,000 if 50 of each type were sold. To make the total sales lower, I reduced the number of Nu 1200 masks sold to 40.)

The Exposition — Presenting the Main Ideas

• Students can read through the exposition independently.

• Display another linear system on the board: $2y = x + 3$ and $3x + 4y = 9$.

Ask students to solve the system, first by substituting the expression $2y – 3$ for x into the second equation and then by substituting the value $x + 3$ for $2y$ into the second equation (and doubling it to represent $4y$). They can see that there is more than one way to use the substitution method in a given situation.

Revisiting the Try This

B. Students have an opportunity to relate their work from **part A** to their new learning about systems of equations.

Using the Examples

- Record the systems of equations displayed in **example 2** and **example 3** on the board. Ask students to try the problems before reading the examples.
- Have students then check their thinking by reading **examples 2 and 3** in the text, along with **example 1.**
- Have students do **example 2** by isolating x in the second equation $x - 2y = -2$.

Practising and Applying

Teaching points and tips

Q 1: Observe whether students realize that substitutions for x or y are both possible, though the first equation makes substituting for y attractive.

Q 2: Some students may need assistance in creating the equations. Partner each of these students with other students who can help them.

Q 3: Some students might want to create the graph. Discuss with them why that is not necessary.

Q 5: Students will need to know that the sum of the angles in a triangle is 180° and that adjacent angles on a straight line sum to 180°.

Common Errors

Some students neglect to multiply by the necessary value when they substitute. For example, substituting $3x + 2$ for $2y$ in the equation $3x + 6y = 13$ requires the student to multiply $3x + 2$ by 3. You might suggest that students write the equation so that the substitution is direct. For example, they could rewrite $3x + 6y = 13$ as
$3x + 3(2y) = 13 \rightarrow 3x + 3(3x + 2) = 13$.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can solve a system of linear equations with integer coefficients by substituting
Question 4	to see if students can set up a system of linear equations to model a problem and solve those equations
Question 5	to see if students can apply their new learning to solving problems in another strand, geometry
Question 7	to see if students can communicate their understanding of the relationship between two equation-solving strategies

Answers

A i) Nu 110,000 (50 × 1000 + 50 × 1200)
ii) Since the amount for 50 masks of each is just less than the actual sales, the number of more expensive masks has to be less than 50 and the number of less expensive masks has to be more than 50.
iii) *Sample response*: About 60 small ones and 40 large ones (60 × 1000 + 40 × 1200 = 108,000).

B. $s + l = 100$; $1000s + 1200l = 108{,}200$; 59 at Nu 1000 and 41 at Nu 1200

1. a) $x = 1$; $y = 3$ **b)** $x = -2$; $y = -1$
c) $x = \frac{1}{2}$; $y = \frac{1}{2}$ **d)** $x = \frac{2}{3}$; $y = \frac{3}{4}$

2. a) $l = d + 5$ **b)** $l + d = 29$
c) Lhamo is 17 years old and Devika is 12 years old.

3. a) (–1, –2) **b)** (5, –2)
c) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ **d)** $\left(\frac{2}{3}, \frac{5}{6}\right)$

4. 150, 62; Equations: $a + b = 212$, $a - b = 88$.

5. a) $x = 60$, $y = 60$;
Equations are $2x + y = 180$ and $x = 180 - 2y$;
Double the second equation and substitute;
$2x = 360 - 4y$; $(360 - 4y) + y = 180$; so $x = 60°$, $y = 60°$.

5. b) $x = 40$, $y = 50$;
Equations are $x + (2y + 40) = 180$ and $x + y = 90$.

6. a) $x = 4$; $y = 6$ **b)** $x = 12$; $y = -8$

c) $x = 2.5$; $y = 3.1$ **d)** $x = -1.7$; $y = 2.3$

7. The comparison strategy requires that the same term be isolated in each equation so that the remaining parts of the equations can be compared. The substitution strategy requires that only one term be isolated in one equation so that it can be substituted into the other equation.

Supporting Students

Struggling students

• Have students work with equations that have x or y on their own (with no numerical coefficient) as in $4y + x = 7$ and prompt them to solve for that variable: $4y + x = 7 \rightarrow x = 7 - 4y$.

• Some students might need to ensure that the substitution is always for $1y$ or $1x$ to make the problem easier. For example, for the system $3x - 2y = 8$ and $4y + 2x = 15$, although some students might substitute the value $(3x - 8)$ for $2y$ from the first equation into the second equation by doubling it, struggling students find it easier to solve the first equation for y as $\frac{3}{2}x - 4$ and then substitute.

Enrichment

• Students can create systems of equations where certain substitutions make sense. For example, a system where it might make sense to substitute $10y - 2$ for $3x$ might be $3x + 2 = 10y$ and $9x + 8y = 25$.

• Students could do **question 4** using only one variable n to represent the smaller number. They could also make up a similar question to solve two ways using one variable and two variables, respectively.

3.2.3 Solving Algebraically — The Elimination Strategy

Curriculum Outcomes	Outcome relevance
10-C8 Systems of Linear Equations: solve • solve linear equations by elimination method **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C13 Equations: rearrange** • transform equations from one form to another	• Many real-life situations are described by mathematical models involving more than one constraint. Systems of linear equations are often used to describe and make predictions about such situations. • There are a number of strategies for solving systems of equations. It is important for students to meet several of these strategies so that they can choose the most appropriate strategy for a given situation.

Pacing	Materials	Prerequisites
1 h	• None	• solving linear equations with decimal or integer coefficients

Main Points to be Raised

• A third strategy for solving a system of two linear equations in two unknowns is elimination. Equations are set up so that either by adding or by subtracting them, one variable is eliminated.

• Once the value for one of the variables has been determined, the value of the other can be determined by substituting into either equation.

Try This — Introducing the Lesson

A. Assign the task to individual students or pairs. Observe while students work. You might ask:

• *What do the variables in your equations represent?* (The number of Bhutanese stamps and the number of other stamps.)

• *Why does the provided information suggest two equations?* (One equation describes Dorji’s stamps and the other describes Deki’s stamps.)

• *How do you know that the solution involves integers?* (You cannot have parts of stamps.)

The Exposition — Presenting the Main Ideas

• Display the system $x + y = 10$ and $x - y = 2$ on the board. Ask students to figure out the solution using one of the strategies they know ($x = 6$ and $y = 4$).

• Write the equations with one below the other so students can observe that when you add the equations, the result is an equation involving only x.

$$\begin{array}{l} x + y = 10 \\ \underline{x - y = \ \ 2} \\ 2x \quad\ = 12 \end{array}$$

Ask students why it is mathematically correct to add equations. (It is like adding the same amount to both sides of a balance.) Students will see doing this yields the same result they offered before. Repeat this using subtraction instead of addition. This time the variables with the same sign (the $+x$ variables) will be eliminated, leaving $2y = 8$ or $y = 4$.

• Work through the exposition with the students. Make sure they realize that they must decide whether to add or subtract equations, and they must also decide whether or not to multiply one or both equation through by some constant(s) to make elimination possible.

• For example, if one equation involves $3x$ and the other involves $-6x$, they might double the first equation and then add the two equations. On the other hand, if the second equation involved $6x$, they would more likely double and then subtract.

Revisiting the Try This

B. This question allows students to make a formal connection between the problem they solved in **part A** and the elimination method for solving systems of linear equations.

Using the Examples

• Have students read through **example 1.** Ask them to explain why the student added the equations instead of subtracting them.

• Work through **example 2** with the students to make sure they understand where the equations came from. Then follow through by explaining the specific elimination method used in **example 2**. Ask students to suggest another way the system could have been solved using elimination and work through the solution that way with them (instead multiplying the first equation by 9).

Practising and Applying

Teaching points and tips

Q 1: Make sure that students realize for **part c)** that they must either double or triple the second equation to eliminate a variable.

Q 2: If students struggle with setting up the equation, you might refer them back to **example 1** in **lesson 3.2.1**.

Q 3: Some students may struggle to decide whether the variables should represent the steel and aluminium or the truck and the car. Encourage them to experiment and see what happens.

Common errors

• One of the most common errors students make in solving systems of equations occurs when they subtract one equation from the other using the elimination strategy. They frequently forget to apply the subtraction all the way across the equation. For example, they might write

$$\begin{array}{r} 5x + 3y = 1 \\ \underline{-\ 2x + 3y = -5} \\ 3x + 6y = -4 \end{array}$$
, but this is incorrect.

• Students are confused because they see the + sign next to the bottom $3y$ and they automatically add.
You might encourage students to change all the signs in the second equation before actually combining them:

$$\begin{array}{r} 5x + 3y = 1 \\ \underline{-\ 2x - 3y = +5} \end{array}$$

Suggested assessment questions from Practising and Applying

Question 1	to see if students can apply the elimination strategy with linear equations involving integer coefficients
Question 6	to see if students can set up a system of linear equations to describe a situation and solve the system
Question 9	to see if students have a good sense of when to apply the various strategies they have learned

Answers

A. i) $b + o = 200$ and $2b + 3o = 450$, where b is Bhutanese stamps and o is stamps from other countries. **ii)** Dorji has 150 and Deki has 300.	**B.** *Sample response*: Change the first equation to $3b + 3o = 600$ and subtract $2b + 3o = 450$. Solve for b and then substitute into one equation to solve for o.
1. a) $x = -1$; $y = 2$ **b)** $x = 1$; $y = 1$ **c)** $x = 3$; $y = -5$ **d)** $x = -2$; $y = -4$ **2. a)** $a + b = 2500$, where a is the amount invested at 4% and b is the amount invested at 5%. **b)** $0.04a + 0.05b = 115$, where a is the amount at 4% and b is the amount at 5%. **c)** Nu 1000 at 4% and Nu 1500 at 5%. **3. a)** $500t + 375c = 125{,}000$ for the amount of steel and $250t + 150c = 55{,}000$ for the amount of aluminium, where c represents the number of cars and t represents the number of trucks. **b)** 100 trucks and 200 cars. **4.** 16 correct (equations: $4c - i = 60$ and $c + i = 20$) **5. a)** $a = 100$, $b = 150$ **b)** $a = -1$, $b = 2$ **c)** $a = 1200$, $b = 3000$	**6.** She can make two batches of deluxe grade and two batches of fine grade (equations: $4.4d + 1.1f = 11.0$ and $19.8d + 16.5f = 72.6$). **7. a)** (10, 12) **b)** (–16, 8) **c)** (–12, –18) **8.** The company can make 1200 batches of light purple dye and 3600 batches of deep purple dye. (equations: $\frac{1}{4}l + \frac{1}{6}d = 900$ and $\frac{1}{6}l + \frac{1}{12}d = 500$) **9.** *Sample response*: **a)** $2x + 3y = 10$ and $4x - 3y = 50$; You can add them and eliminate the $3y$ right away. **b)** $y = 2x + 5$ and $x + y = 15$; The expression for y $(2x + 5)$ is right there for you to substitute into $x + y = 15$. **c)** $2y = 8x + 8$ and $2y = 4x + 15$; You can compare two equivalent expressions for $2y$.

Supporting Students

Struggling students

If students find the systems involving decimals or fractions too difficult, provide alternate systems to solve involving integers. You can create these systems by multiplying the given systems through by an appropriate constant.

Enrichment

Encourage students who are interested to create a problem like those in questions 3, 4, 6, or 8, which can be represented by a system of linear equations and solved using elimination.

CONNECTIONS: Matrix Solution of a Linear System

This connection introduces students to the use of matrices to solve systems of linear equations. This is one of the most valuable uses of matrices. The connection does not introduce formal terminology like *determinant*, but it introduces the concept implicitly.

You may wish to generalize what students discover by asking a question like this at the conclusion of their work on the connection:

Show that the matrix product on the left solves the system of equations on the right.

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix} \qquad \begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

You may even wish to extend the learning to introduce Cramer's rule for solving a system of linear equations with matrices.

Answers

1. $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8x+5y \\ 3x+2y \end{bmatrix}$ so $\begin{bmatrix} 8x+5y \\ 3x+2y \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$;
The top elements in each matrix are equal so $8x + 5y = 11$ and the bottom elements in each matrix are equal so $3x + 2y = 4$.

2. a) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; The top elements in each matrix are equal so $x = 2$ and the bottom elements in each matrix are equal so $y = -1$.

3. a) $\begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$
and $\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} \times \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

c) $x = -4$ and $y = 3$

4. a) $\begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,
so $x = -1$ and $y = 1$.

3.2.4 EXPLORE: Counting Solutions For Different Systems

Curriculum Outcomes	Lesson relevance
10-C8 Systems of Linear Equations: solve • solve linear equations by substitution method, including comparison of equations • solve linear equations by elimination method **10-C10 Equations: solve linear equations and linear inequalities** • encourage proficiency with algebraic manipulation **10-C13 Equations: rearrange** • transform equations from one form to another	In this optional lesson, students explore inconsistent and dependent equations to see how the number of solutions to a system of equations cannot be determined without considering the actual values, coefficients, and constants in the equations.

Pacing	Materials	Prerequisites
1 h	• None	• solving systems of linear equations

Main Points to be Raised

• There is usually only one solution to a system of two linear equations in two unknowns, but in some cases there is more than one solution.

•There are many solutions to a single linear equation in two unknowns.

• If one equation in two unknowns is a multiple of another equation, there are many solutions to that system of two equations.

• There is one solution to a system of three linear equations in two unknowns if exactly one of the equations is a linear combination of the other two, i.e., one equation is a result of combining multiples of the other two equations.

• There can be no solutions to a system of linear equations if the information in the equations is inconsistent.

Exploration

Encourage students to work in pairs. While one student solves one system of equations for a particular question, the other solves the other(s). The students then compare their answers and see what conclusions they can draw.

Observe while students work. You might ask:

• *How could you have rewritten the equation in **part A** to easily determine a number of solutions?* (I could have written y in terms of x and substituted different values for x to get lots of ordered pairs.)

• *What did you notice about the pairs of equations in **part B**?* (The second equation is really the same as the first — you just multiply the first equation by a number.)

•*Why is it easy to see why there cannot be a solution to the first pair of equations in **part E**?* (Because one equation says $2x + 6y$ equals 24 and the other says $2x + 6y$ equals 26 — both cannot be true at the same time.)

Observe and Assess

As students are working, notice:

• Are they relating the new problems to knowledge they already have? For example, are they relating the situations in **part A** to the fact that they would be graphing a line with an infinite number of points?

• Are they summarizing their observations completely and clearly?

Share and Reflect

• Encourage several students to share their summaries.

• Ask them to complete this sentence: A system of two equations in two unknowns can have ___ solutions.

• Ask them to create a system of equations to meet each condition:

- Two equations in two unknowns with many solutions (e.g., $3x - 4y = 8$ and $6x - 8y = 16$)
- Three equations in two unknowns with many solutions (e.g., $3x - 4y = 8$, $6x - 8y = 16$, and $12x - 16y = 32$)
- Three equations in two unknowns with one solution (e.g., $3x - 4y = 8$, $x - y = 3$, and $4x - 5y = 11$)

Answers

A. i) *Sample response*: $x = 1$ and $y = 0$, $x = -2$ and $y = 2$, and $x = -5$ and $y = 4$

ii) *Sample response*: $x = 0$ and $y = -5\frac{1}{3}$, $x = 8$ and $y = 1\frac{1}{3}$, and $x = 10$ and $y = 3$

iii) *Sample response*: $x = 0$, $y = 0$, and $z = 8$; $x = \frac{8}{3}$, $y = 0$, and $z = 0$; and $x = 0$, $y = -1\frac{1}{3}$, and $z = 0$

B. i) *Sample response*: $x = 1$ and $y = 0$, $x = -2$ and $y = 2$, and $x = -5$ and $y = 4$

ii) *Sample response*: $x = 0$ and $y = -5\frac{1}{3}$, $x = 8$ and $y = 1\frac{1}{3}$, and $x = 10$ and $y = 3$

C. i) The only possibility is $x = -2$ and $y = 5$.

ii) The only possibility is $x = 2$ and $y = \frac{1}{2}$.

D. i) The only possibility is $x = -2$ and $y = 5$.

ii) The only possibility is $x = 2$ and $y = \frac{1}{2}$.

iii) The only possibility is $x = 2$ and $y = \frac{1}{2}$.

E. i) If $2x + 6y = 26$, there is no way $2x + 6y$ can also be equal to 24.
ii) If $6x + 4y = 14$, then $3x + 2y = 7$; if $3x + 2y = 7$, there is no way $3x + 2y$ can also be equal to 5.

F. *Sample response*: The solution $x = -2$ and $y = 5$ works for the first two equations, which means $x + y = -2 + 5 = 3$. If $x + y = 3$, then there is no way to also satisfy the third equation because it states that $x + y = 4$.

G. *Sample response*:
- If there is one equation involving two or three variables, it has more than one solution.
- If there are two equations involving two variables, sometimes there is one solution, sometimes there are many solutions, and sometimes there are no solutions.
- If there are three equations involving two variables, sometimes there is a solution, but sometimes there is no solution.

Supporting Students

Struggling students

You could assist students who struggle by suggesting they use a graphical approach rather than an algebraic approach to work through the questions.

Enrichment

You could suggest that students consider solving two or three equations in three unknowns to see what could happen.

UNIT 3 Revision

Pacing	Materials
2 h	• Grid paper (BLM in Unit 6) • Rulers

Question(s)	Related Lesson(s)
1 – 3	Lesson 3.1.1
4 – 8	Lesson 3.1.2
9 – 11	Lesson 3.1.3
12	Lesson 3.1.4
13 – 15	Lessons 3.2.1 to 3.2.3
16	Lessons 3.2.1 and 3.2.2
17	Lesson 3.3.3
18	Lessons 3.2.1 to 3.2.3

Revision Tips

Q 2: This idea was addressed in the lesson but you may need to remind some students that to determine whether something is a function, you might represent it as a set of ordered pairs.

Q 4: Students will need to recall that the sum of the angle in a triangle is 180°.

Q 7: Some students might accidentally represent the mean in terms of the first number instead of the second. Ask them what adjustments would have to be made to their results.

Q 8: Remind students to create the line of best fit by eye and then use first the intercept and then the slope to define the equation.

Answers

1. A; there is only one y-value for each x-value in A, but there are two y-values for some x-values in B.

2. a) No; for any x-value there is only one y-value, 3.

b) Yes; for $x = 3$, there are multiple y-values.

3. a) *Sample response*:

x	$f(x)$
1	7
2	4
3	1
4	–2
5	–5

b) To get the related y-value for each x-value, multiply x by 3 and then subtract the result from 10. For every different x-value you will get a different y-value.

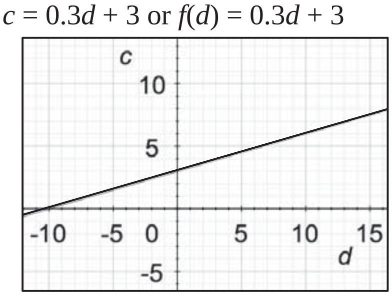
4. a) $f(y) = (180 - y) \div 2$ **b)** $f(x) = 180 - 2x$

c) You could decide on the x-value first, so the value of y would depend on x. Or, you could decide on a y-value first so the value of x would depend on y.

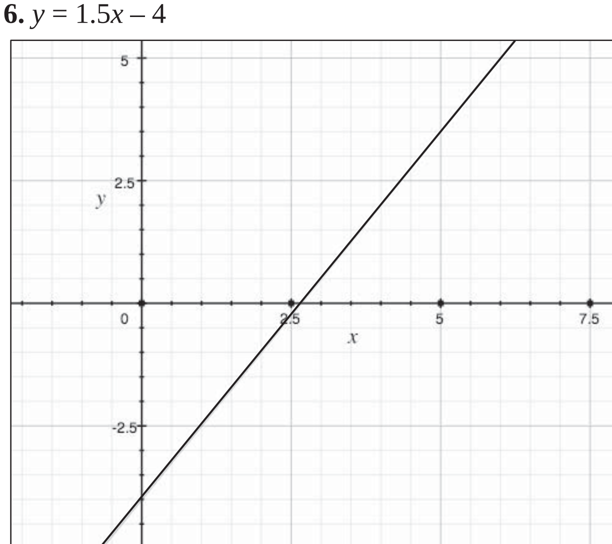
d) $f(20) = 180 - 40 = 140$

5. a) $c = 0.3d + 3$ or $f(d) = 0.3d + 3$

b)



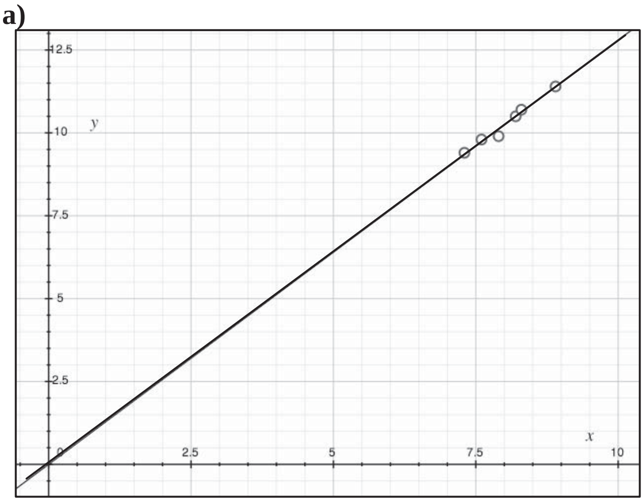
6. $y = 1.5x - 4$



7. $f(x) = x + 1$ $[(x - 1, x, x + 1, x + 2, x + 3) \div 5 = x + 1]$

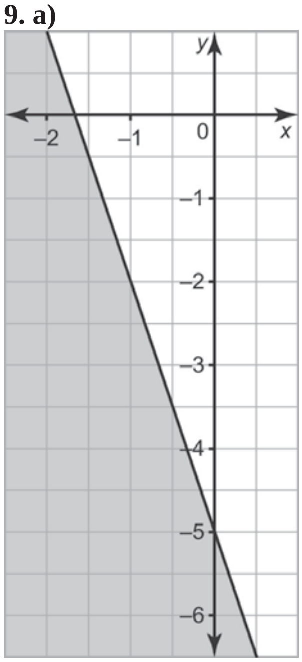
8. *Sample response*:

a)

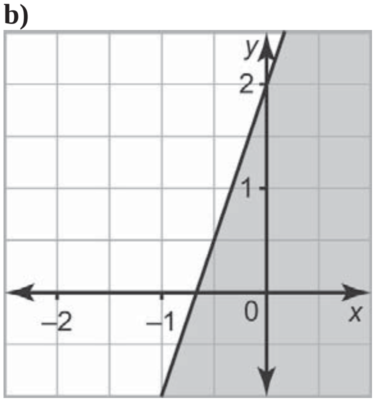


b) $y = 1.3x$

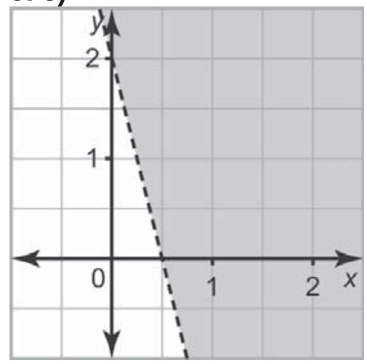
9. a)



b)



9. c)



10. a) $y > 4x - 8$ **b)** $y \leq 4x - 5$

11. a) $3y < 2x + 4$ will have a dashed line because $<$ means values along the line are not included.
$3y \leq 2x + 4$ will have a solid line because the $\leq$ sign means the values along the line are included.
Otherwise they are the same.

b) The dashed line for $4y + 2x < 10$ will be 2 units higher than $4y + 2x < 8$ because the y-intercept is 10, not 8. Otherwise they are the same.

12. a) Shifted (translated) 4 units to the left.

b) Shifted (translated) 4 units to the right.

c) Slope is 4 times as steep and y-intercept is lower.

d) Slope is 4 times as steep but in the opposite direction and the y-intercept is higher.

13. a) (5, 22) **b)** (1, 2.5) **c)** (4, 6) **d)** (2, 17)

14. 21 items (equations: $8c - 2i = 150$ and $c + I = 30$)

15. 39 cm × 21 cm and 78 cm × 7 cm
(equations: $2l + 2w = 120$ and $4l + \frac{2}{3}w = 170$)

16. a) $x = 2$ and $y = 4\frac{1}{3}$ **b)** $y = 4$ and $x = -2$

c) $x = 5$ and $y = 5$

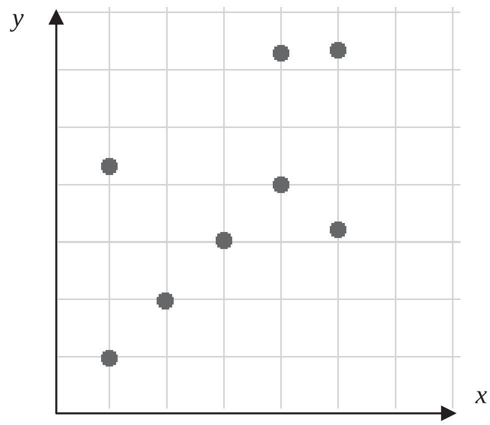
17. a) $x = 5\frac{2}{3}$ and $y = 7$ **b)** $y = 4$ and $x = 2$

c) $x = 3$ and $y = 0.5$

18. *Sample response*:
If you do not need an exact answer and you have grid paper available.

UNIT 3 Linear Functions and Relations Test

1. a) Explain why this graph represents a relation that is not a function.



b) How could you change the graph so that it represents a function?

2. The input for a certain relation is the date. What could the output be, if the relation is a function?

3. a) Create a table of values for the function $f(x) = 2x^2 - 3x$ using at least five values of x.

b) How do you know $f(x)$ is a function?

4. Dorji invests some money and gets 4.25% interest from the bank. Write a function to describe the interest received in terms of the amount invested.

5. a) Write a linear function that describes the sum of the numbers in any L on this calendar in terms of the number at the bottom left of the L.

		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

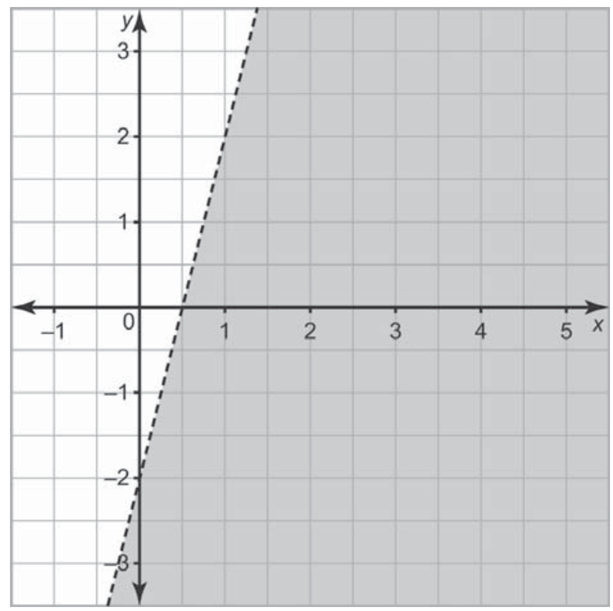
b) If the sum of the numbers in an L is 88, what is the number at the top?

6. Graph each relation.

a) $2y \leq 4x + 9$

b) $4y - 5 > 3x$

7. What inequality does this graph show?



8. For each pair of lines, determine the point of intersection.

a) $y = 2x + 5$ and $y = 3x + 2$

b) $y = 4x - 3$ and $2y + 5x = 72$

c) $3x - 4y = 16$ and $5x + 2y = 44$

9. The perimeter of a rectangle is 148 cm. The length is 12 cm greater than the width. What are the length and width?

10. a) Create a system of two linear equations that you would solve by elimination.

b) Adjust one or two values (but not three) in each of your equations from **part a)** so that the solution is $x = 3$ and $y = 4$.

UNIT 3 Test

Pacing	Materials
1 h	• Grid paper (BLM in Unit 6) • Rulers

Question(s)	Related Lesson(s)
1 – 3	Lesson 3.1.1
4, 5	Lesson 3.1.2
6, 7	Lesson 3.1.3
8, 9	Lessons 3.2.1 to 3.2.3
10	Lesson 3.2.3

Select questions to assign according to the time available.

Answers

1. a) Some x-values correspond to more than one y-value.
b) *Sample responses*:
Remove the points with the three highest y-values from the graph.

2. *Sample response*: How many days are left in the year after that date?

3. a) *Sample response*:

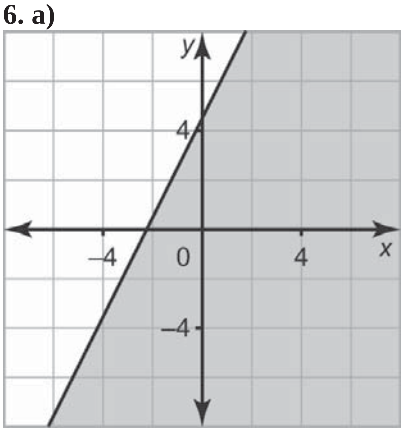
x	$f(x)$
0	0
1	–1
2	2
3	9
4	20

b) For each x-value, there is only one possible value for $2x^2$ minus $3x$, which is y.

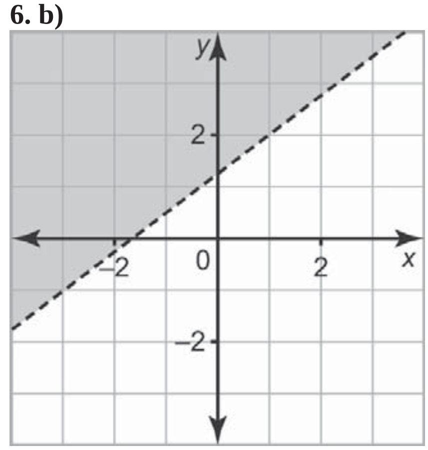
4. $f(i) = 0.0425i$

5. a) $f(c) = 4c - 20$ **b)** 13

6. a)



6. b)



7. $y < 4x - 2$

8. a) (3, 11)

b) (6, 21)

c) (8, 2)

9. Length is 43 cm and width is 31 cm (equations: $2l + 2w = 148$ and $l - w = 12$).

10. a) *Sample response*:
$3x + 2y = 19$ and $8x - 2y = 14$

b) *Sample response*:
$3x + 2y = \mathbf{17}$ and $\mathbf{7}x - 2y = \mathbf{13}$

UNIT 3 Performance Task — When will women run faster?

The winner of the Olympic 100 m sprint is often called the fastest human on earth. For many years, the winning times for this race have been declining for both men and women. Some people claim that the female sprinters' times have been improving more quickly than the male times. They say that eventually the fastest human on earth will be a woman.

Women's 100 m Sprint Winning Times

Year	Name	Country	Time (s)
1928	Elizabeth Robinson	USA	12.2
1932	Stanislawa Walasiewicz	POL	11.9
1936	Helen Stephens	USA	11.5
1948	Fanny Blankers-Koen	NED	11.9
1952	Marjorie Jackson	AUS	11.5
1956	Betty Cuthbert	AUS	11.5
1960	Wilma Rudolph	USA	11.0
1964	Wyomia Tyus	USA	11.4
1968	Wyomia Tyus	USA	11.0
1972	Renate Stecher	GDR	11.07
1976	Annegret Richter	FRG	11.08
1980	Lyudmila Kondratyeva	URS	11.06
1984	Evelyn Ashford	USA	10.97
1988	Florence Griffith-Joyner	USA	10.54
1992	Gail Devers	USA	10.82
1996	Gail Devers	USA	10.94
2000	Marion Jones	USA	10.75
2004	Yuliya Nesterenko	BLR	10.93

Men's 100 m Sprint Winning Times

Name	Country	Time (s)
Percy Williams	CAN	10.8
Eddie Tolan	USA	10.3
Jesse Owens	USA	10.3
Harrison Dillard	USA	10.3
Lindy Remigino	USA	10.4
Bobby Morrow	USA	10.5
Armin Hary	FRG	10.2
Robert Hayes	USA	10.0
Jim Hines	USA	9.9
Valeriy Borzov	URS	10.14
Hassely Crawford	URS	10.08
Allan Wells	GBR	10.25
Carl Lewis	USA	9.99
Carl Lewis	USA	9.92
Linford Christie	GBR	9.96
Donovan Bailey	CAN	9.84
Maurice Greene	USA	9.87
Justin Gatlin	USA	9.85

A. i) Use the data to create a scatter plot for the 100 m winning times for women.

ii) Draw a line of best fit.

iii) Determine the equation of the line of best fit.

B. Repeat **part A** for men on the same grid.

C. i) Use the equations from **parts A and B** to predict the Olympic year in which the fastest human on earth will be a woman.

ii) What do you predict the men's and women's winning times will be that year?

iii) Do you think this is likely to happen? Why or why not?

UNIT 3 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-C2 Patterns and Real-world Relationships: describe, identify and apply **10-C4** Graphs and Tables: construct and analyse **10-C6** Graphs: sketch **10-C7** Graphs: create by constructing a table of values and graphing **10-C8** Systems of Linear Equations: solve	1 h	• Grid paper (BLM in Unit 6) • Rulers

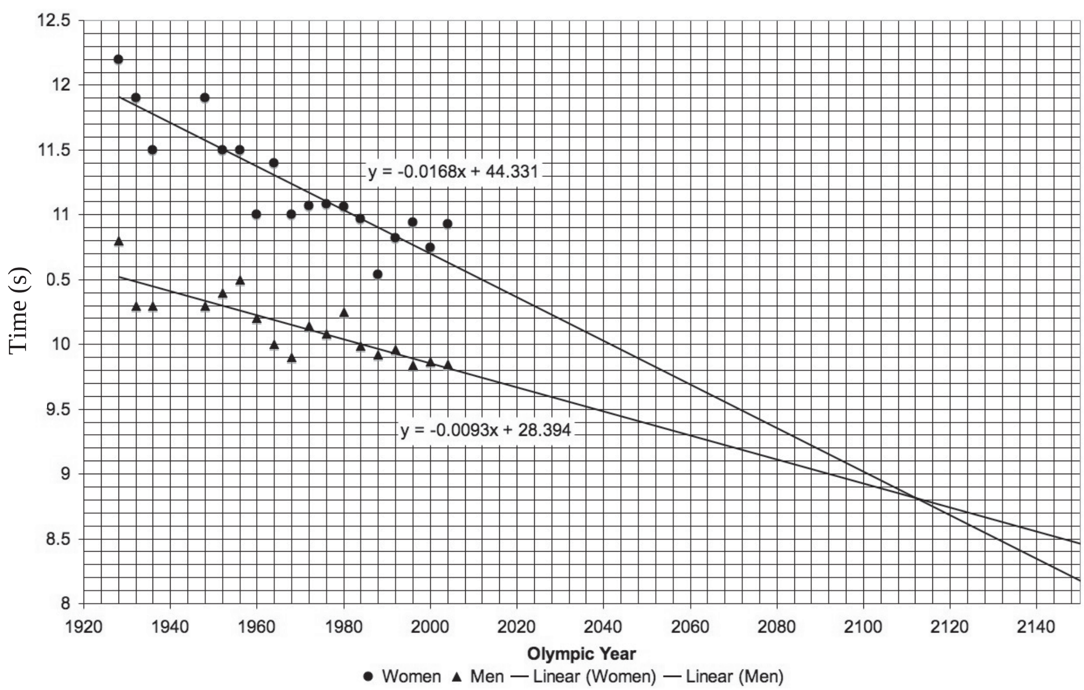
How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used as enrichment material for some students. You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution

A. and B.

Winning Times for Olympic 100 m Sprint (Men and Women)



C. i) $-0.0168x + 44.331 = -0.0093x + 28.394 \rightarrow 15.937 = 0.0075x \rightarrow 2125 \approx x$
Prediction: a woman will become the fastest human on earth sometime around the year 2125, at the Olympic Games held in 2124 or 2128.

ii) In 2124, the predicted women's winning time is $-0.0168(2124) + 44.331 \approx 8.65$ s and the predicted men's winning time is $-0.0093\ (2124) + 28.394 \approx 8.64$ s.
In 2128, the predicted women's winning time is $-0.0168(2128) + 44.331 \approx 8.58$ s and the predicted men's winning time is $-0.0093\ (2128) + 28.394 \approx 8.60$ s.

iii) *Sample response*:
It is not likely to happen because the speed at which a man or woman can run will eventually reach a maximum and this may happen before the women's time is less than the men's. The women's time has only improved from about 12 s to about 11 s since 1920 and the men's time has only improved from about 10.5 s to about 9.75 s.

UNIT 3 Performance Task Assessment Rubric

	Level 4	**Level 3**	**Level 2**	**Level 1**
Determining equations for each line	Determines equations efficiently and correctly	Determines equations correctly, but inefficiently.	Determines one equation correctly	Major errors in relating the equations of the lines to the graphs
Solving the system of equations	Uses efficient, algebraic strategy for solving the system	Uses the graph to estimate the solution, but does some algebraic work to check	Only estimates the correct solution using the graph	Major errors in determining the solution
Interpreting the solution	Insightfully uses the data to relate the mathematical solution to the likely real-world situation in addressing the problem	Considers both the mathematical solution and the real-world situation in addressing the problem	Uses only intuitive knowledge or only the mathematical solution to address the problem	Incomplete or inappropriate use of the mathematical model and/or faulty intuitive knowledge applied to address the problem

UNIT 4 MEASUREMENT

UNIT 4 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 103** **TG p. 112**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Grid paper • Compasses • Rulers (mm) • Protractors	All questions
Chapter 1 Precision				
4.1.1 Precision and Accuracy **SB p. 105** **TG p. 114**	**10-D1 Measurement: accuracy and precision** • understand that accuracy depends upon the usage of the measurement instrument • understand that precision depends on how finely an instrument is graduated • express answers to problems with significant figures	1 h	• Rulers (mm) • Strips of paper to make cm rulers (optional)	Q1, 5, and 6
CONNECTIONS: Precision Instruments **SB p. 112** **TG p. 116**	Learn about the history of precision in measuring instruments and also explore their own environment for measuring instruments	1 h out of class; 30 min in-class discussion	• Measuring instruments at home and in the community	N./A
4.1.2 EXPLORE: Measurement Error (Optional) **SB p. 113** **TG p. 117**	**10-D1 Measurement: accuracy and precision** • address precision issues when performing calculations on measurement data	1 h	• Calculators	Observe and Assess questions
4.2.1 EXPLORE: Regular Polygons with a Constant Perimeter (Optional) **SB p. 115** **TG p. 119**	**10-D1 Measurement: accuracy and precision** • address precision issues when performing calculations on measurement data • express answers to problems with significant figures **10-D2 Perimeter and Area: explore and apply properties** • examine maximizing area while restricting perimeter **10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • understand that areas of regular polygons can be determined by dividing the area into familiar shapes	1 h	• Compasses • Rulers (mm) • Protractors • Grid paper • Calculators	Observe and Assess questions
4.2.2 2-D Efficiency **SB p. 117** **TG p. 121**	**10-D2 Perimeter and Area: explore and apply properties** • examine maximizing area while restricting perimeter • examine minimizing perimeter while restricting area	1 h	• Elastics • Pens • Rope • Rulers	Q1, 3, 4, and 6

UNIT 4 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Efficient Design [Continued]				
4.2.3 3-D Efficiency **SB p. 121** **TG p. 124**	**10-D3 Volume, Capacity, and Surface Area: demonstrate understanding** • understand that surface area, capacity, and volume apply to 3-D shapes • understand the connection between volume and surface area • compare prism measurements **10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • determine capacity using volume	2 h	• Calculators	Q2, 4, 6, and 11
CONNECTIONS: Animal Shapes and Sizes **SB p. 125** **TG p. 126**	• Connect the surface area and volume ratio to animal shapes and sizes • Explore the effect of size on the ratio of surface area to volume	30 min	• Calculators	N/A
UNIT 4 Revision **SB p. 126** **TG p. 127**	Review the concepts and skills in the unit	2 h	• Calculators	All questions
UNIT 4 Test **TG p. 129**	Assess the concepts and skills in the unit	1 h	• Calculators	All questions
UNIT 4 Performance Task **TG p. 131**	Assess concepts and skills in the unit	1 h	• Rulers (mm) • Protractors • Compasses	Rubric provided

Math Background

• In this unit, students start to recognize the effect of precision on the confidence one can have in a reported measurement. Students have implicitly considered these issues earlier, but usually without thinking about them. At this mathematical stage, students are ready to consider the implications of measuring with different degrees of precision.

• Students also explore the relationships between area and perimeter of 2-D shapes and volume and surface area of 3-D objects. They examine which shapes and objects are most efficient. Efficiency is described in two ways — the maximum area for a given perimeter (or volume for a given total surface area) or the minimum perimeter for a given area (or total surface area for a given volume).

• Much of the focus in this unit is on explanation – explanation of accuracy and precision issues, and explanation of reasoning to support predictions of surface area and volume relationships.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in **question 4**, **part d)** in **lesson 4.1.1**, where they must consider all possibilities for a certain situation. Other examples of problem solving are found in **question 5** of **lesson 4.2.2**, where they apply the concept of 2-D efficiency, and in **question 9** of **lesson 4.2.3**, where they apply the concept of 3-D efficiency to a real-world situation.

- They use communication frequently to explain their thinking, for example, in **question 7** of **lesson 4.1.1**, where they discuss the effect of differences in levels of precision when making a comparison, and in **question 9** of **lesson 4.2.2**, where they describe situations in which efficiency might matter.

- They use reasoning in answering questions such as **question 4**, **part b)** of **lesson 4.1.1**, where they think about the relation between the number of digits and the number of significant figures in a number, and in **question 6**, **part a)** of **lesson 4.1.1**, where they reason about the effect of rounding.

- They use visualization throughout **Chapter 2** when they visualize what makes a shape more circle-like or sphere-like. They may also use visualization in **question 3** of **lesson 4.1.1**, where they can use a mental image of a number line to visualize what range of numbers can be rounded to 12 cm.

- Students see connections, for example, in **lessons 4.2.1, 4.2.2, and 4.2.3,** where they relate the concept of efficiency to real-life situations. The **Connections** features connect the concepts of precision and 3-D efficiency, respectively, to manufactured and natural-world situations.

Rationale for Teaching Approach

• This unit is divided into two chapters. Because of the nature of the material being covered, two of the five lessons are exploratory.

- **Chapter 1** deals with measurement accuracy to a limited extent and with measurement precision to a greater extent. The notion of significant figures is raised as a way to talk about measurement precision. It makes sense to talk about measurement accuracy in **Chapter 1** before using the concept in **Chapter 2**.

- **Chapter 2** focuses on the relationship between different measures of a shape or object, whether perimeter and area or volume and surface area. It builds on the students' ability to use measurement formulas, developed in earlier classes.

• The **Explore** feature in **Chapter 1** provides an opportunity for students to calculate the percentage of possible error when measurements are more or less precise. It is not a required lesson, but it does clarify how precision affects the confidence one can have in a reported measurement. The **Explore** feature in **Chapter 2** allows students to consider the areas of 2-D shapes with a fixed perimeter. It is valuable because it makes the next lesson more meaningful, but it is also optional.

• There are also two **Connections** features. The one in **Chapter 1** is historical and allows students to see how measuring instruments have become more precise over time. The feature in **Chapter 2** connects the mathematics being learned to the context of the animal kingdom.

• It is important that discussion of precision and accuracy continue throughout the unit, into **Chapter 2**, and beyond into the trigonometry unit.

Technology in This Unit

• Spreadsheet software, such as Microsoft Excel, can be used effectively in some of the trial and error exploration questions in **lesson 4.2.3**.

• Calculators will be used extensively for formula calculation. It is helpful but not necessary for the calculators to have a π button. If they do not have a π button, students can manually enter an approximation of π, that is, 3.14 or $\frac{22}{7}$.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Volume and Surface Area: estimate and calculate for pyramids, cones, and spheres **9** Volume and Surface Area: estimate and calculate for right prisms and cylinders **9** Volume and Surface Area: estimate and calculate for composite 3-D shapes **9** SI Units: solve measurement problems involving conversion **8** Area (circles): develop formula **8** Pythagorean Relationships: application **7** Angles: sum of interior angles in a triangle **7** Relationships: triangles **7** Angles: estimate and measure using a protractor **7** Circle: solve problems with diameter, radius, circumferences	Students will experience more success in this unit if they review what they already know about volume, surface area, and metric measurement units.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6) • Compasses • Rulers (mm) • Protractors	• volume and surface area formulas for pyramids, cones, spheres, right prisms, and right cylinders • SI units • area and circumference formulas for circles • sum of the interior angles is 180° for a triangle • measuring and constructing angles

Main Points to be Raised

• Shapes can have the same area, but different perimeters.

• Shapes that are more "compact" or circle-like have the same area but with a lesser perimeter.

Use What You Know — Introducing the Unit

• Students can try this activity alone or with a partner.

• Observe students as they work. You might ask:

• *How did you figure out the missing dimensions on the second shape?* (I knew it had to be 3 all the way across; the bottom width had to be 1 since the rest of it was the 2 that was already shown. I knew the left top edge had to be 1 since the total had to be 4 and there was already 3 for the bottom part.)

• *Notice that the right triangle had twice the height of the rectangle, but the same width. Why did that happen?* (When you use the formula for the area of a triangle, you take half, so if the bases are equal and you want the areas to be the same, you need to double the height of the triangle.)

•*How did you create other shapes with the same area?* (I took the original rectangle, cut off a corner and attached it to a different side.)

• *How many different kinds of shapes can you make with the same area? Why?* (I can make as many as you want; I just take any part of it, cut it up and rearrange the parts into a new shape.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually, but it will be helpful if each student has a partner with whom he or she can discuss answers or difficulties.

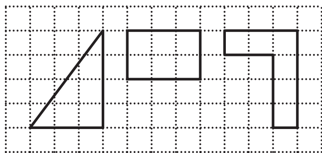
Common Errors

• Some students might find the answers in the answer guide to be different from their own answers. This is likely to occur because of rounding. If this issue comes up, praise the students for noticing it and inform them that this issue is what **Chapter 1** in this unit is all about. It is a very important issue in measurement.

• In **question 3, part c)**, students who do not pay close attention to the vertex names may not realize that to construct the triangle they should find angle K by subtracting the total of the other angles from 180 degrees. Encourage students to label their vertices and be sure that the labels match the measured sides and angles.

Answers

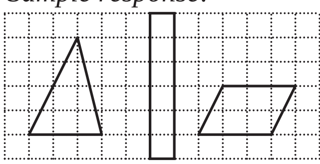
A. i)



ii) 6 square units

B. *Sample response*:

i)



Triangle: 3 units (base) by 4 units (height)
Rectangle: 1 unit (width) by 6 units (length)
Parallelogram: 3 units (base) by 2 units (height)

A. iii) *Sample response*:
I predict the rectangle because its perimeter looks longer since the side lengths are farther apart.
iv) Perimeter of triangle is 12 units; perimeter of rectangle is 10 units; perimeter of L-shape is 14 units.

B. ii) *Sample response*: (*Students might apply measurement formulas, as shown below, or count grid squares to find* areas)

Area of triangle: $A = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6$ square units
Area of rectangle: $A = bh = (1)(6) = 6$ square units
Area of parallelogram: $A = bh = (3)(2) = 6$ square units

iii) Rectangle's perimeter is longest at 14 units (the parallelogram perimeter is $6 + 2\sqrt{5}$, which is about 10.5 units and the perimeter of the triangle is $3 + \sqrt{17} + \sqrt{20}$, which is about 11.6 units); Long, skinny shapes usually have longer perimeters.

1. a) Volume: $4 \times 1.45 \times 1.45 \approx 8.4\text{ cm}^2$.
Total *SA*: $SA = 2 \times (1.45 \times 1.45 + 4 \times 1.45 + 4 \times 1.45) \approx 27.4\text{ cm}^2$.

b) Volume: Base octagon is 8 triangles, each with $A = \frac{1}{2}bh = \frac{1}{2}(4.1)(5) = 10.3$, so $V = \frac{1}{3}Ah = \frac{1}{3}(10.25 \times 8)(12) \approx 328\text{ m}^3$.
Total *SA*: Base octagon is 8 triangles, each with $A = \frac{1}{2}bh = \frac{1}{2}(4.1)(5) = 10.25$, so area of base is $8 \times 10.25 = 82\text{ m}^2$. The area of each lateral triangle is $A = \frac{1}{2}bh = \frac{1}{2}(4.1)(13) = 26.65\text{ m}^2$. So, $SA = 82 + 8 \times 26.65 \approx 295.2\text{ m}^2$.

c) Volume: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6)^2(8) = 96.0\pi \approx 301.6\text{ cm}^3$.
Total *SA*: $SA = \pi r^2 + \pi rs = \pi(6)^2 + \pi(6)(10) = 36\pi + 60\pi = 96\pi \approx 301.6\text{ cm}^2$.

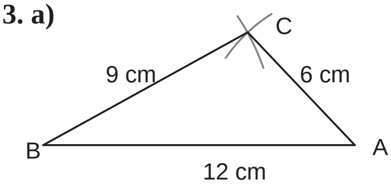
d) Volume: $V = s^3 + \pi r^2 h = (3)^3 + \pi(1)^2(2) \approx 33.3\text{ cm}^3$.
Total *SA*: $SA = 6s^2 + 2\pi(1)(2) + \pi(1)^2 - \pi(1)^2 = 6(3)^2 + 4\pi \approx 66.6\text{ cm}^2$.

e) Volume: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 = 113.1\text{ cm}^3$.
Total *SA*: $SA = 4\pi r^2 = 4\pi(3)^2 = 113.1\text{ cm}^2$.

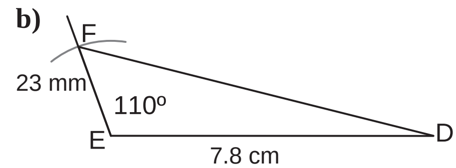
1. f) Volume: $V = \pi r^2 h = \pi(3)^2(2) = 18\pi \approx 56.5\text{ cm}^3$.
Total surface area: $SA = 2\pi r^2 + 2\pi rh = 2\pi(3)^2 + 2\pi(3)(2) = 18\pi + 12\pi = 30\pi \approx 94.2\text{ cm}^2$.

2. a) 3.2 cm **b)** 1700 g **c)** 0.270 L **d)** 20,000 cm^2
e) 7 cm^3 **f)** 4300 cm^3 **g)** 8200 mm^3

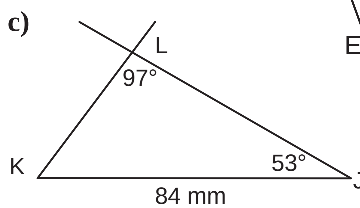
3. a)



b)



c)



4. a) circumference: 10π cm; 31.4 cm
area: 25π cm; 78.5 cm^2
b) circumference: 6.4π cm; 20.1 cm
area: 10.24π cm; 32.2 cm^2

5. a) 40° **b)** 66°

6. a) 3.01 cm **b)** 5.13 cm

Supporting Students

Struggling students

If students are struggling with formulas, give them a list of relevant formulas. Avoid telling them which formula is used in which case. They should be able to recognize the formulas.

Chapter 1 Precision

4.1.1 Precision and Accuracy

Curriculum Outcomes	Outcome relevance
10-D1 Measurement: accuracy and precision • understand that accuracy depends upon the usage of the measurement instrument • understand that precision depends on how finely an instrument is graduated • express answers to problems with significant figures	Students will face issues relating to precision and accuracy throughout their further work in mathematics (in trigonometry, for example). Knowledge of these issues will also help them be critical, informed citizens as they face measurements outside of school (where they will read scientific, statistical, and other reports that impact their lives).

Pacing	Materials	Prerequisites
1 h	• Rulers (mm) • Strips of paper to make cm rulers (optional)	• rounding

Main Points to be Raised

• Precision depends on how finely an instrument is graduated. It is a matter of discretion to then choose a proper level of precision.

• If two people measure the same item and one reports it with more significant figures than the other, the number with more significant figures is more precise. For example, 3.230 m is more precise than 3.23 m. With 3.23 m, the actual value could be between 3.225 m and 3.235 m (a range of 0.01 m). With 3.230 m, the actual value could be between 3.2295 m and 3.2305 m (a smaller range of 0.001 m). The smaller the range, the more precise the measurement.

• Rules for describing significant figures are slightly different for whole numbers than for decimals.

• There are alternate conventions for the precision that should be used in reporting calculated measurements. One convention is to report using the number of significant figures in the least precise measurement. Another convention is to round to the least number of decimal places in the values used for the calculation.

• Accuracy depends upon the correct usage of the measurement tool. A measurement is accurate if it was performed and reported correctly. Sometimes inaccuracy is hard to avoid because a measurement is difficult to take.

Try This — Introducing the Lesson

A. Ask students first to measure alone and then to compare results with a partner. After this, have a class discussion in which students say (or write on the board) the dimensions of their books in millimetres. It can be expected that the results will differ (because not all books are exactly the same and because some students will measure more carefully than others). Ask students to suggest reasons for the differences in measurements.

Observe while students work. You might ask:

• *Which number was greater? Why?* (The millimetre measure was greater since millimetres are smaller.)

• *How did you convert the millimetre measure to centimetres?* (I divided by ten.)

• *Was the converted millimetre measure the same as your centimetre measure?* (No; the number in the tenths place made it different.)

The Exposition — Presenting the Main Ideas

• Ask students to read independently through the first few paragraphs up until the point where significant figures are introduced.

• Guide students through the section on significant figures by recording the values on the board and pointing to the significant figures in each case. Help students understand the relationship between significant figures and measurement precision.

• Students can then read the section on accuracy independently.

• To test understanding, you can do the following:

- Ask students to discuss with a classmate: "What is the difference between accuracy and precision?" Then ask some pairs to report to the rest of the class.

- Allocate four places in the room to represent one, two, three, and four significant figures. Then write a number on the blackboard. Students move to the place that represents the number of significant figures in the number. Ask individuals how they know the given number of significant figures. Do this for various numbers.

- Ask students to suggest examples of inaccurate measurements. You might offer these ideas if they do not come forward: someone might not ensure that he or she is starting at the 0 mark on a ruler, or someone who is reading the number of millilitres in a graduated cylinder might read the scale from too low or high (instead of level with the top of the fluid).

• If time permits, you might also ask these questions:
- *If you measure the distance between two buildings, describe a situation in which you would want to be precise to the nearest metre (or even more precise).* (E.g., if you are running a cable or electrical wire between the buildings.)

- *If you measure the distance between two buildings, describe a situation in which you would not need much precision*. (E.g., If someone asks you how far apart they are, you could say "about 20 or 30 metres.")

Revisiting the Try This

B. This question gives the students an opportunity to use the new terminology (*accuracy* and *precision*) in their description of the issues they encountered measuring the dimensions of their textbooks in **part A**.

Using the Examples

• Have students close their textbooks. Write the questions from **examples 1 and 3** on the board for students to work through. They can then compare their responses to those in the text. They can also read through **example 2**.

• Point out that the Thinking in **example 3** mentions that a time reported as 36 s could have actually been anywhere between 35.5 s and 36.5 s. Conventionally, the 36.5 s is rounded up rather than down, but measurements very, very close to, but slightly lower than 36.5 would be rounded down.

Practising and Applying

Teaching points and tips

Q 1: You might observe whether students realize that the answers for **parts a), d), and j)** would remain unchanged whether or not they are first converted from scientific notation.

Q 3a: Some students might benefit from writing the measurements first in the form [ ].[ ] cm and then as millimetres.

Q 5: Remind students, if necessary, of the difference between precision and accuracy.

Q 6: Students should be considering the range of possible measurements that would have resulted in the two reported measurements.

Common Errors

Some students might have trouble applying the rules for counting significant figures. To help them internalize these rules, it is helpful to ask them to include with their answers an explanation of how they counted the digits.

Suggested assessment questions from Practising and Applying

Question 1	to see if students understand which digits are taken as significant
Question 5	to see if students understand the meanings of precision and accuracy
Question 6	to see if students can use reasoning when they compare measurements reported to different levels of precision

Answers

A. *Sample responses (based on a 18.2 cm by 25.7 cm cover size)*:
i) 18 cm by 26 cm
ii) 182 mm by 257 mm; 18.2 cm by 25.7 cm
iii) You can be more exact with the millimetre ruler.

B. The millimetre ruler lets you be more precise because you can record measurements with a greater number of significant figures. So Dodo's measure is more precise than Dorji's measure.

1. a) 4 **b)** 4 **c)** 4 **d)** 3 **e)** 1
f) 1 **g)** 1 **h)** 3 **i)** 4 **j)** 1

2.a) 0.8 **b)** 4700 **c)** 3.2

3. *Sample response*:
a) 116 mm and 121 mm (Anything between 115 mm (11.5 cm) and 125 mm (12.5 cm) is acceptable.)
b) 7281 mL and 7312 mL (Anything between 7250 mL (7.250 L) and 7350 mL (7.350 L) is acceptable.)

4. a) *Sample response*: 0.200
b) Yes; *Sample response*: 50,300
c) *Sample response*: 0.000002
d) Greatest is 9,000,000; least is 0.000001

5. *Sample response*: The calculator is about 15 cm long. This measurement is not very precise. Trying to be more precise by measuring to the nearest millimetre would be impossible because of the rounded corners and because the calculator is longer in the middle than along the sides, so one measurement to describe its length is not possible. The accuracy of any reported value will depend on the viewer's perspective in reading the ruler.

6. a) No; 65 km might have been reported to the nearest whole kilometre and 70 km to the nearest 10 km. That means that the 65 km could have actually been 65.4 km and the 70 km could have been 65.1 km, making the distance from Paro to Thimphu longer than from Wangdi Phodrang to Thimphu.
b) Novin's measurement is given with more precision than Dodo's, but that does not mean that Novin's measurement is accurate. It could be that one or both measurements comes from a source that reported inaccurately.

7. *Sample response*:
a) Drakpa tells you that her red bucket has a capacity of 10 L. Pema tells you that her blue bucket has a capacity of 9.6 L. Drapka's red bucket appears to hold more.
b) The red bucket may actually have a capacity of 9.51 L but Drakpa rounded to 10 L. The blue bucket could have had a capacity of 9.64 L but was rounded to 9.6 L. Thus the red bucket could be smaller than the blue bucket. A conclusion can not be determined from the given measures.

Supporting Students

Struggling students

Students are likely to struggle with accuracy issues. It is important to emphasize that inaccuracy is not always because a person is careless; there are sometimes valid physical reasons for inaccuracy.

Enrichment

Students might be interested in using the Internet to investigate the various conventions for calculations with significant figures.

CONNECTIONS: Precision Instruments

This feature reminds or suggests to students how technology has allowed people in society to become increasingly precise in measuring. After students have read the feature, you could ask them to suggest other examples of increased precision for the same kind of measurement. (One example is the precision used in determining winners of races in the Olympic Games.) They can explain the advantages and disadvantages of the newer measurement technologies. It is important for students to understand that measurements usually rely on technology and always depend on human choices.

Answers

1. *Sample response*:
- A millimetre ruler is very precise. It measures length.
- The knobs on the stove that control the heat are less precise. They measure heat.
- The cups my mother uses to measure food are probably not very precise; they measure volume.

4.1.2 EXPLORE: Measurement Error

Curriculum Outcomes	Lesson relevance
10-D1 Measurement: accuracy and precision • address precision issues when performing calculations on measurement data	This optional lesson clarifies for students how measurement error in calculated measurements is affected both by the number of significant figures used to report the original measurements and by the size of the measurements.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculating areas and perimeters of rectangles • calculating total surface areas and volumes of rectangular prisms • rounding • percentage calculations

Main Points to be Raised

• If measurements are less precise, the potential error is greater.
• If measurements are smaller the potential error as a percentage is greater than if they are larger.
• Potential error is magnified by multiplication. Hence, potential errors from the same measures will become larger for calculations of area (compared to perimeter) and greater again for volume.

Exploration

• You might wish to model how to proceed with **part A** by working through the first example with the students. Use the format below as a model. Make sure students realize that because only one significant figure is reported, you have to assume that the number was rounded was to the nearest ten centimetres.

Least possible measurements	Reported measurements	Greatest possible measurements
35 cm × 15 cm	40 cm × 20 cm	45 cm × 25 cm

Resulting calculations using the formula P = 2(l + w)

100 cm	120 cm	140 cm

Maximum discrepancy = 20 cm (either 120 – 100 or 140 – 120)

Maximum discrepancy percentage = $\frac{20}{120} \approx 16.7\%$

Note that sometimes the discrepancies will not be the same on either side of the reported measurement. In these cases, students are to use the maximum discrepancy.

• Then ask students to work in pairs through the rest of **part A** and **parts B to D.** They might discuss **part D** with a partner to come up with a response.

• You might write the formulas students need on the board. Recalling formulas is not the point of the lesson.

Observe and Assess

As students are working, notice:

• Do they correctly determine the minimum and maximum possible values, showing an understanding of significant figures?
• Do they correctly calculate the maximum discrepancy?
• Do they use the appropriate values to determine the maximum discrepancy percentage?
• Do they recognize how the error is affected by precision? by size? by type of measurement?

Share and Reflect

• Ask students how the measurements in each of these pairs differ: **i** and **iii** (by size); **i** and **v** (by precision).
• Have students describe another pair of measurements that are essentially the same except for the size of the numbers involved. Repeat, but this time with measurements that are essentially the same except for precision.

• Ask students why they might predict that the discrepancy percentage for the perimeter would be less when the measurements were greater, but measured with the same precision. (It makes sense because the difference between the minimum value and the reported value was the same in both cases, but the denominator was greater when the reported measurement was greater.)

Answers

A. i Calculated value: 120 cm
Least possible value: 100 cm
Greatest possible value: 140 cm
Discrepancy of up to 20 cm, or 16.7%

ii Calculated value: 800 cm^2
Least possible value: 525 cm^2
Greatest possible value: 1125 cm^2
Discrepancy of up to 325 cm^2, or 40.6%

iii Calculated value: 520 cm
Least possible value: 500 cm
Greatest possible value: 540 cm
Discrepancy of up to 20 cm, or 3.8%

iv Calculated value: 16,800 cm^2
Least possible value: 15,525 cm^2
Greatest possible value: 18,125 cm^2
Discrepancy of up to 1325 cm^2, or 7.9%

v Calculated value: 126 cm
Least possible value: 124 cm
Greatest possible value: 128 cm
Discrepancy of up to 2 cm, or 1.6 %

vi Calculated value: 882 cm^2
Least possible value: 850.75 cm^2
Greatest possible value: 913.75 cm^2
Discrepancy of up to 31.75 cm^2, or 3.5%

vii Calculated value: 2800 cm^2
Least possible value: 1550 cm^2
Greatest possible value: 4350 cm^2
Discrepancy of up to 1550 cm^2, or 55.4 %

viii Calculated value: 8000 cm^3
Least possible value: 2625 cm^3
Greatest possible value: 16,875 cm^3
Discrepancy of up to 8875 cm^3 , or 110.9%

ix Calculated value: 90,800 cm^2
Least possible value: 83,550 cm^2
Greatest possible value: 98,350 cm^2
Discrepancy of up to 7550 cm^2, or 8.3 %

x Calculated value: 1,848,000 cm^3
Least possible value: 1,630,125 cm^3
Greatest possible value: 2,084,375 cm^3
Discrepancy of up to 236,375 cm^3, 12.8%

xi Calculated value: 2446 cm^2
Least possible value: 2321.5 cm^2
Greatest possible value: 2573.5 cm^2
Discrepancy of up to 127.5 cm^2, or 5.2 %

xii Calculated value: 7161 cm^2
Least possible value: 6565.125 cm^3
Greatest possible value: 7788.375 cm^3
Discrepancy of up to 627.375 cm^3, or 8.8%

B. *Sample response*:
When the measurements increased, the percentage effect on perimeter was less. The same was true with area in 2-D, total surface area in 3-D, and volume.

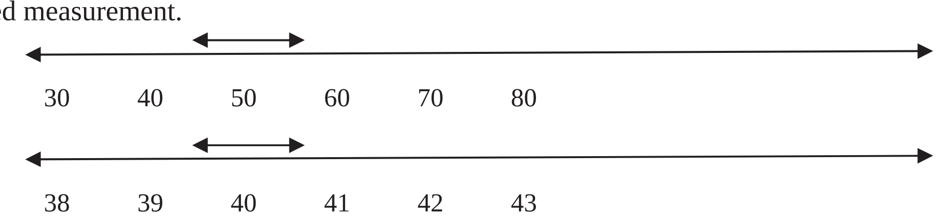
C. *Sample response*:
As the measurements became more precise, the percentage error went down.

D *Sample response*:
I think an error under 10% is pretty good because 10% is quite low. But when errors are 40% or 50% or higher, like they were for **parts ii, vii, and viiii**, you cannot have a lot of confidence in the measurements.

Supporting Students

Struggling students

You might draw a set of number lines like those below on the board to remind students of how to calculate ranges for a reported measurement.



Chapter 2 Efficient Design

4.2.1 EXPLORE: Regular Polygons with a Constant Perimeter

Curriculum Outcomes	Lesson relevance
10-D1 Measurement: accuracy and precision • address precision issues when performing calculations on measurement data • express answers to problems with significant figures **10-D2 Perimeter and Area: explore and apply properties** • examine maximizing area while restricting perimeter **10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • understand that areas of regular polygons can be determined by dividing the area into familiar shapes	Students will have to apply their understanding of precision and accuracy to their work in this optional exploration. The exploration demonstrates 2-D efficiency that is discussed in the next lesson.

Pacing	Materials	Prerequisites
1 h	• Compasses • Rulers (mm) • Protractors • Grid paper (BLM in Unit 6) • Calculators	• constructing a perpendicular bisector • using a protractor to construct an angle • calculating the area of a triangle • recognizing that the sum of the angles in a triangle is 180° • recognizing that the measure of the total angle around a point is 360° • formula for the area and circumference of a circle • knowledge of terms dependent/independent variable

Main Points to be Raised

• Among regular polygons with equal perimeter, the polygons with the most sides have the greatest area.

• A circle with the same perimeter as any polygon has a greater area than that of the polygon.

Exploration

• Before inviting students to work on **parts A to D**, demonstrate the constructions in the introduction.

• Ask students to work with a partner or in a small group. Observe while students work. You might ask:

- *Why do you think this area is larger than this area, even though the perimeters are the same?* (The perimeter is less pointed so the shape is broader and has more area.)

• *What do you notice about the areas?* (They are increasing.)

• *How did you predict the circle area from the graph?* (I noticed that the areas are not growing that much after there are a lot of sides, so I just went a bit higher than for a decagon.)

Observe and Assess

As students are working, notice:

• Are they measuring accurately and carefully?

• Are they measuring with sufficient precision?

• Do they reasonably predict the area of the circle from either the graph or the table?

• Do they provide a good explanation for why they expect the circle to have the greatest area for a given perimeter?

Share and Reflect

Ask students to share their responses to **parts E and F**. You might also ask:

• *Do you think the increasing pattern would be similar if you used a different perimeter, say 120.0 cm instead of 84.0 cm?* (Yes; The pattern would be similar but the values would be proportionally greater.)

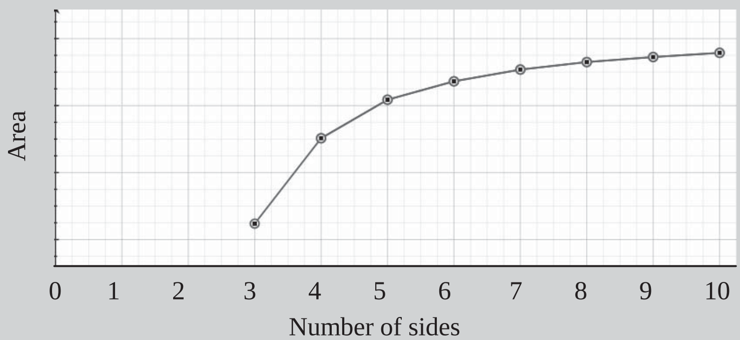
• *How important are precision and accuracy in your work? Explain why.* (Important, because, if the results were wrong, I would not be able to recognize the pattern.)

Answers

A. *Values close to these should be expected.*

Square	Pentagon	Hexagon	Heptagon	Octagon	Nonagon	Decagon
441 cm^2	487 cm^2	509 cm^2	523 cm^2	532 cm^2	538 cm^2	543 cm^2

B.



C. As the number of sides increases, the area increases, but the increase is smaller each time.

D. i) About 550 cm^2 because the area should be a little greater than the area of the decagon.
ii) 561 cm^2

E. A circle is like a many-sided polygon.

F. No; you cannot increase the number of sides of a circle.

G. A circle.

Supporting Students

Struggling students

For some students, you might want to provide a sheet with the regular shapes already drawn. In this way, they could avoid the drawing the constructions.

Enrichment

Some students might want to explore different types of triangles to decide if the equilateral triangle provides the greatest area among triangles of given perimeter, much like the circle does for regular polygons.

4.2.2 2-D Efficiency

Curriculum Outcomes	Outcome relevance
10-D2 Perimeter and Area: explore and apply properties • examine maximizing area while restricting perimeter • examine minimizing perimeter while restricting area **10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • understand that areas of regular polygons can be determined by dividing the area into familiar shapes	Many real-life problems involve determining the minimum fencing, or enclosure for a given area or the maximum space (area) for a given boundary.

Pacing	Materials	Prerequisites
1 h	• Elastics • Pens • Rope • Rulers	• perimeter of a rectangle • circumference of a circle

Main Points to be Raised

• The more circular a 2-D shape is, the more efficient it is, i.e., it gives the greatest area for a given perimeter or the least perimeter for a given area.

• In a subset of shapes of a particular type, e.g., triangles or quadrilaterals, the shape that is most circle-like has the greatest area for a given perimeter or the least perimeter for a given area.

Try This — Introducing the Lesson

Demonstrate Seldon's experience by asking a student to hold seven or more pens flat on a desk. You put an elastic around them and then ask the student to let go of the pens. This will help students visualize the situation.

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *Why is the first shape like a rectangle? What are its dimensions?* (The two flat sides, at the top and bottom, are two of the sides of the rectangle and the width of the pens describes the other two dimensions.)

• *What formula did you use to estimate the length of the elastic the second time?* (Circumference of a circle.)

If many students have trouble seeing the rectangle and circle formed by the elastics, have one student draw a sketch on the board.

The Exposition — Presenting the Main Ideas

• If **lesson 4.2.1** was completed, remind students about their work. Ask them to describe the main thing they learned.

• Invite students to read through the exposition, even if they completed **lesson 4.2.1**.

• Once they have finished, ask them to imagine a real situation in which someone might try to minimize perimeter or maximize area. Point out that in these situations a circle is often not practical (e.g., if fence pieces cannot bend) but a shape most like a circle would be the optimal shape.

Revisiting the Try This

C. You could apply the situation with the pens to a different context. Ask a group of seven or more students to stand in a row. Then tie a rope around them, and tighten the rope more and more. The students will be grouped into a circle. This will reinforce the connection between the **Try This** and the main ideas presented in the exposition.

Using the Examples

Draw the diagrams for **examples 1 and 2** on the board. In each case, ask students to predict which shape has a greater area and to share their thinking with a partner. Observe whether they recognize that they are looking for the shape that is fuller (rounder). Then invite them to read the presented solutions.

Practising and Applying

Teaching points and tips

Q 1: You may have to draw students' attention to the fact that both shapes have the same perimeter.

Q 2: Observe whether students realize that the shape that is most like a regular hexagon has the greater area.

Q 5: Students first have to realize that for every side they make, they will need one post costing Nu 300, and 6 m of wire costing Nu 120, for a total cost of Nu 420 per side. If they divide 3000 by 420, they realize they can only afford a seven-sided shape. Once students realize that the shape they want can only have seven sides, observe whether they realize that the regular seven-sided shape is the most circle-like and so will have the greatest area for a fixed perimeter.

Q 7: Students must generalize to realize that a circle-like shape is what they are looking for. Therefore, an isosceles right triangle is the correct shape. Some students might start with an isosceles right triangle with sides in a 1-1-$\sqrt{2}$ ratio and scale up by dividing 100 by the total perimeter of the smaller triangle.

Common Errors

Some students might use the same (or similar) words again and again to explain why one shape is more efficient than another. Encourage students to explain in different ways. This will help them form explanations in the future, and it will help them understand the concept now.

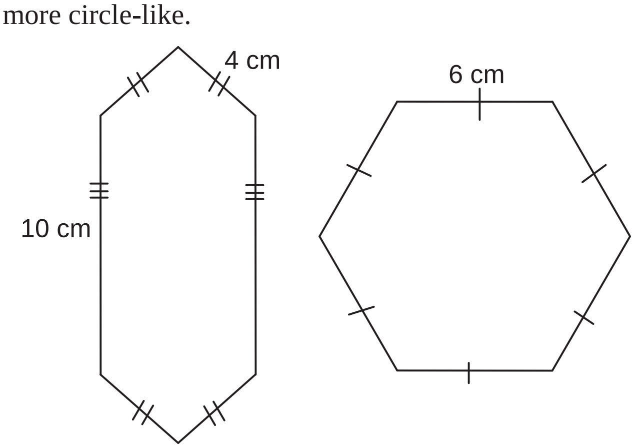
Suggested assessment questions from Practising and Applying

Question 1	to see whether students identify that the most circular of two shapes with the same perimeter will have the greater area
Question 3	to see if students understand minimizing perimeter with restricted area
Question 4	to see if students understand maximizing area with restricted perimeter
Question 6	to see if students can solve a real-world problem involving the concepts of the lesson

Answers

<table>
<tr><td>A. About 96 mm ($7 \times 6 + 7 \times 6 + 6 + 6 = 96$ mm)

B. About 60 cm ($\pi \times 20 \approx 60$ cm)</td><td>C. When the shape was longer and thinner, it had a greater perimeter than when it was more circular, even though it enclosed the same amount.</td></tr>
<tr><td>1. The hexagon's area is greater (about 10 cm^2) than the triangle's (about 7 cm^2).

2. Sample response:
The regular hexagon has a greater area because it is more circle-like.

</td><td>3. a) Sample response:
12 m × 9.0 m and 10 m × 10.8 m

b) Sample response:
I knew I needed a more square-like rectangle so I tried different lengths and widths that were close in size and had a product of 108.

4. a) Sample response:
A rectangle measuring 9 cm × 3 cm and a parallelogram of length 10 cm and slant height 2 cm

b) Sample response:
I knew I needed a less square-like shape so I tried different lengths and widths that were farther apart in size but added to 12.</td></tr>
</table>

5. The most efficient area would be a regular heptagon (seven-sided polygon) with each side 3 m and a perimeter of 21 m. It would cost Nu 420 for each side of the shape they create. 3000 ÷ 420 ≈ 7.1, so Therchu can buy 7 posts and 42 m of wire: 7 × 300 + 21 × 20 + 21 × 20 = 2100 + 420 + 420 = Nu 2940. **6. a)** 580 cm ($4 \times \sqrt{21{,}000} \approx 580$) **b)** About 514 cm (circle perimeter would be $2\pi\sqrt{\frac{21{,}000}{\pi}}$) **7. a)** 29.3 cm, 29.3 cm, and 41.4 cm **b)** It makes the two leg dimensions as close to each other as possible in length so the shape is the most like a circle	**8.** No; If they have the same area, then the regular polygon with more sides has a shorter perimeter. (No; If they have the same perimeter, then the regular polygon with more sides has a larger area.) **9. a)** *Sample response*: You only have a certain amount of fencing, but you want to enclose a large area. **b)** *Sample response*: You do not want to require as much material to fence a garden, so you want to minimize the perimeter of the garden.

Supporting Students

Struggling students

• If students are struggling with understanding what characteristics make one shape more circular than another, ask them to talk with others about the way they recognize the difference. Tell them that they should not be discouraged because it is often an intuitive decision.

• If students are struggling, you might help them with **question 7** by suggesting the shape needs to be isosceles and letting them continue from there. For **question 5**, you might suggest that they first calculate the cost to create one side of a shape.

Enrichment

Students might write another set of examples to support fellow students who want further examples. They should do these neatly, showing their thinking on the side, and then they should have a classmate check their work.

4.2.3 3-D Efficiency

Curriculum Outcomes	Outcome relevance
10-D3 Volume, Capacity, and Surface Area: demonstrate understanding • understand that surface area, capacity, and volume apply to 3-D shapes • understand the connection between volume and surface area • compare prism measurements **10-D4 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • determine capacity using volume	Many real-life problems involve determining the maximum volume for a given amount of covering (surface area).

Pacing	Materials	Prerequisites
2 h	• Calculators	• calculating surface area and volume • comparing ratios

Main Points to be Raised

• For shapes that have the same total surface area, the most spherical shape will have the greatest capacity or volume and is therefore most efficient.

• For shapes with the same capacity or volume, the most spherical shape will have the least total surface area and is therefore most efficient.

• A cube is the most efficient rectangular prism because it is the rectangular prism that is most like a sphere.

• A cylinder with a height equal to its diameter is the most efficient cylinder because it is the cylinder that is most like a sphere.

Try This — Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *How did you calculate the surface area?* (For Option 1, because it was a cube with 6 congruent square faces, I just squared 4.0 m to find the area of one face and then multiplied by 6. For Option 2, because of the dimensions of the rectangular prism, I knew there were three different rectangles on the surface area and two of each so I found the area of each and doubled it; then I added.)

• *Which would you predict to have the larger volume and why?* (The cube because it is kind of like a square, with a greater area than that of a longer rectangle with the same perimeter.)

If students predict that the more spherical shape is more efficient, generalizing from the last lesson about 2-D shapes, encourage them to explain why they see the situations as parallel.

The Exposition — Presenting the Main Ideas

• Have students read the exposition.

• Ask them for examples of things they see in school and outside of school that seem to be shaped a certain way to maximize volume and minimize surface area (e.g., rooms are often more cube-like than long and narrow).

• Ask them why, in each case, it would be impractical to make the item spherical in shape.

• An interesting design idea: In Southern Africa someone invented a spherical water container for fetching water from a well and carrying it far. Instead of carrying the water, the people roll the sphere filled with water.

Revisiting the Try This

C. This question allows students to connect the language of maximizing volume and the concept of spherical shapes to their work in **parts A and B**.

Using the Examples

• Put the students in groups of three. Each person in each group can be responsible for reading through one of the examples and explaining it to the others in the group.

• Point out that spreadsheet software, if available, can be very helpful doing trial and error work like in **example 1.** You would still do one calculation first and then write formulas for the spreadsheet cells to apply the same calculations to the other trials. Here is an example of applying formulas (in row 4) in a Microsoft Excel spreadsheet. The row with the formulas can be copied onto other rows further down, carrying the formulas with it, and you can enter different trial values in column A.

	A	B	C	D	E	F
1	diameter	radius	area of base	height	lateral surface	total surface
2	20	10	314.1592654	1.527887454	96	724.3185307
3	10	5	78.53981634	6.111549815	192	349.0796327
4	8	=A4/2	=PI()*(B4^2)	=480/C4	=2*PI()*B4*D4	=E4+2*C4

Practising and Applying

Teaching points and tips

Note: Many of the calculations in this section are complex. This is why 2 h have been allocated to this lesson.

Q 1 and 4: You may want to remind students of the formula for volume of a cylinder.

Q 3: Make sure students put the surface area first in the ratio so that the least value is more obvious.

Q 4: For this and other questions, students will continue to need to visualize what a compact shape would look like for the circumstances.

Q 7: Encourage students to explain why they expect the volume of the sphere will be greater.

Common Errors

Because of the complexity of the calculations, there is lots of potential for error. Encourage students to use their calculators, but also to record their steps along the way so they can go back and find where an error might have occurred.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can apply their knowledge of a cylinder having the least surface area for a given volume to a comparable situation
Question 4	to see if students can minimize surface area while restricting volume
Question 6	to see if students can maximize volume while restricting surface area
Question 11	to see how well students communicate their understanding of 3-D efficiency

Answers

A. i) Option 1: $SA = 96\ \text{m}^2$; Option 2: $SA = 96\ \text{m}^2$ **ii)** Option 1: $V = 64\ \text{m}^3$; Option 2: $V \approx 45\ \text{m}^3$ **B.** Option 1 because it will hold more fruit using the same amount of insulation.	**C.** I would have predicted the cube because I now know that a cube is the most efficient rectangular prism.
1. a) 79.6 cm **b)** 62.9 cm **c)** 50.9 cm **d)** 42.1 cm **2.** I predict barrel **c)** because its dimensions (height and radius) are the closest to being equal, making it the most sphere-like. The surface areas are **a)** 12,516.1 cm^2 **b)** 12,073.1 cm^2 **c)** 11,922.3 cm^2 **d)** 12,026.0 cm^2	**3.** Barrel **c)** because all of them have the same volume and **c)** has the least surface area. **4.** A base of about 8.8 cm × 8.8 cm and a height of about 12.4 cm. **5. a)** 6.5 cm **b)** 4.0 cm **c)** 2.3 cm **d)** 1.0 cm

Answers [Continued]

6. I predict prism **b)** because its height is the same as its width, which makes it the most sphere-like.
The volumes are
a) 59 cm^3 **b)** 64 cm^3 **c)** 58 cm^3 **d)** 36 cm^3

7. a) 2714 cm^2 **b)** 15 cm **c)** 22.6% greater

8. a) 21 cm

b) *Sample response*:
The volume of the cube will be smaller than both the volumes of the cylinder and the sphere.

c) The cylinder volume is 10,857 cm^3 and the sphere volume is 13,306 $cm^{3.}$ The cube volume is 9528 cm^3, which is less than both the others.

9. Nu 237

10. a) V = 3217 cm^3; SA = 1206 cm^2

b) SA = 1053 cm^2

c) Sphere

d) *Sample response*:
The surface area of the sphere was smaller and the volumes were the same.

11. If the volume of two shapes is the same, the lower the value of the ratio, the more efficient the shape.
A lower ratio means that less surface area is required for the same volume.

Supporting Students

Struggling students

If students are struggling to apply the different formulas from Class IX, they may need extra revision before this lesson.

Enrichment

Students might want to find examples of different 3-D shapes that they experience every day, and to write a small report that compares the surface to volume ratio of each shape.

CONNECTIONS: Animal Shapes and Sizes

The effect of size on the total surface area to volume ratio is not explicit in the curriculum but it is relevant to the exploration of issues relating to total surface and volume. Work on this feature may alleviate some confusion for students who happen upon a surprising result in volume and surface area exploration.

To explore the shapes of animals, you could have students develop rough models of familiar animals using shapes they are familiar with (cylinders, cones, etc). They could use their models as composite shapes to compare surface to volume ratios for the different animals.

Answers

1.

Radius (cm)	20.0	10.0	5.0	2.0	1.0	0.5
Surface area (cm^2)	5026.5	1256.6	314.2	50.3	12.6	3.1
Volume (cm^3)	33,510.3	4188.8	523.6	33.5	4.2	0.5
Surface area ÷ volume (SA:V ratio)	0.15	0.30	0.60	1.50	3.00	6.2

2.

Edge length (cm)	20.0	10.0	5.0	2.0	1.0	0.5
Surface area (cm^2)	2400	600	150	24	6.0	1.5
Volume (cm^3)	8000	1000	125	8.0	1.0	0.13
Surface area ÷ volume (SA:V ratio)	0.300	0.600	1.2	3.0	6.0	12

UNIT 4 Revision

Pacing	Materials
2 h	• Calculators

Question(s)	Related Lesson(s)
1 – 5	Lesson 4.1.1
6 – 9	Lesson 4.2.2
10 – 12	Lesson 4.2.3

Revision Tips

Q 3: Many students might automatically answer yes to this without thinking about precision (since the term *precision* was not mentioned). You might need to remind these students to think about what actual values the 70 and 67 could represent.

Q 5: Although these ideas were addressed directly in optional **lesson 4.1.2**, students could address them using what they learned in **lesson 4.1.1.**

Q 6: Students will need to use visualization skills to consider which shape is most circle-like.

Q 8d: Some students might exclude a circle since its boundary is often called a circumference rather than a perimeter, but it is an acceptable answer. A regular shape with more than four sides with a perimeter of 24 cm is also acceptable.

Q 10 to 12: You may want to write the formulas for the volume of a cylinder and a sphere on the board.

Answers

1. a) 2 **b)** 4 **c)** 3

2. *Sample responses*:
a) 20.4 **b)** 12,000,000 **c)** 0.02134

3. No. Dawa's mass was given with only one significant figure. This means that even if Dawa measured his mass accurately, he could be as light as 65 kg (the margin of error for 70 kg is 65 kg to 75 kg), so Nima cannot be sure he is lighter.

4. a) 70 cm
b) *Sample response*:
727 mm (Any value from 725 mm to 734 mm would be acceptable.)

5. a) *Sample responses*:
Due to precision, the 20 cm measure could be actually as much as 5 cm off the actual measure. It is unlikely that the actual measure is precise.

20 cm × 34 cm might be 680 cm^2 but the actual measurements could be as low as 15 cm × 33.5 cm or as high as 25 cm × 34.5 cm.

b) 502.5 cm^2 (15 cm × 33.5 cm)
c) 862.5 cm^2 (25 cm × 34.5 cm)

6. The shape on the right since it is most like a circle.

7. The middle shape since it is most like a circle.

8. a) Side length: 24 ÷ 3 = 8 cm;
Height of triangle using Pythagorean theorem:
$4^2 + h^2 = 8^2$, $h = 6.9$ cm;
$A = \frac{1}{2}bh = \frac{1}{2}(8)(6.9) \approx 28\text{ cm}^2$.

b) Side length: 24 ÷ 4 = 6 cm; $A = 6 \times 6 = 36\text{ cm}^2$

c) The square has more sides, so it is more like a circle and therefore has more area for the same perimeter.

d) *Sample response*:
Circle with circumference 24 cm.

9. 3164.06 m^2

10. a) 9.0 cm **b)** 6.5 cm **c)** 4.7 cm **d)** 3.2 cm

11. I predict cylinder **b)** because the height is closest to the diameter, and so that cylinder is the most like a sphere. The volumes are
a) 175.9 cm^3 **b)** 183.8 cm^3
c) 180.5 cm^3 **d)** 158.8 cm^3

12. a) 1.15 m^3 **b)** 1.05 m sides

c) The cube; the less sphere-like a shape, the greater its surface area for the same volume.

d) *SA* of cube is 6.62 m^2, *SA* of sphere is 5.31 m^2.

UNIT 4 Measurement Test

1. How many significant figures in each?

a) 400

b) 4.208

c) 0.030

2. Buthri heard on the radio that someone had grown a squash with circumference 80 cm. She wondered how her largest squash compared in size. She measured it and found that its circumference was 83 cm. Can she conclude that her squash is larger than the squash described on the radio? Explain using the terms precision and accuracy.

3. Drakpa measured a side of a parallelogram to be 26 cm when he used a centimetre ruler.

a) If he had used a ruler that measures to the nearest 10 cm, what might he have found the length to be?

b) If he had used a millimetre ruler, what might he have found the length to be?

4. Why might it be misleading to say that the area of a 5.1 m by 2.7 m rectangle is 13.77 m^2?

5. a) Determine the area of a circle with circumference 42 cm.

b) Determine the area of a square with a perimeter of 42 cm.

c) Describe another shape with a perimeter of 42 cm and an area between that of the square and the circle.

6. Four square-based prisms each have a capacity of 180 mL. Determine the height of the square-based prism with each base:

a) 6.0 cm × 6.0 cm

b) 7.0 cm × 7.0 cm

c) 8.0 cm × 8.0 cm

d) 9.0 cm × 9.0 cm

7. Use your results from **question 6** to predict which prism has the least surface area. Explain your prediction. Check it by calculating the surface areas.

8. a) Determine the surface area of a cube with edge lengths 2.6 mm.

b) Determine the radius of a sphere with the same surface area as the cube.

c) An octagonal prism has the same surface area as the cube. How will its volume compare with that of the cube? Explain your answer.

9. It costs Nu 30.0 to fill a cubic container of sweets measuring 10.0 cm along each edge. Determine the cost of filling a spherical container of sweets with the same surface area.

UNIT 4 Test

Pacing	Materials
1 h	• Calculators

Question(s)	Related Lesson(s)
1 – 3	Lesson 4.1.1
4	Lessons 4.1.1. and 4.1.2
5	Lesson 4.2.2
6 – 9	Lesson 4.2.3

Select questions to assign according to the time available.

Answers

1. a) 1 **b)** 4 **c)** 2

2. No. The circumference of the squash reported on the radio has only one significant figure. The margin of error for 80 cm is 75 cm to 85 cm. Buthri's squash is reported to 2 SFs. The margin of error for 83 cm is 82.5 cm to 83.5 cm. That means it is possible that the squash reported on the radio is larger. There is also a chance that Buthri or the person who measured the other squash (or both of them) did so inaccurately because of sloppiness or, because of difficulties with measuring the circumference due to the not so perfect spherical shapes of squash.

3. a) 30 cm

b) *Sample response*:
256 mm (Accept any answer from 255 mm to 264 mm)

4. A 5.1 m by 2.7 m rectangle could be anywhere from 5.05 m by 2.65 m to 5.15 m by 2.75 m, which means the area could be between 13.38 m^2 and 14.16 m^2.

5. a) A = 140 cm^2

b) A = 110 cm^2

c) Any regular polygon with more than four sides and with perimeter 42 cm would enclose more area than the square but less than the circle.

6. a) 5.0 cm **b)** 3.7 cm

c) 2.8 cm **d)** 2.2 cm

7. I predict **a)** has the least surface area because the height is closest to the length of the sides of the base, and thus the prism is closest to the shape of a cube. The surface areas are
a) 192 cm^2 **b)** 200.8 cm^2
c) 217.6 cm^2 **d)** 241.2 cm^2

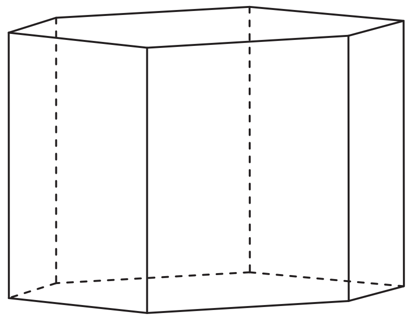
8. a) 40.6 mm^2 **b)** 1.8 mm

c) The octagonal prism will have a greater volume than the cube (and lesser volume than the sphere) because it is more sphere-like than the cube and they have the same surface area.

9. Nu 41.3
The radius of the spherical container is 6.9 cm.
The volume of the spherical container is 1376.1cm^3.
The cost is therefore 30 × 1376.1 ÷ 1000 = Nu 41.3

UNIT 4 Performance Task — An Efficient Hexagon-Based Prism

Your task is to find the dimensions of a regular hexagon-based prism with the least surface area that will hold 1.0 L.

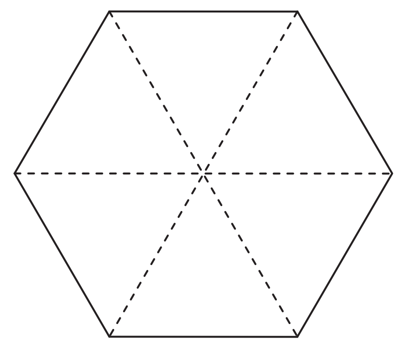


1. *Show all your calculations for this question.*

a) Determine the area of a regular hexagon with sides 5.0 cm.

b) Determine the height of a prism that will hold 1.0 L. Use the base area from **part a)**.

c) Determine the surface area of the regular hexagon-based prism.



5.0 cm

2. Enter your results from **question 1** into the chart below, and repeat for the various side lengths. (You do not have to show your work for each prism.)

Base side length (cm)	5.0	6.0	7.0	8.0
Area of base (cm^2)				
Height (cm)				
Surface area (cm^2)				

3. i) Which regular hexagon-based prism in **question 2** has the least surface area?

ii) Explain why you might have predicted this result.

UNIT 4 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-D1 Measurement: accuracy and precision **10-D2** Perimeter and Area: explore and apply properties **10-D3** Volume, Capacity, and Surface Area: demonstrate understanding **10-D4** Area, Perimeter, Surface Area, Capacity, and Volume: determine	1 h	• Rulers (mm) • Protractors • Compasses

How to Use This Performance Task

Before assigning the task, you could remind students how they found the areas of regular polygons in **lesson 4.2.1** (if it was done) or **lesson 4.2.2**. It would be especially helpful if you demonstrated how to find the area of an octagon.

Ask the students to read the task and the assessment rubric, and ask if they have any questions before they begin the task.

Assign the performance task. You could invite the students to do the calculations and constructions for **questions 1 and 2** in small groups, and have each student write up his or her own submission.

For an interesting variation, you could ask different groups to use different kinds of regular polygon prisms. Some groups could do regular pentagons, some regular hexagons, some regular heptagons, etc. The various results may be of interest to you and the students.

Note: If this task were done with further trials and perfect accuracy, the prism with side lengths 6.06 cm and height 10.5 cm would have the least surface area, 572 cm^2 (which is marginally smaller than when the side lengths are 6.00 cm, but it is still rounded to 572 cm^2).

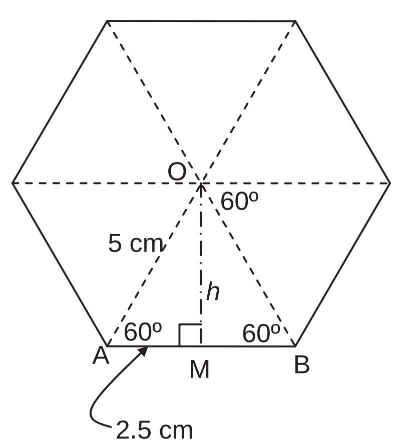
Sample Solution

1. a) The hexagon is made up of six congruent equilateral triangles. I can calculate the area of one and multiply by 6 to get the area of the hexagon.

$$\begin{aligned} 2.5^2 + h^2 &= 5^2 \\ 6.25 + h^2 &= 25 \\ h^2 &= 18.75 \\ h &= 4.33 \text{ cm} \end{aligned}$$

$A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2}(5)(4.3) = 10.8 \text{ cm}^2$

$A_{\text{hexagon}} = 6 \times 11 = 64.6 \text{ cm}^2$



b) $V = 1.0 \text{ L} = 1000 \text{ mL} = 1000 \text{ cm}^3$

$$\begin{aligned} V &= Ah \\ 1000 &= 64.6h \\ h &= 15.4 \text{ cm} \end{aligned}$$

c) I remember that I can calculate total surface area by multiplying the perimeter of the base by the height and then adding on the areas of the two bases: $SA = Ph + 2A = (6 \times 5.0)(15.4) + 2(64.6) = 591.2 \text{ cm}^2$.

Sample Solution [Continued]

2.

Base side length (cm)	5.0	6.0	7.0	8.0
Area of base (cm^2)	64.6	93.5	127.3	166.3
Height (cm)	15.4	10.7	7.9	6.0
Surface area (cm^2)	591.2	572.2	586.4	620.6

3. i) The prism with the least surface area is the one with base side length of 6.0 cm and height 10.7 cm.
ii) I expected the most efficient regular hexagon-based prism would be as spherical or cube-like as possible. If it has a base side length of 6.0 cm that means its width is going to be close to but less than 12 cm. With a height of 11 cm, it is almost like a cube or a sphere.

OR

I expected the most efficient regular hexagon-based prism to have the same width as height because the most efficient cylinder does. If it has a base side length of 6.0 cm that means it has a width that approaches 12 cm, which is almost the same as its height of 11 cm.

UNIT 4 Performance Task Assessment Rubric

	Level 4	**Level 3**	**Level 2**	**Level 1**
Finding the height and surface area of the various prisms	Calculates all required measurements accurately with appropriate precision; all work shown for the first trial.	Calculates and measures reasonably accurately and with reasonable precision; enough work shown to indicate a correct method.	Mostly correct but with some • inaccuracy in measuring, • flaw in method, • incorrect calculations, and/or • poor judgment in reporting precisions.	Two or more of these problems show in the work: • inaccuracy in measuring, • flaw in method, • incorrect calculations, and • very poor judgment in reporting precisions.

UNIT 5 NON-LINEAR FUNCTIONS AND EQUATIONS

UNIT 5 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 129** **TG p. 138**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Cubes (optional)	All questions
Chapter 1 Graphing Functions				
5.1.1 Forms of Quadratic Functions **SB p. 129** **TG p. 140**	**10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in vertex or factored form • translate among tabular, written, symbolic, and graphical representations of functions **10-C7 Graphs: create by constructing a table of values and graphing** • construct a graph from a table of values	1 h	• Grid paper (BLM in Unit 6)	Q3, 6, and 7
5.1.2 Graphs of Quadratic Functions in Factored Form **SB p. 135** **TG p. 144**	**10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in vertex or factored form • translate among tabular, written, symbolic, and graphical representations of functions	1 h	• Grid paper (BLM in Unit 6)	Q1, 2, and 4
5.1.3 EXPLORE: Transforming Quadratic Function Graphs (Optional) **SB p. 139** **TG p. 148** [Continued]	**10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • translate among tabular, written, symbolic, and graphical representations of functions **10-C7 Graphs: create by constructing a table of values and graphing** • construct a graph from a table of values	2 h	• Grid paper (BLM in Unit 6)	Observe and assess questions

UNIT 5 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 1 Graphing Functions [Continued]				
5.1.3 EXPLORE: Transforming Quadratic Function Graphs (Optional) [Continued]	**10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function			
5.1.4 Relating Graphs of Quadratic Functions **SB p. 141** **TG p. 151**	**10-C1 Transformations: express algebraically or using mapping rules** • express transformations from a graph using algebraic expressions or using mapping rules • express mapping rules algebraically and vice versa **10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics • interpret characteristics in relation to given contexts **10-C5 Graphs and tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • translate among tabular, written, symbolic, and graphical representations of functions **10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function	1 h *Additional time might be required if **lesson 5.1.3** was omitted.*	• Grid paper (BLM in Unit 6)	Q2, 3, and 5
CONNECTIONS: Parabolas and Paper Folding **SB p. 148** **TG p. 155**	Explore a geometric interpretation of the quadratic function	20 min	• Grid paper (BLM in Unit 6)	N/A
5.1.5 EXPLORE: The Absolute Value Function (Essential) **SB p. 149** **TG p. 156**	**10-C1 Transformations: express algebraically or using mapping rule** • express transformations from a graph using algebraic expressions or using mapping rules • express mapping rules algebraically and vice versa **10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics	1 h	• Grid paper (BLM in Unit 6)	Observe and assess questions

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
	10-C5 Graphs and Tables: explore dynamics of change • determine how changes in one variable affect another through the analysis of tables/graphs **10-C6 Graphs: sketch** • translate among tabular, written, symbolic, and graphical representations of functions **10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the absolute value function			
Chapter 2 Solving Non-Linear Equations				
5.2.1 Factoring Quadratic Expressions **SB p. 150** **TG p. 159**	**10-C16 Quadratic Equations: solve by factoring** • develop factoring strategies for polynomials in one variable that are products of degree one binomials	2 h	• Algebra tiles	Q3, 5, 7, and 8
5.2.2 EXPLORE: Roots of Quadratic Equations (Essential) **SB p. 157** **TG p. 163**	**10-C9 Non-linear Equations: evaluate and interpret** • determine the roots of quadratic equations from the corresponding graph **10-C12 Non-linear equations: evaluate and interpret** • determine the roots of quadratic equations from the corresponding graph **10-C14 Equations: solve using graphs** • use the x-intercept to determine the solution of quadratic equations	1 h	• Grid paper (BLM in Unit 6)	Observe and assess questions
5.2.3 Solving Quadratic Equations by Factoring **SB p. 158** **TG p. 165** [Continued]	**10-C4 Graphs and Tables: construct and analyse** • analyse graphs to determine mathematical characteristics • interpret characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in factored form **10-C9 Non-linear Equations: evaluate and interpret** • determine the roots of quadratic equations from the corresponding graph **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C13 Equations: rearrange** • transform equations from one form to another	1 h	• Grid paper (BLM in Unit 6) (optional)	Q3, 5, 7, and 12

UNIT 5 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Solving Non-Linear Equations [Continued]				
5.2.3 Solving Quadratic Equations by Factoring [Continued]	**10-C16 Quadratic equations: solve by factoring** • understand the zero product rule: if $ab = 0$, then either $a = 0$ or $b = 0$ • apply the zero product rule to solve quadratics by factoring • convert a quadratic equation to two linear equations by the factoring method • solve equations including those which involve common factors, regular equations, perfect square trinomials, and difference of squares			
5.2.4 EXPLORE: Absolute Value Equations (Essential) **SB p. 162** **TG p. 168**	**10-C4 Graphs and Tables: construct and analyse relating two variables** • analyse graphs and tables to determine mathematical characteristics **10-C10 Equations: solve for linear and simple radical, exponential, and absolute value equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C13 Equations: rearrange** • transform equations from one form to another	1 h	• Grid paper (BLM in Unit 6)	Observe and assess questions
GAME: Get the Points **SB p. 162** **TG p. 171**	Create and graph quadratic and absolute value functions in a game situation	20 min	• Grid paper (BLM in Unit 6)	N/A
UNIT 5 Revision **SB p. 163** **TG p. 172**	Review the concepts and skills in the unit	2 h	• Grid paper (BLM in Unit 6)	All questions
UNIT 5 Test **TG p. 177**	Assess the concepts and skills in the unit	1 h	• Grid paper (BLM in Unit 6)	All questions
UNIT 5 Performance Task **TG p. 180**	Assess concepts and skills in the unit	1 h	• Grid paper (BLM in Unit 6)	Rubric provided
UNIT 5 Assessment Interview **TG p. 182**	Assess concepts and skills in the unit	20 min	• Graph on **page 182** in this *Teacher's Guide*	All questions

Math Background

• In this unit, students explore two types of non-linear functions: quadratics and absolute value functions.

• Working with quadratic functions, they learn about the various forms in which a quadratic can be presented and how each form makes it easy to describe certain characteristics of the quadratic. They learn how graphs of quadratics are related using transformations. They learn how to solve quadratics using both graphs and algebraic methods, including factoring.

• They also learn how to graph absolute value functions by relating each member of the absolute value function family to the basic form of $y = |x|$ using transformational concepts. They solve absolute value equations by relating the equations to a corresponding absolute value function graph.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- They use problem solving in **question 8** of **lesson 5.1.1** by describing a situation algebraically and then graphing the function created to help them solve the related problem. Other examples of problem solving are found in **question 5** of **lesson 5.1.2** and in **question 9** of **lesson 5.2.3**, where they use a quadratic model to solve problems.

- They use communication to explain their thinking, for example, in **question 9** of **lesson 5.1.1**, where they describe why three input values are required to test for equivalence of quadratic functions, and in **question 5** of **lesson 5.1.4**, where they relate what they have learned about transformations of quadratics to explain why a certain form of the quadratic makes sense to call vertex form.

- They use reasoning in answering questions such as **question 2** of **lesson 5.1.2**, where they consider the need to pull out common factors to write a quadratic in factored form and in **question 5** in **lesson 5.1.4**, where they use transformational concepts to reason about the coordinates of the vertex of a quadratic. They also use reasoning in **parts C and F** of Explore **lesson 5.2.2** where they reason about the relationship between the graph of a quadratic and the number of roots.

- They use visualization extensively in this unit. For example, in **question 7** of **lesson 5.1.1**, they use a graph instead of an equation to determine a maximum value. Other examples are the Explore **lessons 5.1.3** (optional) **and 5.1.5**, where they visualize the effect of transformations on graphs of quadratics and absolute value functions respectively. They also solve absolute value equations using graphical representations in **lesson 5.2.4.**

- Students see connections, for example, in the optional Explore **lesson 5.1.3** where they connect the representations of different graphs to one another using transformations. They also use connections in **question 8** of **lesson 5.2.3,** where they use techniques for solving quadratic equations involving the Pythagorean theorem. The **Connections** feature uses paper folding to expose students to an alternative approach to defining a parabola.

Rationale for Teaching Approach

• This unit is divided into two chapters.

- **Chapter 1** deals with quadratic and absolute value functions. The focus is on linking the algebraic and graphical representations of each type of function.

- **Chapter 2** follows naturally, dealing with the solution of equations involving quadratics and absolute values by relating back to the appropriate functions.

• There are many **Explore** features in this unit. By using transformations, students gain first hand experience to help them understand how different functions are related. They also explore the relationship between the roots of a quadratic and its graph and the solution of absolute value equations using graphs of the related functions.

• The **Connections** feature is not essential to the unit but it provides an interesting look at parabolas, which have a central place in the unit.

Technology in This Unit

Students will not be required to use much technology in this unit. If graphing calculators are available, students might use them to explore transformations of graphs.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Patterns and Relationships: determine non-algebraic representations **9** Volume and Surface Area: estimate and calculate for right prisms and cylinders **9** Single Variable Equations: solve algebraically and graphically **9** Polynomial Expressions: interpreting **9** Polynomial Products and Quotients: concretely, pictorially, and symbolically **10** Transformations: express algebraically or using mapping rules	Students will experience more success in this unit if they review what they already know about describing quadratics, representing quadratic situations, multiplying and dividing polynomials, and function and mapping notation.

Pacing	Materials	Prerequisites
1 h	• Cubes (optional)	• surface area of rectangular prisms • recognition of quadratic relationships using second differences or graphs • describing patterns algebraically • solving linear equations • function and mapping notation • multiplication and division of polynomials

Main Points to be Raised

• If the second differences from a table of values are constant and not zero, the relationship being described is quadratic.

• A quadratic relationship is a relationship where the second power of a variable is involved.

Use What You Know — Introducing the Unit

• Students can try this activity alone or with a partner.

• Observe students as they work. You might ask:

- *What was the area when the wall was 4 cubes high? How did you figure that out?* (48 square units; There were two faces that were 4-by-4, so that makes 32 units. Then there were four faces that were 1-by-4 and that makes another 16. 32 + 16 = 48.)

- *How did you calculate the first and second differences?* (I subtracted the numbers in the area column in order to get the first differences. I subtracted the first differences in order to get the second differences.)

- *How did you know that the function was not linear?* (I could tell before I even did any calculations. If it were linear, then after 6 and 16, the number of painted faces would be 26 and not 30.)

- *What does the variable in your equation represent? How do you know your equation makes sense?* (My variable represents the height since I based the total surface area on the height. I substituted using heights of 3 and 5 to check it.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually, but it will be helpful if each student has a partner with whom he or she can discuss answers or difficulties.

Common Errors

Many students will confuse the answers for **parts c) and d)** of **question 2**. Help them understand that for **part d)**, you substitute the value $x + 2$ for x before applying the function rule and that for **part c)**, the function rule is applied to x only and then 2 is added.

Answers

A. and B. i)

Height of wall (number of cubes)	Area of painted surface (number of painted cube faces)	First differences	Second differences
1	6		
2	16	16 – 6 = 10	
3	30	30 – 16 = 14	14 – 10 = 4
4	48	48 – 30 = 18	18 – 14 = 4
5	70	70 – 48 = 22	22 – 18 = 4

B. ii) The relationship is quadratic because the second differences are constant and not zero.

C. i) If n represents the height of the wall in cubes and p represents the area of the painted surface in number of painted cube faces, the equation is $p = 2n^2 + 4n$.

ii) $2n^2 + 4n$ is in the form $ax^2 + bx + c$ ($c = 0$).

1. A; *Sample response:* Because it is a parabola.

2. a) -7 **b)** $-9x - 6$ **c)** $3x + 4$ **d)** $3x + 8$

3. a) $x = 4$ **b)** $x = 2$ **c)** $x = -\frac{2}{7}$

4. a) (2, 3) **b)** (4, –2) **c)** (4, –3) **d)** (1, –3)

5. a) $(3x + 1)(2x - 3)$ **b)** $6x^2 - 7x - 3$

6. a) $x^2 + 7x + 12$ **b)** $6x^2 + 13x + 6$

c) $x^2 - 9x + 20$ **d)** $-6x^2 + 13x + 5$

7. a) $y + 3$ if $y \neq 0$ **b)** $x + 5$ if $x \neq -2$

Supporting Students

Struggling students

- For the main activity, you may need to provide cube models for struggling students so they can build the shapes 4 units high and 5 units high.
- For **questions 6 and 7**, you may want to provide algebra tiles for students to use.

Enrichment

You might challenge students to look at the **Use What You Know** activity as a model and then use cubes to create a different problem that will result in a quadratic relationship.

Chapter 1 Graphing Equations

5.1.1 Forms of Quadratic Functions

Curriculum Outcomes	Outcome relevance
10-C4 Graphs and Tables: construct and analyse • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in vertex or factored form • translate among tabular, written, symbolic, and graphical representations of functions **10-C7 Graphs: create by constructing a table of values and graphing** • construct a graph from a table of values	Graph of parabolas are helpful for describing functions, particularly because the vertex and roots are more obvious in graphical form than in equation form.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• graphing from a table of values on a coordinate grid • function notation • formula for area of a rectangle • meaning of the terms *x-intercept* and *y-intercept* • describing a situation using variables • reading a graph

Main Points to be Raised

• A quadratic function is a polynomial of degree 2.

• There are three common forms for a polynomial: standard form, ax^2 + bx + c; factored form, a(x – p)(x– q); and vertex form, a(x – h)^2 + k. A quadratic described in factored or vertex form can be described in standard form by expanding.

•If a polynomial is quadratic, the value for *a* in any form cannot be zero. The other parameters (b, c, p, q, h, k) can each be zero or non-zero.

• When a quadratic function is in factored form, it is easy to read off its roots, namely p and q.

• When a quadratic function is in vertex form, it is easy to read off the coordinates of its vertex, namely (h, k).

• By substituting into the function rule, a quadratic can be plotted on a graph. The graph is a parabola.

• Two different quadratic functions can go through one or two common points, but once three points of a parabola have been plotted, the graph is determined; there is only one possible graph.

Try This — Introducing the Lesson

A. and B. Students can solve the problem alone or with a partner.

Observe while students work. You might ask:

• *What would the picture of the pen look like if w is the width?* (A rectangle with two sides marked *w*.)

• *What equation would you use to describe the perimeter of the pen? How did that help you with* ***part A?*** (I knew that 2*w* + 2*l* = 100. That means I could solve for *l* in terms of *w*.)

• *Do you think the equation represents a linear situation? Explain.* (I do not think so since area is usually in square units, which makes it quadratic.)

The Exposition — Presenting the Main Ideas

• Bring the students' attention to the description of the three forms of a quadratic on **page 129**. Record a few expressions on the board to make sure students can identify the form, for example $f(x) = 3x^2 – 2x$, $f(x) = \frac{(x-2)^2}{4}$, and $f(x) = 3(x-4)^2 - 7$.

• Model how to move from one form to an equivalent form. For example, show how $f(x) = 3(x – 2)(x + 3)$ can be expanded to form the equivalent expression $f(x) = 3x^2 + 3x – 18$ in standard form. Verify the equivalence by substituting several values for x into each form to show that the outputs are the same.

• Demonstrate how to graph either form of the equation using a table of values. You might use values of $x = 0, 1, 2,$ and 3. Point out the vertex on the graph.

• Ask students to read through the exposition to review what you have presented and to learn that it takes three points to uniquely define a parabola. You may want to follow this up by showing how you can create several parabolas that go through any two given points, but that only one parabola fits when the third point is given.

For example for two parabolas to go through (–1, 1) and (1, 1), you can substitute the first coordinate for x and the second coordinate for $f(x)$ and know that if $y = ax^2 + bx + c$, then $a – b + c = 1$ and $a + b + c = 1$. That means $b = 0$ and $a + c = 1$ would work. So, for example, both points are on the parabolas $y = 3x^2 – 2$ and $y = 2x^2 – 1$. But as soon as you add a third point, e.g., (0, –2), only one parabola works, in this case the first parabola. Later in their studies, the students will learn that the reason three points define a parabola is that a system of three linear equations in three unknowns normally has only one solution. They do not need to know this now.

Revisiting the Try This

C. and D. Students have the opportunity to apply to the goat pen problem the skills they learned in the exposition.

Using the Examples

• Write the problems posed in the two examples on the board. Encourage pairs of students to attempt the problems. They can then compare their solutions to those on **pages 131 to 133**.

• Discuss students' work by asking how many of them solved the problem in **example 1 b)** as is demonstrated in **solution 1**, how many used the approach in **solution 2**, and how many used the approach in **solution 3**. Point out the fact that all approaches are equally valid.

• In **example 2**, some students might notice the change from the form $A = w(25 – w)$ to $a(w) = w(25 – w)$ and finally to $y = w(25 – w)$ on the graph. Make sure they know that all of these expressions are acceptable to use. The use of the lower case a for the function is simply convention and is not required.

Practising and Applying

Teaching points and tips

Q 1: Most students will solve this by simply looking at the power of the polynomial, but students might choose instead to graph or to look at first and second differences.

Q 3: Remind students that to expand, they must use the distributive property for $f(x)$ and $h(x)$, multiplying four pairs of terms to include all partial products. For **part a)**, they need only one value of x where the functions differ to know they are not equivalent.

Q 4: Students might assume they have to substitute a negative value, zero, and a positive value for x, but any three values will suffice to sketch a graph. In this case, it is easier to sketch the parabola if some negative and some positive values are used.

Q 5: Students should try to graph the three functions on the same set of axes to feel confident in their determination of equivalence.

Q 7: Reference to this problem is made in the next lesson, so you may wish to ensure students attempt it.

Q 8: Some students might need a reminder to multiply the price by the number of items sold to answer **part b)**. **Part d)** is designed to help students see the importance of the vertex in determining the maximum or minimum of a quadratic function.

Common Errors

Many students do not really understand that to prove two functions are not equivalent, you only need one counterexample, but that to prove they are equivalent, you need to be certain they are the same for all values of x. It turns out that for quadratic functions, that means they have to be identical for three values of x.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can use algebraic techniques to show two quadratic functions are equivalent
Question 6	to see if students can graph a quadratic function from an algebraic description and use the graph to describe the quadratic function
Question 7	to see if students can graph and interpret the graph to solve a problem

Answers

A. *Sample response*:
If w represents the width, l represents the length, and the perimeter is 100 m, the equation $100 = 2w + 2l$ would represent the situation. If you solve for l you get $l = 50 – w$.

B. $A = w(50 - w)$ or $A = 50w – w^2$ or $A = –w^2 + 50w$

C. The algebraic expression $A = –w^2 + 50w$ is a degree 2 polynomial in standard form.

D. i)

w	A
5	225
10	400
15	525
20	600
25	625

ii) *Sample response*:
The right side of a parabola that goes through (5, 225) and (25, 625).

1. A, B, and E

2. a)

x	$f(x)$	$g(x)$
–2	1	1
–1	4	4
0	9	9
1	16	16
2	25	25

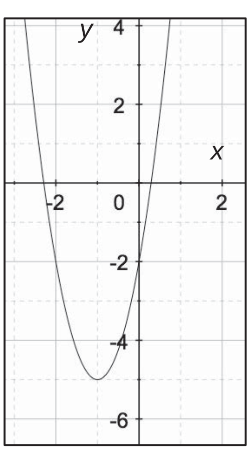
b) The functions produce the same output value for at least three values of x, so they are equivalent.

3. a) *Sample response*:
If $x = 0$, $g(0) = –12$, but $h(0) = –9$.

b) $f(x) = 3(x + 3)(x – 1) = 3(x^2 + 2x – 3) = 3x^2 + 6x – 9$

$$\begin{aligned} h(x) = 3(x + 1)^2 – 12 &= 3(x^2 + 2x + 1) – 12 \\ &= 3x^2 + 6x + 3 – 12 \\ &= 3x^2 + 6x – 9 \end{aligned}$$

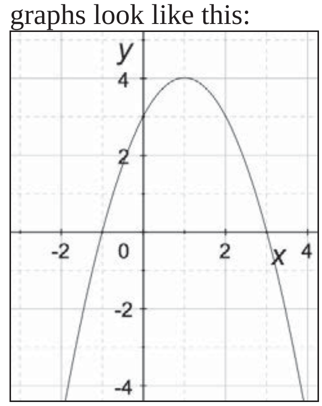
4. a)



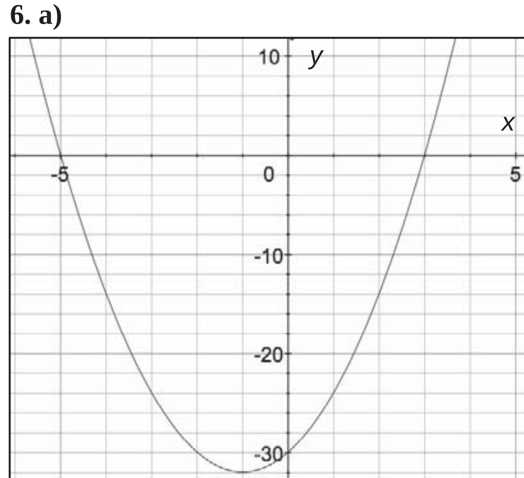
4. b) *Sample response*: (–1, –5); Minimum

c) *Sample response*:
The x-intercepts are about 0.25 and –2.25 and the y-intercept is about –2.

5. All of $f(x)$, $g(x)$, and $h(x)$ are equivalent. The three graphs look like this:

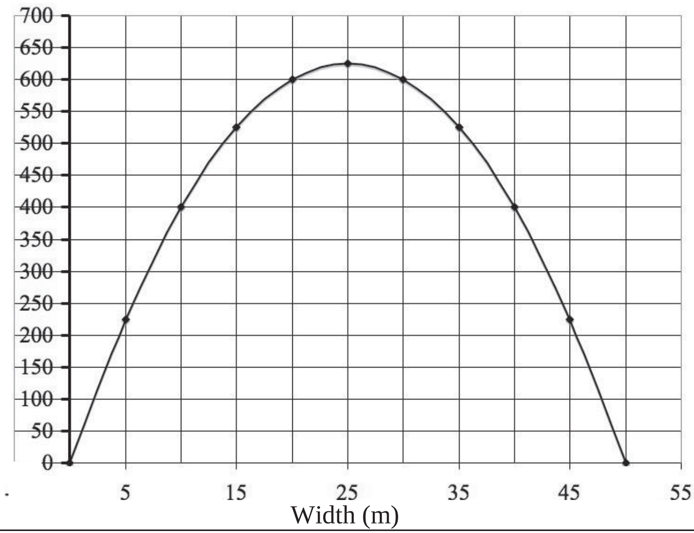


6. a)



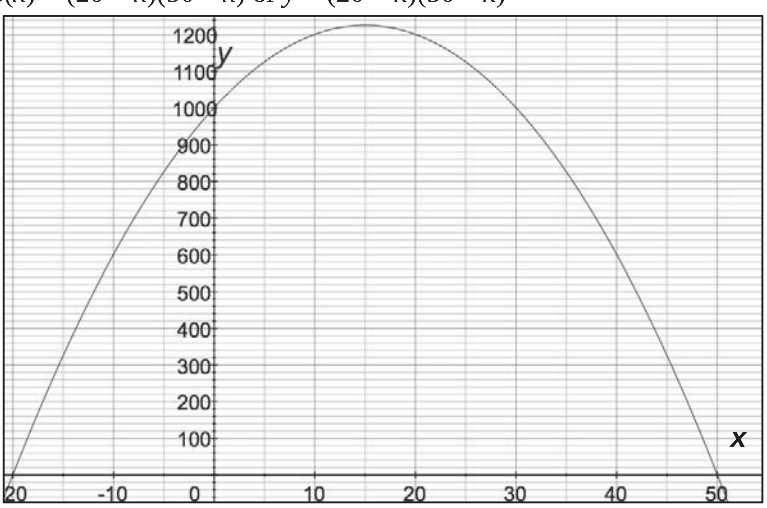
b) *Sample response:* (–1, –32)

Answers [Continued]

<table>
<tr>
<td>7. a) Vertical axis is Area (m^2)
</td>
<td>7. b) The y-coordinate of the vertex represents the greatest area and the x-coordinate of the vertex represents the width that relates to this area.

c) 25 m by 25 m; if the area is 625 m^2 at a width of 25 m, then the length has to be 25 m because $l = 50 – w$.</td>
</tr>
<tr>
<td>8. a) i) $20 + x$ ii) $50 – x$

b) $s(x) = (20 + x)(50 – x)$ or $y = (20 + x)(50 – x)$

c)
</td>
<td>8. d) about Nu 35; about Nu 1225

9. Sample response:
The parabolas $y = 2x^2 – x + 1$ and $y = 3x^2 – 5x + 1$ both go through (0, 1) and (2, 3), but if I add the point (–1, 4), only $y = 2x^2 – x + 1$ works.</td>
</tr>
</table>

Supporting Students

Struggling students

Some students may need practice in expanding quadratics in factored or vertex form to get to standard form. Remind them of the format for multiplying two binomials:

$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd$$

Enrichment

Students might attempt to explain algebraically how to determine the parabola that goes through three specific points. You need to solve a system of three equations in three unknowns to determine the values for a, b, and c in $y = ax^2 + bx + c$. For example, for (0, 3), (1, 2), and (2, 5), c = 3, a + b + c = 2, and 4a + 2b + c = 5.

5.1.2 Graphs of Quadratic Functions in Factored Form

Curriculum Outcomes	Outcome relevance
10-C4 Graphs and Tables: construct and analyse • analyse graphs and tables to determine mathematical characteristics • interpret the characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in vertex or factored form • translate among tabular, written, symbolic, and graphical representations of functions	Students need to recognize that different forms of an algebraic expression may provide different insights into that expression.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• graphing on a coordinate grid • substitution into an algebraic expression

Main Points to be Raised

• It is easy to determine the x-intercepts of a quadratic function when it is in factored form. The x-intercepts are also called the zeros.

• When a function is in the form $f(x) = a(x - p)(x - q)$, the zeros are p and q.

• The x-coordinate of the vertex of the parabola is always halfway between the two zeros. The y-coordinate of the vertex can be found by substituting the x-coordinate into the equation for the parabola.

Try This — Introducing the Lesson

A. You may want to take this up with the whole class by drawing the graph on the board. You might ask:

• *Why is there more than one answer to* ***part i)****?* (There are two x-intercepts for the graph.)

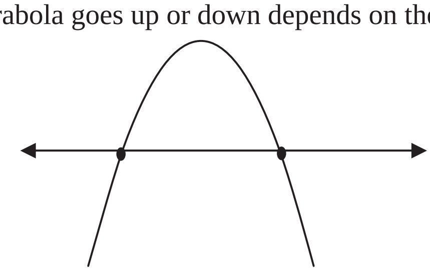
• *What does it mean when the area is zero?* (There is really no pen since there is either no length or no width.)

• *Where is the value for the maximum area located?* (The y-coordinate of the vertex.)

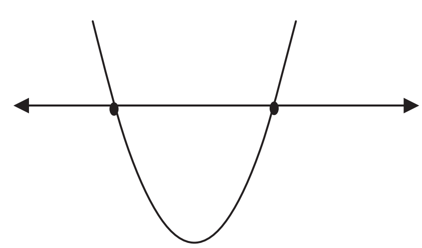
The Exposition — Presenting the Main Ideas

• Write the equation $y = 2(x - 4)(x - 3)$ on the board. With the students, work out a table of values and sketch a graph. Have students observe that the x-intercepts are 4 and 3, corresponding to the values in the equation. Talk about why this makes sense: if $x = 4$, the first bracketed expression is zero and so $y = 0$. A similar thing happens if $x = 3$ with the second bracketed expression. Also have them observe that the vertex appeared at 3.5, halfway between the two zeros.

• Point out how once the zeros of a quadratic are known, you can easily sketch the parabola. Whether the parabola goes up or down depends on the sign of the coefficient of x^2.



OR



Encourage students to use the exposition for reference.

Revisiting the Try This

B. Students have the opportunity to relate what they learned about vertex form to the questions they answered in the **Try This.** They should recognize that the y-intercept has to be zero since one of the roots is zero.

Using the Examples

• Record the problems for **examples 1 and 2** on the board. Assign students to work in pairs, one working on the first problem and the other on the second problem. When students have finished, they can check their answers against those in the student text. Then they can share their work with the other student with whom they are paired.

• Answer any questions students might still have after working through the two examples.

Practising and Applying

Teaching points and tips

Q 1: Students need to remember to average the values of the zeros to determine the x-coordinate of the vertex.

Q 3a: Some students might benefit from creating a small table of values to help them write the expressions required.

Q 3d: Observe whether students use the factored form to help them quickly locate the zeros to sketch the graph or whether they use a different approach.

Q 4: Students must remember to include the piece of fence that divides the yard into two sections.

Q 6: Students can respond to this question by thinking about the geometric shape of the parabola, as opposed to referring to its equation. You may need to remind them that the mean is the average.

Common Errors

In solving **question 4**, many students will forget to include the piece of fence in the middle. The formula for using the 210 m will need to include $3w$, where w represents the width of the yard. You might suggest that students make a sketch of the diagram in the book and use a coloured pencil to mark all the fencing

Suggested assessment questions from Practising and Applying

Question 1	to see if students can use the factored form of a quadratic function to quickly determine the zeros and the vertex
Question 2	to see if students recognize the need to factor out the coefficient of x^2 to use the factored form
Question 4	to see if students can represent a situation with a quadratic function and use it to solve a problem

Answers

A. i) (0, 0) **ii)** (0, 0) and (50, 0) **iii)** (25, 625)

B. i) (0, 0) and (50, 0); $A(w) = w(50 - w)$, so $w = 0$ or 50; $-w = 0$, so $w = 0$ or 50.

B. ii) (25, 625); If $w = 0$ or 50, the x-coordinate of vertex is $(0 + 50) \div 2 = 25$, so the y-coordinate is $25(50 - 25) = 625$.

iii) (0, 0); If $w = 0$ or 50, the coordinates of the y-intercept are $(0, 0 \times 50) = (0, 0)$.

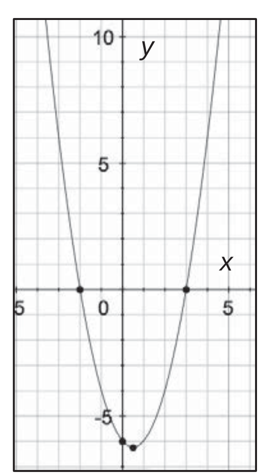
1. a)

	Zeros	y-intercept
i)	–2, 3	–6
ii)	5, –5	–25
iii)	–1.5, –0.5	6
iv)	2, –2	–12
v)	–1, –2	–4

b) i) (0.5, –6.25) **ii)** (0, –25) **iii)** (–1, –2)

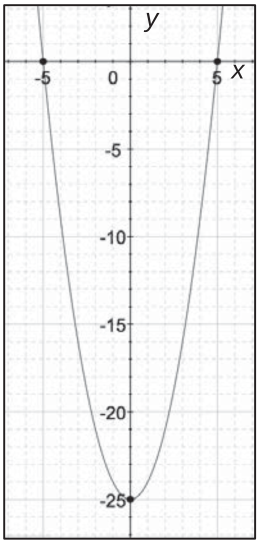
iv) (0, –12) **v)** (–1.5, 0.5)

c) i)

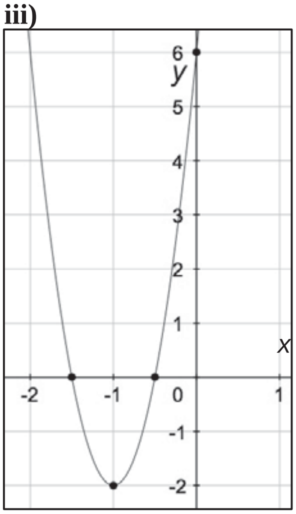


Answers [Continued]

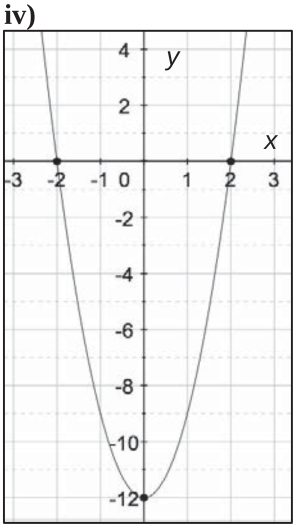
1.c) ii)



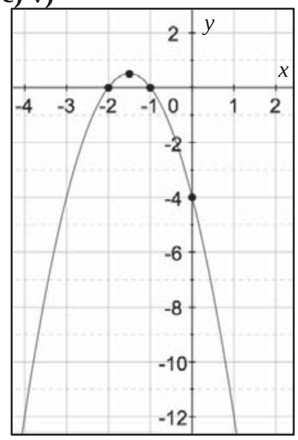
iii)



iv)



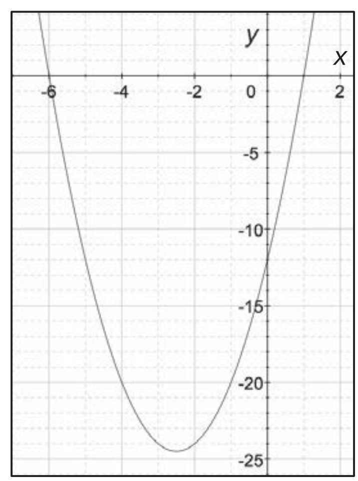
1.c) v)



2. a) $2(x-1)(x+6) = 2(x^2 + 5x - 6) = 2x^2 + 10x - 12$

b) When the function is in factored form, Mindu can determine the zeros easily in order to find the coordinates of the *x*-intercepts, the vertex, and the *y*-intercept.

c)

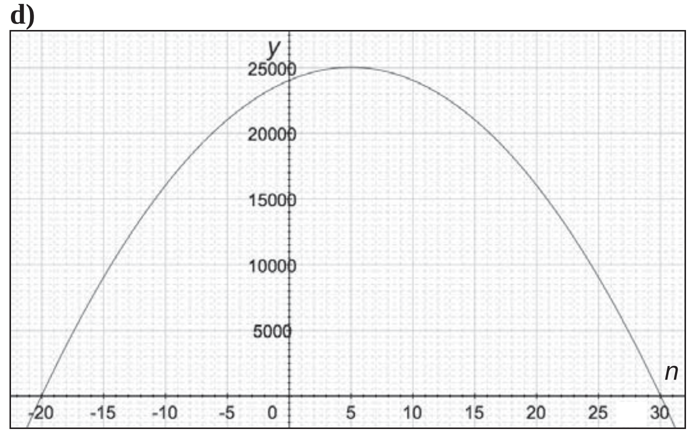


3. a) i) $800 + 40n$ **ii)** $30 - n$

b) $f(n) = (800 + 40n)(30 - n)$

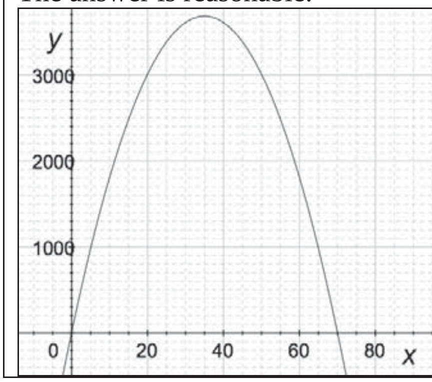
c) Five price increases of Nu 40 on Nu 800, or a total price of Nu 1000

d)



4. a) $a(w) = w(210 - 3w)$

b) 35 m; the vertex of the graph is at about (35, 3600). The answer is reasonable.



5. The function that represents the situation is $f(x) = (50 + 5x)(100 - 2x)$ where x represents the number of Nu 5 increases in price per kilogram; Nu 150

6. The vertex is on the axis of symmetry. The axis of symmetry is halfway between the x-intercepts and is perpendicular to the x-axis. Any point on the axis of symmetry will have an x-coordinate that is halfway between the x-coordinates of the x-intercepts, which is the same as the mean of the zeros.

Supporting Students

Struggling students

Students who are struggling may have difficulty representing the situations in **questions 3, 4, and 5** algebraically. Encourage them to create a table to help guide their thinking. For example, for **question 5**:

Number of price increases	Selling price/kg	Expected sales (kg)
0	50	100
1	55	98
2	60	96
3	65	94

They can then see that the variable x is the number of price increases and that the selling price is $(50 + 5x)$ and the expected sales is $(100 - 2x)$.

Enrichment

Students might investigate many different parabolas that go through two particular x-intercepts, for example, (2, 0) and (5, 0), to observe the effect of the multiplier, a, in the vertex form $y = \text{a}(x - 2)(x - 5)$.

5.1.3 EXPLORE: Transforming Quadratic Function Graphs

Curriculum Outcomes	Lesson relevance
10-C4 Graphs and Tables: construct and analyse • analyse graphs and tables to determine mathematical characteristics **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • translate among tabular, written, symbolic, and graphical representations of functions **10-C7 Graphs: create by constructing a table of values and graphing** • construct a graph from a table of values **10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function	This optional lesson will build on students' knowledge of transformations to help them predict the graphs of other quadratic functions based on the graph of $y = x^2$. Although the material is covered in the next lesson, personal exploration will probably make it more meaningful to students.

Pacing	Materials	Prerequisites
2 h	• Grid paper (BLM in Unit 6)	• coordinate graphing • meaning of the term *dilatation* • mapping notation for transformations

Main Points to be Raised

• The graph of $y = ax^2$ is a vertical stretch of $y = x^2$. If a < 0, the graph is reflected in the *x*-axis. If 0 < a < 1, the graph becomes wider; this is sometimes called a vertical compression. If |a| > 1, the graph becomes narrower; this is called a stretch. These changes are actually positive or negative dilatations of the basic graph.

• The graph of $y = (x - h)^2$ is a horizontal translation of $y = x^2$, h units to the right.

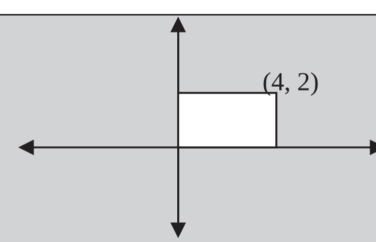
• The graph of $y = x^2 + v$ is a vertical translation of $y = x^2$, v units up.

• The graph of $y = a(x - h)^2 + v$ is a dilatation (vertical stretch or compression) followed by a translation (horizontal or vertical).

Exploration

• Before inviting students to work on the activity, you may want to review what they know about dilatations (stretches) of shapes and about mapping notation. For example, ask them what a dilatation of × 3 from the origin would do to the rectangle shown at the right. You might then ask what a mapping of $(x, y) \rightarrow (2x, y + 3)$ would do to the rectangle.

• Then encourage students to work with a partner on **parts A to F.** Partners might share the work on **parts B, C, and D.**



Observe and Assess

As students are working, notice:

• Do they substitute correctly into the functions?

• Are they comfortable with mapping notation?

• Do they generalize from the examples they try?

• Do they make predictions based on prior experiences?

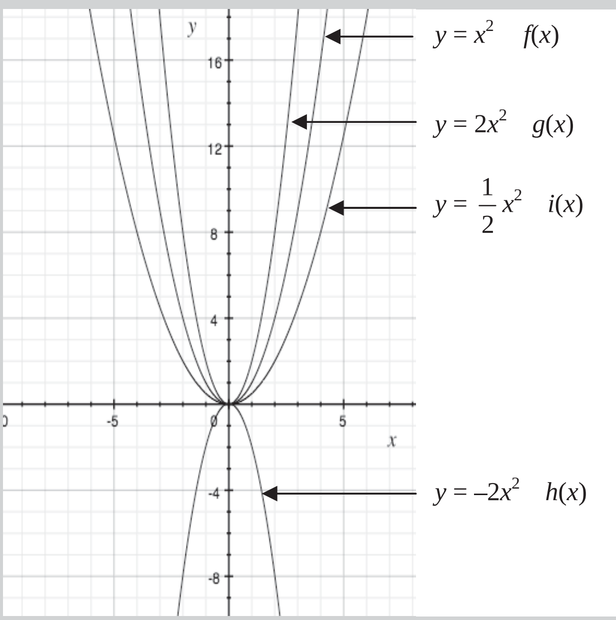
Share and Reflect

Ask students to look at the function in **part F**. Ask them to describe how they predicted what the function would look like and how each part of the function rule played a role in that prediction.

Answers

A. and B. i)

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = -2x^2$	$i(x) = \frac{1}{2}x^2$
–3	9	18	–18	4.5
–2	4	8	–8	2
–1	1	2	–2	0.5
0	0	0	0	0
1	1	2	–2	0.5
2	4	8	–8	2
3	9	18	–18	4.5

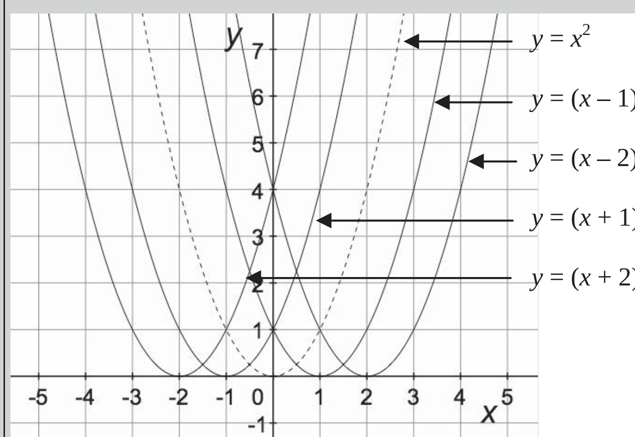


B. ii)
- The graph of $g(x)$ is steeper than $f(x)$ on both sides of the axis of symmetry. That makes sense since the values in the table for $g(x)$ go down and then up much more quickly than for $f(x)$.
- The graph of $h(x)$ is below the x-axis whereas $f(x)$ is above the x-axis. That makes sense because $h(x)$ has all negative values except for 0. But $f(x)$ has all positive values except for 0. The graph of $h(x)$ is also steeper than $f(x)$ for the same reason $g(x)$ is steeper than $f(x)$.
- The graph of $i(x)$ is less steep than $f(x)$. That makes sense since the values in the table for $i(x)$ do not go down and then up as quickly as those of $f(x)$.

iii) Each graph looks similar to $f(x)$ but is either stretched ($g(x)$) or compressed ($i(x)$); $h(x)$ has not only been stretched but it has also been reflected in the x-axis — it is the reflection image of $g(x)$.

C. i)

x	$f(x)$	$f(x-1)$	$f(x-2)$	$f(x+1)$	$f(x+2)$
–3	9	16	25	4	1
–2	4	9	16	1	0
–1	1	4	9	0	1
0	0	1	4	1	4
1	1	0	1	4	9
2	4	1	0	9	16
3	9	4	1	16	25



ii) A translation to the right or left (a horizontal translation).

iii) I could have predicted whether the translation would be left or right and the number of units it would have translated left or right.

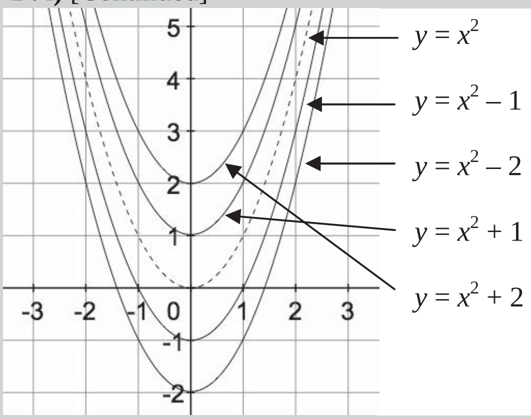
iv) Each would be a horizontal translation of the original function.

D. i)

x	f(x)	$f(x) - 1$	$f(x) - 2$	$f(x) + 1$	$f(x) + 2$
–3	9	8	7	10	11
–2	4	3	2	5	6
–1	1	0	–1	2	3
0	0	–1	–2	1	2
1	1	0	–1	2	3
2	4	3	2	5	6
3	9	8	7	10	11

Answers [Continued]

D. i) [Continued]



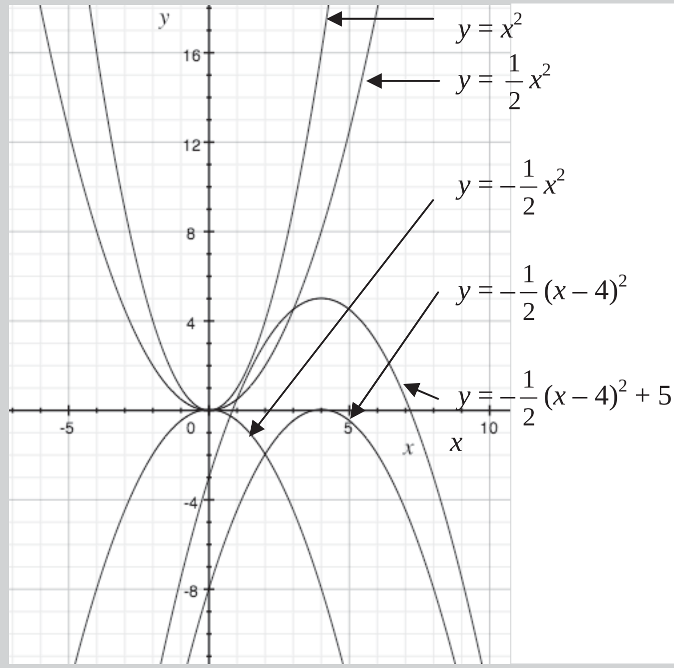
ii)

- A translation up or down (a vertical translation).

- I could have predicted whether the translation would be up or down and the number of units it would have translated up or down.

iii) Each would be a vertical translation of the original function.

E.



$y = -\frac{1}{2}x^2$, $y = -\frac{1}{2}(x-4)^2$, and $y = -\frac{1}{2}(x-4)^2 + 5$ are negative dilatations, which is the same as a reflection in the x-axis.

F. Translate right (horizontally) 4 units:

$f(x) = -\frac{1}{2}(x - \mathbf{4})^2 + 5$

Reflect in the x-axis (negative dilatation):

$f(x) = \mathbf{-}\frac{1}{2}(x-4)^2 + 5$

Dilatate (compress vertically) by half:

$f(x) = -\frac{\mathbf{1}}{\mathbf{2}}(x-4)^2 + 5$

Translate up (vertically) 5 units: $f(x) = -\frac{1}{2}(x-4)^2 \mathbf{+ 5}$

Supporting Students

Struggling students

• There is a lot of graphing in this lesson. Some students might work on just one of the **parts B, C, or D** and share results with other students who worked on other parts.

• Mapping notation will be revisited in the next lesson so if students are struggling with it in this lesson, do not insist that they complete **parts C iv) and D iii)**.

Enrichment

Some students might investigate what happens when the graph $y = x^2$ is transformed to $y = (kx)^2$.

5.1.4 Relating Graphs of Quadratic Functions

Curriculum Outcomes	Outcome relevance
10-C1 Transformations: express algebraically or using mapping rules • express transformations from a graph using algebraic expressions or using mapping rules • express mapping rules algebraically and vice versa **10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics • interpret characteristics in relation to given contexts **10-C5 Graphs and tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • translate among tabular, written, symbolic, and graphical representations of functions **10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function	Following the initial optional exploration of transformations in the previous lesson, it is important to solidify student understanding of the role of the parameters when graphing functions of the form $y = \mathrm{a}(x - \mathrm{h})^2 + \mathrm{k}$. This will allow students to solve problems involving quadratics more efficiently.

Pacing	Materials	Prerequisites
1 h *Additional time might be required if* ***lesson 5.1.3*** *was omitted.*	• Grid paper (BLM in Unit 6)	• use of a coordinate grid • mapping notation • familiarity with the terms *dilatation*, *vertical*, and *horizontal*

Main Points to be Raised

• The graph of $y = (x - \mathrm{h})^2$ is a horizontal translation of $y = x^2$ to the right by h units. In mapping notation, you can write $(x, y) \rightarrow (x + \mathrm{h}, y)$. The y-coordinate is not affected. If h is negative, the move is to the left rather than to the right.

• The graph of $y = x^2 + \mathrm{v}$ is a vertical translation of $y = x^2$ up by v units. In mapping notation, you can write $(x, y) \rightarrow (x, y + \mathrm{v})$. The x-coordinate is not affected. If v is negative, the move is down rather than up.

• The graph of $y = \mathrm{a}x^2$ is a stretch of $y = x^2$. If $|\mathrm{a}| > 1$, the graph is narrower; this is called a vertical stretch. If $0 < \mathrm{a} < 1$, the graph becomes wider. This is called a vertical compression. If $\mathrm{a} < 0$, the graph is reflected in the x-axis. This is called a negative dilatation since the sign of the y-coordinate is reversed.

• Vertical stretches and reflections can be described using the mapping notation: $(x, y) \rightarrow (x, \mathrm{a}y)$. The x-coordinate is not affected.

• The graph of $y = \mathrm{a}(x - \mathrm{h})^2 + \mathrm{v}$ is a dilatation or stretch followed by horizontal and vertical translations.

• A vertical stretch or compression should always be performed before a vertical translation. A horizontal translation can happen anywhere in the sequence.

Try This — Introducing the Lesson

A. Students can solve the problem alone.

Observe while students work. You might ask:

• *Why does it make sense that the graph is this shape?* (The ball goes up and comes down again.)

• *How high was the ball 1 s after it was kicked?* (15 m up.)

• *Why are there no values for x after 4 s?* (Once the ball hits the ground again, it is meaningless to continue to measure its height.)

The Exposition — Presenting the Main Ideas

• If students have completed **lesson 5.1.3**, they can read through the exposition on their own to confirm what they have learned and to become familiar with the new vocabulary, *vertical stretch* and *vertical compression.*

• If students have not completed **lesson 5.1.3**, lead them through the exposition on **pages 141 to 143**.

• Rather than memorizing the rules for what happens for various values of a, h, and v, students should develop mental representations of the various transformations. For example, they might think of $y = 2x^2$ as a narrow parabola pointing up to remind them of the effect of a in $y = ax^2$.

• You might put up a poster with some sample parabolas for $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$, $y = (x - 2)^2$, $y = (x + 2)^2$, $y = x^2 + 2$, and $y = x^2 - 2$ so students can refer to them.

Revisiting the Try This

B. Students have the opportunity to apply what they learned in the exposition to the situation they became familiar with when completing the **Try This** problem**.**

Using the Examples

• Write the function from the problem in **example 1** on the board. Ask students to predict what the graph will look like and to compare their predictions with the worked example on **pages 144 and 145**.

• Lead students, as a group, through **example 2**. Reinforce these notions:

- A narrow graph means |a| >1 whereas a wider graph means |a| < 1.
- A graph pointing downwards suggests that a < 0.
- A graph moved up or down has undergone a vertical translation.
- A graph that has moved right h units suggests that h > 0. The mapping notation is always $(x + h, y)$ whether h is positive or negative.
- If a < 0, you can consider the effect as one transformation (a single negative dilatation) or as a composite transformation (a stretch or compression and a reflection in the *x*-axis).

Practising and Applying

Teaching points and tips

Q 1: Observe whether students realize that they need to look only at the coefficient of x^2 for **part b**, but that they need to consider the other parameters in the function descriptions for the other parts of the question.

Q 2: Make sure students remember that the mapping diagram showing a horizontal translation has a sign that is opposite to the related sign in the equation.

Q 3: Students might refer back to **example 2** for this question.

Q 4: Some students might struggle with the idea that we call a narrower parabola a stretched version of the basic parabola.

Common Errors

• As was noted earlier, many students will assume that if the basic parabola moves left, the equation involves $(x - h)$, whereas actually it involves $(x + h)$. Help students by using this example: Imagine starting with $y = x^2$ where the vertex is (0, 0). If h = 3, you have $y = (x - 3)^2$, so the vertex (0, 0) moves to (3, 0); that means the mapping notation is $(x, y) \rightarrow (x + h, y)$. If h = –3 you have $y = (x + 3)^2$, so the vertex (0, 0) moves to (–3, 0). That means the mapping notation is $(x, y) \rightarrow (x + h, y)$ since 0 + (–3) = –3.

• Students may mix up the terms *stretch* and *compression*. Encourage students to think of the transformation not in terms of what happened to the first coordinate, but in terms of what happened to the second coordinate.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can go from a mapping to an equation of a parabola
Question 3	to see if students can recognize the transformations from a graph and use them to represent an equation algebraically
Question 5	to see if students can communicate their understanding of vertex form of a parabola in the context of what they have learned about transformations

Answers

A. i) 4 s **ii)** 20 m **iii)** 2 s

B. i)

1. A horizontal translation right of 2 units because $h = -5(t - 2)^2 + 20 = -5(t - (\mathbf{+2}))^2 + 20$.
2. A vertical stretch (dilatation) by 5 and a reflection in the x-axis because $h = \mathbf{-5}(t - 2)^2 + 20$.
3. A vertical translation up of 20 units because $h = -5(t - 2)^2 \mathbf{+ 20}$.

Other possible orders are: 2, 3, 1 and 2, 1, 3.

ii) The vertex of $h = t^2$ is (0, 0), so the vertex of $h = -5(t - 2)^2 + 20$ is (0 + 2, 0 + 20) = (2, 20).

1. a) i) (0, 4) **ii)** (8, 0) **iii)** (3, –2)

iv) (–1, –1) **v)** (1, 1) **vi)** (2, 2)

b) i) Up; not dilatated

ii) Up; not dilatated

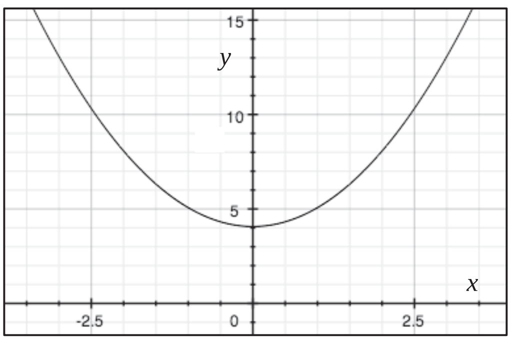
1. b) iii) Up; not dilatated

iv) Down; dilatated by a factor of –2

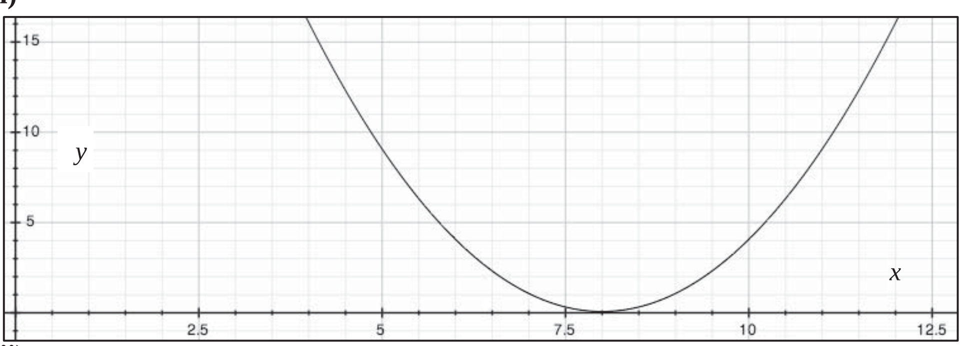
v) Up; dilatated by a factor of $\frac{1}{5}$

vi) Down; dilatated by a factor of $-\frac{1}{5}$

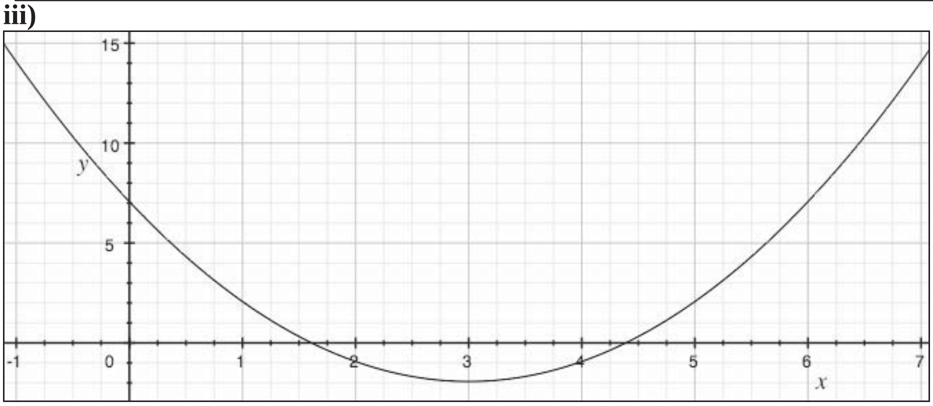
1. c) i)



ii)

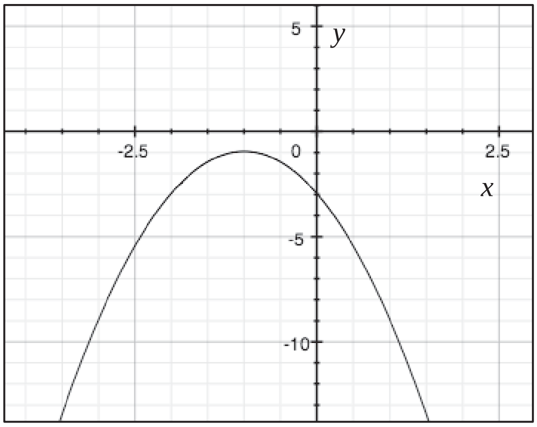


iii)

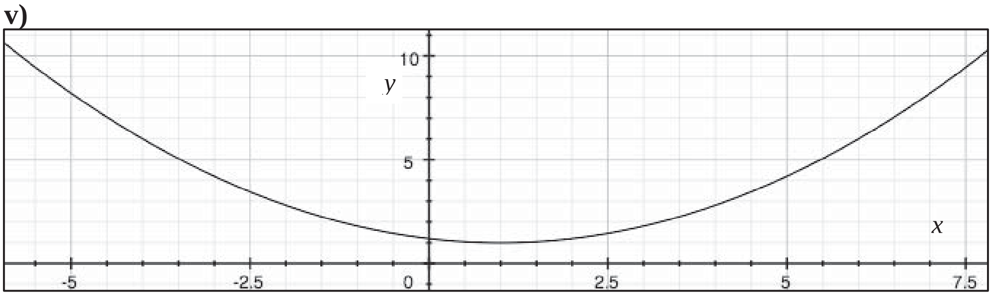


Answers [Continued]

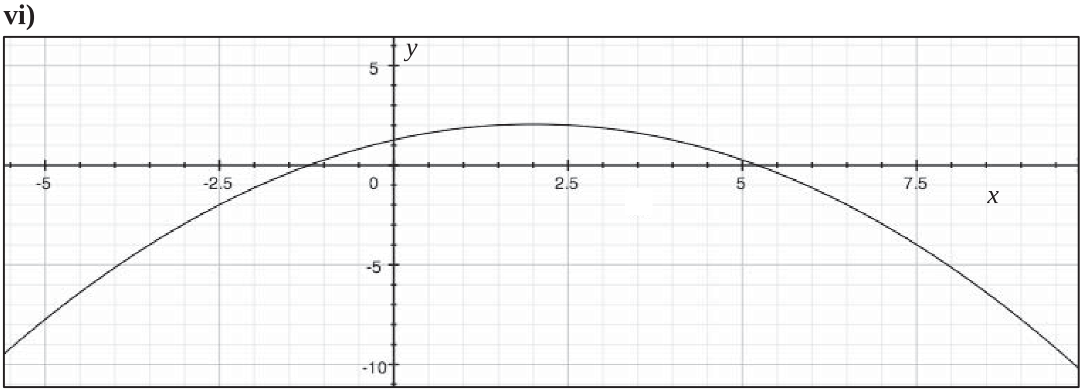
1. c) iv)



v)



vi)



2. a) $y = (x - 4)^2$ **b)** $y = -\frac{1}{2}x^2$

c) $y = (x + 4)^2 - 3$ **d)** $y = \frac{1}{4}(x - 4)^2$

e) $y = -5(x - 4)^2$ **f)** $y = -3(x - 4)^2 + 6$

3. $y = -2(x + 2)^2 - 1$

4. a) When the dilatation factor is, for example, –1, the *y*-coordinate of every point on the parabola is multiplied by –1 because $(x, y) \rightarrow (x, -1 \times y)$. The *x*-coordinate stays the same but the *y*-coordinate becomes its opposite so every point is reflected across the *x*-axis, resulting in a parabola that is a refection of the original parabola.

4. b) The word vertical applies to vertical axis (the *y*-coordinate of each point), so when the *y*-coordinate of each point on a parabola is multiplied by a factor of 2 while the *x*-coordinate stays the same, the parabola is stretched vertically.

5. It is a good name because the vertex is easy to figure out when it is in the form $f(x) = a(x - h)^2 + k$; the vertex is (h, k). For example, the coordinates of the vertex of $y = -3(x - 4)^2 + 6$ are (4, 6).

Supporting Students

Struggling students

For **question 2**, encourage struggling students to use tables of values to help them apply mapping diagrams. They can then graph the functions and observe the transformations directly from the graph.

Enrichment

Students can explore mapping diagrams like these: $(x, y) \rightarrow (2x + 4, y)$ or $(x, y) \rightarrow (4x + 4, y)$ to see the effect of the coefficient of x.

CONNECTIONS: Parabolas and Paper Folding

This **Connections** feature provides an alternative way for students to think of parabolas. This activity is built on the definition of a parabola as the locus of points that are equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

This activity also exposes students to the concept of an *envelope* of a curve, which defines a curve through its tangents.

Answers

1. The vertex is halfway between the Focus and the line.

2.

- The distance between the Focus and the line influences the width of the parabola.
- If the focus point and the line are the same distance from the x-axis, the vertex of the parabola is at the origin. Otherwise, the vertex is above or below the x-axis.
- If the Focus is not on the y-axis, the vertex will not have an x-coordinate of zero.

5.1.5 EXPLORE: The Absolute Value Function

Curriculum Outcomes	Lesson relevance
10-C1 Transformations: express algebraically or using mapping rule • express transformations from a graph using algebraic expressions or using mapping rules • express mapping rules algebraically and vice versa **10-C4 Graphs and Tables: construct and analyse** • analyse graphs and tables to determine mathematical characteristics **10-C5 Graphs and Tables: explore dynamics of change** • determine how changes in one variable affect another through the analysis of tables or graphs **10-C6 Graphs: sketch** • translate among tabular, written, symbolic, and graphical representations of functions **10-C18 Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions** • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the function • apply graphical transformations (reflections, stretches, and translations) resulting from changes in the parameters of the absolute value function	This essential lesson is the students' first exposure to the absolute value function. It helps them generalize what they have learned about quadratics to other non-linear functions.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• coordinate graphing • function notation • mapping notation

Main Points to be Raised

• The absolute value function has a V shape. This is because $|x| = |-x|$.

• The graph of $y = |x - \text{h}|$ is a horizontal translation of the graph $y = |x|$ to the right by h units. In mapping notation, you can write $(x, y) \rightarrow (x + \text{h}, y)$. The y-coordinate is not affected. If h is negative the move is to the left.

• The graph of $y = |x| + \text{v}$ is a vertical translation of the graph $y = |x|$ up by v units. In mapping notation, you can write $(x, y) \rightarrow (x, y + \text{v})$. The x-coordinate is not affected. If v is negative, the move is down rather than up.

• The graph of $y = \text{a}|x|$ is a stretch of the graph $y = |x|$. If $|\text{a}| > 1$, the graph becomes narrower. This is called a vertical stretch. If $-1 < \text{a} < 1$, the graph becomes wider. This is sometimes called a vertical compression. If $\text{a} < 0$, the graph is reflected in the x-axis. This is called a negative dilatation since the sign of the y-coordinate is reversed.

• Vertical stretches and reflections can be described using the mapping notation $(x, y) \rightarrow (x, \text{a}y)$. The x-coordinate is not affected.

• The graph of $y = \text{a}|x - \text{h}| + \text{v}$ is a dilatation or stretch followed by horizontal and vertical translations.

• A vertical stretch or compression should always be performed before a vertical translation. A horizontal translation can happen anywhere in the sequence of motions.

Exploration

Introduce absolute value notation to students. Relate it to the distance from zero on a number line.

Make sure students understand that $|x|$ is x if $x \geq 0$, but that it is $-x$ if $x < 0$.

Show several examples. For example, $|2 - 3| = |-1| = 1$.

Students might read the introduction on **page 149** and then work on **parts A to D** in pairs.

Observe and Assess

As students are working, notice:

- Do they apply what they already know about transformations related to quadratic functions?
- Do they use reasonable values when they create their tables of values in **part B**?
- Are they as comfortable creating mapping notation as applying it?

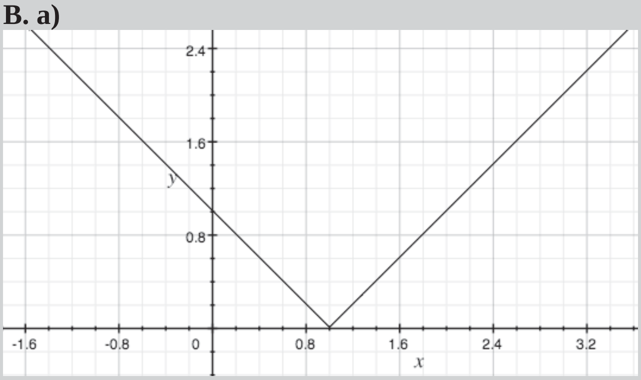
Share and Reflect

Ask students to look at the function in **part D**. Ask them to describe how they predicted what the function would look like and how each part of the function rule played a role in that prediction.

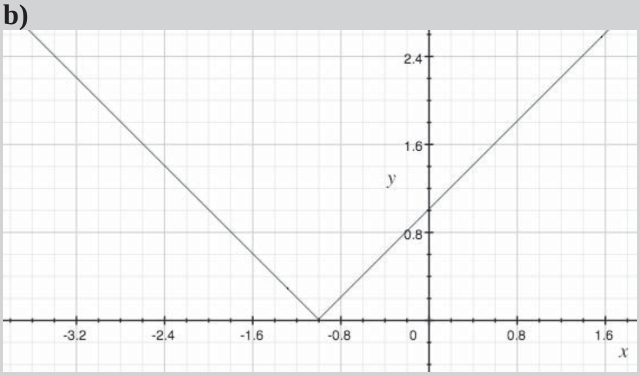
Answers

A. Every value of *x* and its opposite, $-x$, have the same *y*-value.

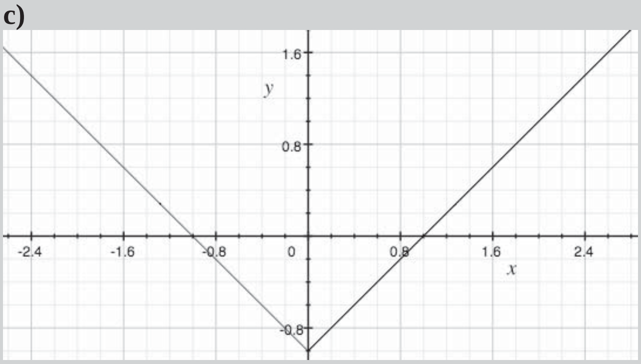
B. a)



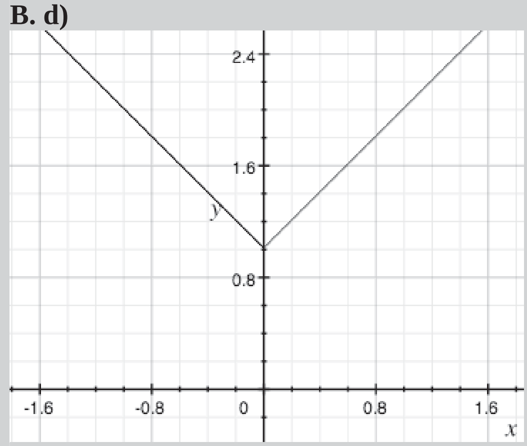
b)



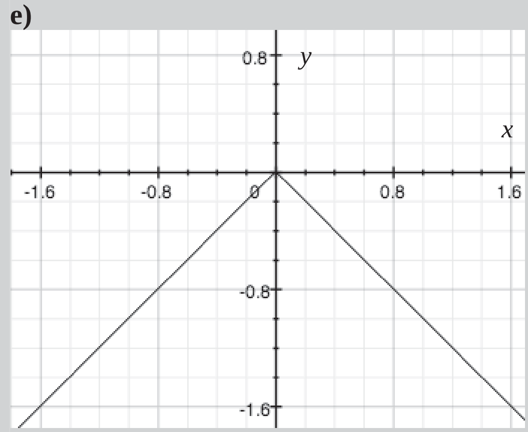
c)



B. d)



e)



Answers [Continued]

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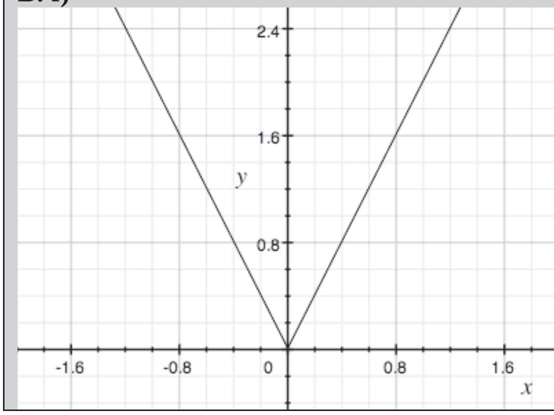
B. f)



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C. a) $(x, y) \rightarrow (x + 1, y)$ **b)** $(x, y) \rightarrow (x - 1, y)$

c) $(x, y) \rightarrow (x, y - 1)$ **d)** $(x, y) \rightarrow (x, y + 1)$

e) $(x, y) \rightarrow (x, -y)$ **f)** $(x, y) \rightarrow (x, 2y)$

D.
1. A horizontal translation right by 4 units.
2. A reflection in the *x*-axis and a compression by $\frac{1}{2}$.
3. A vertical translation up by 5 units.
Other possible orders are: 2, 3, 1 and 2, 1, 3

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Supporting Students

Struggling students

If students are uncomfortable with mapping notation, encourage them to use tables of values to create graphs that will help them answer **part C**.

Enrichment

Some students might investigate what happens when the graph $y = |x|$ is transformed to $y = |\mathrm{k}x + \mathrm{h}|$.

Chapter 2 Solving Non-Linear Equations

5.2.1 Factoring Quadratic Expressions

Curriculum Outcomes	Outcome relevance
10-C16 Quadratic Equations: solve by factoring • develop factoring strategies for polynomials in one variable that are products of degree one binomials	Factoring polynomials is a traditional part of the mathematics curriculum. It is more useful for higher level mathematics than for solving everyday problems, but the ability to factor will help students solve some of the quadratic application problems in this chapter.

Pacing	Materials	Prerequisites
2 h	• Algebra tiles	• multiplying two linear binomials • zero principle for integers (recognizing that 0 can be represented as $x + (-x)$ • representing polynomial products with algebra tiles

Main Points to be Raised

• The product of two linear binomials is the area of a rectangle where the linear binomials represent the length and the width.

• Factoring is the reverse of multiplying; in this case, you know the product instead of the factors. When talking about factoring, it is conventional to talk about factoring over the set of integers, i.e., a quadratic is factorable if binomials with integer coefficients can be found to describe the polynomial as a product.

• If $x^2 + \mathrm{b}x + \mathrm{c}$ can be factored into $(x + \mathrm{r})(x + \mathrm{s})$, then $\mathrm{rs} = \mathrm{c}$ and $\mathrm{r} + \mathrm{s} = \mathrm{b}$. To find these values, it makes sense first to look for factors of c in order to get candidates for r and s.

• Factoring as $(-x - \mathrm{r})(-x - \mathrm{s})$ is equivalent to factoring as $(x + \mathrm{r})(x + \mathrm{s})$. So if r and s have been checked to see if they are helpful in factoring, there is no need to also check –r and –s.

• One way to factor is first to pull out a common factor and then to factor the remaining expression further.

• To factor $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, look for two numbers r and s such that $\mathrm{rs} = \mathrm{ac}$ and $\mathrm{r} + \mathrm{s} = \mathrm{b}$. If you do, you can rewrite $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$ as $\mathrm{a}x^2 + \mathrm{r}x + \mathrm{s}x + \mathrm{c}$. The first two terms can be factored, as can the last two, and they always share a common factor. This allows you to factor the original quadratic completely.

[The reason this works is that $\mathrm{a}x^2 + \mathrm{s}x + \mathrm{r}x + \mathrm{c} = \mathrm{s}x(\frac{\mathrm{a}}{\mathrm{s}}x + 1) + \mathrm{c}(\frac{\mathrm{r}}{\mathrm{c}}x + 1)$. Since $\mathrm{rs} = \mathrm{ac}$, $\frac{\mathrm{r}}{\mathrm{c}} = \frac{\mathrm{a}}{\mathrm{s}}$. There is a common factor of $(\frac{\mathrm{a}}{\mathrm{s}}x + 1)$ and the original quadratic is factored as $(\mathrm{s}x + \mathrm{c})(\frac{\mathrm{a}}{\mathrm{s}}x + 1)$.] To find the two values, r and s, you might take the value ac and successively divide it by candidate factors. When you have found the prime factors, you can use values that can be added to create the middle coefficient. [The discussion in the above paragraph is background information for you, not an explanation to be provided to students.]

• Certain quadratics may be easier to factor if you can find a pattern. These include the difference of two squares, $x^2 - y^2$, which can be represented as $(x + y)(x - y)$, and the square of a linear binomial, which always takes the form of $x^2 + 2xy + y^2$ for $(x + y)^2$.

Try This — Introducing the Lesson

A. Students can solve the problem alone or with a partner. You may wish to remind students of the distributive principle with numbers before starting the **Try This**.
For example, show how $(20 + 3)(30 + 5) = 20 \times 30 + 20 \times 5 + 3 \times 30 + 3 \times 5$.

Observe while students work. You might ask:

• *Which parts will you multiply to get the coefficient of x^2?* (I will multiply the triangle and the square since they both go with an x. When I multiply them, I will get x^2.)

• *Why are there only three terms when there are four partial products?* (There are two terms involving a number multiplied by x. When I collect like terms, these combine to become one term.)

• *How will you get the –6 ?* (I will multiply the circle and the diamond.)

The Exposition — Presenting the Main Ideas

• If algebra tiles are available, you may want to use them to model the binomial multiplication in the text. You can use the material from the Class IX text on polynomial multiplication to remind students of how polynomial multiplication works. Whether or not you use tiles, students need to understand that you can represent the factors of a polynomial by thinking of the polynomial as the area of a rectangle and by looking for expressions for the length and width as the factors.

• If you use tiles, it is particularly important to review how negative values are represented and multiplied. You will notice that the tiles are never used in the text to multiply two negatives; tiles are more helpful when a positive is being multiplied by either another positive or a negative.

• Work through the exposition with students to make sure they understand the important concepts.

• It will probably take a full hour to go through the exposition and some of the examples.

Revisiting the Try This

B. Students have the opportunity to think explicitly about the relationship between polynomial multiplication and factoring.

Using the Examples

• Write the problem from **example 1** on the board. Ask students to work in pairs to try to solve the problem. Then work through the solution in the text with them. Make sure they understand why it is possible to have only one pair of factors. (If it were possible to write $x^2 + \mathrm{b}x + \mathrm{c}$ as $(x + \mathrm{r})(x + \mathrm{s})$ and as $(x + \mathrm{p})(x + \mathrm{q})$ then rs = pq and r + s = p + q. This means that a certain pair of factors of c would add to the same amount, and this is only possible if they are the same factors.)

• Make sure they understand that the process used to factor $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, often called decomposition, can be summarized as follows:

- Multiply a and c.
- Find two factors of the product ac that add to b. Call these factors r and s.
- Break up $\mathrm{b}x$ into $\mathrm{r}x + \mathrm{s}x$.
- Factor the first pair of terms, $\mathrm{a}x^2 + \mathrm{r}x$, and then factor the second pair, $\mathrm{s}x + \mathrm{c}$, pulling out the greatest common factor for each pair.
- Factor again, pulling out the binomial common factor of the two terms.

For example, for $4x^2 + x - 18$:

- ac = –72
- Two factors of –72 that add to 1 are 9 and –8.
- Rewrite as $(4x^2 - 8x) + (9x - 18) = 4x(x - 2) + 9(x - 2) = (4x + 9)(x - 2)$

• One way to find the factors is to factor the product ac as a product of primes and combine the primes appropriately, e.g., $-72 = -2 \times 2 \times 2 \times 3 \times 3$. Another way is to simply list possible pairs of factors.

• Allow students to read through **examples 2, 3, and 4** on their own or in pairs.

• You may want to support **example 4** by looking at patterns:

$(x + 3)^2 = x^2 + 6x + 9$; $(x + 4)^2 = x^2 + 8x + 16$; $(x + 5)^2 = x^2 + 10x + 25$; $(x + 6)^2 = x^2 + 12x + 36$

Students will notice that the first and last terms are perfect squares. The middle term is always even and is the double of the constant in the binomial. They could generalize to patterns like these:

$(x + 3)^2 = x^2 + 6x + 9$; $(2x + 3)^2 = 4x^2 + 12x + 9$; $(3x + 3)^2 = 9x^2 + 18x + 9$

Students will notice that the first and last terms are perfect squares and the middle term is now double the product of the coefficient of x and the constant term. By looking at these patterns, they are more likely to recognize perfect squares when they encounter them.

• Make sure students have understood the material by asking them to factor these quadratics. Students can work in groups as these are challenging quadratics to factor. Note that students should be encouraged to remove a common factor of 2 in the third expression before factorising.

$2x^2 - 5x + 3$, $15x^2 - 4x - 32$, $25x^2 - 81$, $9x^2 + 48x + 64$, and $50x^2 + 40x + 8$.

Practising and Applying

Teaching points and tips

Q 1: Some students will benefit from using *guide tiles* to represent the linear dimensions of the rectangle. For example, in this case they might line up an x-tile next to a 1-tile on one dimension and an x-tile next to two 1-tiles on the other dimension.

Q 2: Students who place the x^2-tile and the two –1-tiles will realize they need either two $-x$-tiles or one $-x$-tile to go with the two –1-tiles. They will need to add x and $-x$ to make this work.

Q 5: It will help for students to refer to **example 3 part b)**.

Q 9: Observe whether students use a variety of approaches or a single approach to deal with the variety of quadratics presented.

Common Errors

• Students may have difficulty working with algebra tiles that have negative coefficients. Make sure they recall that as they form rectangles with the tiles, all the negative tiles must form a rectangle that extends completely from one side to the other or from the top to the bottom within the full rectangle.

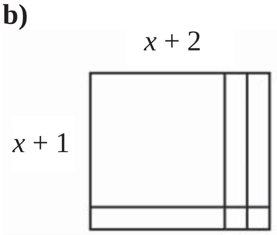
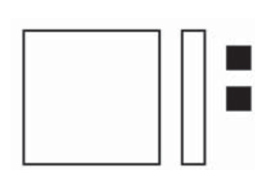
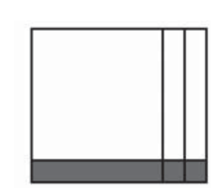
• Many students overlook possible factor combinations when they look for factor pairs with a particular sum. Encourage them to be systematic. For example, to look for pairs of factors that multiply to 7 × 8, 56, they might start at 1 and systematically go up:

$$1, 56 \rightarrow 2, 28 \rightarrow 4, 14 \rightarrow 7, 8$$

Suggested assessment questions from Practising and Applying

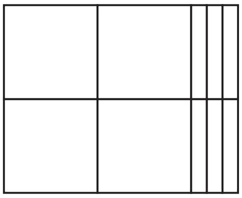
Question 3	to see if students can relate factoring to multiplication of polynomials using a concrete or pictorial model
Question 5	to see if students recognize a difference of squares pattern and apply what they learned about factoring such quadratics
Question 7	to see if students recognize perfect binomial squares and apply what they learned about factoring such quadratics
Question 8	to see if students can apply a variety of factoring methods

Answers

A. i) 2; –6; 1 $(▲x + ●)(■x + ♦) = ▲x \times ■x + ▲x \times ♦ + ● \times ■x + ● \times ♦$ $= (▲ \times ■)x^2 + (▲ \times ♦ + ● \times ■)x + ● \times ♦$ $= \quad 2x^2 \quad + \quad x \quad - \quad 6$ If $(▲ \times ■)x^2 = 2x^2$, then $▲ \times ■ = 2$. If $(▲ \times ♦ + ● \times ■)x = 1x$, then $▲ \times ♦ + ● \times ■ = 1$. $● \times ♦ = 6$	**A. ii)** ▲is 1; ● is 2; ■ is 2; ♦ is –3. Note that ▲is 2; ● is -3; ■ is 1; ♦ is 2 is also acceptable. **B. i)** Multiplying two binomial factors is the reverse of factoring a trinomial into two binomials.
1. a) $x^2 + 3x + 2$ **b)**  **c)** $(x + 1)(x + 2)$	**2. a)**  $x^2 + x - 2$ **b) and c)** $(x - 1)(x + 2)$ 

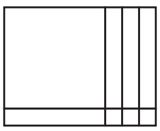
Answers [Continued]

3. a) $2x(2x + 3)$

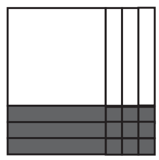


b) $(x - 3)(x + 1)$

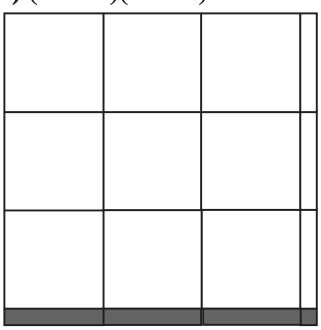
c) $(x + 3)(x + 1)$



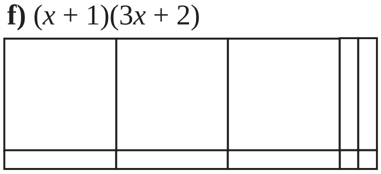
d) $(x + 3)(x - 3)$



e) $(3x + 1)(3x - 1)$



f) $(x + 1)(3x + 2)$



4. a) $(x + 3)(x + \mathbf{4}) = x^2 + \mathbf{7}x + 12$

b) $(x + \mathbf{9})(x - \mathbf{9}) = x^2 - 81$

c) $(\mathbf{1}x + 2)(\mathbf{5}x - \mathbf{1}) = 5x^2 + 9x - 2$

d) $(2x + \mathbf{3})(\mathbf{3}x - 2) = 6x^2 + \mathbf{5}x - 6$

5. a) $(x + 11)(x - 11)$ **b)** $(x + 20)(x - 20)$

c) $(5x + 1)(5x - 1)$ **d)** $(6x + 5)(6x - 5)$

6. a) $6x(2x + 3)$ **b)** $5x(3x - 5)$ **c)** $ax(x + 1)$

7. a) $(x + 3)^2 = x^2 + \mathbf{6}x + \mathbf{9}$

b) $(x + \mathbf{6})^2 = x^2 + \mathbf{12}x + 36$

c) $x^2 - 12x + \mathbf{36} = (x - \mathbf{6})^2$

d) $x^2 + \mathbf{14}x + 49 = (x + \mathbf{7})^2$

8. a) $(x + 3)(x + 5)$ **b)** $(x - 6)(x - 1)$

c) $(x + 4)(x - 3)$ **d)** $(x + 2)(x - 8)$

e) $(x - 4)^2$ **f)** $(x + 8)(x - 8)$

9. a) $(2x - 3)^2$ **b)** $(2x + 3)(x - 2)$

c) $(3x + 1)(x - 4)$ **d)** $(5x - 1)(2x + 1)$

e) $(3x - 2)(x - 3)$ **f)** $(7x + 10)(7x - 10)$

10. *Sample response*:
Similar: If you think of factoring using the area model, you are still finding the side lengths of a rectangle with a particular area that you know.
Different: You have to juggle more pieces of information to figure out the factors of a quadratic compared to factors of a number.

Supporting Students

Struggling students

• If students struggle with factoring abstractly, allow them to use algebra tiles.

• Focus on factoring expressions where the coefficient of x^2 is 1 and allow students to gain success with these before moving on to more general quadratic functions.

Enrichment

• Students might try to create quadratic expressions that cannot be factored and compare them with those that can be factored.

• They might also try to figure out why the decomposition method always works if a quadratic is factorable.

5.2.2 EXPLORE: Roots of Quadratic Equations

Curriculum Outcomes	Lesson relevance
10-C9 Non-linear Equations: evaluate and interpret • determine the roots of quadratic equations from the corresponding graph **10-C12 Non-linear equations: evaluate and interpret** • determine the roots of quadratic equations from the corresponding graph **10-C14 Equations: solve using graphs** • use the x-intercept to determine the solution of quadratic equations	This required lesson focuses on the relationship between the graph of a parabola and the roots of the related equation.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• concept of an equivalent equation • term *x-intercept*

Main Points to be Raised

• The roots of a quadratic equation are also called its zeros. They are described by the x-intercepts of the graph of the related function.

• Once a quadratic expression has been factored, the roots of the related equation can be determined by setting each factor equal to zero.

• A quadratic equation $ax^2 + bx + c = 0$ has only one root if the expression on the left can be factored into a perfect square.

• There are equivalent forms of quadratic equations, but it is easiest to determine the roots when the equation is in factored form.

• Looking at a graph, you can see that there can be zero, one, or two roots of a quadratic equation.

Exploration

Students might work on **parts A to F** in pairs. The pairs can share the work for **parts A and D.**

Observe and Assess

As students are working, notice:

• In **part A**, do they graph by using tables of values or by reasoning about what the intercepts have to be?

• Do they recognize when equations are equivalent?

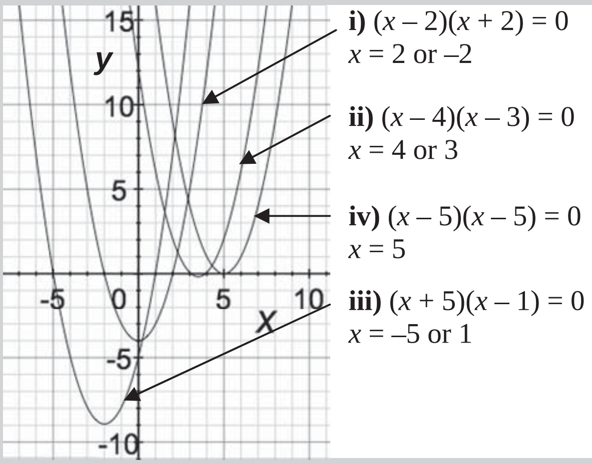
• Do they have more than one strategy for determining roots?

Share and Reflect

Ask students to share their answers to **part F**. Ask them how they could do a vertical translation of the function $f(x) = x^2 + 9$ so that there would be either one root or two roots.

Answers

A. and B.
The graphs are shown for **part A**. For **part B**, the roots are the numbers you must substitute for x to make one of the factors 0. These roots can be read directly from the graphs as they are the x-intercepts.



D. i) $(x – 2)(x + 2) = 0$ is equivalent to **i)**, **vi)**, and **viii)**.

ii) $(x – 4)(x – 3) = 0$ is equivalent to **iii)** and **vii)**.

iii) $(x + 5)(x – 1) = 0$ is equivalent to **x)** and **iv)**.

iv) $(x – 5)(x – 5) = 0$ is equivalent to **ii)**, **v)**, and **ix)**.

E. i) $(x – 6)(x + 1) = x^2 – 6x + x – 6 = x^2 – 5x – 6$, so $(x – 6)(x + 1) = 0$ and $x^2 – 5x – 6 = 0$ are equivalent. $x^2 – 5x – 6 = 0 \rightarrow x^2 – 5x – 6 + 6 = 6$ (add 6 to both sides) $\rightarrow x^2 – 5x = 6$, so $x^2 – 5x – 6 = 0$ and $x^2 – 5x = 6$ are equivalent.

ii) $x = 6$ or $–1$

iii) $(x – 6)(x + 1) = 0$, since the roots can be determined immediately by examining the factors.

F. The parabola opens up and its vertex is above the x-axis, so there are no x-intercepts and therefore there are no roots.

C. i) Equation **iv)**, $(x – 5)(x – 5) = 0$, has only one root, 5.

ii) The graph touches the x-axis at only one point, so there is only one x-intercept.
The two factors are identical so the same value will make both factors zero.

Supporting Students

Struggling students

To help students recognize equivalent equations, encourage them to begin with an equation and apply the same operation to both sides of it. For example, if you start with $x^2 – 4x + 6 = 2$, all of the equations below are equivalent.

$x^2 – 4x + 6 = 2$
$x^2 – 4x + 6 – 6 = 2 – 6$, so $x^2 – 4x = –4$
$x^2 – 4x + 6 + 4x = 2 + 4x$, so $x^2 + 6 = 2 + 4x$

5.2.3 Solving Quadratic Equations by Factoring

Curriculum Outcomes	Outcome relevance
10-C4 Graphs and Tables: construct and analyse • analyse graphs to determine mathematical characteristics • interpret characteristics in relation to given contexts **10-C6 Graphs: sketch** • create graphs given information in a variety of formats • sketch the graph of a quadratic function in factored form **10-C9 Non-linear Equations: evaluate and interpret** • determine the roots of quadratic equations from the corresponding graph **10-C12 Problems: express in terms of equations** • analyse and interpret a variety of situations and model algebraically as equations **10-C13 Equations: rearrange** • transform equations from one form to another **10-C16 Quadratic equations: solve by factoring** • understand the zero product rule: if $ab = 0$, then either $a = 0$ or $b = 0$ • apply the zero product rule to solve quadratics by factoring • convert a quadratic equation to two linear equations by the factoring method • solve equations including those which involve common factors, regular equations, perfect square trinomials, and difference of squares	Factoring polynomials is a useful way to solve problems involving simple quadratic functions.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6) (optional)	• solving linear equations • factoring quadratics • Pythagorean theorem

Main Points to be Raised

• The zero product rule says that when a product of numbers is zero, one of them must be zero. This is useful when a quadratic has been factored into a product of binomials. If the quadratic is set equal to zero, one of the factors must be zero.

• If a quadratic equation is in the form $ax^2 + bx + c = d$, it is useful to use an equivalent equation of the form $ax^2 + bx + e = 0$. The reason it is useful is that if $ax^2 + bx + c$ can be factored and set equal to d, it is not possible to know what each factor might be; there are an infinite number of combinations of numbers that multiply to d. But if $ax^2 + bx + e$ can be factored and set equal to zero, the zero product rule applies.

• The solutions of a quadratic equation are called its roots or zeros. If the related quadratic function were graphed, the solutions or roots would correspond to the x-intercepts.

• There are never more than two roots of a quadratic since there are at most two x-intercepts of the related function. Sometimes there are fewer roots. Some people describe the roots saying, "*The roots are ... and …*"; other people say "*The roots are ... or …*". They mean the same thing.

• Certain problems can be represented and solved using quadratic equations.

Try This — Introducing the Lesson

A. Allow students to try this in pairs. Observe while students work. You might ask:

• *Could the other number be greater than 7?* (No, because then the answer would have to be more than 56, since the number and the square are both greater than 49 and 7.)

• *How do you know the number cannot be a positive number less than 7?* (The number and the square together would be too small.)

• *Why might a negative number lower than* $-\sqrt{56}$ *work?* (Because the square would be a large positive but you could bring it back down to 56 by adding the negative.)

The Exposition — Presenting the Main Ideas

- Allow students to read through the exposition on their own.
- They should recognize that the last statement summarizes their learning in **lesson 5.2.2**.

Revisiting the Try This

B. Students have an opportunity to think about how the problem they solved in **Try This** relates to what they have just learned about quadratic equations.

Using the Examples

Pair students up and have them work together on **example 1**. Then each student can work on either **example 2** or **example 3**. The students can then teach their partners about how the assigned example works.

Practising and Applying

Teaching points and tips

Q 2: Students should be applying a variety of factoring strategies to solve the parts of this question.

Q 5: There is more than one way to represent the situation. For example, if the lesser number were called n, the equation would be $n^2 + (n + 1)^2 = 221$, but if the greater number were called n, the equation would be $(n – 1)^2 + n^2 = 221$.

Q 6: The variable used to set up the equation might represent either the length or the width.

Common Errors

Students often have trouble using an algebraic expression to represent a problem. For example, for **question 5**, many students might write $(n + n + 1)^2 = 221$. They will not get the correct answer using this equation.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can solve a quadratic equation by factoring
Question 5	to see if students can represent a numerical problem with a quadratic equation and solve the equation to solve the problem
Question 7	to see if students can represent a geometric problem with a quadratic equation and solve the equation to solve the problem
Question 12	to see if students can set up an equation for an applied problem and solve the equation to find both solutions to the problem

Answers

A. $7 + 7^2 = 7 + 49 = 56$;
$–8$ is also possible $(–8 + (–8)^2 = –8 + 64 = 56)$.

B. i) $x^2 + x = 56$; there is an x^2-term in it and no higher powers of x.

B. ii) $x = 7$ or $–8$; yes

iii) *Sample response:*
A parabola has only two x-intercepts.

1. a) 4, 2 **b)** –3, 9 **c)** $\frac{-5}{2}, \frac{1}{3}$

d) $\frac{-4}{5}, \frac{9}{2}$ **e)** 0, 10 **f)** –2, 2

2. a) –3, –5 **b)** 11, –10 **c)** 0, –3

d) 6, –6 **e)** 10 **f)** –2

3. a) 5, $\frac{-5}{2}$ **b)** $\frac{1}{4}, \frac{-1}{2}$

c) –3, $\frac{1}{2}$ **d)** 2, –1

4. a) –1, 7 **b)** 5, 2 **c)** 3, 4 **d)** 5, –5

5. a) $x^2 + (x + 1)^2 = 221$ **b)** $x = 10$ and $–11$

c) The solutions of 10 and –11 are the lesser of the two consecutive integers in each pair since the other would be $x + 1$. For $x = 10$, we get $10^2 + 11^2 = 221$; for $x = –11$, we get $(–11)^2 + (–10)^2 = 221$.

6. 7 cm by 13 cm (Possible equation: $w(w + 6) = 91$)

7. 6 cm (Possible equation: $l^2 + (l + 2)^2 = 100$)

8. 17 m by 34 m (Possible equation $(2w + 4)(w – 3) = 532$)

9. 8 m (Possible equation: $d^2 + 225 = (2d + 1)^2$) **10.** 5 m, 12 m, and 13 m (Possible equation: $l^2 + (l - 7)^2 = (l + 1)^2$) **11.** 11 and 13, or –11 and –13 (Possible equation: $x^2 + (x + 2)^2 = 290$)	**12.** 10 s and 30 s; at 10 s the rocket is on the way up and at 20 s it is on the way down. (Possible equation: $-5t^2 + 200t = 1500$) **13.** *Sample response:* Some situations involve areas of rectangles, relationships of sides of a right triangle and relationships involving squares of numbers.

Supporting Students

Struggling students

• If students are still struggling with factoring, you may wish to delay assigning the problems that apply quadratics (**questions 5 to 12)** until they are better able to work with the equations.

• You might need to remind students of the Pythagorean theorem for several questions in the exercise set.

Enrichment

Students might create problems that can be solved using quadratic equations. They could share their problems with other students who might try to solve them.

5.2.4 EXPLORE: Absolute Value Equations

Curriculum Outcomes	Lesson relevance
10-C4 Graphs and Tables: construct and analyse relating two variables • analyse graphs and tables to determine mathematical characteristics **10-C10 Equations: solve for linear and simple radical, exponential, and absolute value equations and linear inequalities** • encourage proficiency with algebraic manipulation • use strategies to check answers for reasonableness within the problem context **10-C13 Equations: rearrange** • transform equations from one form to another	This essential lesson provides the only opportunity for students to see how absolute value equations are solved. These will be useful in mathematical situations in higher mathematics courses.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6)	• meaning of *absolute value* • solving linear equations

Main Points to be Raised

• You can solve an absolute value equation by graphing a related absolute value function and seeing where the y-coordinate takes on a certain value.

• You can interpret the equation $|x - \mathrm{k}| = \mathrm{d}$ to mean: *What are the values of x that are* d *units away from* k *on a number line?*

Exploration

Students might work on **parts A to C** in pairs.

Observe and Assess

As students are working, notice:

• Are they able to graph absolute value functions? What techniques do they use?

• Do they recognize they should always be looking for more than one solution?

• Do they effectively use “working backwards” or “opposite operations” in **part C**?

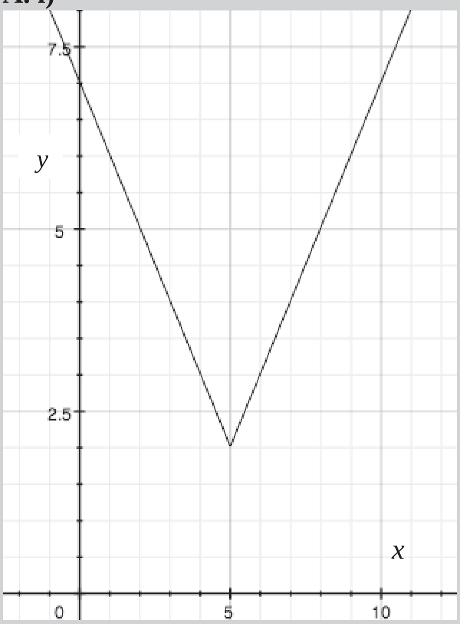
Share and Reflect

• Ask students to share their answers to **part C**. Ask them how solving these equations is similar to thinking about analysing transformations of functions.

• You might point out that where it was important to get one side of the equation to be zero with quadratic equations because of the zero product rule for multiplication, there is no equivalent reason for doing so with absolute value equations.

Answers

A. i)



A. ii) 0 and 10

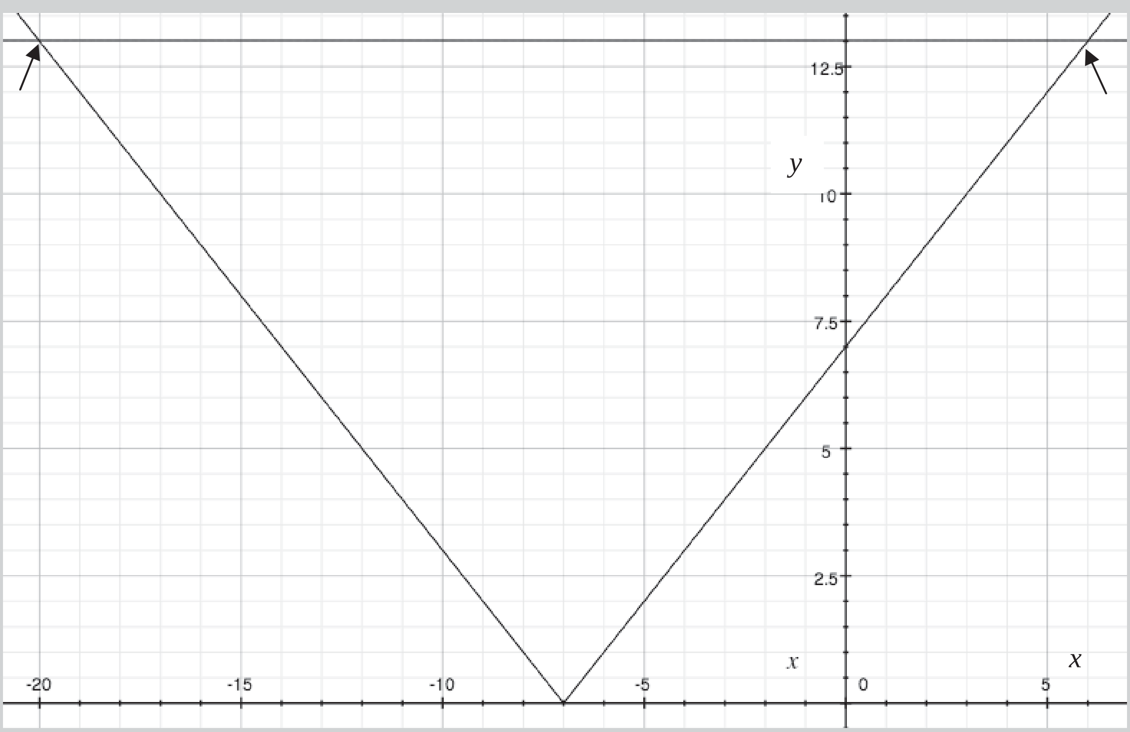
ii) Because they make the equation true.

B. i) 5 because 5 + 2 = 7

ii) $x - 5$ could be –5 or + 5 because the absolute value of $x - 5$ is being used in the equation. Both $|5|$ and $|-5|$ are equal to 5.

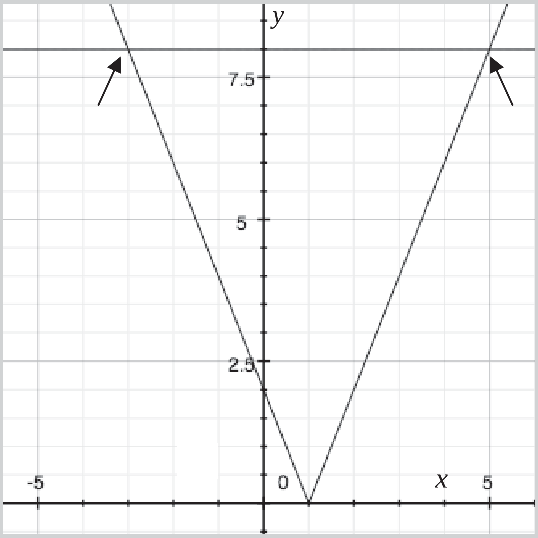
iii) 0 and 10 because $|0 - 5| = |-5| = 5$ and $|10 - 5| = |5| = 5$

C. i) 6 and –20

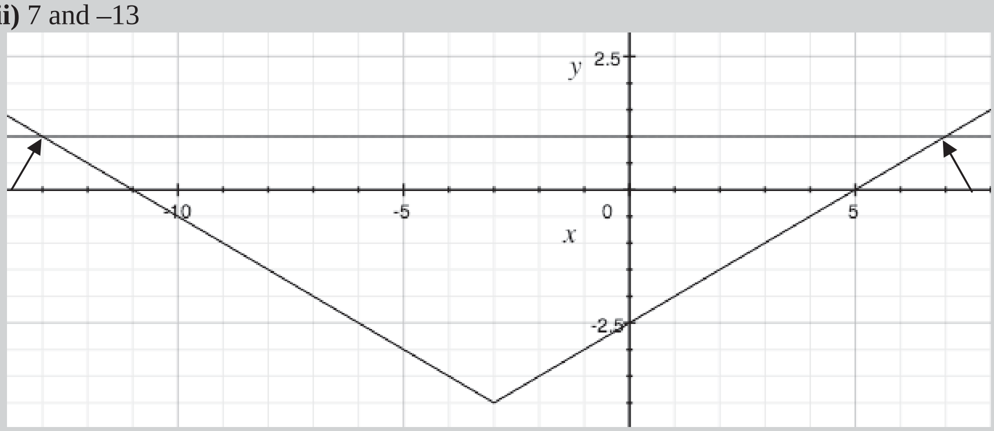


Answers [Continued]

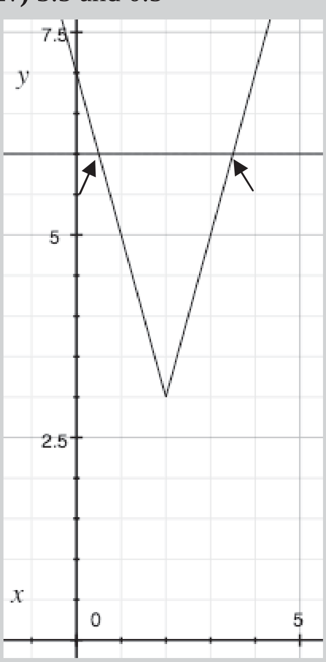
ii) 5 and –3



iii) 7 and –13



iv) 3.5 and 0.5



Supporting Students

Struggling students

You might put a poster on the wall with sample absolute value functions as reference for students.
You might include these samples:

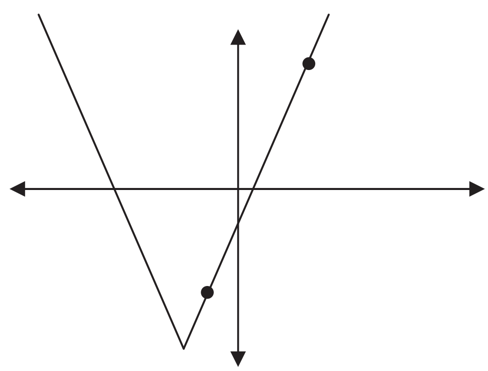
$y = |x|$ $y = |x + 3|$ $y = |x| + 3$ $y = 2|x|$ $y = |2x|$

GAME: Get the Points

This game allows students to relate quadratic and absolute value functions to their graphs.
For example, if a student rolls these combinations: 2 and 4, and 3 and 6, and 1 and 3, the points (2, 4), (–3, 6), and (–1, –3) would be included on the grid.

They might use the equation $y = x^2$, but it will only go through (2, 4).
If they use the equation $y = \frac{7}{3}x^2 - \frac{16}{3}$, it will go through two points, (2, 4) and (–1, –3).
If they use the equation $y = |2x|$, or $y = 2|x|$, it will go through (2, 4) and (–3, 6).

To find the points, for example, for the last equation shown, they might simply determine the equation of the line that goes through the points and then use what they know about absolute value.



UNIT 5 Revision

Pacing	Materials
2 h	• Grid paper (BLM in Unit 6)

Question(s)	Related Lesson(s)
1 – 3	Lesson 5.1.1
4 – 6	Lesson 5.1.2
7 – 9	Lesson 5.1.4
10	Lesson 5.1.5
11 – 13	Lesson 5.2.1
14	Lesson 5.2.3
15	Lesson 5.2.2
16, 17	Lesson 5.2.3
18	Lesson 5.2.4

Revision Tips

Q 3: Students might think about the relevant values. If x is negative, all the values are positive, so the sum is positive. If the value of x is a high positive, the $4x^2$ term will outweigh the other terms. So it is only for small positive values of x that the function might have a chance of being zero. In those cases, the +8 will outweigh anything else.

Q 4: Students will probably use the variable to represent the number of increases in price.

Q 6: Students will probably convert from factored form to standard form to answer the question. Identifying the intercepts from factored form is easier.

Q 7: Recall that there are several correct orders for most composite transformations.

Q 8: You may need to remind some students how the mapping that shows x + h as the first coordinate results in an equation involving (x – h).

Q 9: Students should analyze the graphs in terms of transformations of the basic graph $y = x^2$.

Q 11: If students want to factor without the tiles, allow them to do so.

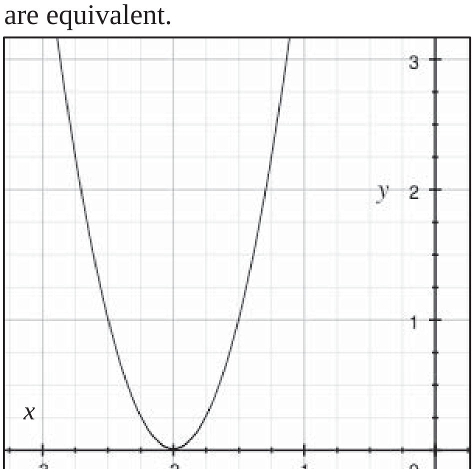
Q 13: Students will be more successful with this if they recall the pattern for squaring a linear binomial.

Q 15: Observe whether students realize how much easier it would be to answer the question using a factored form of a quadratic.

Answers

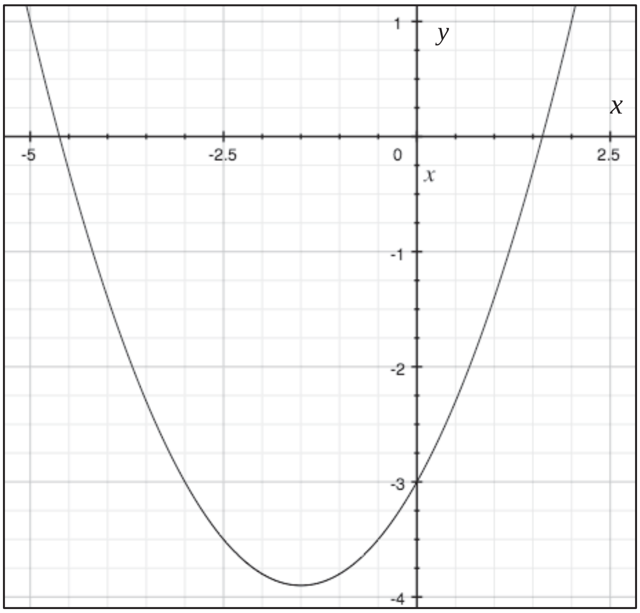
1. a) C and D

b) The graph is a parabola, so the functions are quadratic; The graphs are identical, so the functions are equivalent.

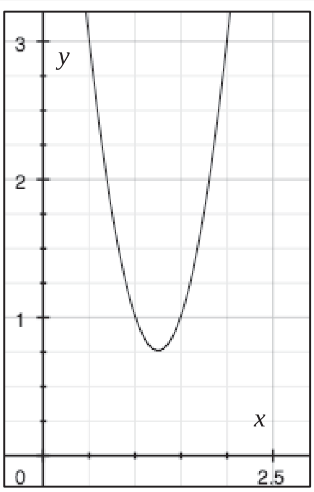


1. c) *Sample responses:*
- Substitute three values of x into each function and see if I get the same results each time (need only three points on each to be the same to be sure they are equivalent since they represent quadratic functions).
OR
- Expand C to standard form and it would look like D. (Factor D and it will look like C.)

2. a) i)



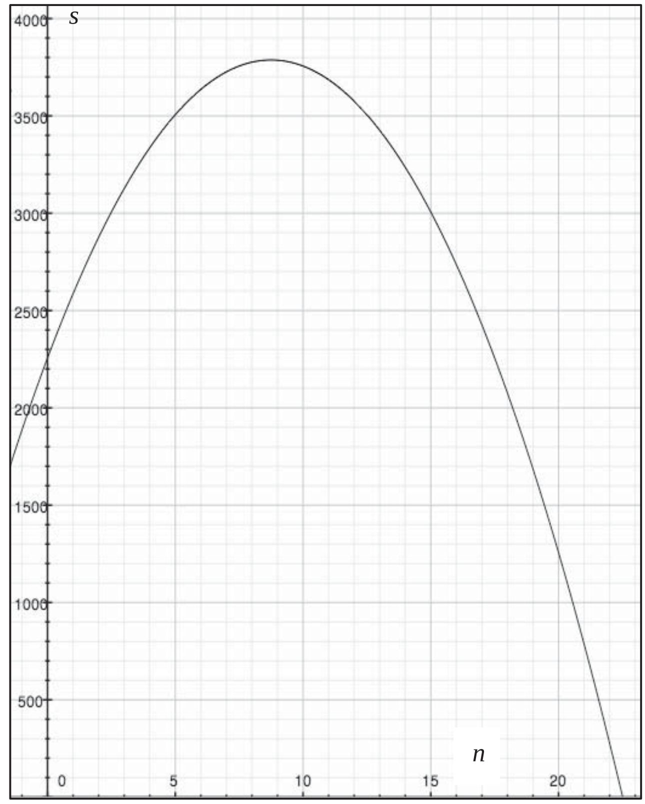
ii)



b) *Sample responses*:
i) Minimum: about –3.9; x-intercepts: –4.6 and 1.6.
ii) Minimum: about 0.8; no x-intercepts.

3. For negative values of x, each term is positive (because $-x$ would be positive and $4x^2$ is never negative) so the sum is positive and not 0.
I think $4x^2$ will outweigh $-x$ for all positive values except maybe for very small values, so I only checked small positive values.
I checked $x = 1$ and 2 and the values were positive.
I am sure $f(x)$ will always be positive, because for small positive values, the +8 will outweigh anything else.

4. a) $f(x) = (50 + 10x)(45 - 2x)$
OR
$s = (50 + 10n)(45 - 2n)$, where s is total sales and n is the number of Nu 10 price increases.
b)



c) The best price to charge is Nu 140.
The vertex shows you the maximum value of the function (maximum sales at Nu 3780) and the number of Nu **10** increases (**9**) over the original price of Nu **50** at that level of sales. Nu **50** + **9** × Nu **10** = Nu 140.

5. a) i) 4, – 3 **ii)** 6, 2 **iii)** 2.5, –1

iv) 0.5, –2.5 **v)** 0.6 , –5

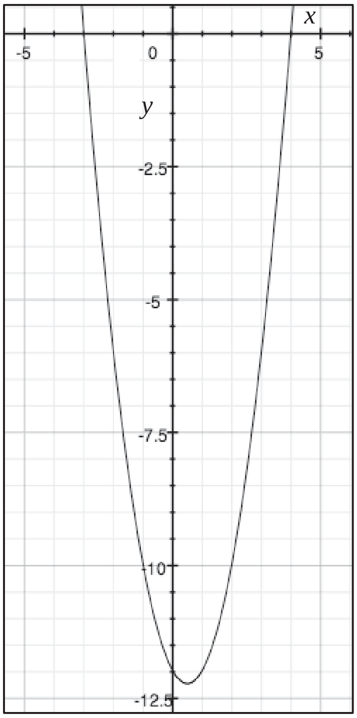
b) i) (0.5, –12.25) **ii)** (4, –12)

iii) (0.75, –18.375) **iv)** (–1, –2.25)

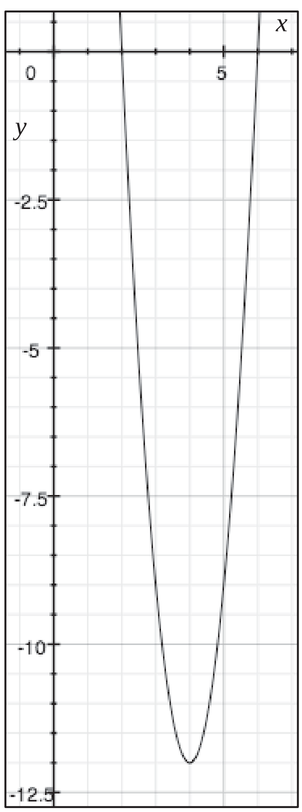
v) (–2.2, –23.52)

Answers [Continued]

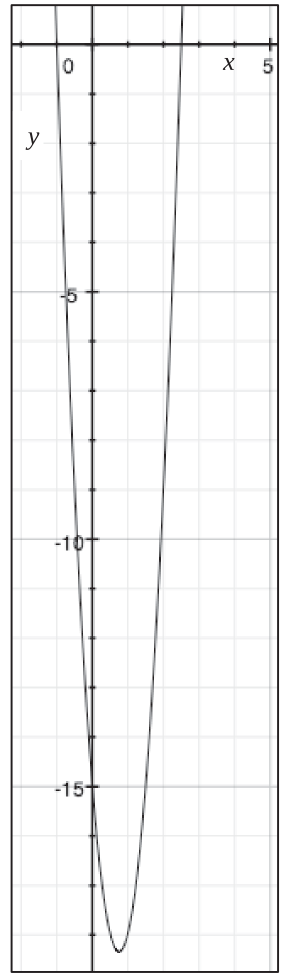
c. i)



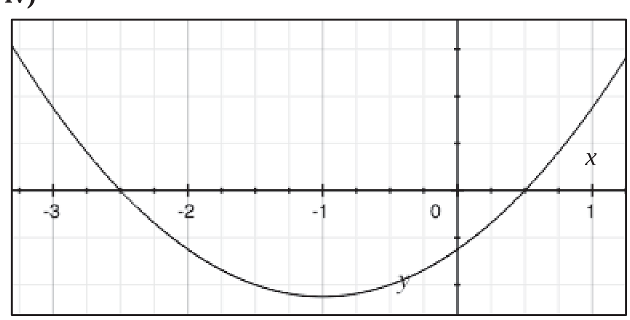
ii)



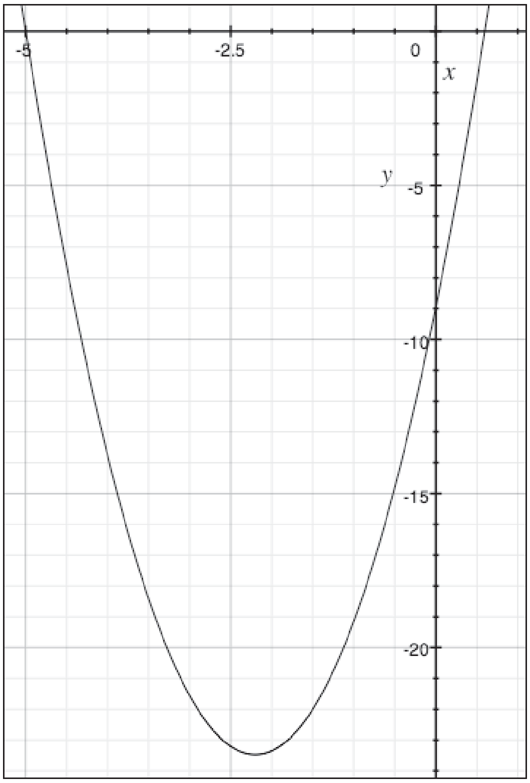
iii)



iv)



5. c) v)



6. h > –4; The *x*-intercepts are 4 and h. The *x*-coordinate of the vertex occurs at (4 + h) ÷ 2. This is only positive if 4 + h is positive. That means h > –4.

7. a) i) (Vertical) translation down 30 units.
ii) (Horizontal) translation left 30 units.
iii) Reflection in *x*-axis, (vertical) stretch by 2, and then (vertical) translation down 30 units.
iv) (Horizontal) translation right 2 units and then (vertical) stretch by 3 (or reverse order).
v) (Horizontal) translation left 30 units and then (vertical) translation down 7 units (or reverse order).
vi) Reflection in the *x*-axis, (vertical) compression by 10, and then (vertical) translation up 8 units.

b) i) (0, –30) **ii)** (–30, 0) **iii)** (0, –30)
iv) (2, 0) **v)** (–30, –7) **vi)** (0, 8)

c) i) $(x, y) \rightarrow (x, y - 30)$
ii) $(x, y) \rightarrow (x - 30, y)$
iii) $(x, y) \rightarrow (x, -2y - 30)$
iv) $(x, y) \rightarrow (x + 2, 3y)$
v) $(x, y) \rightarrow (x - 30, y - 7)$
vi) $(x, y) \rightarrow (x, -0.1y + 8)$

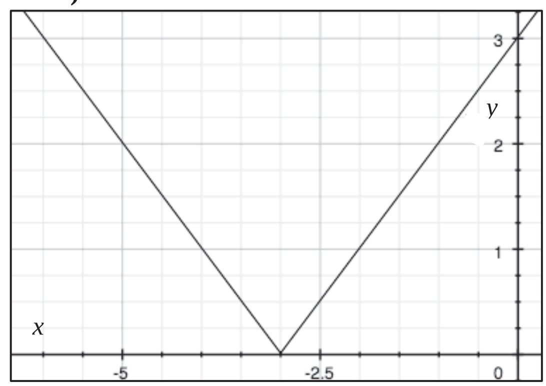
8. a) $f(x) = (x - 4)^2 + 3$ **b)** $f(x) = 0.5x^2 + 6$

c) $f(x) = -3(x + 3)^2$ **d)** $f(x) = (x + 8)^2 + 2$

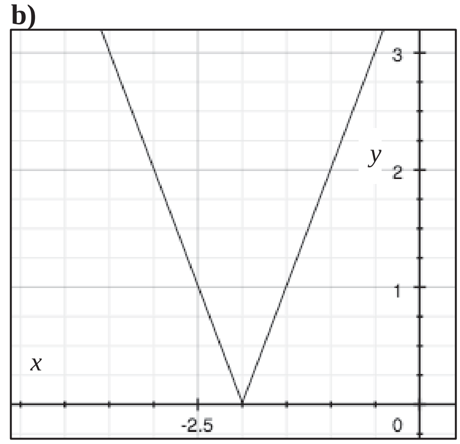
e) $f(x) = 3(x + 8)^2 + 2$

9. a) $y = 3(x - 2)^2$ **b)** $y = 2(x + 3)^2$ **c)** $y = x^2 - x - 2$

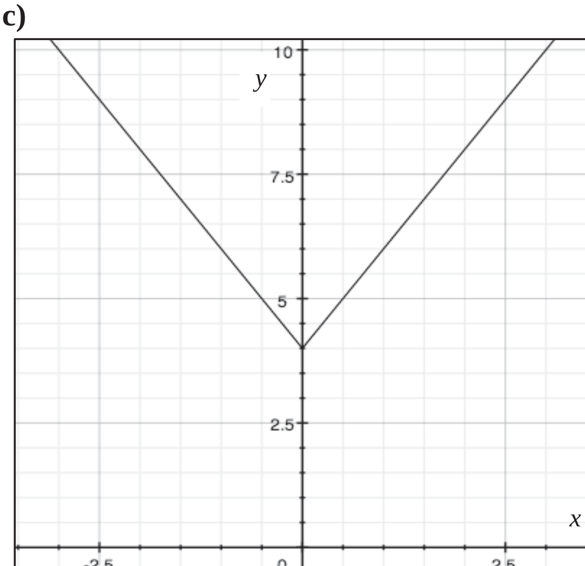
10. a)



b)

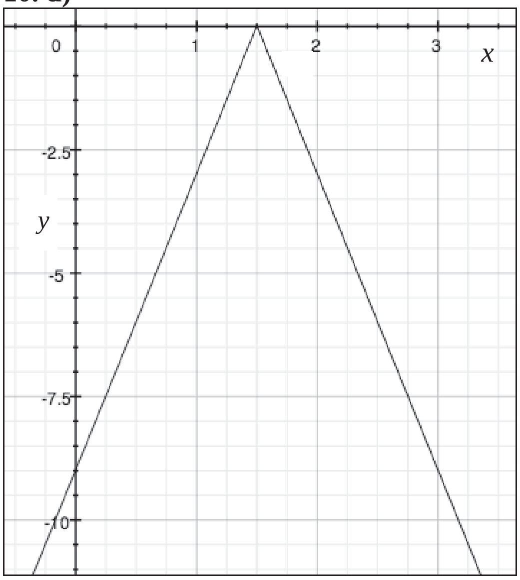


c)

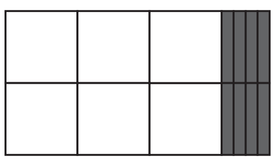


Answers [Continued]

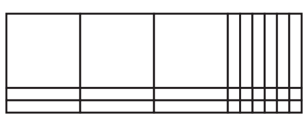
10. d)



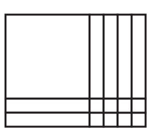
11. a) $2x(3x - 4)$



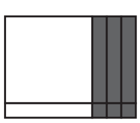
b) $(3x + 6)(x + 2)$



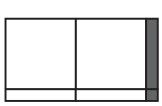
c) $(x + 4)(x + 2)$



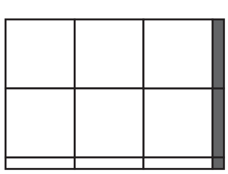
d) $(x + 1)(x - 3)$



e) $(2x - 1)(x + 1)$



f) $(3x - 1)(2x + 1)$



12. a) $4(2x + 1)(2x - 1)$ **b)** $4x(4 - 3x)$

c) $(5x + 3)^2$ **d)** $(x + 11)(x - 8)$

13. a) $x^2 + \mathbf{10}x + 25 = (x + \mathbf{5})^2$

b) $(3x - \mathbf{4})^2 = \mathbf{9}x^2 - 24x + \mathbf{16}$

14. a) $x = 2, \frac{1}{2}$ **b)** $x = \frac{3}{2}, -\frac{1}{3}$

c) $x = -2, 5$

15. a) $x^2 - 11x + 24 = 0$ **b)** $x^2 + 9x + 14 = 0$

c) $20x^2 - 31x + 12 = 0$ **d)** $25x^2 - 45x + 18 = 0$

16. Equation is $d^2 + 9.4^2 = (d + 6.6)^2$; $d = 3.39$ m

17. 17 and 18

18. a) $x = 2, -2$

b) $x = \frac{8}{3}, -\frac{4}{3}$

c) $x = \frac{5}{3}, -\frac{7}{3}$

Supporting Students

Struggling students

Encourage students who are struggling to go back to the expositions and examples for the relevant lesson involving the items that are causing difficulty.

UNIT 5 Non-Linear Functions and Equations Test

1. Use two methods to show that $f(x)$ and $g(x)$ are equivalent.

$f(x) = 3x^2 – 5x – 28$

$g(x) = (3x + 7)(x – 4)$

2. Manju currently sells about 33 small baskets per month for Nu 50 each. She predicts that for every increase in price of Nu 8 per basket, she will sell four fewer baskets.

a) Write a function to represent her total sales in terms of n, the number of price increases of Nu 8.

b) Sketch a graph of the function.

c) What is the best price for Manju to charge? What would be the total sales? How do you know?

3. a) What are the zeros of each?

i) $f(x) = (2x – 4)(x + 6)$

ii) $f(x) = 6(x – 4)(x + 5)$

b) What are the coordinates of the vertex for each function?

c) Sketch the graph of each function.

4. a) Which geometric transformations, in what order, should be applied to $f(x) = x^2$ to result in each function?

i) $f(x) = 3(x – 2)^2 – 30$

ii) $f(x) = 12 – 0.75x^2$

b) What are the coordinates of the vertex of each function?

c) Describe the transformations using mapping notation. For a composite transformation, use one mapping.

5. If you were to apply each composite transformation to the function $f(x) = x^2$, what would be the final function?

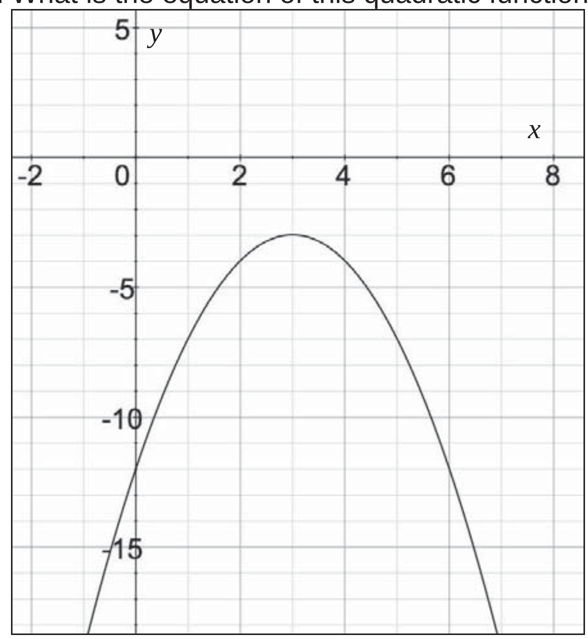
a) stretch vertically by 3 and translate 5 down

b) translate 2 left, reflect in the x-axis

c) $(x, y) \rightarrow (x – 10, 3y + 2)$

d) $(x, y) \rightarrow (x + 4, 0.5y – 2)$

6. What is the equation of this quadratic function?



7. Sketch each graph, if $f(x) = |x|$.

a) $f(2x – 3)$ **b)** $f(0.5x + 4)$

8. Factor each.

a) $36x^2 – 49$ **b)** $24x – 8x^2$

c) $2x^2 – 18$ **d)** $x^2 + 4x – 32$

9. Solve each.

a) $2x^2 – 7x – 15 = 0$

b) $4x^2 – 17x = 15$

c) $5x – 6 = x^2$

10. The hypotenuse of a right triangle is 18 units longer than the shorter leg of the triangle. The longer leg is 3 units longer than triple the length of the shorter leg. How long is the hypotenuse?

11. Solve each.

a) $6 + |3x + 5| = 14$

b) $|8x – 11| = 5$

UNIT 2 Test

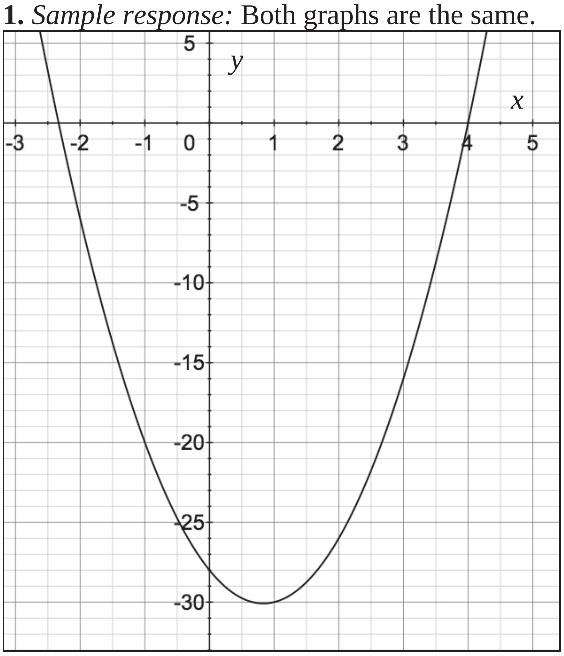
Pacing	Materials
1 h	• Grid paper (BLM in Unit 6)

Question(s)	Related Lesson(s)
1, 2	Lesson 5.1.1
3	Lesson 5.1.2
4 – 6	Lesson 5.1.4
7	Lesson 5.1.5
8, 9	Lesson 5.2.1
10	Lesson 5.2.3
11	Lesson 5.2.4

Select questions to assign according to the time available.

Answers

1. *Sample response:* Both graphs are the same.

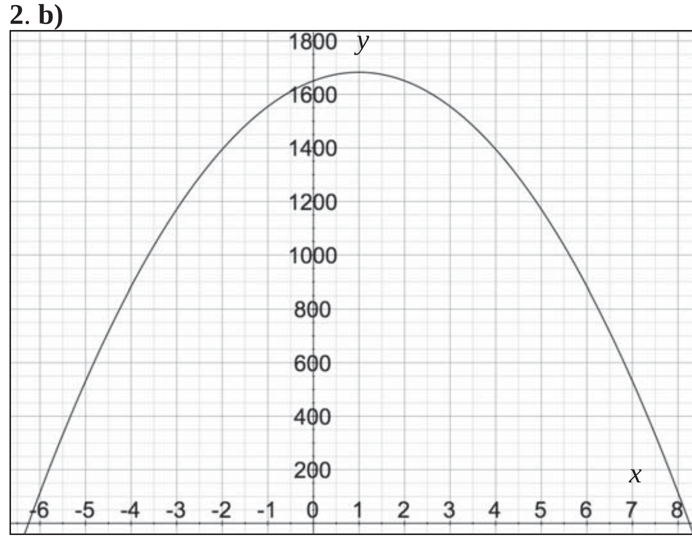


I expanded $g(x)$ to get the same form as $f(x)$.

$$\begin{aligned}(3x + 7)(x - 4) &= 3x^2 + 7x - 12x - 28 \\ &= 3x^2 - 5x - 28\end{aligned}$$

2. a) $f(n) = (33 - 4n)(50 + 8n)$

2. b)

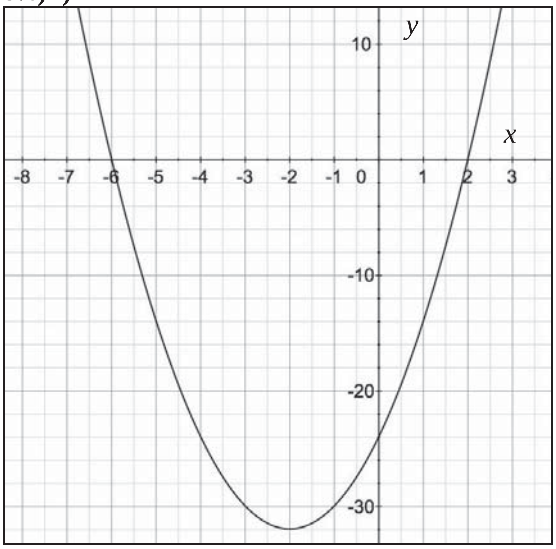


c) Nu 58 and Nu 1682; maximum sales are represented by the *y*-coordinate of the vertex. The *x*-coordinate of the vertex is 1 representing one price increase of Nu 8. The resulting maximum sales are 29 × 58 = Nu 1682.

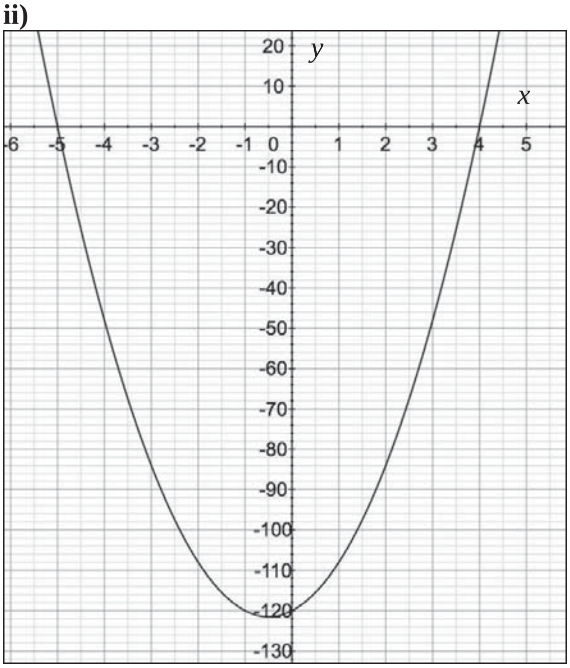
3. a) i) 2, –6 **ii)** 4, –5

b) i) (–2, –32) **ii)** (–0.5, –121.5)

3.c) i)



ii)



4. a) i) Horizontal translation 2 to the right, stretch by 3, vertical translation of 30 down.

ii) Reflection in x-axis, compression by 0.75, vertical translation of 12 up.

b) i) (2, –30) **ii)** (0, 12)

c) i) $(x, y) \rightarrow (x + 2, 3y - 30)$

ii) $(x, y) \rightarrow (x , -0.75y + 12)$

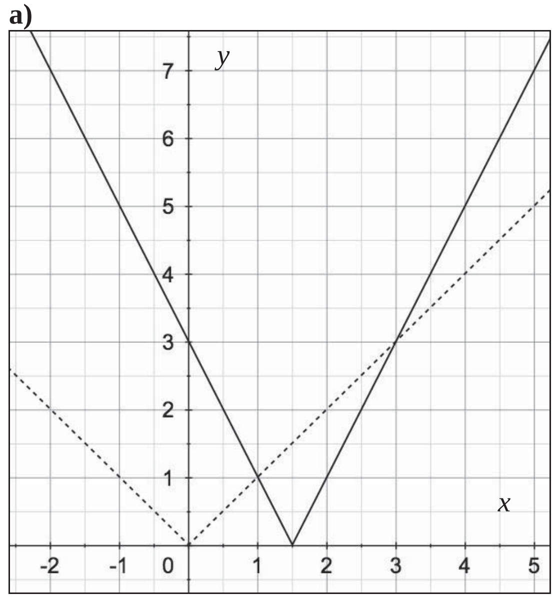
5. a) $y = 3x^2 - 5$ **b)** $y = -(x + 2)^2$

c) $y = 3(x + 10)^2 + 2$ **d)** $y = 0.5(x - 4)^2 - 2$

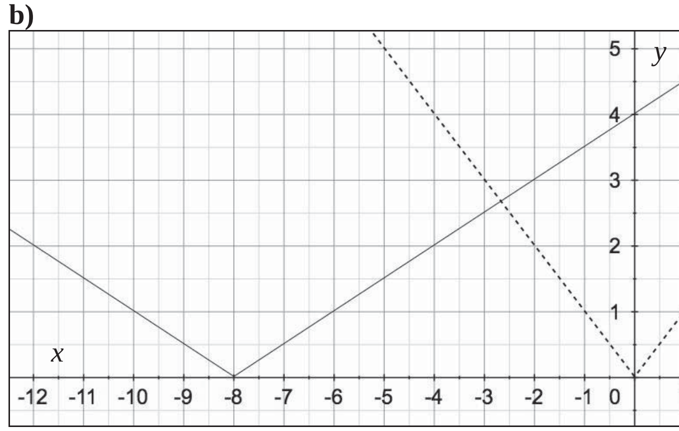
6. $y = -(x - 3)^2 - 3$

7. Note: the solid graphs are the answers

a)



b)



8. a) $(6x + 7)(6x - 7)$ **b)** $8x(3 - x)$

c) $2(x + 3)(x - 3)$ **d)** $(x + 8)(x - 4)$

9. a) 5, $\frac{-3}{2}$ **b)** 5, $\frac{-3}{4}$ **c)** 2, 3

10. 25 units
(Solve the equation $x^2 + (3x + 3)^2 = (x + 18)^2$ where x is the length, in units, of the shorter leg.)

11. a) 1, $\frac{-13}{3}$ **b)** $\frac{3}{4}$, 2

UNIT 5 Performance Task — Firecracker Function

A firecracker is fired from the ground. The height of the firecracker at a given time is modelled by the following function, where h is the height in metres and t is the time in seconds:

$$h(t) = -5t^2 + 50t$$

1. Predict how the graph of this function will compare to the graph of $y = x^2$.

2. a) Represent this function both in factored form and in vertex form.

b) What does each form quickly tell you?

3. State the coordinates of:

a) the x-intercepts **b)** the vertex

4. What do the zeros of the function represent in this situation?

5. Create and solve two problems related to this situation.

- One problem should be easy to solve using the graph.
- The other problem should be solved using a quadratic equation.
- One problem should have only one solution and the other should have two solutions.

6. Suppose the height of a different firecracker were given by the following function:

$$h(t) = -6t^2 + 48t - 54$$

a) How would the graph of the path look different from the graph of the first firecracker?

b) How is the situation different from the first situation?

UNIT 5 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-C4 Graphs and Tables: construct and analyse **10-C5** Graphs and Tables: explore dynamics of change **10-C6** Graphs: sketch **10-C9** Non-linear Equations: evaluate and interpret **10-C12** Problems: express in terms of equations **10-C16** Quadratic Equations: solve by factoring **10-C18** Non-linear functions: analyse and describe transformations and apply them to quadratic and absolute value functions	1 h	Grid paper (BLM in Unit 6)

How to Use This Performance Task

Ask students to read the task and the assessment rubric, and ask if they have any questions before allowing them the time they need to perform the task. Then assign the performance task. If you are using this task for the purpose of recording a mark, students should work alone. Otherwise, they might work with a partner.

Sample Solution

1. It will be stretched, reflected in the x-axis and translated horizontally.
First I factored it to be $h(t) = –5t(t – 10)$. I could see the negative stretch and the horizontal translation in this form.

2. a) Factored form: $h(t) = –5t(t – 10)$;
Vertex form: $–5(t – 5)^2 + 125$.
b) The factored form makes it easy to know where the roots are. The vertex form makes it easy to figure out the coordinates of the vertex.

3. a) (0, 0) and (10, 0) **b)** (5, 125)

4. The zeros tell when the firecracker is at ground level; they represent the time when it is fired and when it returns to the ground.

5. *Sample response:*
Problem 1: When is the firecracker at 100 m?
Solution 1: I used the graph to see where the y-coordinate was 100. There were two solutions, about 7.2 s and about 2.8 s.

Problem 2: At what time is the firecracker at 125 m?
Solution 2: I solved the equation $–5t(t – 10) = 125$

$$\begin{aligned} t(t-10) &= -125 \\ t^2 - 10t + 100 + 125 &= 0 \\ t^2 - 10t + 225 &= 0 \\ (t-5)^2 &= 0 \\ t &= 5 \end{aligned}$$

6. *Sample responses*:
a) It would be stretched more than the first graph, going up higher. The graph is also shifted to the left.
b) There is only one time when the firecracker was at ground level since the second zero involves a negative time and that is not possible. It also did not start at ground level. It started at 54 m.

UNIT 5 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Graphing	Uses graphs accurately and effectively to solve problems, uses transformations to relate two graphs, and interprets a graph in the context of a problem.	Uses graphs reasonably accurately to solve problems, uses transformations to relate two graphs, and interprets a graph in the context of a problem.	Uses graphs to solve a problem and uses transformations to relate some aspects of two graphs. Has difficulty relating the graph to the context of a problem	Has difficulty with one or both of relating two graphs using transformations and solving a problem using a graph.
Algebra	Insightfully relates the forms of a quadratic and recognizes the advantages of each. Sets up and solves a quadratic efficiently to solve a problem.	Relates the forms of a quadratic by manipulating one form to get another form. Sets up and solves a quadratic correctly to solve a problem.	Relates the forms of a quadratic by manipulating one form to get at least one other form. Sets up and solves a quadratic with minimal errors to solve a problem.	Struggles in performing the algebraic manipulations to move from one form of a quadratic to another and experiences difficulty in setting up the appropriate quadratic to solve a problem.

UNIT 5 Assessment Interview

You may want to take the opportunity to interview selected students to assess their understanding of the work of this unit. The results can be used as formative assessment or, if you wish, as a piece of summative assessment data. As each student works, ask him or her to explain his or her thinking.

Display the graph below and ask:

• *How do you know that this is the graph of a quadratic function?* (It is a parabola. It is symmetric and curved.)

• *Suppose the equation were in standard form, $y = ax^2 + bx + c$:*

- *What do you know about the value of a?* (It is negative and I think it is less than –1.)
- *Which other value in the standard form can you be pretty sure of? How?* (c; it is the *y*-intercept and it is –12.)

• *Suppose the equation were in factored form:*

- *What would the two factors be?* ((x – 1) and (x – 3))

- *How would you figure out the multiplier?* (Substitute 2 for x and 4 for y in $y = a(x – 1)(x – 3)$ and solve for a. Note that the y-intercept of $x = 0$, $y = –12$ could also be used.)

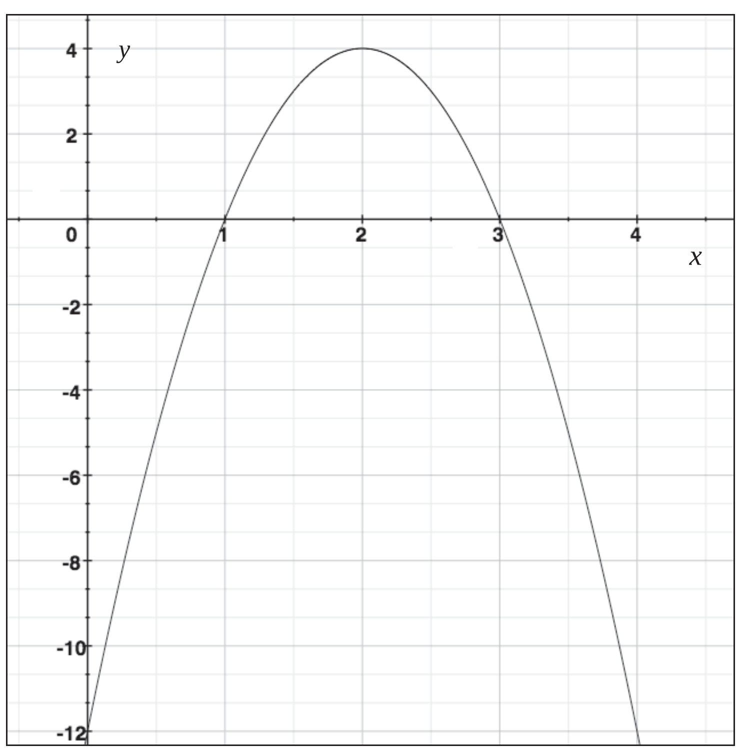
• *Suppose the equation were in vertex form of $y = a(x – h)^2 + k$. How can you use the graph to figure out the values for a, h, and k?* (I know the vertex is at (2, 4) so those are the values for h and k. I would solve for a by using the values $x = 1$ and $y = 0$ in the equation $y = a(x – 2)^2 + 4$.)

• *Which transformations of the graph $y = x^2$ would have resulted in this graph? How do you know?*
(1) Reflection in the x-axis; I knew the reflection by seeing it pointed down instead of up.

2) Stretch by a factor of 4; I knew it was a stretch of 4 because the value of a is –4.

3) Translation horizontally 2 to the right and translation vertically 4 up; I knew the translations by looking at the position of the vertex.)

• *Which quadratic equations could you solve with this graph? Give one example. How would you solve it?*
(Any equation of the form $y = –4x^2 + 16x + 12 = d$. Solve by looking for values on the graph when y is d.)



UNIT 6 DATA, STATISTICS, AND PROBABILITY

UNIT 6 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 165** **TG p. 187**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Grid paper • Rulers	All questions
Chapter 1 Data Involving One Variable				
6.1.1 Histograms and Stem and Leaf Plots **SB p. 167** **TG p. 190**	**10-F5 Displaying Data: construct and interpret** • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: stem and leaf plots and histograms	2 h	• Grid paper • Rulers	Q3, 6, and 8
6.1.2 EXPLORE: Investigating Bin Width in Histograms (Optional) **SB p. 177** **TG p. 194**	**10-F5 Displaying Data: construct and interpret** • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: histograms	1 h	• Grid paper • Rulers	Observe and Assess questions
6.1.3 Histograms and Box and Whisker Plots **SB p. 178** **TG p. 196**	**10-F4 Data Analysis: distribution of data** • understand that box and whisker plots are useful when comparing data **10-F5 Displaying Data: construct and interpret** • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: stem and leaf, box and whisker plots, and histograms	2 h	• Grid paper • Rulers	Q1, 6, 8, and 10
6.1.4 Data Distribution **SB p. 186** **TG p. 202**	**10-F3 Normal Curves: explore measurement issues** • understand that a frequency polygon is created by joining the midpoints of the top of each bar in a histogram • identify situations that give rise to common distributions (e.g., U-shaped, skewed, and normal); • demonstrate an understanding of the properties of the normal distribution (e.g., the mean, median, and mode are equal; the curve (and data) is symmetric about the mean) • understand that a normal curve is based upon a certain type of histogram with infinitely small bins	1 h	None	Q1, 2, 6, and 8
CONNECTIONS: Normal Distribution and Sample Size **SB p. 194** **TG p. 207**	Explore how the shape of a histogram and its frequency polygon change as the sample size increases.	1 h	• Tape measures • Grid paper	

UNIT 6 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Data Involving Two Variables				
6.2.1 Correlation and Lines of Best Fit **SB p. 195** **TG p. 208**	**10-C19 Data: gather, plot, and demonstrate understanding of independent and dependent variables, and domain and range** • make decisions regarding independent and dependent variables **10-F1 Correlations: develop an intuitive understanding** • understand that a correlation coefficient is a description of how well data fits a linear pattern • identify the difference between a strong and weak correlation and between a negative and positive correlation based on the scatter plot and the value of the correlation coefficient	1 h	• Grid paper • Rulers	Q1, 3, 5, and 7
6.2.2 Non-Linear Data and Curves of Best Fit **SB p. 205** **TG p. 212**	**10-F2 Curves of Best Fit: non-linear data** • explore curve fitting for non-linear data • understand that non-linear models often show a better relationship than linear models	1 h	• Grid paper • String	Q1, 2, and 6
CONNECTIONS: Data Collection by Census **SB p. 212** **TG p. 216**	Investigate what a census is and the particulars about the most recent census conducted in Bhutan	Time outside of class and 30 min in class to share	None	N/A
Chapter 3 Probability				
6.3.1 Dependent and Independent Events **SB p. 213** **TG p. 217**	**10-G1 Theoretical Probability: independent and dependent events** • distinguish between two events that are dependent or independent using reasoning and calculations	2 h	• Coins, dice, and spinners (optional)	Q2, 3, and 4
6.3.2 Calculating Probabilities **SB p. 218** **TG p. 220**	**10-G1 Theoretical Probability: independent and dependent events** • distinguish between two events that are dependent or independent using reasoning and calculations	2 h	None	Q3, 4, and 5
UNIT 6 Revision **SB p. 223** **TG p. 223**	Review the concepts and skills in the unit	2 h	• Grid paper • Rulers	All questions
UNIT 6 Test **TG p. 227**	Assess the concepts and skills in the unit	1 h	• Grid paper • Rulers	All questions
UNIT 6 Performance Task **TG p. 232**	Assess concepts and skills in the unit	1 h	• Grid paper • Rulers	Rubric provided
UNIT 6 Blackline Masters **TG p. 236**	BLM 1 Grid Paper (0.5 cm by 0.5 cm) BLM 2 Grid Paper (1 cm by 1 cm) BLM 3 Fraction Circle Spinners			

Math Background

• In previous classes students have created and interpreted many different types of graphs, including pictographs, bar graphs, circle graphs, line graphs, stem and leaf plots, box and whisker plots, histograms, and scatter plots. The focus in Class X is on further exploring the usefulness of some of these displays. Stem and leaf plots, box and whisker plots, and histograms are examined for their usefulness in representing large sets of data, grouped in intervals. The approach to scatter plots is based on describing the data they represent with lines and curves of best fit for the purpose of prediction.

• In **Chapter 1**, students create graphs, justify their choice of a particular type of graph, and draw conclusions from graphs. They also examine how the distribution of data changes as the bin width changes and how this affects the conclusions that can be drawn from the graphs. They use histograms to examine shapes of different types of data distributions. They focus on the properties of the normal distribution.

• In **Chapter 2**, attention shifts to data sets that involve two variables, more specifically sets that show a relationship between the independent and dependent variables. From their work in Class IX, students already know how to judge the correlation of linear relationships as positive or negative and strong or weak. They extend their use of models to include curves of best fit for non- linear situations and they use numerical values to represent correlations.

• The focus in **Chapter 3** is on probability. Students explore criteria for testing whether two events are dependent and are introduced to a computational method to accomplish this.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in the **Try This** in **lesson 6.2.2**, where they look for a pattern in data and in **question 3** of **lesson 6.3.2** where they calculate probabilities.

- They use communication frequently as they create graphs throughout the unit, choosing appropriate scales and labelling axes. They also explain their thinking in answering questions, for example, throughout **lesson 6.1.4**, students are asked to identify, describe, and reason about the distributions of data sets. They also use communication, for example, in **question 2** of **lesson 6.1.3**, where they compare graphs. You might notice that the last question in a lesson often requires an element of communication.

- Students use reasoning in answering questions such as **question 6** in **lesson 6.1.1**, where they draw conclusions from a stem and leaf plot. Other examples are **question 3** in **lesson 6.3.1**, where they reason out why events are dependent and **question 5** in **lesson 6.2.1**, where they decide whether a line of best fit is appropriate.

- They use representation and visualization and make connections as they represent numerous data sets graphically throughout the unit. In particular, they examine how a stem and leaf plot turned sideways resembles a histogram and how superimposing a box plot on a histogram provides more information on how the data set is distributed. They also use the pattern of points in scatter plots in **lessons 6.2.1 and 6.2.2** to help them determine the best model to use to represent the relationship.

- They connect math to everyday events in many questions. For example, in **question 2** of **lesson 6.1.1**, they describe how graphs are useful in real world situations. There are two **Connections** features. One highlights a real life example of the normal curve and the other describes data collection procedures that are important to governments and their citizens.

Rationale for Teaching Approach

This unit is divided into three chapters.

- **Chapter 1** reviews various types of graphs that can be used to display sets of data that involve a single variable, are relatively large, and must be grouped into intervals. The chapter deals mainly with justifying the choice of types of graphs used, interpreting these graphs and drawing conclusions from them. The focus on one variable naturally precedes work with two variables.

- **Chapter 2** looks at how scatter plots can be used to help determine an appropriate model for data sets that exhibit a relationship between the variables. There is a natural progression from linear to non-linear models.

- **Chapter 3** switches to probability so that the unit flows from least complex to most complex concepts.

- **Lesson 6.1.2** is an **Explore** lesson, where students are required to organize data into frequency tables and create corresponding histograms — the bin widths vary and students investigate how this affects possible conclusions drawn from the graph.

[Continued]

Rationale for Teaching Approach [Cont'd]

• There are two **Connections** in the unit.

- In **Chapter 1**, students collect height data of students in their class and compare this distribution to the height data of all the students in their school. The larger data set should be close to a normal distribution, while the class data may resemble a U-shaped distribution.

- In **Chapter 2**, students do some research to discover what a census is and to investigate the background of the Population and Housing Census of Bhutan conducted in 2005.

Technology in This Unit

• A spreadsheet program would be very useful for creating graphs in **Chapter 2**. Most spreadsheets can easily graph data involving two variables and create scatter plots. If you have access to computers and a spreadsheet program, refer to the user manual for the program to get detailed instructions on how to create these graphs. Spreadsheets do not create histograms or box and whisker plots.

• Some programs, like Microsoft Excel, have additional applications that can be downloaded from the Internet to allow the construction of graphs. There are also some useful Internet sites that have graphing applets.

• Use of a calculator should be encouraged in **lesson 6.1.3**, where students estimate the 5-number summary needed to create a box plot for a data set that is grouped into intervals. A calculator is also recommended in **lesson 6.1.4** to estimate the mean and median of a data set grouped into intervals.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Displaying Data: draw inferences and make predictions **9** Scatter Plots: characteristics of relationships **9** Lines of Best Fit: sketch and determine equations	Students will be more successful in this unit if they review what they already know about constructing and interpreting a variety of types of graphs.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM)) • Rulers	• creating and interpreting box and whisker plots, double bar graphs, stem and leaf plots, and scatter plots • correlation • discrete vs. continuous variables • line of best fit • mean, median, and mode • independent and dependent variables

Main Points to be Raised

• It is essential to determine the maximum value, the minimum value, the median, and the upper and lower quartiles in order to create a box plot.

• The maximum and minimum values are the endpoints of the whiskers. The lower and upper quartiles are used to make the box. The data values must be arranged in ascending order to determine the median and the lower and upper quartiles

• The box plot divides the data into fourths:

- 50% of the data values lie below the median and 50% lie above the median
- 50% of the data values lie between the lower and upper quartiles
- 25% of the data values lie below the lower quartile
- 25% of the data values lie above the upper quartile

Generally, we use the term *quartile* to mean a quarter of the data set and *first*, *second*, *third*, and *fourth quartile* to indicate which quarter of the data we are talking about. The terms *lower and upper quartiles* the values at the top end of the first quartile and the bottom end of the fourth quartile.

Use What You Know — Introducing the Unit

• This activity revisits the information needed from a set of data to create a box and whisker plot. It also requires students to recall how a box plot distributes the data.

• This activity can be completed individually. If students cannot recall how to find any of the required information, they should be encouraged to ask their classmates.

• You may need to remind students to include the median in calculating the upper and lower quartiles.

• A number of students may be confused that the numbers in the lower quartile actually represent the fastest time. You may need to clarify this for them.

• Observe students as they work. You might ask:

- *How did you determine the range of the data?* (I subtracted the minimum value from the maximum value.)
- *What did you do to find the median and lower and upper quartiles?* (I arranged the data in order from least to greatest. Then I looked for the halfway point and then for the halfway point in each half.)
- *Where does Pema's time lie in relation to the median and lower and upper quartiles of your box plot? What does this mean?* (Pema's time of 12.4 s is in the first quartile. It is one of the fastest times in the class.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually. They can check their answers with a classmate when they are uncertain.

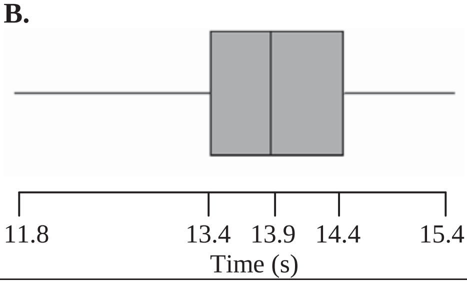
NOTE

When calculating the lower and upper quartiles when there is an odd number of data values, some people include the median value when calculating the lower and upper quartiles while others do not. The median has been included in the answers in this unit.

Answers

A. Maximum 15.4, minimum 11.8, range 3.6, median 13.9, lower quartile 13.4, and upper quartile 14.4.

B.



C. i) 50%

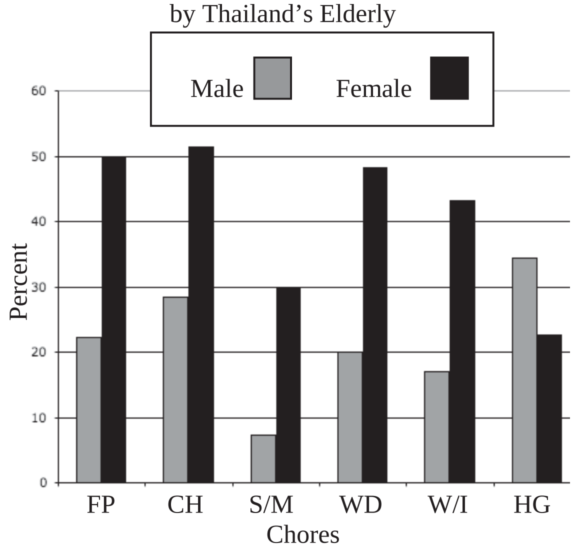
ii) 50%

iii) 25%

D. *Sample response*:
Pema's time of 12.4 s is in the first quartile. It is one of the fastest times in the class.

1. a)

Household Chores Performed
by Thailand's Elderly



b) *Sample response*:
Females perform the majority of household chores, with the exception of gardening.

2. Mean 59.4, median 58.9, modes 66.0 and 53.5.

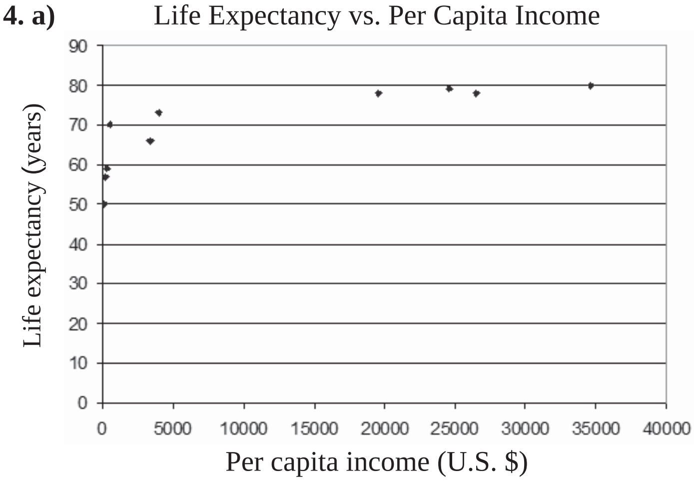
3. Maximum Temperatures

Stems	Leaves
18	.7
19	.4 .6 .7
20	
21	.1 .2
22	.9
23	.1 .5 .5
24	.8 .9
25	.0 .1 .4
26	.6 .7 .8
27	.1 .2
28	.4
29	.1
30	.0 .3

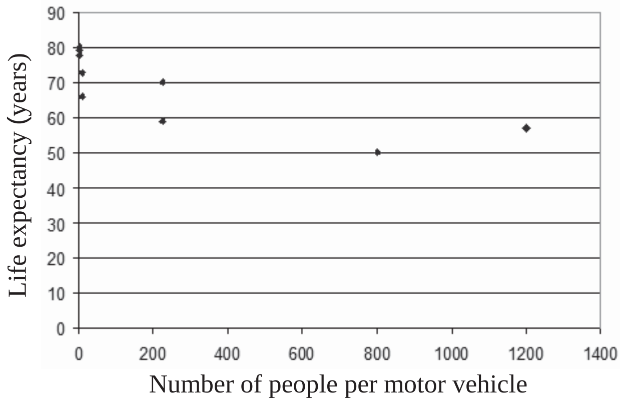
NOTE: The decimal can also be placed with the stems, i.e.,

Stems	Leaves
18.	7
19.	4 6 7

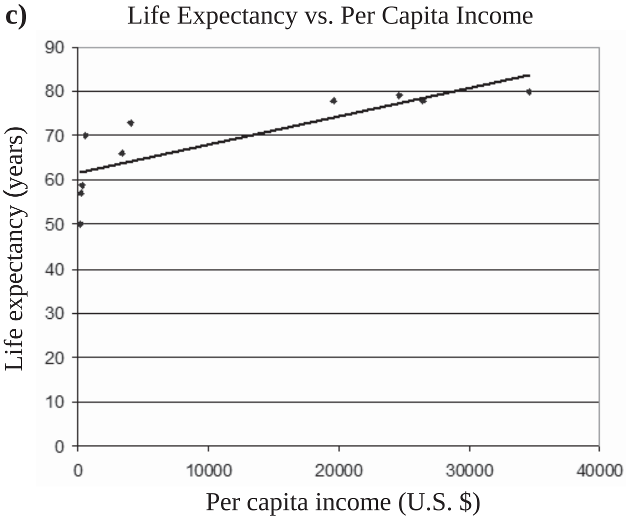
4. a) Life Expectancy vs. Per Capita Income



4. b) Life Expectancy vs.
Number of People Per Motor Vehicle

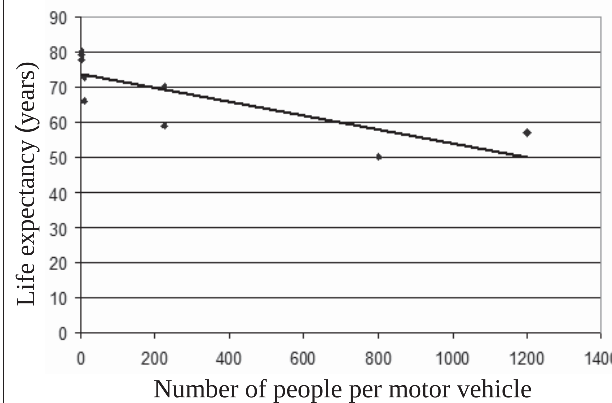


c) Life Expectancy vs. Per Capita Income



The correlation is positive because it slopes up and weak because many of the points are far from the line.

Life Expectancy vs. Number of People Per Motor Vehicle



The correlation is negative because it slopes down. It is weak because many of the points are far from the line.

d) *Sample responses*:
For Life Expectancy vs. Per Capita Income
A positive correlation make sense because if you have more money, it means you are better able to afford better health care and a healthy diet, so life expectancy increases. However, because there are other factors that affect life expectancy, the correlation is not very strong.

For Life Expectancy vs. Number of People Per Motor Vehicle
A negative correlation make sense because the more people per vehicle in a country, the less affluent are its people, and less money means they are less able to afford better health care and a healthy diet, so life expectancy decreases. However, because there are other factors that affect life expectancy, the correlation is not very strong.

5. First graph: discrete; one variable, countries
Second graph: continuous; two variables, independent variable is initial speed and dependent variable is skid length

Supporting Students

Struggling students

• You may need to review strategies for creating box and whisker plots, stem and leaf plots, and double bar graphs. For some students, a simple reminder of what those graphs look like might be sufficient.

• It may be useful to suggest students use the glossary to recall the meaning of the terms *discrete, continuous, independent,* and *dependent.*

6.1.1 Histograms and Stem and Leaf Plots

Curriculum Outcomes	Outcome relevance
10-F5 Displaying Data: construct and interpret • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: stem and leaf plots and histograms	By revisiting and analysing the choices involved in creating histograms and stem and leaf plots, students consider why each graph is appropriate in a given situation. Students also consider the usefulness of a double stem and leaf plot.

Pacing	Materials	Prerequisites
2 h	• Grid paper (BLM) • Rulers	• creating stem and leaf plots • creating histograms • creating frequency tables

Main Points to be Raised

• The type of graph used in a situation depends on the nature of the data and the purpose of the graph.

• If you are given a data set for a group that measures a single characteristic (variable), then a stem and leaf plot or a histogram may be appropriate.

• Histograms look very similar to bar graphs because they both use the heights of bars to represent the frequency of the variable or categories being measured. However, bar graphs are generally used for categorical data that is discrete. Histograms are used for data sets represented by a single variable (or category) whose data is continuous. For example, heights of students would be continuous data, whereas, the number of boys and girls in the class would be discrete data.

• Histograms are easiest to interpret when there are 5 to 12 bins.

• It is important that the bars used in a histogram be of the same width. Otherwise, the graph can be misleading.

• Stem and leaf plots use bin widths based on place value whereas histograms use bin width chosen by the graph's creator. These bin widths could be organized by place value for a histogram, but they do not have to be.

• All graphs should have a title and axis labels.

Try This — Introducing the Lesson

Allow students to try these alone and then to compare their answers with a partner. The questions provide an opportunity for students to discuss how unorganized data is difficult to use for drawing conclusions. Make sure students understand what the phrase *at random* means; i.e., every plant is equally likely to be selected and you cannot be sure which ones will be selected; there is no pattern for the selection.

A. to C. Observe while students work. You might ask:

• *What did you do to the data to make it easier to compare the yield of tomatoes grown in the two different locations?* (I arranged the data in order from the least value to the greatest.)

• *Why do you think that a bin width of 5 was chosen?* (I would not want too many bins and it is easy to work with intervals that are multiples of 5 and 10)

• *If a different bin width were used in the frequency table, do you think that your conclusions would differ?* (I would have to try it to be sure, but since some of the numbers are 5 and 10, they would be in different groups and it could change things.)

The Exposition — Presenting the Main Ideas

• You may have students read along with you as you introduce the terminology in the exposition box, specifically the terms *bins,* its synonym *intervals,* and *frequency.* Review the terminology related to the two types of graphs being examined. Or, you could use the tomato data given in the **Try This** to go through the steps with the class to create two single stem and leaf plots. You could use these to create two histograms and engage the class in a discussion of how the stem and leaf plots are similar to the histograms.

• You could also use the frequency table they created in **part B** to go through the steps required to create a histogram. Point out what can and cannot be determined from these different types of graphs.

• It is important to discuss the way to determine the bin width or bin size. Students must realize that this formula is not set in stone and that they have some flexibility in determining the size. Intervals of 5, 10, 20, and 25 are easy for people to understand, so rounding the bin width up or down to the closest of these values is often a good idea.

Revisiting the Try This

D. This question allows students to make a formal connection between what was done in the **Try This** and the stem and leaf plots and procedures used to create them, as presented in the exposition. You might approach **part D** as a whole class. You could create a double stem and leaf plot here to introduce this new idea.
The answer to **part ii)** will depend on the nature of the conclusion made in **part A**. If students were able to mentally organize the original data, then their answer should be yes. If they just made some casual observations from the original data, their conclusions could vary.

Using the Examples

Allow students time to read the examples, one at a time.

• Ask them what the similarities and differences are between the two graphs used in **example 1.**

• In **example 2**, discuss the advantage of using a double stem and leaf plot for comparing two sets of data. Point out that this is only possible if both sets of data contain numbers of relatively the same size. This enables the data to share the same stems.

• The purpose of **example 3** is to introduce the notion of what happens to the distribution of data when different bin widths are used. (This is further explored in the optional Explore **lesson 6.1.2**.) Ask students to compare the shapes of both histograms and to compare the conclusions made from each of the graphs. You could ask students to organize the data using a third bin width and to create the histogram. Ask them to compare their results with the two histograms shown in the example.

Practising and Applying

Teaching points and tips

Q 1: You might do this question orally with the class. Ask students to justify why they chose true or false.

Q 2, 3, 4, and 8: In these questions, students should be encouraged to work with a partner to compare the graphs they create and the conclusions they draw. They should identify any differences in their graphs or conclusions and discuss the reasons for these.

Q 4: It might be helpful if you discuss with the class the choice of stems for each set of data.

Q 5: Speeds on Canadian highways are much faster than those in Bhutan. Most major highways have a posted speed limit of 100 km/h.

Q 6: Make sure students understand how to read the double stem and leaf plot. For example, the first number in the first row on the left represents 15 and the numbers on the right represent 11, 15, etc.

Q 9: This question gives students an opportunity to compare histograms created from the same data set using different bin widths to see how the choice might affect the conclusions.

Common Errors

Students often struggle with choosing an appropriate stem for a stem and leaf plot and with choosing bin width for a histogram. Spend some time in this lesson discussing the reasons why a certain stem or bin width was chosen as you work through the examples.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can determine an appropriate bin width and organize data in a frequency table and then create the corresponding histogram
Question 6	to see if students can draw conclusions from a double stem and leaf plot
Question 8	to see if students can create a stem and leaf plot and a histogram, use the graphs to draw conclusions, and contrast the information that can be drawn from each

Answers

• *You will notice that most frequency tables and histograms are prefaced with "Sample response". This is because, unless specified, the interval or bin widths chosen for frequency tables and histograms may vary.*
• *Also see the note at the beginning of the answers to Getting Started.*

A. There are more tomatoes per plant in the South.

B. i)

Number of tomatoes/plant	Frequency (Southern Bhutan)	Frequency (Central Bhutan)
1 – 5	0	1
6 – 10	8	13
11 – 15	10	6
16 – 20	2	0

ii) Yes

C. It is easier to draw conclusions from the frequency table since the data has been grouped into intervals so you can compare the four intervals instead of 40 individual numbers.

D. i)

Southern Bhutan

Stems	Leaves
0	6 7 8 8 9 9
1	0 0 1 1 2 2 2 2 3 3 4 5 6 7

Central Bhutan

Stems	Leaves
0	5 6 6 7 7 7 7 8 8 8 9 9 9 9
1	0 1 1 1 1 2 2

ii) Yes; I can see that there are more data values in the 10 to 19 interval than the 0 to 9 interval for the Southern tomatoes, whereas there are more data values in the 0 to 9 interval than the 10 to 19 interval for the Central tomatoes. This means the tomato plants grown in the South generally have more tomatoes per plant

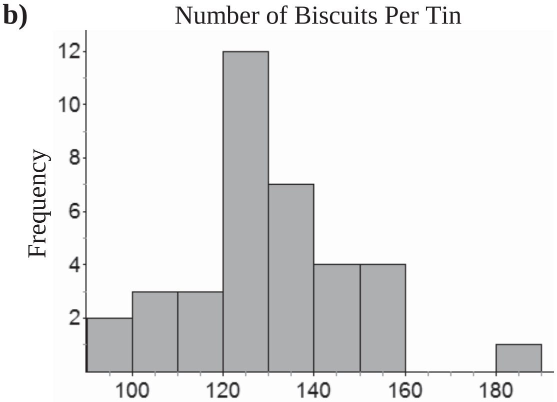
1. a) T **b)** F **c)** F

d) T **e)** T **f)** F

2. a) Number of Biscuits Per Tin

Stems	Leaves
9	6 9
10	7 8 8
11	7 9 9
12	2 2 3 3 3 5 6 7 7 7 8 9
13	2 2 3 4 5 7 7
14	2 3 4 5
15	0 0 1 4
16	
17	
18	8

b) Number of Biscuits Per Tin



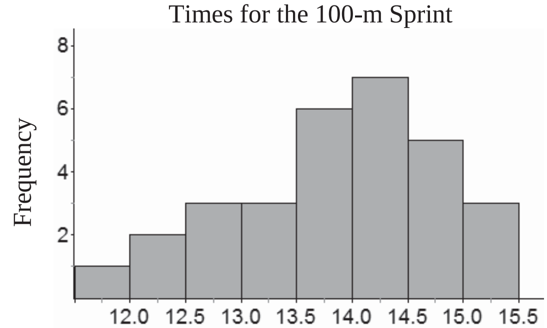
Biscuits per tin

2. c) The manager would want to make sure each tin contained about the same number of biscuits. Both graphs show that only about half (19 out of 36) of the tins contained between 120 and 140 biscuits. There were 9 tins with more than 120 and 8 tins with fewer. The manager would use this data to support efforts to change the way the tins are filled.

3. a) *Sample response*:

Time (s)	Frequency
11.5–12.0	1
12.0–12.5	2
12.5–13.0	3
13.0–13.5	3
13.5–14.0	9
14.0–14.4	8
14.5–15.0	5
15.0–15.5	3

Times for the 100-m Sprint



Time (s)

b) *Sample response*:
The majority of students (18 out of 34) ran the race in a time between 13.5 s and 15.0 s. Nobody ran faster than 11.5 s or slower than 15.5 s.

4.

Group 2		Group 1
99 89	1	50 63 68 83
99 90 77 76 56 55	2	46 87 98
89 85 24 24 10	3	55 55 76
12	4	22 78
	5	05 99

5. a) 37 **b)** 11 **c)** 26 **d)** 29 km/h **e)** 96 km/h

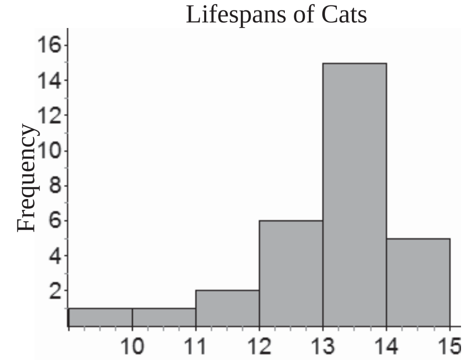
6. Yes; Even though there are more data values on the male side (because the plot shows data for 11 males and 7 females), the data values are clustered at the higher end for the females but not for the males. If you look at the total number of hours for each group and compare them — males at 240 h and females at 243 h — you notice that the females in this group watched more TV in a week .

7. a) i) *Sample response*:
Most heights are between 155 and 165 cm. There are 30 students in the class. The range is less than 30 cm.
ii) *Sample response*:
There are usually between 100 and 130 passengers on the train in a six-week period. There are never more than 151 or fewer than 95. The mode is 129.
b) i) Intervals of 5 are appropriate because they are easy to work with; six intervals are appropriate since there are data values in each interval but not too many. There are differences in the frequencies of the intervals.
ii) Place value intervals of 10 are appropriate because the data values range from 95 to 151. The only other reasonable choice is intervals of 100, and that would mean only 2 intervals, which would not be enough.

8. a) Lifespans of Cats (in years)

Stems	Leaves
9	.6
10	.4
11	.5 .8
12	.8 .8 .8 .9 .9 .9
13	.0 .1 .1 .2 .2 .3 .3 .4 .5 .5 .6 .6 .6 .8 .9
14	.1 .4 .5 .6 .8

Lifespans of Cats



Lifespan (years)

b) The stem and leaf plot; It shows every piece of data and they are arranged in increasing size so it is fairly easy to count from the least or greatest number to find the one in the middle.
c) Same: They both use intervals of 1 so the frequencies for each interval are the same. This results in the same shape, even though the stem and leaf plot is turned sideways.
Different: The stem and leaf plot shows every piece of data, whereas the histogram does not.

9. a) Three age intervals occur most frequently:
1–10, 21–30, and 41–50.
b) Two age intervals occur most frequently:
10–19 and 30–39.
c) The conclusions are different because the graphs use different intervals. If there were a lot of 10-year-olds, Yuden's first bar would be tall, but Maya's second bar would be tall. If there were a lot of 30-year-olds, Yuden's third bar would be tall, but Maya's fourth bar would be tall. It looks like there were a number of 10-year-olds and 30-year-olds in the group and that is why the graphs look different.

10. Organizing the data into equal intervals, or bins, reduces a set of data from a number of single values to a smaller and more manageable number of intervals. Then all you have to do is compare the frequencies of the various intervals to see how the data values are distributed.

Supporting Students

Struggling students

Students who struggle to remember the necessary procedures for constructing each type of graph might benefit from a poster displayed in the class that lists the procedures for creating stem and leaf plots, histograms and box plots (**lesson 6.1.3**).

Enrichment

Students might create questions that other students could answer about the graphs created in **questions 2**, **3**, **4**, **and 8**. For example, as an extension of **question 2**, a student might create questions such as: How would the stem and leaf plot change if the 135 were 185 and the 99 were 59?

6.1.2 EXPLORE: Investigating Bin Width in Histograms

Curriculum Outcomes	Lesson relevance
10-F5 Displaying Data: construct and interpret • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: histograms	This optional lesson helps students to realize that not only do they need to know how to construct histograms from data, but they also need to understand that changing the bin width affects the way the data values are distributed. This in turn may change the nature of the conclusions drawn from the histograms.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM) • Rulers	• constructing frequency tables and histograms • interpreting histograms

Main Points to be Raised

• Changing the bin width can change the distribution of the data. This can change the shape of the histogram and lead to different interpretations of the data.

• Because the choice of bin width affects the conclusions made about the data, histograms are graphs that can be manipulated to support a specific argument or point of view. For this reason, students need to be able to judge whether a histogram has been created to mislead the reader.

Exploration

Ask students to work on **parts A and B** with a partner. One student could work on **part A** while the other student works on **part B**. They could then compare their frequency table, histogram, and conclusions. The students could then discuss their answer to **part C** before answering this part individually.

Observe and Assess

As students are working, notice:

• Do they construct the frequency table correctly and use the correct numbers in each bin?

• Do they draw reasonable conclusions from their graphs?

Share and Reflect

After students have completed **parts A to C**, engage the class in a discussion about what they found. Listen to the points the students make and summarize them in a list.

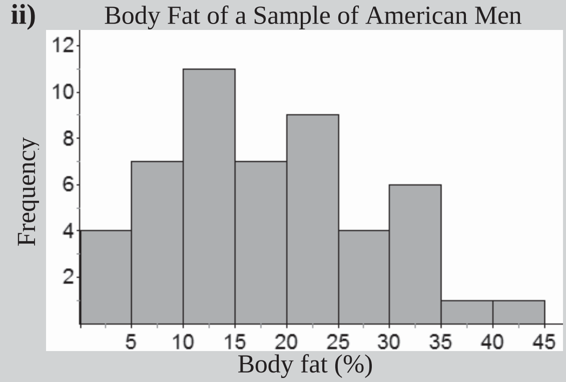
Answers

See the notes at the beginning of the answers to Getting Started and Lesson 6.1.1.

A. i) Body Fat in American Men

Body fat (%)	Frequency
0–4.9	4
5–9.9	7
10–14.9	11
15–19.9	7
20–24.9	9
25–29.9	4
30–34.9	6
35–39.9	1
40–44.9	1

ii) Body Fat of a Sample of American Men

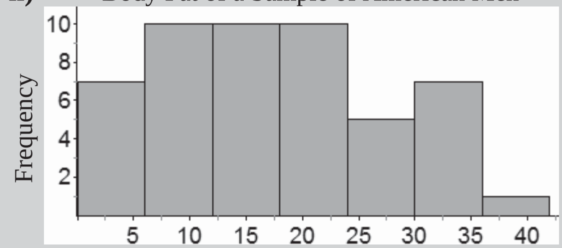


iii) The most frequent interval is 10–14.9% body fat.

B. i) Body Fat in American Men

Body fat (%)	Frequency
0–5.9	7
6–11.9	10
12–17.9	10
18–23.9	10
24–29.9	5
30–35.9	7
36–41.9	1

ii) Body Fat of a Sample of American Men



Body fat (%)

iii) Three groups have the same frequency: 6–11.9%, 12–17.9%, and 18–24.9%.

C. In this situation, the different interval sizes might lead to slightly different conclusions about the same set of data. With intervals of 5%, it seemed that one category was most likely, but with intervals of 6%, it seemed that three categories were most likely.

Supporting Students

Enrichment

This exploration deals with only one set of data. Challenge the students to find another set of single variable data and have them experiment with organizing the data using different bin widths to see whether the bin width seems to affect the distribution of the data and the shape of the histogram in that situation.

6.1.3 Histograms and Box and Whisker Plots

Curriculum Outcomes	Outcome relevance
10-F4 Data Analysis: distribution of data • understand that box and whisker plots are useful when comparing data **10-F5 Displaying Data: construct and interpret** • compare various methods of displaying data which are grouped in intervals and evaluate their effectiveness: stem and leaf, box and whisker plots, and histograms	Histograms show how data values are distributed, but the choice of bin width can seriously affect the conclusions drawn from them. Box plots show how the data values are distributed about the median. Combining these two types of graphs provides the student with more information about the distribution of data. Box plots are also useful when it is necessary to compare two or more data sets.

Pacing	Materials	Prerequisites
2 h	• Grid paper (BLM) • Rulers	• creating and interpreting histograms • creating box plots from ungrouped data

Main Points to be Raised

• With histograms, the way data values appear to be distributed depends on the bin width. This is not the case with box plots.

• Histograms offer choice for the size of each bin, whereas box plots do not. The bin sizes for a box plot are determined by the number of pieces of data in each quartile.

• The lower quartile is the median of the lower half of the data, including the median if the number of pieces of data is odd. The upper quartile is the median of the upper half of the data, including the median if the number of pieces of data is odd.

• Box plots are also called box and whisker plots.

• Box plots are effective for comparing two or more sets of data. By creating box plots on the same scale, you can easily see how the medians and quartiles compare with each other. The range of each set of data is also clearly visible from the whiskers.

Try This — Introducing the Lesson

A. and B. Encourage students to solve **part A** independently and then to discuss their answers with a partner. Have them solve **part B** with a partner. Observe while students work. You might ask:

• *What is the most common lifespan of the light bulbs tested?* (from 300 hours to 350 hours)

• *How many light bulbs had a lifespan less than 150 hours?* (4)

• *How many light bulbs had a lifespan greater than 350 hours?* (7)

• *If a bin width of 100 was used to organize this data set, would the histogram look different?* (Yes, the distribution of the data would change: 0 – 99: 2, 100 – 199: 7, 200 – 299: 7, 300 – 399: 10, 400 – 499: 4)

The Exposition — Presenting the Main Ideas

• Have students read through the exposition. They should already be familiar with the procedures used to create box plots. The term *5-number summary* is new.

• Stress that the 5-number summary divides the data values into four bins. Each of these bins contains the same amount of data. The width of each bin is entirely determined by the numbers in the data set. This is different than with a histogram, where the number of bins varies but the width of the bins does not.

Revisiting the Try This

C. and D. These questions allow students to make a formal connection between the histogram used in **parts A and B** and the box plot created in **part C**. Students can do this on their own or with a partner. They should note how the distribution of data is represented differently in each case. They should also observe that the conclusions drawn from each graph can be different.

Using the Examples

• Put the question from **example 1** and the frequency table on the board and ask students to close their textbooks. Work through this question together. Take time to explain how the numbers in the 5-number summary are being estimated. Students may find this confusing the first time through. You should point out that the best box plot is one constructed from the original values in the data set and that a box plot created from a frequency table is only an estimate.

• Put the question from **example 2** a on the board and ask students to close their textbooks. Work through this question together. Focus their attention on the widths of the boxes and the lengths of the whiskers in the box plots, as well as on the way the data sets are compared at the end of the solution.

Practising and Applying

Teaching points and tips

Q1: Because the histogram is created to the same scale as the box plot, it becomes easier to combine the two graphs to deduce more information.

Q 2: The most important aspect of this question is **part d)**. Discuss the conclusions drawn from these graphs and the justification for them. Since each piece of data is evident in the stem and leaf plot, students should realize that in this case they can construct the actual box plot and not an estimate (as in **example 1**).

Q 3: If students are not familiar with breakfast cereal, a typical North American breakfast, you might spend a few minutes discussing it (e.g., it is sold in cardboard boxes, served cold with milk and eaten with a spoon, and there are many varieties available ranging from those with a high sugar content to a more nutritious low sugar content).

Q 5: In order to estimate the location of the quartiles, students need to recognize that there are about 225 pieces of data in all.

Q 6 and Q 7: Ask the students whether their conclusions would have been as obvious before they drew the graphs.

Common Errors

Some students will find it confusing to perform the calculations to determine the estimates of the 5-number summary for a box plot created from a frequency table. It will help if you can clearly explain why the numbers used in each calculation were chosen.

Suggested assessment questions from Practising and Applying

Question 1	• to see if students can organize data into a frequency table and create an appropriate histogram • to see if students can construct a box plot from data and describe the distribution of data using both graphs.
Question 6	to see if students can use box plots to compare two sets of data and draw appropriate conclusions
Question 8	to see if students can construct an estimate of a box plot from grouped data
Question 10	to see if students can express their understanding about how box plots and histograms differ

Answers

See the note at the beginning of the answers to Lesson 6.1.1.

A. i) 50

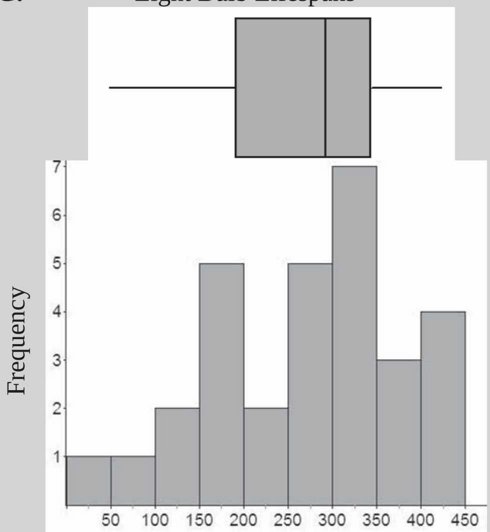
ii) The range is about 400 so a bin width of 50 will give about eight bins of 50. Eight is a good number of intervals and the number 50 is easy to work with.

B. The claim appears to be valid because 19 (5 + 2 + 5 + 7 = 19) of the 30 data values (63%) are between 150 h and 350 h.

C. Minimum = 45, maximum = 425, lower quartile = 189, median = 293.5, upper quartile = 345

Answers [Continued]

C. Light Bulb Lifespans



D. The histogram shows that more of the bulbs had lifespans above 250 h than below 250 h. The box plot shows that 50% of the light bulbs had a lifespan between 189 h and 345 h. The median lifespan is 293.5 h.

1. a)
minimum: 4 ppm
Q1: 29 ppm
Q2 (median): 48.5 ppm
Q3: 80 pp,
maximum: 141 ppm

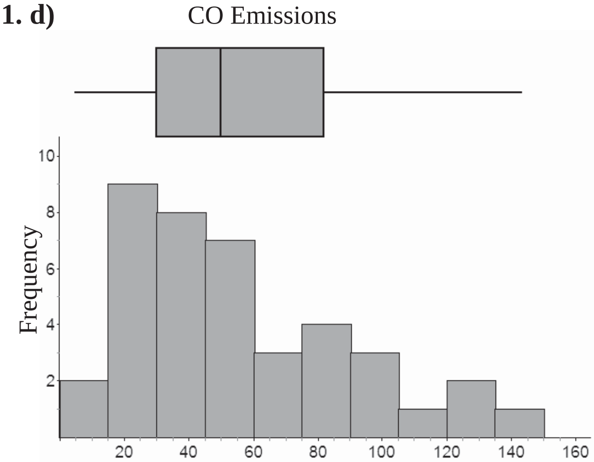
b) *See answer to* ***part c)****.*

c) *Sample response*:

Daily CO Measures

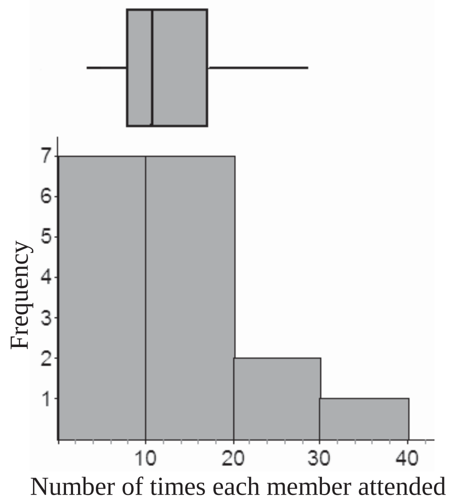
CO emissions (ppm)	Frequency
0 – 15	2
15 – 30	7
30 – 45	10
45 – 60	7
60 – 75	3
75 – 90	4
90 – 105	3
105 – 120	1
120 – 135	2
135 – 150	1

1. d) CO Emissions



d) *Sample response*:
Box plot: median CO emission is about 48.5 ppm and 50% of the emission data values are between about 29 ppm and about 80 ppm
Histogram: the majority of CO emissions ($\frac{24}{40}$ or 60%) fall between 15 ppm and 59.9 ppm

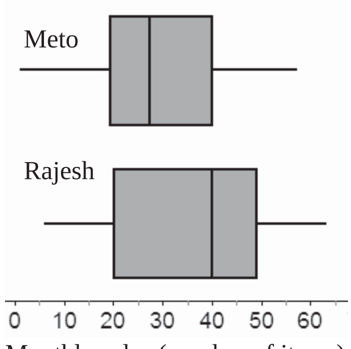
2. a) and c)
Number of Times Running Club Members Met in October



b) Minimum: 4, Q1 is 8, Q2 or Median is 11, Q3 is 18, Maximum is 30
d) *Sample response*:
- The box of the box plot is above the two highest bars in the histogram. The box plot shows that 50% of the data values are found between 8 and 18 but the histogram shows that 14 out of 17 (82%) of the data values are between 0 and 20.
- The box plot makes it easy to see that, even though the histogram shows that most of the data values are between 0 and 20, 50% of them are actually between 8 and 18.
- The box plot also shows the median, 11. Because it is above the histogram, it shows where the median occurs in the histogram.

3. a) Median is 34%
b) Between 10% and 43%
c) *Sample response*:
There are 2 types of breakfast cereal. One type has brands that are low in sugar and the other type has brands that are high in sugar. This explains the two tall bars, one at each end of the graph.

4. a) Monthly Sales Comparison



4. b) Rajesh seems to be the better salesman because of his significantly higher median. The range is similar for both though Rajesh's sales are consistently above the corresponding sales for Meto.

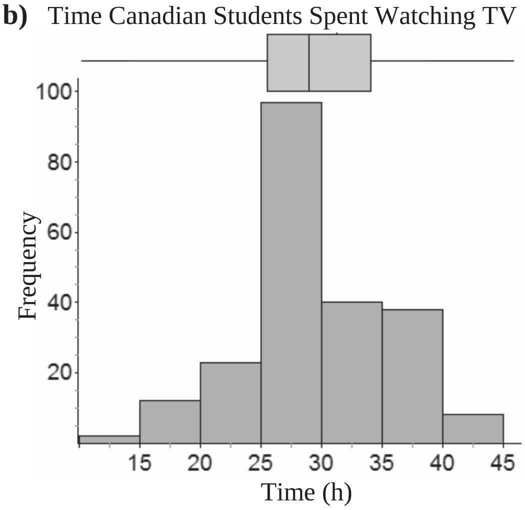
5. a) 220 students were surveyed.
Calculate lower quartile (mean of 55th and 56th values)
- The 55th and 56th values are in the bin 25 – 30. There are 37 numbers in the bins below this and 55 – 37 = 18.
- Halfway between the 55th and 56th values can be estimated by the 18.5th value in this interval.
- Lower quartile = $25 + \frac{18.5}{97} \times 5 = 25 + 0.95 = 25.95$.

Calculate median (mean of 110th and 111th values)
- The 110th and 111th values are in the bin 25 – 30. There are 37 numbers in the bins below this and 110 – 37 = 73.
- Halfway between the 110th and 111th values can be estimated by the 73.5th value in this interval.
- Median = $25 + \frac{73.5}{97} \times 5 = 25 + 3.79 = 28.79$.

Calculate upper quartile (mean of 165th and 166 values)
- The 165th and 166th values are in the bin 30 – 35. There are 134 numbers in the bins below this and 165 – 134 = 31.
- Halfway between the 165th and 166th values can be estimated by the 31.5th value in this interval.
- Upper quartile = $30 + \frac{31.5}{40} \times 5 = 30 + 3.94 = 33.94$.

Minimum = 10 and maximum = 44.

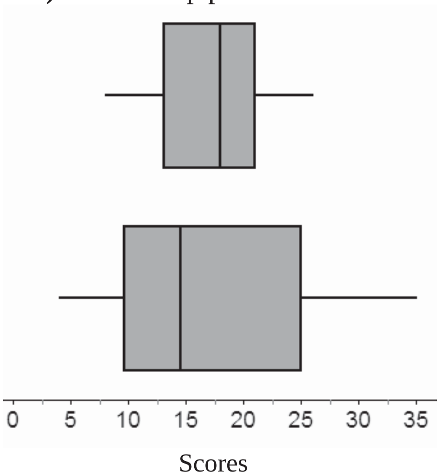
b) Time Canadian Students Spent Watching TV



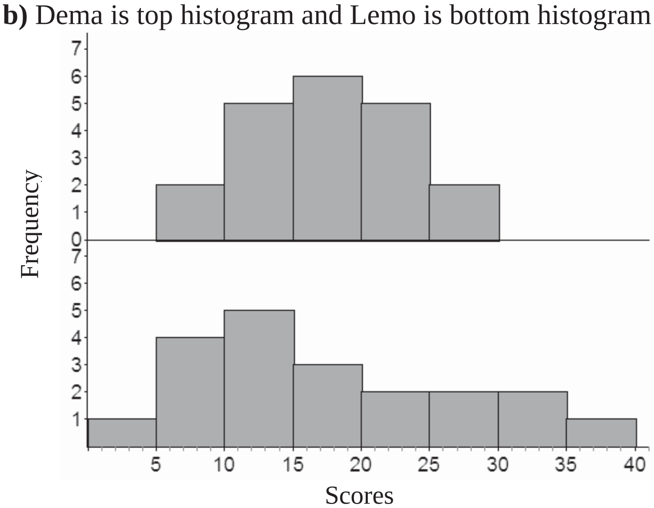
Answers [Continued]

5. c) Histogram: the majority of the students ($\frac{175}{220}$ or about 80%) watch TV between 25 h and 40 h each week.
Box plot: median time is about 29 h/week and half the students watch between 26 h and 34 h each week.

6. a) Dema is top plot and Lemo is bottom plot

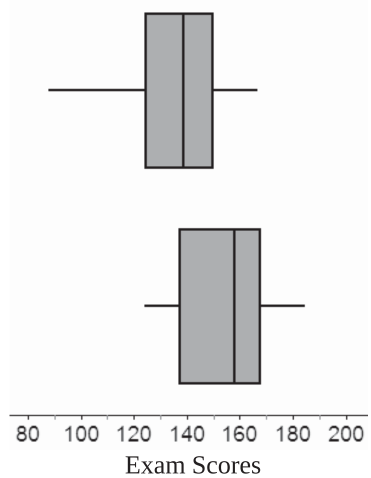


b) Dema is top histogram and Lemo is bottom histogram



c) From the box plot, you can tell that Dema has a higher median number of points. From both graphs, you can tell that Lemo is capable of scoring more points than Dema (above 28) but also fewer points because Lemo has a greater range with a minimum that is below Dema's and a maximum that is above Dema's.
Conclusion: Dema is the stronger player overall because of the higher median score and more consistent results.

7. a) Group 1 is top plot and Group 2 is bottom plot.



b) Yes; Group 1 had a lower median score and lower maximum and minimum scores, indicating that they did not do as well as the Group 2 students. Group 1's wider range also indicates less consistent results.

8. a) There are 50 values.
Calculate lower quartile (the 13th value)
The 13th value is in the bin 1775 – 1800.

$1775 + \frac{13}{16} \times 25 = 1775 + 20.3 = 1795.3.$

Calculate median (mean of 25th and 26th values)
- The 25th and 26th values are in the bin 1825 – 1850. There are 24 numbers in the bins below this and 25 – 24 = 1.
- Halfway between the 25th and 26th values can be estimated by the 1.5th value in this interval.

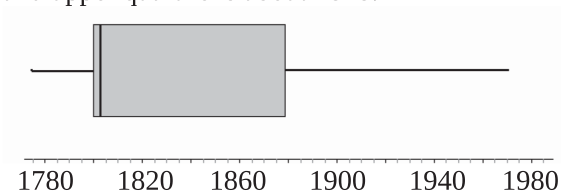
$1825 + \frac{1.5}{6} \times 25 = 1825 + 6.3 = 1831.3.$

Calculate upper quartile (38th value):
The 38th value is in the bin 1875 – 1900. There are 37 numbers in the bins below this and 38 – 37 = 1.

$1875 + \frac{1}{8} \times 25 = 1875 + 3.1 = 1878.1.$

Minimum = 1775 and maximum = 1974

8. a) Minimum is about 1775, maximum is about 1974, median is about 1831, lower quartile is about 1795, and upper quartile is about 1878.



b) Maybe, but only if the estimated values used for this box plot were the same as the actual values. For example, if no state joined in 1974, the whisker would be different.

9. Count the total number of pieces of data and divide that number by 4; call it *d*. Keep a running total of the frequencies starting at the lowest interval and stop once you have reached (or passed) *d*. Suppose this happened in the interval with the star. Calculate the total number of pieces of data in the intervals before that interval and call it *x*. Call the frequency of the interval with a star *y*. Subtract *x* from *d*. Divide *d* by *y*. Multiply that fraction by the size of the interval and add it to the lowest value in the interval.

Interval	Frequency
*	*y*

9. [Cont'd] For example:
If there were 40 pieces of data, you would look for the 10th piece (so $d = 10$). If there were 7 pieces of data in the first two intervals and 5 pieces of data in the next interval, *x* would be 7 and you would calculate $10 - 7 = 3$ to find *y*. Multiply $\frac{3}{5}$ by the size of the interval and add it to the low value in the interval column for that row.

10. The box plot shows the median and how the data values are clustered around the median, where 50% of the data values lie. It also shows the location of the extreme values. You cannot directly determine these exact values from a histogram.

Supporting Students

Struggling students

Some students may have more difficulty drawing conclusions from several box plots drawn on the same scale than from just one plot. However, box plots are ideal for comparing distributions because the centre, spread, and overall range are immediately apparent. Explain to students that they should compare the following:
- the location of the median in each (centre)
- the width of the box in each (spread)
- the length of the whiskers in each (range)

Each of these characteristics gives information about how the data sets are related to each other.

Enrichment

Some students might start with a given histogram, rather than with the frequency table or data, to estimate a box plot that might represent the same data. They could then explain why it might represent that same set of data.

6.1.4 Data Distribution

Curriculum Outcomes	Outcome relevance
10-F3 Normal Curves: **explore measurement issues** • understand that a frequency polygon is created by joining the midpoints of the top of each bar in a histogram • identify situations that give rise to common distributions (e.g., U-shaped, skewed, and normal); • demonstrate an understanding of the properties of the normal distribution (e.g., the mean, median, and mode are equal; the curve (and data) is symmetric about the mean) • understand that a normal curve is based upon a certain type of histogram with infinitely small bins	The normal curve is an important model for describing both mathematical and real world phenomena. Students will benefit by becoming familiar with its properties.

Pacing	Materials	Prerequisites
1 h	• None	• interpreting histograms and box plots

Main Points to be Raised

• Histograms can take on a variety of different shapes. The shape of a histogram depends on how the data values are distributed and this in turn depends on the bin width.

• A data set can be left or right skewed (negatively or positively skewed, respectively). This happens when the majority of the pieces of data occur at either the high end or the low end of the range. This causes the median to be pulled to the left or to the right. The description of whether a set of data is left or right skewed is based on the tail of the data and not on the location of the mound.

• Data values taken from natural phenomena are often distributed in a symmetrical manner about the mean and median like the normal curve.

• Some types of data distributions are more common than others. The normal distribution has many applications beyond the scope of this course and is the most frequent and important distribution of those discussed in this section.

Try This — Introducing the Lesson

A. Students can work individually or with a partner.

Observe while students work. You might ask:

• *The histogram shows two modes. What could these account for in a class composed of boys and girls?*
(The lower mode could represent the most common height of the female students in the class and the higher mode could represent the most common height of the male students.)

The Exposition — Presenting the Main Ideas

• Before the students read the exposition, you could draw a picture of each of the different types of distributions presented in the exposition on the board and ask students to:

- describe their shape
- suggest an example of a data set that might be associated with each type.

They could then read the exposition individually and compare the data sets in the examples with those they suggested earlier.

• Review the meaning of the term *frequency polygon* to make sure students can distinguish it from the histogram it is built from. Talk about how the frequency polygon is sometimes easier to draw and that is why these polygons have been defined. Review the terms *normal distribution*, *skewed positively*, *skewed negatively*, *U-shaped*, and *uniform.*

• Make sure students understand the relationship between the skew of a distribution and the shape of the associated box plot.

Revisiting the Try This

B. This question allows students to formally classify the type of distribution shown in **part A.**

Using the Examples

• Draw the frequency table from **example 1** on the board and ask students to create the histogram and frequency polygon for the data in their notebooks. Have them discuss the shape of the graph. They can then compare their answers with the solution presented in the book.

• Present the students with each of the scenarios in **example 2** and engage the class in a discussion about which distribution they feel is best suited for each data set. Again, they can compare their answers to those presented in the text.

• Work through **example 3** with the class. Explain how the mean, median, and mode are determined from grouped data. This will likely be new material for most students.

Practising and Applying

Teaching points and tips

Q 1: You may discuss this question as a class. It will help to save time as students will not need to draw six histograms but can use the diagrams in the book.

Q 2: You might want to model one example, for example, **part c)**, or refer students back to the exposition.

Q 3: Observe whether students assume that the mode of the ungrouped data has to appear in the highest bar in the histogram. (It doesn't always.)

Q 4b: Students should calculate the mean from the original data set and not from the histogram.

Q 6: This question helps students focus on how box plots and histograms are related. It also confirms their understanding of what the different types of skewed distributions represent.

Q 7: Consistency is indicated by the box plot with the smallest range.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can sketch a frequency polygon and classify the type of distribution given
Question 2	to see if students can sketch an appropriate box plot based on the type of distribution given
Question 6	to see if students can classify types of distributions based on box plots
Question 8	to see if students recognize the relationship between the probability that a single piece of data is included in a particular sample and its related histogram

Answers

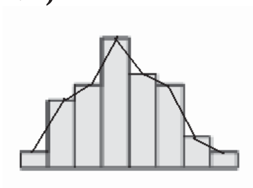
A. i) The graph has two equal tallest bars which indicate that the most common or frequent masses are 29 kg to 31 kg and 37 to 39 kg. The bar heights increase, then decrease, then increase again, and then decrease again.

ii) *Sample response*:
There are two groups within the class, females and males. The females have a different average height than the males. The first tall bar consists primarily of the heights of the female students and the second tall bar is primarily male heights.

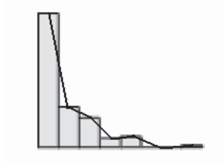
B. U-shaped or bimodal distribution

Answers [Continued]

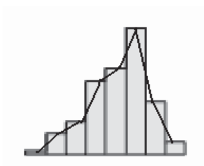
1. a) Normal



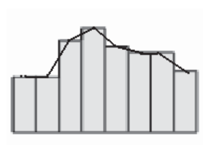
b) Positively or right skewed



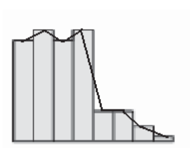
c) Negatively or left skewed



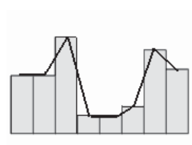
d) Uniform



e) Positively or right skewed

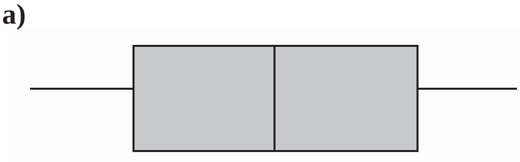


f) U-shaped

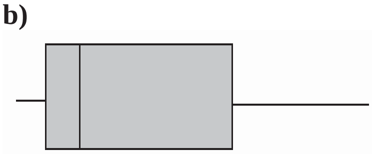


2. *Sample responses*:

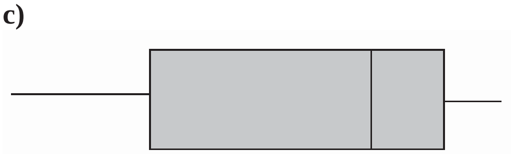
a)



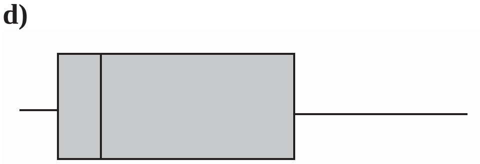
b)



c)

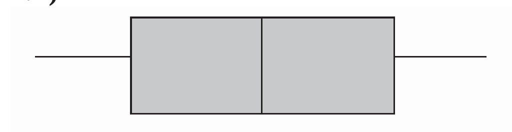


d)



e)

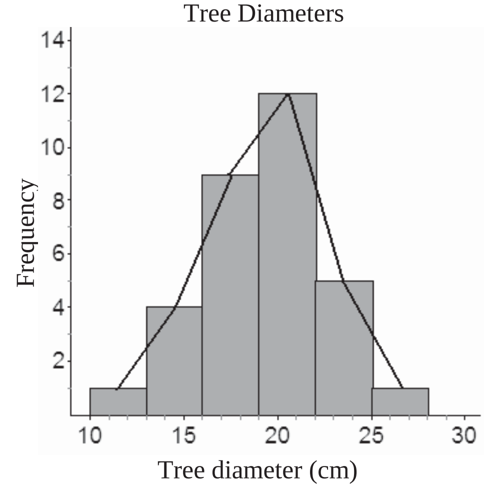
2. f)



3. a) Mean: 19.1 cm, median: 19.35 cm, modes: 18.7 cm and 19.5 cm.

b) *Sample response*:

Tree Diameters

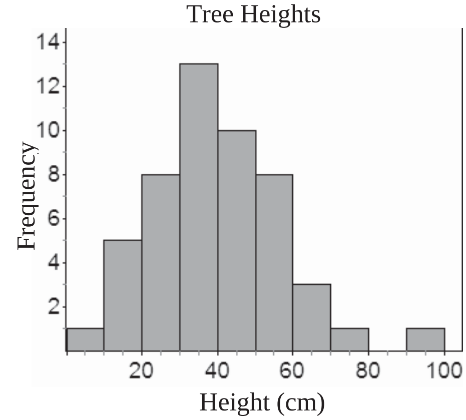


c) It is almost a normal distribution; it has an almost symmetrical mound shape.

d) Some trees may be smaller due to disease, or due to their poorer location with respect to sunlight, moisture, and soil.

4. a) It is close to a normal distribution.
Sample response:

Tree Heights

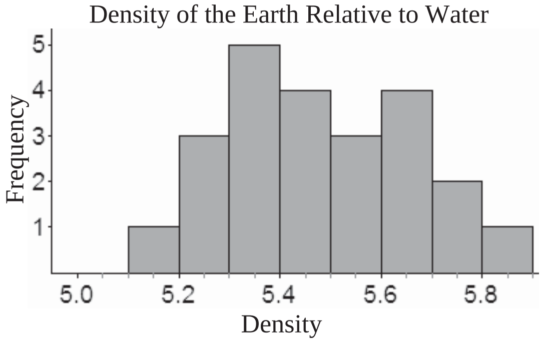


4. b) 38.8 cm; the median height will be close to the mean height since this is close to a normal distribution.

5. a) Mean: 5.48, median: 5.46, modes: 5.29 and 5.34. The mean and median are very close to each other but the modes differ. Both modes are lower than the mean and median.

b) *Sample response*:

Density of the Earth Relative to Water



c) It is close to a normal distribution.

6. a) The Granny Smith sample is close to being a normal distribution because the distribution is symmetrical and the median is in the centre of the box.

b) The Red Delicious sample is left skewed because most of the values are large and the median is to the right of the centre of the box.

c) The McIntosh sample is right skewed because most of the values are small and the median is to the left of the centre of the box. The right whisker is longer.

d) Granny Smith: A lot of apples are close to the median mass while a few are heavier and a few more are light.
Red Delicious: There are a lot of heavier apples in the sample, with a few very heavy ones and no really light ones.
McIntosh: There a lot of lighter apples in the sample, with no very heavy ones and some very light ones.

7. a) The 323 has the most consistent ratings; this is shown by the short whiskers and narrow box width

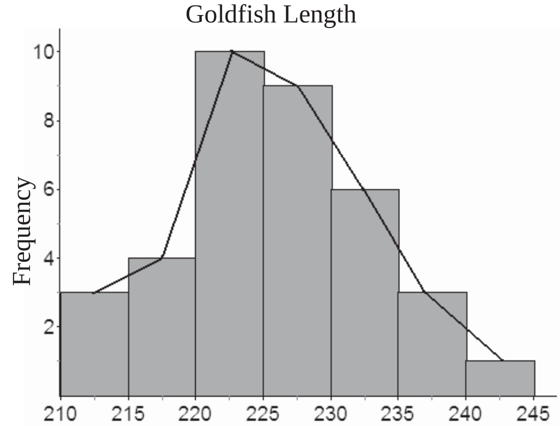
b) Mirage; The median is close to the centre of the box and the whiskers are about the same length.

c) Tracer and Festiva; There are long whiskers on the right with very small whiskers on the left.

d) The data set clearly shows that the car the consumers were most satisfied with was the Civic because it had the highest median, minimum, and maximum values.

8. *Sample response*:
a)

Goldfish Length



b) It is close to a normal distribution.

c) 210 to 220 mm: Unlikely; The data set shows that only 7 out of 36 (less than 20%) of the goldfish in the pond were within this range.
220 to 230 mm: Likely; The data set shows that 19 out of 36 (more than 50%) of the goldfish in the pond are within this range.

9. Table A is close to a normal distribution because the middle two intervals have the greatest frequencies and the frequencies of the intervals on either side decrease.
Table B is right skewed because the first three bins have the greatest frequencies.

Supporting Students

Struggling Students

In order to clarify how the box plot and histogram are related, it might be useful to have some simple normal, right skewed, left skewed, and uniform data sets for students to use. They could create both histograms and box plots and use their results as a reference for working through the exercises. For example, the data sets might be:

Set 1: 10, 20, 20, 30, 30, 30, 40, 40, 50
Set 2: 10, 10, 10, 20, 20, 20, 30, 40, 50
Set 3: 10, 20, 30, 40, 40, 40, 50, 50, 50
Set 4: 10, 20, 30, 40, 50, 60, 70, 80, 90

Enrichment

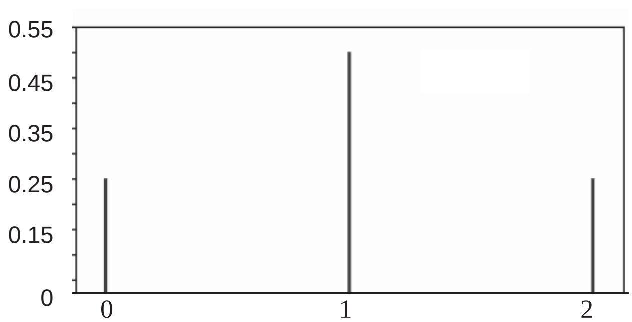
There are many uses of the normal distribution in mathematics courses at higher levels. It originates from studies of probability.

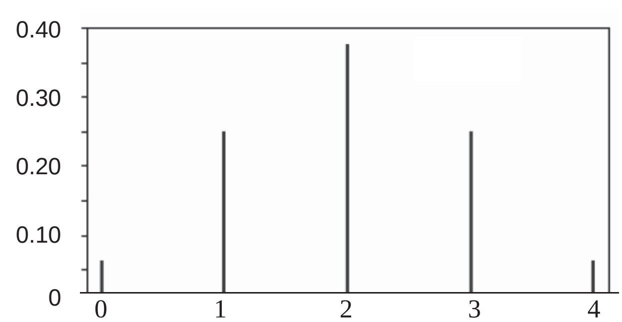
• You might pose this question for some students:

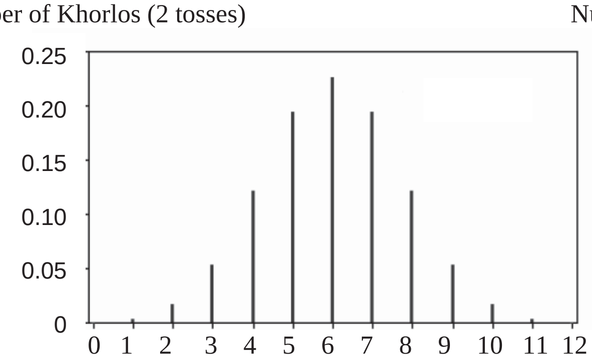
What is the probability of getting the Khorlo 60 or more times if a coin is tossed 100 times?

These were the kinds of questions the 18th century mathematician and statistician Abraham de Moivre spent considerable time studying. He answered this question by examining similar problems using fewer tosses. He conducted experiments where he tossed a coin a predetermined number of times and recorded the number of times the Khorlo landed facing upwards. He then calculated the experimental probability for each outcome and created a graph that compared the probabilities to each outcome. Graphs similar to what de Moivre would have created for 2, 4, and 12 tosses are shown below.

The vertical axis in each graph is the probability expressed as a decimal.







He noticed that each time the distribution formed a mound shape. Drawing a smooth curve through the tops of the bars created what is now called the normal curve. Then de Moivre went further and was able to determine an equation for this curve. The probability of exactly x Khorlos out of N tosses is computed using the formula $P(x) = \dfrac{N!}{x!(N-x)!} a^x (1-a)^{N-x}$, where x is the number of Khorlos (60), N is the number of tosses (100), and a is the probability of a Khorlo (0.5). Therefore, to solve this problem, you compute the probability of 60 Khorlos, then the probability of 61 Khorlos, 62 Khorlos, and so on, and add up all these probabilities. Imagine how long it must have taken to compute binomial probabilities before the advent of calculators and computers.

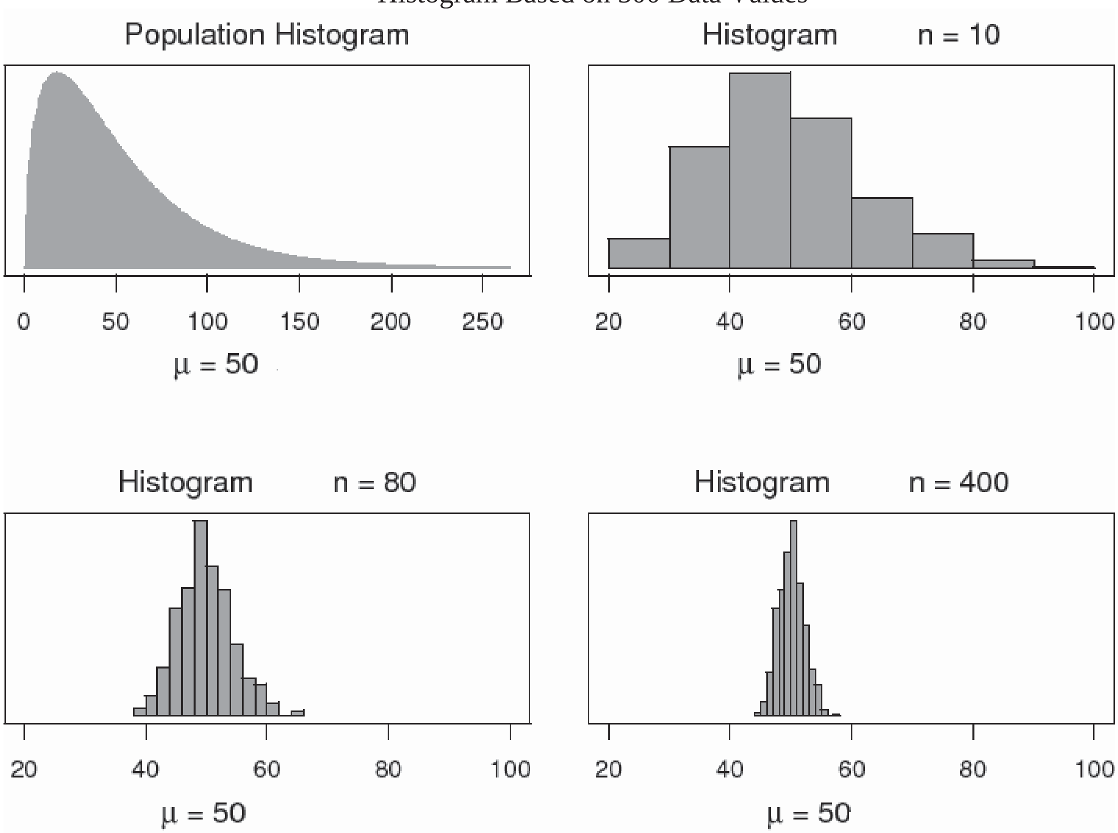
• Many other famous mathematicians did work with the normal curve. You could ask students to do some research to find out who they where and what their work with the normal curve involved.

• Students might also research the many uses of the normal curve.

CONNECTIONS: Normal Distribution and Sample Size

Students have already examined the characteristics of the normal curve. In this Connection, they will look at how sample size can influence the shape of the histogram even the same type of data is used. As more data values are collected the shape of the histogram should approach that of the normal curve. This is often referred to as the Central Limit theorem. As the sample size increases, the shape of the histogram approaches a symmetrical distribution.

Histogram Based on 500 Data Values



μ represents the mean; n represents the number of data values

Answers

1. a) The data set will vary from class to class.

b) The frequency table created will depend on the data collected and the bin width used to organize the data. The histogram and frequency polygons will vary from student to student unless the class uses a common table to represent the data set.

c) The shape depends on the results of **parts a) and b)**. It may be skewed, bi–modal, or some other shape.

2. a) Data will vary from school to school

b) The frequency table created will depend on the data collected and the bin width used to organize the data.

c) Both histograms are similar in that they show the heights of students. The scale on both axes and the units used will be the same. The range of data will be much greater for the data set of the entire school. This data set will also contain many more numbers. Its histogram should be close to a normal distribution. The class data set contains fewer numbers and its range will be much smaller. Its shape should look different than the data displayed for the entire school. This does not mean that the class data will not be symmetrical; it could be, but it is less likely.

Chapter 2 Data Involving Two Variables

6.2.1 Correlation and Lines of Best Fit

Curriculum Outcomes	Outcome relevance
10-C19 Data: gather, plot, and demonstrate understanding of independent and dependent variables, and domain and range • make decisions regarding independent and dependent variables **10-F1 Correlations: develop an intuitive understanding** • understand that a correlation coefficient is a description of how well data fits a linear pattern • identify the difference between a strong and weak correlation and a negative and positive correlation based on the scatter plot and the value of the correlation coefficient	When a data set involves two variables, you often want to know whether there is a linear relationship between the independent and dependent variables so you can predict one value from the other. Correlation is a measure of the strength of the relationship and the confidence you can have in using one variable to predict the other. The correlation coefficient quantifies relationships using a numerical value. It also takes into consideration whether the relationship is positive or negative.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM) • Rulers	• constructing scatter plots and drawing lines of best fit • interpolating and extrapolating • describing the type of correlation between the variables as strong/weak and positive/negative

Main Points to be Raised

• A solid line of best fit can only be drawn when the data set shows a linear trend and it is continuous.

• If the relationship is strong, the correlation coefficient is close to 1 or –1. Only strong relationships should be used to predict the value of one variable if information about the other variable is known. Weak relationships are not conducive to making predictions.

• The absolute value of the correlation coefficient indicates how well a data set demonstrates a linear relationship.

• A low absolute value of the correlation coefficient (close to 0) indicates that a linear relationship between the variables is unlikely. This does not mean that other types of non-linear relationships (examined in the next lesson) are not possible.

Try This — Introducing the Lesson

A. and B. Students can work individually or with a partner. Observe while students work. You might ask:

• *What patterns do you notice in the data?* (As altitude increases, temperature decreases.)

• *Which quantity is the independent variable and which is the dependent variable?* (The temperature depends on the altitude. Altitude is the independent variable and temperature is the dependent variable.)

• *Describe the arrangement of points in your scatter plot.* (The points form a linear pattern falling from left to right.)

The Exposition — Presenting the Main Ideas

• The concepts of variables, scatter plots, and correlation are not new. Have students read the first two pages of the exposition (up until the paragraph where the correlation coefficient is redefined) on their own or work through the pages with them.

• Next, engage the class in a discussion about the correlation coefficient and how this number is related to the amount of scatter in the points of a scatter plot. On the board, reproduce the correlation pictures shown in the last part of the exposition so that you can draw attention to their characteristics. Define the range of possible values for the correlation coefficient (1 to –1) and ask students to estimate what the value could be for each of the pictures shown. Clarify and correct as required.

Revisiting the Try This

C. This question allows students to make a formal connection between what was done in **part B** and the numerical value of the correlation coefficient.

Using the Examples

• Students can work through **example 1** in pairs. Have them compare and discuss their answers to **part a) and part b)**. For **part c)**, they can position rulers on the graph to help them make their predictions.

• Engage the class in a discussion of **example 2**.

• Suggest that students answer the questions in **example 3** with a partner before reading through the solution. They can then compare their own solutions to the solution in the text. To check their understanding, you might draw a different scatter plot on the board and go through the same questions.

Practising and Applying

Teaching points and tips

Q 2: For some parts of this question, students might need to either talk through the situation or to try actual values. For example, for **part c),** it might not be obvious to some that cold drink sales will go up with hot weather, but they might realize this by talking it over with another student. For **part e)**, they might want to use actual values.

Q 4: Stress that even though the data values are discrete and a solid line of best fit should not be used, it is okay to use a dotted line as a model of the relationship, provided there is a linear pattern. They can use that dotted line to make predictions by interpolating. They need to understand that the line of best fit is only appropriate for a limited domain of values.

Q 6c: This question should cause students to think before they respond.

Common Errors

• Some students will automatically draw a line of best fit without considering whether a linear pattern is present or considering the type of data being represented. Stress that a line of best fit can be drawn only if the pattern of the dots is close to linear and the data values are continuous.

• Students often want to use lines of best fit to extrapolate. This is not good practice. Making predictions beyond the data set involves the assumption that the trend will continue. This is only speculation. Predictions made beyond the given data set are subject to error.

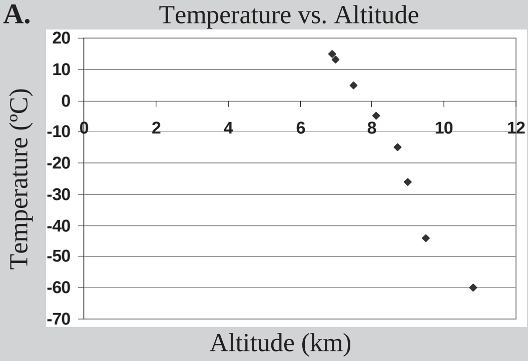
Suggested assessment questions from Practising and Applying

Question 1	to see if students can identify the type of correlation present in given scatter plots and estimate the value of the correlation coefficient
Question 3	to see if students can create a scatter plot, use it to make predictions with continuous data, and relate its value to the level confidence possible in the prediction
Question 5	to see if students can create a scatter plot and use it to make predictions with discrete data
Question 7	to see if students recognize the effect of scatter on the correlation coefficient

Answers

See the note at the beginning of the answers to Getting Started.

A.



A. A line of best fit is appropriate since the data values are continuous and the points in the scatter plot clearly form a linear pattern.

B. Strong negative correlation

C. It is close to –1 since the correlation is negative (the line of best fit would have a negative slope) with almost no scatter around a line passing through most of the points.

Answers [Continued]

1. a) I) Negative; the data values are clustered around a line that goes from the upper left to the lower right.
II) Positive; the data values are clustered around a line that goes from the lower left to the upper right.
III) No correlation; the data values are not clustered around a line; they are scattered in all directions.
IV) Negative; the data values are clustered around a line that goes from the upper left to the lower right.

b) I) r is about –1 II) r is about 1
III) r is about 0 IV) r is about –0.5

2. a) Negatively correlated; as you go up in altitude, the air temperature decreases.

b) Not correlated

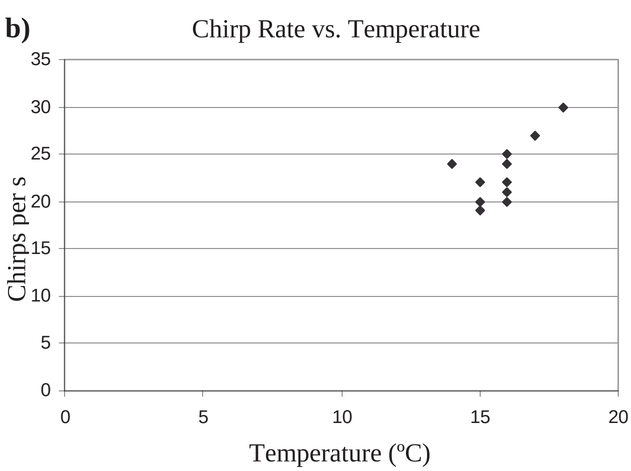
c) Positively correlated; as the outside temperature increases and people get hot, there is a greater need for cold drinks so sales would increase.

d) Not correlated.

e) Perfectly negatively correlated; as you read through a book, the number of pages you have read increases and the number of pages you have not yet read decreases.

3. a) The independent variable is temperature and the dependent variable is chirps per second. I suspect this because the table of values lists the independent variable in the left column and the dependent variable in the right column. Also, the number of chirps a cricket makes depends on the temperature. This means that the chirp rate must be the dependent variable.

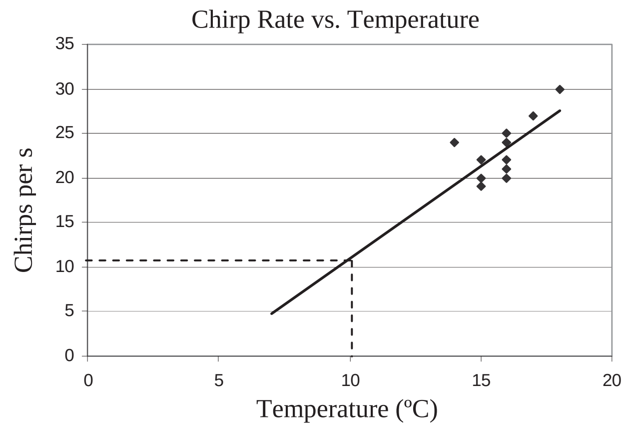
b)



A line of best fit is appropriate since the data values are continuous and show a linear relationship — the points are in a reasonably linear pattern.

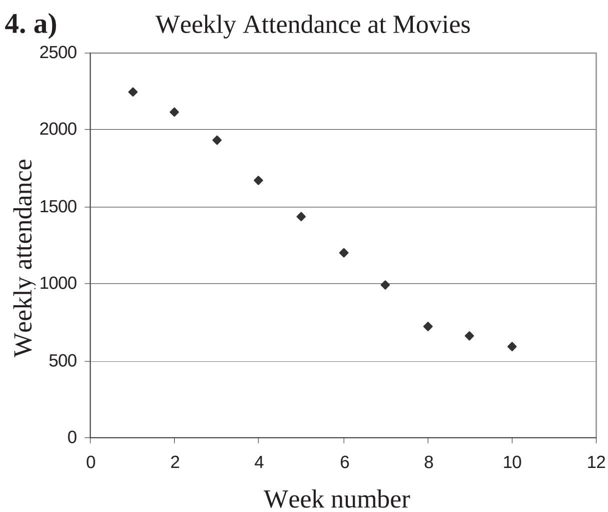
3. c) About 0.5; weak positive and linear

d) About 11 times/s; I drew a line through the dots and extended it beyond 10ºC. Then I used the line to predict the chirp rate for 10ºC.



e) I am not very confident in this prediction. Crickets might stop chirping altogether when the temperature drops below, for example, 14ºC. I do not have enough data to know for sure.

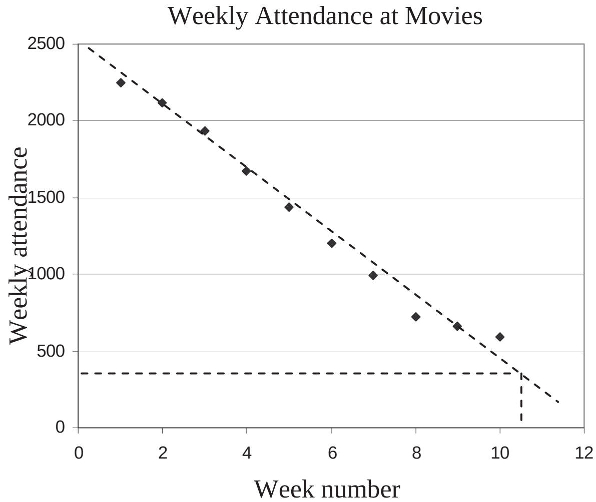
4. a)



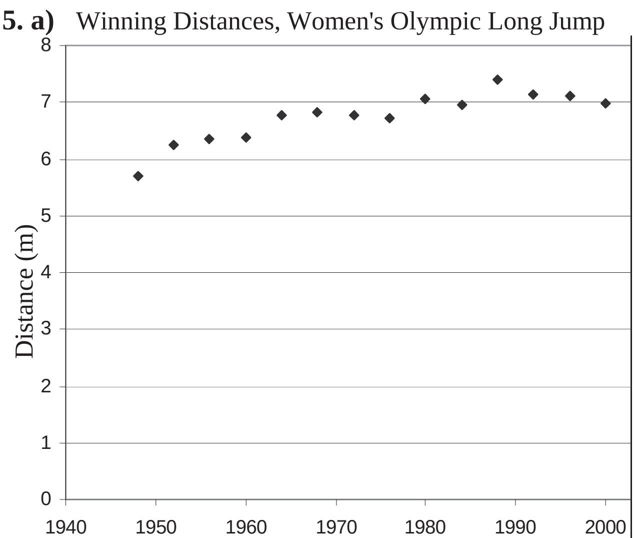
A line of best fit is not appropriate since the data values are discrete, although they do follow a linear pattern. There are no data values for dates in between those plotted.

b) r is close to –1; the data values are closely scattered about a line that falls from left to right.

4. c) It will close around the 11th week



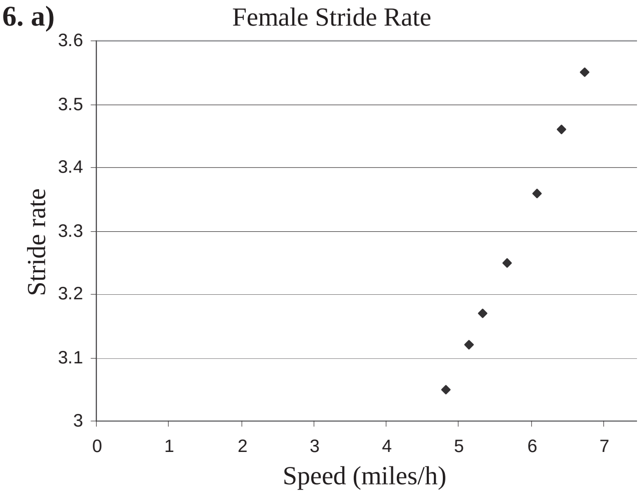
5. a)



A line of best fit is not appropriate as the Olympics are held every four years; the data values are discrete.

5. b) *r* is close to 1; the data values are closely clustered around a line that rises from left to right.
c) About 5.5 m
d) Since 1976, the distance has been up and down.

6. a)



It is a linear relationship.

b) A line of best fit is appropriate since the data values are continuous and the points fall along a line.

c) No; Humans cannot run at a speed of 50 km/h as it is physically impossible.

7. a) Figure A: *r* is close to 1;
Figure B: *r* is close to 0.5

b) Both show positive correlations but the correlation in A is stronger. Its points are closely scattered along a line rising from left to right. The points in B are more scattered about a line but still rising from left to right.

Supporting Students

Struggling students

• Some students might find it easier to think of the correlation coefficient in terms of percentages. A correlation of 100% (or a correlation of 1 of –1) allows you to predict with 100% confidence. If there is no linear relationship, you would have 0% confidence in making predictions so the correlation coefficient would be 0. The confidence level of other predictions increases as the correlation coefficient moves away from 0 toward 1 or –1.

• Some students will struggle with the concept that a correlation that is negative can still be high in terms of its predictive value. Emphasize that the sign only tells whether one variable increases or decreases with respect to an increase in the other; it has no bearing on the predictive value.

Enrichment

Some students may be interested to know how the correlation coefficient is calculated. It involves a measure called standard deviation. Ask students to do some research to find out what standard deviation measures and how this relates to the normal distribution. There are a number of Internet sites that define the exact calculation of the correlation coefficient. Some are interactive and allow the student to see how the coefficient changes with changes in the data. Try doing an Internet search using the terms “calculate correlation coefficient.”

6.2.2 Non-Linear Data and Curves of Best Fit

Curriculum Outcomes	Outcome relevance
10-F2 Curves of Best Fit: non-linear data • explore curve fitting for non-linear data • understand that non-linear models often show a better relationship than linear models	Students need to be able to find a suitable graphical model for a data set that displays non-linear relationships and use the model to solve problems by interpolating and extrapolating where appropriate.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM) • String	• creating and interpreting scatter plots. • recognizing the general shapes of some familiar non-linear graphs (quadratic, exponential and other types of relations).

Main Points to be Raised

• A curve of best fit passes through or near as many points on the scatter plot as possible. It is helpful to use a piece of string that can be manipulated around the points to determine an appropriate curve.

• Once the curve has been drawn, you can use it to make predictions by interpolating and extrapolating.

• As with lines of best fit, if you make predictions by extrapolating, you assume the non-linear pattern will continue in the same manner. This is a poor assumption and may lead to erroneous predictions.

Try This — Introducing the Lesson

A. and B. Suggest that students work in pairs. Observe while they work. You might ask:

• *What patterns do you see in the number of intersection points column in your table?* (The values increase by 3, then 4, then 5, so then the next value must increase by 6 and so on.)

• *What are the independent and dependent variables in the table?* (The independent variable is side length and the dependent variable is the number of intersection points.)

• *Do the points in the scatter plot fall in or close to a straight line?* (No)

The Exposition — Presenting the Main Ideas

• Before students read the exposition, hold a brainstorming session where the students try to recall as many different non-linear relationships as they can. On the board, create a table that lists the name of the relationship in the left column and a sketch of its typical graph on the right. As students offer their suggestions, add them to the list.

• Have the students read through the exposition. Make sure you stress that the curve does not have to pass through all the points; you do not draw the curve by connecting the points. Be aware that students are likely to be unfamiliar with the term *cubic*, but will be able to relate to the volume of the cube.

Revisiting the Try This

C. This question allows students to make a formal connection to the procedures that they are familiar with from working with lines of best fit.

D. This question requires students to think about the relationship between the numbers in the table and the resulting graph. The relationship cannot be linear since the number of intersection points increases by a different (greater) value each time. If the relationship were linear, this increase would be constant.

Using the Examples

Have students work through the examples in pairs. One student can be responsible for each example. They should then explain what they learned to each other. Creating scatter plots should be familiar to most students. Observe as they attempt to draw curves of best fit. Help those who are having difficulty.

Practising and Applying

Teaching points and tips

Q 2: Engage students in a discussion about why it might not be a good idea to use the graph to predict too far into the future.

Q 3: This question is designed to let students see that making predictions using the curve of best fit is easier than making predictions from a table of values.

Q 5: This is really a counting problem that deals with combinations. For n points, the number of lines possible is ${}_nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$. This is a quadratic relationship defined by $f(n) = 0.5n^2 - 0.5n$.

Q 6: Although the actual quadratic model is $S_n = 4n^2$, students should be predicting from the graph rather than from the model.

Common Errors

Many students still try to make sure every point is on the curve and create curves which are really not helpful for predicting. Emphasize the need to make sure the curve is one that would make prediction possible.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can identify the type of relationship based on its graph
Question 2	to see if students can create a scatter plot and exponential curve of best fit and use it to solve problems
Question 6	to see if students can create a scatter plot and quadratic curve of best fit and use it to solve problems

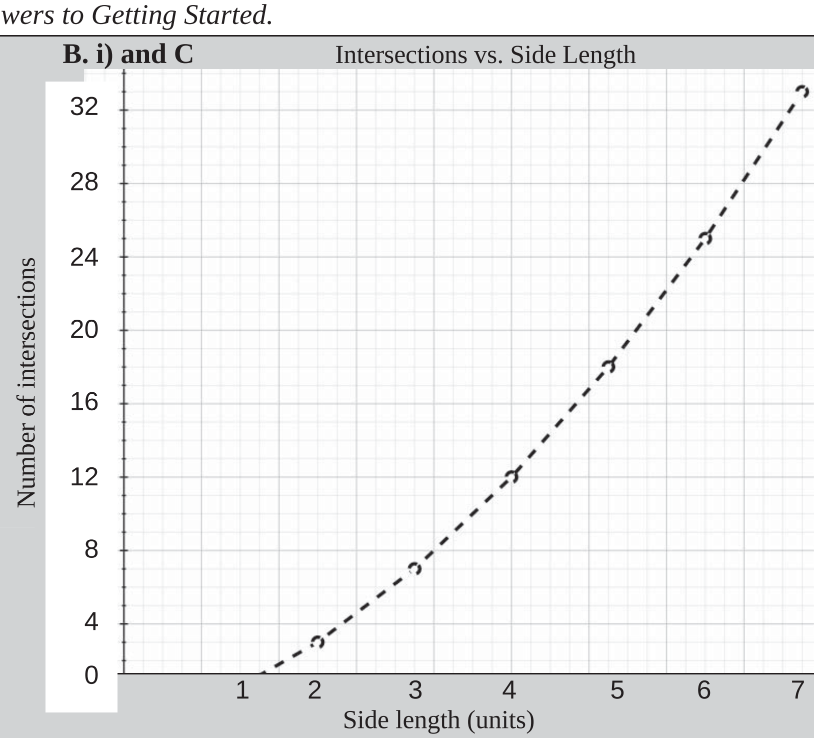
Answers

See the note at the beginning of the answers to Getting Started.

A.

Side length	Number of intersection points
1	0
2	3
3	7
4	**12**
5	**18**
6	**25**

B. i) and C



ii) No; the plotted points appear to form a curve

Answers [Continued]

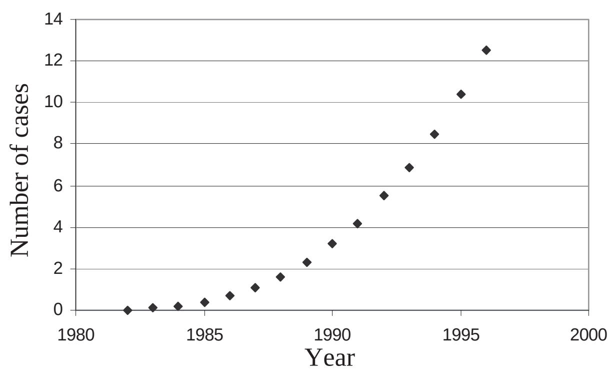
C. There will be 33 intersection points.
Table: I used the pattern in the second column:
3 + **3** = 3, 3 + **4** = 7, 7 + **5** = 12, 12 + **6** = 18,
18 + **7** = 25, 25 + **8** = 33
Graph: I draw a curve of best fit and extended it to about 33, although I wasn't sure it was exactly 33 until I checked using the pattern in the table.

D. Each time the side length increases by the same amount, 1 unit, the number of intersection points increases by one more than the time before: 3, then 4 more, then 5 more, and so on. The rate of change is not constant, but continually increasing more and more each time, resulting in a curve instead of a line.

1. a) Exponential **b)** Linear

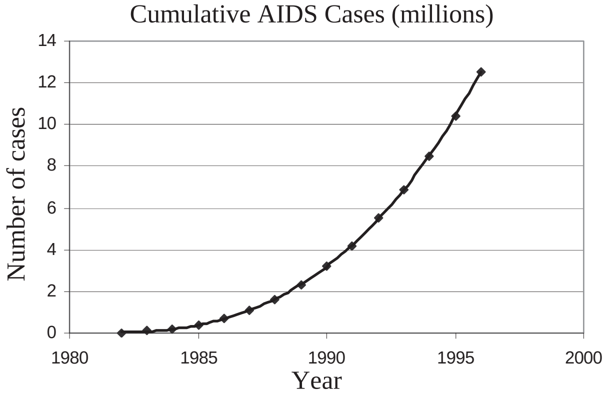
c) Quadratic **d)** None of these

2. a) Cumulative AIDS Cases (millions)



Sample response:

Cumulative AIDS Cases (millions)



b) This appears to be an exponential curve or an exponential relationship. But, it could be a really wide quadratic or cubic curve instead.

c) The number of AIDS cases is increasing rapidly. When this happens an exponential curve is a good model to use.

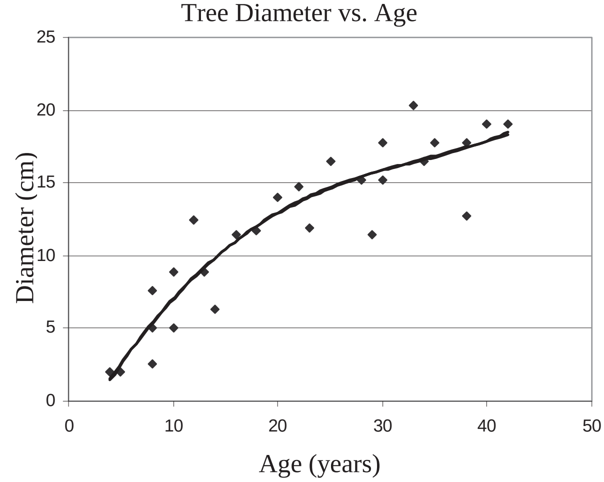
d) About 25 million; My prediction makes sense because the number of AIDS cases is increasing rapidly so a jump from 12.5 million to 25 million between 1996 and 2000 is not surprising.

3. a) About 19.2 cm

b) I am fairly confident based on the trend in the table.

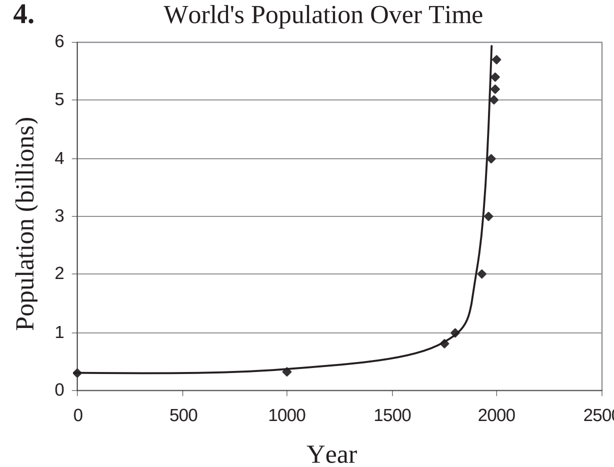
c) *Sample response*:

Tree Diameter vs. Age



d) From the curve of best fit, the diameter of a 32-year-old tree should be about 16 cm. I am more confident in this prediction than in my first prediction since this one is based on the graph that takes all the data values into account and not just the values that fall between 30 years and 33 years.

4. World's Population Over Time

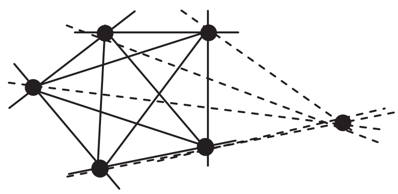


a) About 0.5 billion 0.3

b) About 0.8 billion 0.7

c) About 5.5 billion 7.0

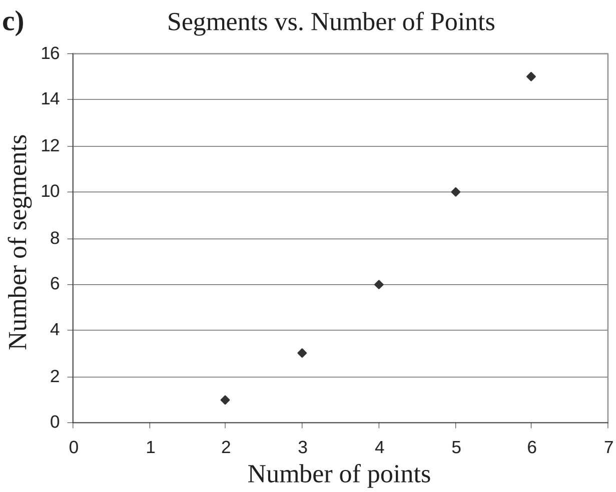
5. a)



b)

Points	Segments
2	1
3	3
4	6
5	10
6	15

c)



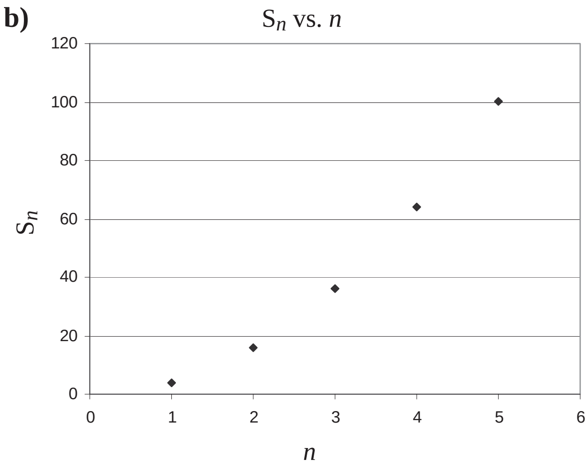
d) The data set is nonlinear; the points lie on a curve, not a line.

e) 21 segments; I saw that the pattern in the segments column was +2, +3, +4, +5, so the next number had to be 15 + 6 = 21.

6. a) 8 by 8 array of dots; 10 by 10 array of dots

n	1	2	3	4	5
S_n	4	16	36	64	100

b)



c) Nonlinear; the points lie on a curve, not on a line.

d) 3600; S_n is the square of double n, so if n is 30, then S_n is $(2 \times 30)^2 = 60^2 = 3600$

7. a) To look for a linear pattern, check to see if the data values are clustered closely around a line.

b) To look for a quadratic pattern, check to see if the data values are closely clustered in a parabolic form or around a portion of a parabola that either increases from left to right or decreases from left to right.

c) For an exponential model, look to see if the dependent variable is increasing or decreasing very rapidly as the independent variable is increasing. The data values should be closely clustered around an exponential curve.

Supporting Students

Struggling students

It might help for some students to use a piece of string to get a better sense of the curve of best fit if they have trouble visualizing it. They could position the string in the desired location and adjust it until they are happy with the fit. They could then use this as a guide to draw the curve.

Enrichment

If students have access to graphing software, a spreadsheet program, or the Internet, you could introduce them to the idea of regression. These programs have the ability to determine curves of best fit using quadratic and exponential regression.

CONNECTIONS: Data Collection by Census

Information on the Bhutan Census of 2005 is available at http://www.bhutancensus.gov.bt/

Answers

Note**:** *All the information below has been taken directly from Wikipedia and from the Bhutan Census website.*

<table>
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1. A census is the process of obtaining information about every member of a population (not necessarily a human population). It can be contrasted with sampling, in which information is obtained only from a subset of a population. It is a method used for accumulating statistical data, and it is vital to democracy (voting). Sets of census data are commonly used for research, business marketing, and planning purposes.

Key people involved in creating the Census:
Dasho (Dr). Sonam Tenzin — Census Commissioner
Mr. Kunzang Dorji — Head, Census Publicity
Mr. Sangay Tempa — Chief Statistical Officer
Mr. Jigme Thinley — Chief Demographer, Project Manager and Head AFD
Mr. Thinley Jameson — Head, Rural Mapping
Mr. Hindu Dorji — Statistical Officer
Mr. Dada — Statistical Officer
Mr. Ushering Jamb yang — Statistical Officer
Mr. Sonam Punish — Head, Urban Mapping
Mr. Dorji Manchu — Survey Engineer
Ms. Decent Tahoma — Asst. Finance Officer
Mrs. Paden — Data Programmer
Ms. Mina Decoy Sharpe — Data Programmer

Data sets were collected by:
- Teachers
- RNR Extension workers
- Health workers
- NFE instructors
- Students of Sherbets College, NIE, NRTI, RBIT, RIM

• Data sets were collected about population characteristics, migration, health, education, labour and employment, and household and housing characteristics.

• The government wants to know this information so that it can adjust its programs. The records also form the basis for taxation.

</td>
<td>

2. One of the earliest documented censuses taken was in the year 500–499 BC by the military of the Persian Empire for the purpose of issuing land grants and taxation. Many countries conduct a census at either 5- or 10-year intervals

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Chapter 3 Probability

6.3.1 Dependent and Independent Events

Curriculum Outcomes	Outcome relevance
10-G1 Theoretical Probability: independent and dependent events • distinguish between two events that are dependent or independent using reasoning and calculations	Before students can perform calculations to determine probabilities of situations that involve several events, they must understand the difference between events that are independent and events that are dependent, that is, whether or not two events are related to each other.

Pacing	Materials	Prerequisites
2 h	• Coins, dice, and spinners (BLM) (optional)	• theoretical probability • probability notation

Main Points to be Raised

The key to understanding whether two events are independent or dependent is to look at how the events are related to each other. If the outcome of the first event does not affect the outcome of the second event, then the events are independent. Otherwise, they are dependent.

Try This — Introducing the Lesson

A. and B. Students can work these questions with a partner.

Observe while students work. You might ask:

• *How many notes are in the bag? What does this represent in terms of the probability you were asked to find?* (Six notes. This is the total number of possible outcomes.)

• *How many Nu 20 notes are in the bag? What does this represent in terms of the probability you were asked to find?* (Two notes. This is the number of ways the event can occur.)

• *How does drawing and replacing a Nu 20 note change the probability of drawing a Nu 20 note the next time?* (It does not change the probability because the number of Nu 20 notes in the bag will still be two and there will still be six notes in the bag to draw from.)

• *How does drawing and not replacing a Nu 20 note change the probability of drawing a Nu 20 note the next time?* (It changes the probability because the number of Nu 20 notes in the bag will be one less and there will be only five notes to draw from in the bag.)

The Exposition — Presenting the Main Ideas

• You may want to use an actual coin and a die to illustrate how tossing a coin and rolling a die are independent events. You can also use a bag with the eight indicated blocks (or pieces of paper) to show how not replacing a block that has been drawn affects the probability of getting a particular type of block on the next draw. Discuss the points from **Main Points to be Raised** as you do the demonstrations with your class.

• Have students read through the exposition to review the ideas presented.

Revisiting the Try This

C. This question allows students to make a formal connection between what was done in **parts A and B** and the concept of independent and dependent events. It also enables the teacher to assess whether or not most students in the class understand the difference between independent and dependent events.

Using the Examples

• For **example 1**, have the students work in groups of four. One pair of students can discuss whether the events in Pair A are dependent or independent while the other pair of students discusses whether the events in Pair B are dependent or independent. The two pairs of students can then take turns explaining to the other pair how they made their decision. The group can then come to a consensus and discuss what the correct decision is for each situation. The remaining parts of the example can then be completed together.

• For **example 2**, you can make a spinner by drawing a circle and dividing it into the required number of sectors using a protractor**.** A pointer can be made from a paper clip and a pencil. Open one end of the paper clip. Put the pencil through the round end of the paper clip and hold it in place it at the centre of the spinner, as shown.

Creating a spinner using a pencil, paper clip, and a fraction circle

Spin the spinner several times and engage the class in a discussion about whether or not two spins are dependent or independent. Discuss how to make an outcome chart.

Practising and Applying

Teaching points and tips

Q 1 and Q 2: Replacement or no replacement defines whether the described events are dependent or independent.

Q 2: Students will see that in the case of dependent events, the probabilities change for the second event depending on the outcome of the first event.

Q 3 and Q 4: Dependence is based on whether or not the first event influences the likelihood of the second event occurring. For example, initially spinning a 3 makes it impossible (probability 0) to get a total of 3. If a 2 is spun the first time, the probability of spinning 1 the second time and getting a total of 3 is $\frac{1}{5}$.

Q 4: Students may need to calculate the probabilities in order to determine dependence or independence, particularly with Pair A.

Common Errors

Some students fail to see how the outcome of the first event influences or affects the likelihood of the second event occurring. They should compare the probability that the event would occur without the knowledge of the outcome of the first event to the probability they would use with knowledge of results of the first event.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can relate independence to the concept of replacement
Questions 3 and 4	to see if students can judge independence based either on the nature of the situation or on the probabilities involved

Answers

A. P(Nu 20 note) = $\frac{2}{6}$ or $\frac{1}{3}$

B. i) P(Nu 20 note again with replacement) = $\frac{2}{6}$ or $\frac{1}{3}$ **ii)** P(Nu 20 note again without replacement) = $\frac{1}{5}$

C. i) There was a different number of Nu 20 notes and a different total number of notes to draw from when the second note was drawn in each of the two situations.
ii) When you do not return the note, is the events are dependent since the probability depends on what happened the first time; it could be $\frac{1}{5}$ if you drew a Nu 20 note the first time but $\frac{2}{5}$ if you drew a different note.

1. a) $\frac{2}{11}$ **b)** $\frac{1}{10}$ **c)** They are different. In **part a)**, you had 11 cards to draw from and two were B. In **part b)**, you had ten cards in the bag and only one was a B. **d)** The events are dependent in **part b)** because the probability of drawing the letter B the second time was affected by what happened in the first draw. **2. a)** $\frac{3}{7}$ **b)** $\frac{2}{6}$ or $\frac{1}{3}$ **c)** $\frac{3}{7}$ **d)** $\frac{3}{6}$ or $\frac{1}{2}$ **3. a)** The events are independent because the probability of a second spin of 4 is $\frac{1}{5}$ whether you get a 4 the first time or not. **b)** The events are independent because the probability of spinning an odd number the second time is $\frac{3}{5}$ no matter what happens on the first spin. **c)** The events are dependent because Event B can happen but it becomes impossible if Event A happens.	**3. d)** The events are dependent because the probability of spinning a number that results in a difference of 1 on the second spin depends on the result of the first spin. For example, a first spin of 2 gives two possibilities with a difference of 1, whereas, a first spin of 5 would not. **4.** Pair A are dependent since the probability of rolling a number that would result in a total of 5 or more on the second roll, if a 2 is rolled the first time, is $\frac{4}{6}$ or $\frac{2}{3}$. If a 2 is not rolled, the probability of rolling a number that would result in a total of 5 or more is $\frac{26}{30}$ or $\frac{13}{15}$. Pair B is independent because the probability of rolling a number less than 5 on the second roll is $\frac{4}{6}$ or $\frac{2}{3}$, no matter what happens on the first roll. **5.** *Sample response*: **a)** Drawing two red cards in a row from a deck of playing cards if the first card is not replaced. These are dependent events because the probability of drawing a second red card will be affected by what happens in the first draw. **b)** Drawing two red cards in a row from a deck of playing cards if the first card is replaced. These are independent events because the probability of drawing a second red card will not be affected by what is drawn the first time.

Supporting Students

Enrichment

Students could create additional descriptions of events that are independent and dependent, particularly events involving materials not already presented in the exercises.

6.3.2 Calculating Probabilities

Curriculum Outcomes	Outcome relevance
10-G1 Theoretical Probability: independent and dependent events • distinguish between two events that are dependent or independent using reasoning and calculations	Sometimes it is easier to use calculations rather than reasoning to determine whether two events are dependent or independent.

Pacing	Materials	Prerequisites
1 h	• None	• theoretical probability • probability notation • Venn diagrams

Main Points to be Raised

• A second way to determine whether two events are independent or dependent is to calculate the product of the two separate probabilities. If the product is equal to the product of the joint probability (the probability that both events occur), the events are independent, that is, if P(A and B) = P(A) × P(B). Otherwise, they are not.

• A conditional probability P(B|A) means the probability that B occurs if you know that A has occurred. If B and A are independent, then P(B|A) = (B).

• When exploring the independence of events, you might look at two sequential events, for example, selecting a card after having selected a different card. Or, you can look at one event in two different ways, for example, selecting a card from a deck of cards and comparing the events that the card is red and that the card is a 7.

• A Venn diagram is sometimes a useful tool for clarifying the sample space to determine conditional and joint probabilities.

Try This — Introducing the Lesson

A. Students can work with a partner.

Observe while students work. You might ask:

• *How many elements are there in the sample space for all the outcomes for two spins? How do you know?* (Nine. If I made a tree diagram, there are three original branches and each branch has three smaller branches.)

• *Which outcomes are associated with having the second spin greater than the first?* (It could be (1, 2), (1, 3), or (2, 3) so the probability is $\frac{1}{3}$.)

• *Which outcomes are associated with a sum of four?* ((1, 3), (2, 2), or (3, 1); the probability is also $\frac{1}{3}$.)

The Exposition — Presenting the Main Ideas

• Tell students to imagine rolling a die. Ask for the probability that the result is 2 ($\frac{1}{6}$). Then tell them that you have rolled a die and the result is even. Now ask for the probability that the result is 2 ($\frac{1}{3}$). Ask why these two probabilities are different. Explain that because values are different, the two events are dependent rather than independent.

• Remind students that they already know that events are dependent in a situation where, for example, a card is drawn and not replaced.

• On the board, draw the sample space for rolling a die twice. Show all 36 outcomes. Ask for the individual probability of A: rolling a 3 on the first die, and B: rolling a sum of 4. Ask for the probability of B if you know that A has already occurred. Discuss how this shows that events A and B are dependent.

• Remind students that they learned in Class IX that if two events are independent, their probabilities can be multiplied to get the probability that both occur. Observe that this is not the case with the example above.

• Ask students to read through the exposition. Answer any questions they might have.

Revisiting the Try This

B. This question allows students to make a formal connection between what was done in **part A** and the concepts introduced in the exposition.

Using the Examples

• Lead students through **example 1**, re-introducing the process of interpreting Venn diagrams if necessary.

• Students can read through **example 2** independently.

Practising and Applying

Teaching points and tips

Q 1: Some students may interpret **parts i) and ii)** as only brothers or only sisters, but the intent is to include families with both in each of those calculations.

Q 2: Make sure students pay attention to the note about how to read the ordered pairs in the Venn diagram. Otherwise, it will be difficult for them to solve the problem.

Q 4: For **part c) i)**, students need to realize that P(A|B) = 1 since A has to happen if B happens.

Q 5: To perform this calculation, students need to solve the equation P(A|B) = 0.25 ÷ 0.6.

Q 6: For **part c),** if students are not familiar with playing cards and/or the notion that Aces can be high or low, take the time to explain that each suit of 13 cards has Ace, 2, 3, …, 10, Jack, Queen, and King. The Ace can be thought of as the number 1, or “low,” i.e., the least valuable card, or it can be thought of as being higher than the King, or “high,” i.e., the most valuable card.

Common Errors

Some students will confuse the meaning of B|A with A|B. Remind students that the letter following the bar symbol represents the event that definitely occurred.

Suggested assessment questions from Practising and Applying

Question 3	to see if students can determine whether two events are dependent using any strategy
Question 4	to see if students can use a computational strategy to determine whether events are dependent
Question 5	to see if students can determine a conditional probability using a computational strategy

Answers

A. i) $\frac{1}{3}$; There are three equally likely outcomes and one of them is 3.

ii) $\frac{1}{3}$; There are nine equally likely outcomes and three have a greater second spin than the first.

	1	2	3
1	1, 1,	2, 1	3, 1
2	**1, 2**	2, 2	3, 2
3	**1, 3**	**2, 3**	3, 3

iii) $\frac{3}{9}$ or $\frac{1}{3}$; There are nine equally likely outcomes and three have a sum of 4.

	1	2	3
1	1, 1,	2, 1	**3, 1**
2	1, 2	**2, 2**	3, 2
3	**1, 3**	2, 3	3, 3

B. i) Dependent; Spinning a 3 and then spinning a number greater than 3 in the second spin is impossible, so the probability is 0, not $\frac{1}{3} \times \frac{1}{3}$.

ii) Independent; The probability of spinning a 3 on the first spin and a sum of 4 on both spins is $\frac{1}{9}$.

	1	2	3
1	1, 1,	2, 1	**3, 1**
2	1, 2	2, 2	3, 2
3	1, 3	2, 3	3, 3

$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$, so the events are independent.

Answers [Continued]

1. a) i) $\frac{29}{40}$ **ii)** $\frac{27}{40}$ **iii)** $\frac{19}{40}$ **iv)** $\frac{3}{40}$ **b)** Dependent; $\frac{29}{40} \times \frac{27}{40} \neq \frac{19}{40}$ **2. a)** Count the number of pairs that have two odd numbers (in the left circle) and compare that number to the total number of pairs. **b)** Count the number of pairs that have a total of 6 (in the right circle) and compare that number to the total number of pairs. **c)** Multiply the probabilities from **parts a) and b)** and compare that product to the ratio of the number of pairs that have both two odd numbers and a total of 6 (in the intersection region of the two circles) to the total number of pairs. **3. a) i)** $\frac{16}{50}$ **ii)** $\frac{10}{50}$ **iii)** $\frac{3}{50}$ **3. b)** Dependent; $\frac{16}{50} \times \frac{10}{50} \neq \frac{3}{50}$	**4. a) i)** $\frac{1}{2}$ **ii)** $\frac{1}{4}$ **iii)** $\frac{1}{4}$ **b)** Dependent; $\frac{1}{2} \times \frac{1}{4} \neq \frac{1}{4}$ **c) i)** $\frac{1}{4} = 1 \times \frac{1}{4}$ **ii)** $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$ **5.** About 0.42 (or $\frac{5}{12}$) **6. a**) By calculating: P(rolling a 4 first and a total of 10) = $\frac{1}{36}$; $\frac{1}{36} \neq \frac{1}{6} \times \frac{3}{12}$ Using reasoning: The events are dependent because, to get a total of 10, the first roll must be at least 4. That means the result of the first roll affects the probability. **b)** By calculating: P(red, then blue) = $\frac{5}{10} \times \frac{5}{9}$; $\frac{25}{90} \neq \frac{5}{10} \times \frac{5}{10}$ Using reasoning: The probability of selecting a blue marble second would be $\frac{5}{9}$ if a red marble was first drawn and not replaced and $\frac{5}{10}$ if it was replaced. Since the probability was affected, this means the events are dependent.

Supporting Students

Struggling students

Some students find it difficult to keep track of each of the separate probabilities. Encourage them to use some sort of diagram, whether a Venn diagram or a sample space chart, to keep track of the outcomes they are counting.

Enrichment

Students might set up situations with particular conditional probabilities. For example, if $P(B|A) = \frac{1}{2}$, the situation might be about the integers from 1 to 20. Event A might be the probability of a selected number being even and Event B might be the probability of the selected number being greater than 10.

UNIT 6 Revision

Pacing	Materials
2 h	• Grid paper (BLM) • Rulers

Question(s)	Related Lesson(s)
1 – 3	Lessons 6.1.1 and 6.1.3
4 – 6	Lesson 6.1.3
7, 8	Lesson 6.1.4
9 – 12	Lesson 6.2.1
13	Lesson 6.2.2
14 – 19	Lessons 6.3.1 and 6.3.2

Revision Tips

Q 1: Remind students of some possible contexts, e.g., length or area measurements, monetary values, time measurements.

Q 3: Ask students why **part d)** did not ask them to use the same interval size.

Q 4: Ask why it was easier to answer **part b)** after the graph was drawn rather than before.

Q 7: Remind students of what the possible distributions are.

Q 10: Ask students to justify their choices.

Q 13: Observe whether some students recognize immediately that area relationships are probably not linear.

Q 18: You may have to remind students to use the equation P(A and B) = P(A) P(B|A).

Answers

See the notes at the beginning of the answers to Getting Started and Lesson 6.1.1.

1. *Sample responses*:

a) Examine the distribution of the masses of 100 dogs.

b) Compare the masses of 50 female and 50 male dogs.

c) Examine the distribution of the monthly income of all the residents in a community.

d) Compare the distribution relative to the median of the monthly income of all the residents in one community to that of residents in another community.

2. *Sample responses*:

a) Advantages: It shows how the data values are distributed. It is easy to group data into place value intervals and there is no scale to create. It is easy to find the extremes, range, median, and mode since it shows all the data and the data values are ordered.
Disadvantages: Interval size is limited to place value intervals.

2. b) Advantages: It shows how the data values are distributed. You can select the interval size. It results in a simple visual image you can examine to determine the distribution.
Disadvantages: It does not show the actual data values. Changing the interval size can lead to different shapes and different conclusions.

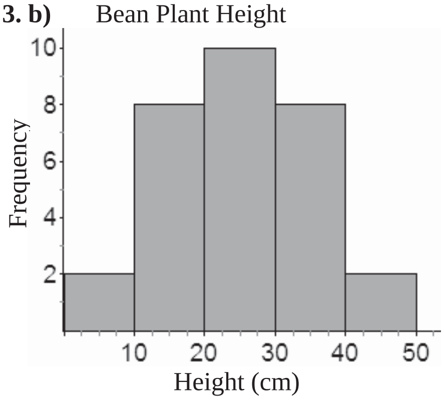
c) Advantages: It shows how the data values are distributed. It is good for examining the distribution of the data relative to the median and extremes. It can be easy to create depending on the data values.
Disadvantages: It does not show the actual data values. It can be difficult to create depending on the data values.

3. a)

Stem	Leaves
0	5 7
1	0 0 1 3 5 6 7 8
2	1 2 2 4 5 5 5 6 7 9
3	2 3 6 6 7 7 8 9
4	5 8

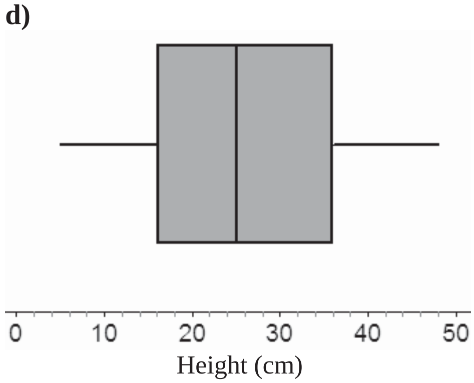
Answers [Continued]

3. b) Bean Plant Height



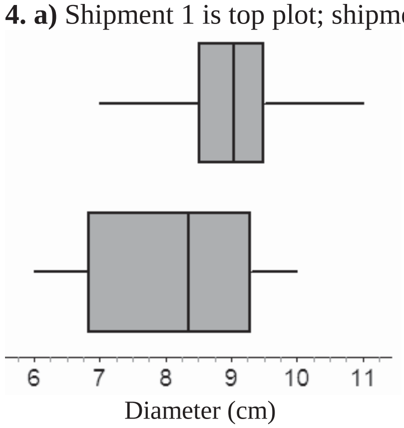
c) Normal distribution

d)



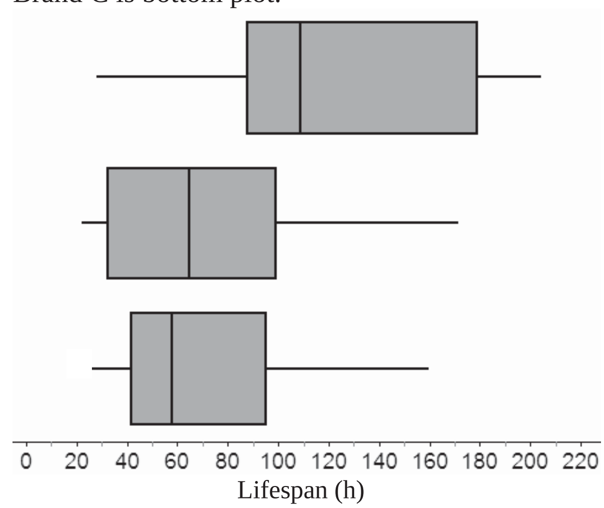
e) *Sample response*:
I prefer the stem and leaf plot because it is easy to construct and I do not have to choose an interval size or create a scale. If I want to find values such as the median or extremes, I can use the plot without having to go back to the original data.

4. a) Shipment 1 is top plot; shipment 2 is bottom plot



b) Shipment 2; The median and extremes in the box plot are lower than for Shipment 1, which means the oranges are generally smaller. The box in the box plot is wider than for Shipment 1, which means the oranges are less uniform in size.

5. a) Brand A is top plot; Brand B is middle plot; Brand C is bottom plot.



b) Brand A; It has the highest median lifespan. 50% of the time Brand A batteries last 90 h to 180 h whereas 50% of the time Brand B batteries only last about 30 h to 100 h, and 50% of the time Brand C batteries only last about 40 h to 95 h.

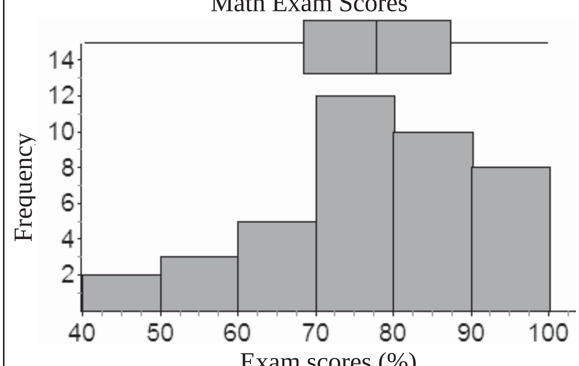
6. a) Box plot calculations:

Lower quartile (9.5th value) = $60 + \frac{9.5-5}{5} \times 10 = 69$

Median (20.5th value) = $70 + \frac{20.5-10}{12} \times 10 = 78.75$

Upper quartile (30.5th value) = $80 + \frac{30.5-22}{10} \times 10 = 88.5$

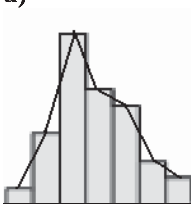
Math Exam Scores



b) Both graphs show that the data set is skewed left, as the majority of the students scored above 70%.
The box plot shows that the median mark was about 79% and half the class scored between about 69% and 89%, but these are only estimates.

7. SET I

a)

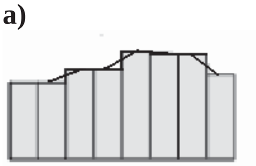


b) Rises quickly then tails off to the right;
The distribution is right or positively skewed.

c) *Sample response*:
Marks on a test where most students did not perform well.

SET II

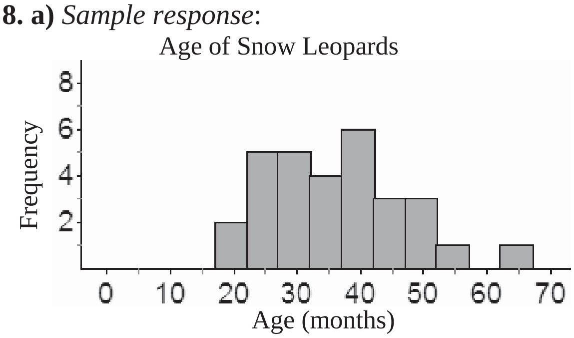
a)



b) Relatively flat; It is close to a uniform distribution.

c) *Sample response*:
Heights of tomato plants grown inside a greenhouse under controlled conditions and planted at the same time.

8. a) *Sample response*:



b) *Sample response*:
Close to mound-shaped with a normal distribution.

c) Mean: about 35.37 months; median: 35 months; mode: 37 months.

d) Very likely, since 28 out of the 30 leopards were under 50 months old.

9. a) IV **b)** I and IV **c)** III

d) I **e)** II

9. f) I: r is close to –0.5 II: r is close to 1
III: r is close to 0 IV: r is close to –1

10. a) Relationship A: very strong negative correlation; close to –1

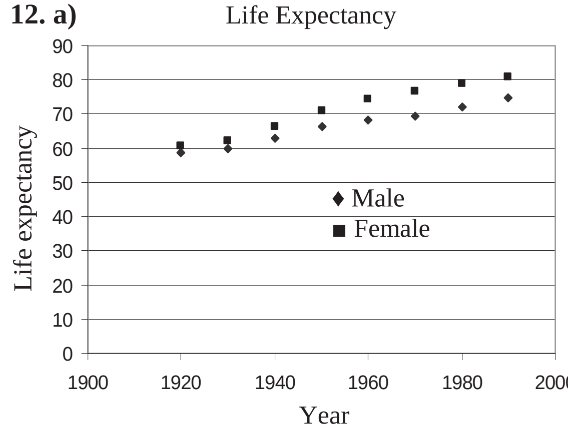
b) Relationship B: weak positive correlation; close to 0.5

11. *Sample responses*:
a) The relationship between the age of a child up to age 18 and his/her height.

b) The number of sips you take from a drink and the volume of liquid remaining in the glass.

c) Shoe size and number of siblings.

12. a) Life Expectancy



b) Strong positive correlation

c) Close to 1 for both data sets

d) Yes; Both sets of data are continuous and have a linear trend.

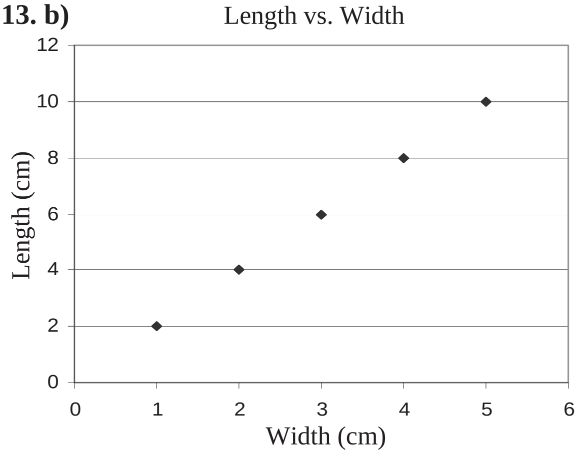
e) Males: about 81 years; Females: about 91 years

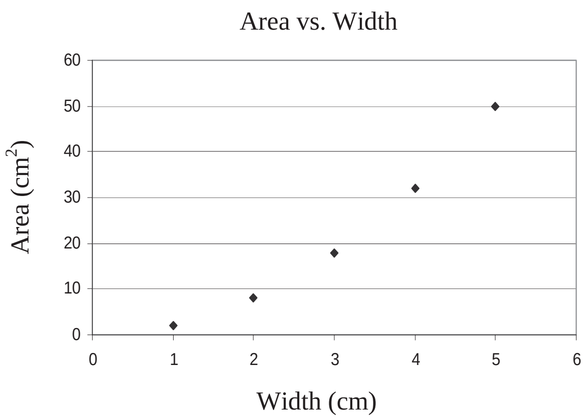
13. a) 4 row by 8 column array;
5 row by 10 column array

Width (cm)	Length (cm)	Area (cm^2)
1	2	2
2	**4**	**8**
3	**6**	**18**
4	**8**	**32**
5	**10**	**50**

Answers [Continued]

13. b)





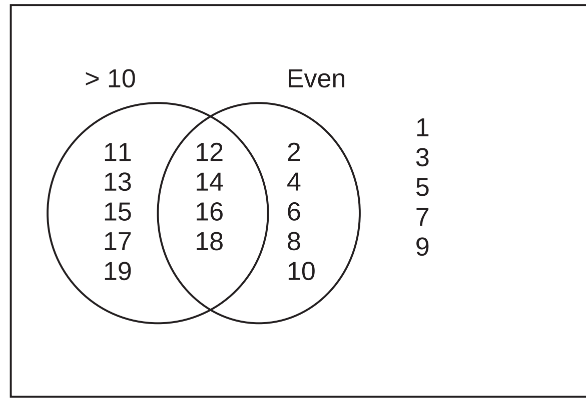
c) A line of best fit is appropriate for length vs. width because the data values are continuous and form a linear pattern. A curve of best fit is appropriate for area vs. width because the data values are continuous and form a non- linear pattern.

14. Not replacing the item changes the number of items in the bag and therefore changes the probability for the next draw.

15. The events are dependent because the outcome of the first roll affects the second roll. Novin has treated them as independent events and multiplied the probabilities of the two events together to find the probability of both events happening.

16. a) $\frac{5}{14}$ **b)** $\frac{4}{14}$ **c)** $\frac{5}{14}$ **d)** $\frac{4}{14}$

17. a) If the number selected has to be both greater than 10 and even, only four numbers in the sample space of 19 meet these criteria (12, 14, 16, and 18) so the probability is $\frac{4}{19}$.



b) P(even) = $\frac{9}{19}$, P(> 10) = $\frac{9}{19}$, so

P(even) × P(>10) = $\frac{9}{19} \times \frac{9}{19} = \frac{81}{361}$

The probability from **part a)** is $\frac{4}{19}$ or $\frac{76}{361}$, which is not equal to $\frac{81}{361}$, so the events must be dependent.

18. $\frac{1}{2}$; You solve the equation
0.3 = 0.6 × P(second blue|first blue)

19. a) $\frac{1}{3}$

b) $\frac{5}{36}$

c) No, since $\frac{5}{36} \times \frac{1}{3} \neq \frac{1}{36}$ (the probability that you roll 4, 2)

UNIT 6 Data, Statistics, and Probability Test

1. A stem and leaf plot, a histogram, and a box plot can all be used to show how data values are distributed. For each graph, describe one aspect that makes it unique.

2. The data below shows the ages of the siblings of the students in Dorji's class.

6	9	13	18	21	14
26	30	24	26	11	7
14	16	10	23	26	20
10	17	21	22	24	24
14	22	5	25	12	23
21	20	17	18	11	9

a) Create a stem and leaf plot of the data.

b) Determine the mean, median, and mode.

c) Make three conclusions based on the data.

3. a) Use the data in **question 2** to create a frequency table with a bin width of 5.

b) Create a histogram.

c) Make three conclusions based on the data.

4. The following data set shows the class sizes at the start and at the end of the school year in Yuden's class.

Class Sizes

First Day of School

15	23	38	12	36
26	14	23	34	34
23	17	30	23	24
31	43	26	28	17
37	29	31	26	21

Last Day of School

20	23	35	22	36
26	25	23	27	31
20	22	35	26	29
23	36	29	33	26
28	27	23	25	21

a) Make a box plot for each data set using the same scale.

b) Use your plots to write two statements comparing class sizes on the first day and on the last day of school.

5. The frequency table shows the number of space shuttle launches over the last 25 years. Construct a histogram and use it to discuss how the data values are distributed.

Years	Number of launches
1980–1984	14
1985–1989	18
1990–1994	34
1995–1999	30
2000–2004	17

6. A manufacturer of batteries for laptop computers tested a sample of laptops to see how long the charge in the batteries would last. Here are the results of the test:

Battery life (min)	Frequency
260 – 270	2
270 – 280	3
280 – 290	5
290 – 300	7
300 – 310	5
310 – 320	4

a) Create a histogram and a frequency polygon.

b) Discuss the shape of the frequency polygon and identify the distribution.

c) State three conclusions based on the data.

d) If you buy a new laptop, how likely is it that the battery will hold its charge for

i) up to 4.5 h? **ii)** over 5 h?

7. Draw a sketch of a histogram that displays each distribution and describe a situation that could lead to each graph.

a) a positively skewed distribution

b) a uniform distribution

c) a normal distribution

d) a U-shaped distribution

8. This table shows the age of a sample of people and how many hours each person engages in physical activity each week.

Age	Hours of activity
20	15
22	11
30	6
30	7
34	6
26	14
26	8.5

Age	Hours of activity
18	16
36	3
36	6
28	11
30	9
40	3
35	4

a) Identify the independent and dependent variables. Explain how you know.

b) Create a scatter plot of the data.

c) What type of correlation is shown?

d) Estimate the value of the correlation coefficient.

e) Is a line of best fit appropriate? Explain.

8. [Cont'd] **f)** Estimate the number of hours of weekly physical activity a 25-year-old person engages in.

g) Estimate the age of a person who spends 20 hours each week on physical activity.

9. Sketch a scatter plot with each of the following correlation coefficients.

a) close to –1

b) close to 0.5

c) close to 0

10. Researchers at a School of Veterinary Medicine deposited 25 female and 10 male fleas in the fur of a cat in order to study the egg production of the fleas. This table shows the number of eggs produced by the fleas over 27 consecutive days.

Day	No. of eggs	Day	No. of eggs
1	436	15	550
2	495	16	487
3	575	17	585
4	444	18	549
5	754	19	475
6	915	20	435
7	945	21	523
8	655	22	390
9	782	23	425
10	704	24	415
11	590	25	450
12	411	26	395
13	547	27	405
14	584		

a) Create a scatter plot of the data.

b) Draw a curve of best fit.

c) Describe what happened to the egg production over the 27 days.

11. When you drive a car, the cost of petrol depends on the speed at which the car travels. The table below shows an example of this relation.

Speed (km/h)	Cost of petrol/km (Nu)
20	3.7
40	3.2
60	2.9
80	2.9
100	3.2
129	3.7

a) What type of relationship exists between speed and cost of petrol?

b) Create a scatter plot. Draw a curve of best fit.

c) Which speed is the most cost-efficient?

d) On a 600 km trip, how much money would be saved by driving at the most cost-efficient speed rather than at 100 km/h? Show your work.

12. Determine which pairs of events are dependent and explain why.

a) *Event A* Rolling a 5 on a die

Event B Rolling an even number on a die

b) *Event A* An integer from 1 to 50 is a multiple of 3

Event B An integer from 1 to 50 has a 2 as one of its digits

c) *Event A* Choosing a white chip from a bag of 10 white and 2 black chips

Event B Choosing a second white chip from the bag (if the first chip is not returned)

13. Calculate the probability of winning both of two races if the probability of winning the first race is 0.3 and the probability of winning the second race if you have already won the first race is 0.8.

UNIT 6 Test

Pacing	Materials
1 h	• Grid paper (BLM) • Rulers

Question(s)	Related Lesson(s)
1	Lessons 6.1.1 and 6.1.3
2, 3	Lesson 6.1.1
4	Lesson 6.1.3
5	Lessons 6.1.1 and 6.1.4
6, 7	Lesson 6.1.4
8, 9	Lesson 6.2.1
10, 11	Lesson 6.2.2
12	Lesson 6.3.1 and 6.3.2
13	Lesson 6.3.2

Select questions to assign depending on the time available.

Answers

1. *Sample response*:
- Stem and leaf plots make it possible to reconstruct the original data set, but histograms and box plots do not.
- You can manipulate a histogram by changing the interval sizes to give different impressions of the data.
- Box plots give a quick and easy picture of where most of the data values are located.

2. a) Ages of Siblings in Dorji's Class

Stems	Leaves
0	5 6 7 9 9
1	0 0 1 1 2 3 4 4 4 6 7 7 8 8
2	0 0 1 1 1 2 2 3 3 4 4 4 6 6 6 6
3	0

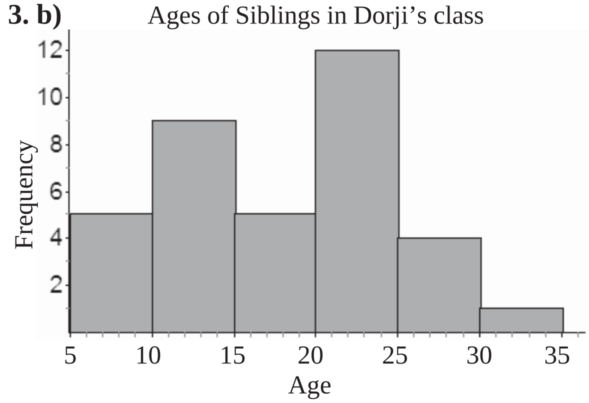
b) Mean is 17.5, median is 18, and mode is 26.

c) *Sample response*:
- Less than half the siblings are over 20 years old.
- Only $\frac{5}{36}$ or about 14% are under the age of 10.
- Only one sibling is over 26 years old.

3. a) Ages of Siblings in Dorji's Class

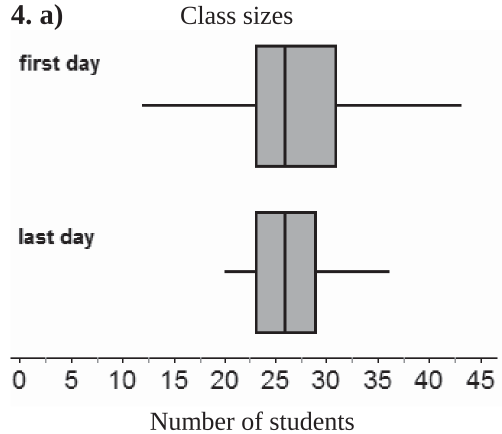
Age	Frequency
5 – 9	5
10 – 14	9
15 – 19	5
20 – 24	12
25 – 29	4
30 – 34	1

3. b) Ages of Siblings in Dorji's class



c) *Sample response*:
- The most frequent age of siblings is 20 to 24 years old.
- There are almost equal numbers of siblings in the age ranges of 5 – 9, 15 – 19, and 25 – 29.
- Only one sibling is 30 or older.

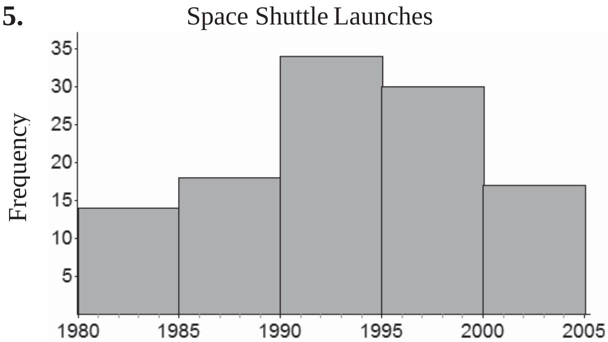
4. a) Class sizes



Answers [Continued]

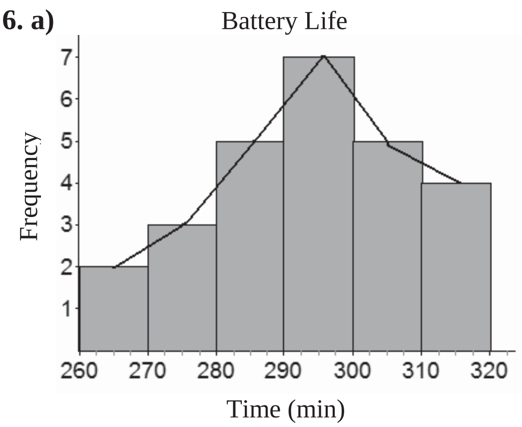
4. b)
• The median class size on the first day is about the same as the median class size on the last day of school.
• There was a wider range in class size on the first day of school compared to the last day.

5.



The histogram shows that the space shuttle was launched most often from 1990 through 1999.

6. a)



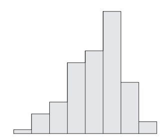
b) *Sample response*:
The frequency polygon rises then falls. It is shaped like a hill. This is close to a normal distribution.

c)
• The majority of the laptop batteries ($\frac{17}{26}$ or 65%) lasted between 280 and 310 min.
• None of the batteries held a charge for over 320 min.
• All of the batteries held a charge for at least 260 min.

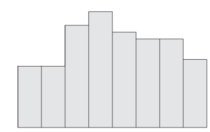
d) i) 4.5 h is 270 min. Only $\frac{2}{26}$ of the batteries did not last this long so it is likely the battery will last up to 4.5 h.

6. d) ii) 5 h is 300 min. Only $\frac{9}{26}$ batteries of the lasted over 300 min. It is unlikely the battery will last over 300 min.

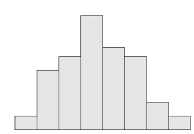
7. *Sample responses*:



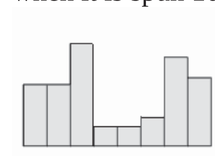
a) The heights of the players on a basketball team.



b) The number of times a spinner divided into eighths labelled 1 to 8 lands on each number when it is spun 100 times.



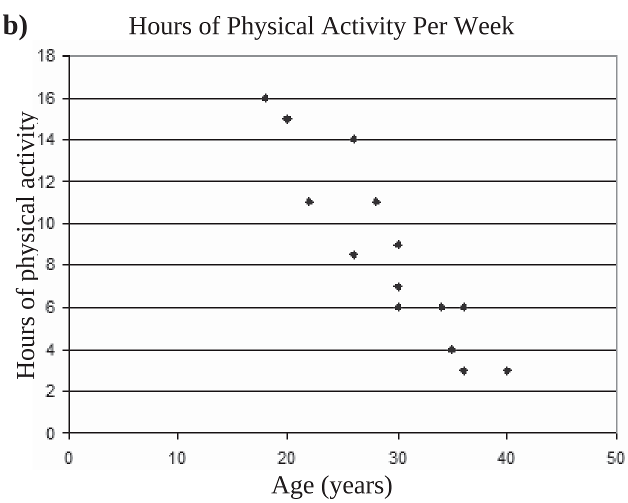
c) The age of all the residents in my village.



d) The heights of the adults in my village.

8. a) The independent variable is age and the dependent variable is hours of activity per week. The independent variable is usually listed in the left column of a table while the dependent variable is listed on the right.
It also appears from the data that the time spent in physical activity depends on your age.

b)



c) Negative correlation

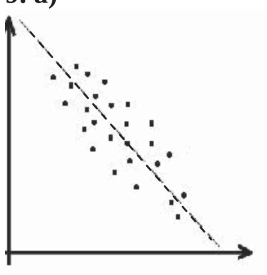
8. d) Close to –1

e) Yes; There appears to be a linear trend and the data values are continuous.

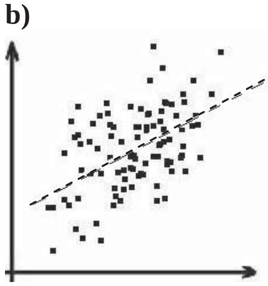
f) About 11 h/week

g) About 10 years old

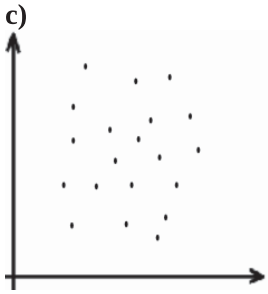
9. a)



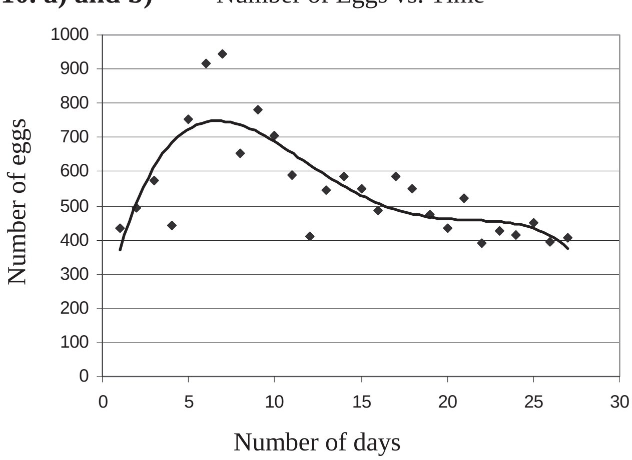
b)



c)



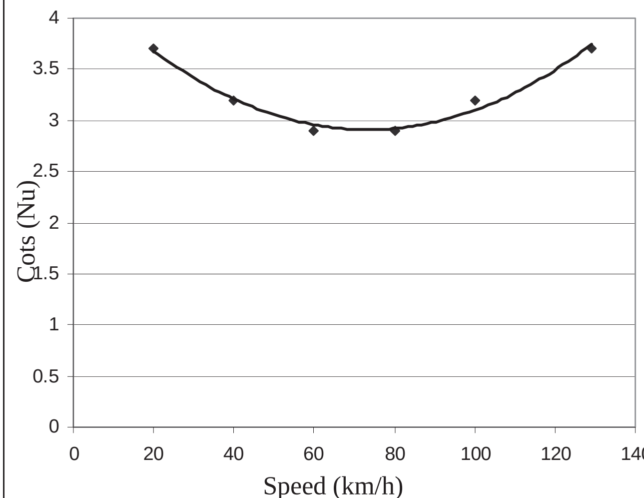
10. a) and b) Number of Eggs vs. Time



10. c) In the first few days the number of eggs increased quickly, then at about day 7 the number of eggs declined. It almost levelled off on about the 22nd day.

11. a) Non-linear, quadratic

b) Cost of Petrol (Nu) vs. Speed



c) 70 km/h

d) Nu 240;
Cost at 100 km/h = 600 × 3.2 = 1920 Nu;
cost at 70 km/h = 600 × 2.8 = 1680 Nu;
the difference is Nu 240.

12. a) Dependent: if you roll a 5, then rolling an even number is impossible. Since Event A affects the probability of Event B, the events are dependent.

b) Dependent; P(A) = $\frac{16}{50}$, P(B) = $\frac{14}{50}$, and

P(A and B) = $\frac{5}{50}$ which does not equal P(A) × P(B).

c) Dependent; not replacing the first white chip chosen makes it less likely to choose another white chip on the second selection, so the events are dependent.

13. 0.24

UNIT 6 Performance Task — Health Benefits of Fruits and Nuts

People around the world have become more health-conscious over the last 25 years. They have become better educated about the benefits of healthy eating and an active lifestyle. Fruits and nuts are a very important part of a healthy diet. The chart below shows the amount of carbohydrates and dietary fibre in a variety of fruits and nuts as well as in some common North American snack foods. Nutritionists and doctors have found that a healthy diet tends to be low in carbohydrates and high in dietary fibre.

How do fruits and nuts compare to snack foods with respect to health benefits?

Nutritional Information for Selected Fruits and Nuts (per serving)

Fruits and Nuts	Carbo-hydrates (g)	Dietary fibre (g)
Apples	22	5
Almonds	20	3
Bananas	29	4
Blueberries	27	3
Cashews	28	1
Cherries	22	3
Grapefruit	16	6
Grapes	24	1
Hazelnuts	17	4
Kiwi Fruit	24	4
Lemons	5	1
Limes	7	2
Mangos	17	1
Nectarines	16	2
Oranges	21	7
Peaches	10	2
Peanuts	21	5
Plums	19	2
Strawberries	12	4
Tangerines	15	3
Walnuts	14	5

Nutritional Information for Selected Snack Foods (per serving)

Snack Foods	Carbo-hydrates (g)	Dietary fibre (g)
Chocolate bars	31	2
Chocolate cookies	10	1
Crackers	2	0
Cup cakes	36	1
Noodles	42	2
Tortilla chips	11	2
Granola bars	16	2
Molasses cookies	11	0
Pie	42	2
Popcorn	10	4
Chocolate Bars (nuts)	35	1.7
Doughnuts	23	0.8
Muffins	10	0.2
Sugar cookies	10	0.2
Fig Bars	11	0.7
Pretzels	4	0.1
Potato chips	10	0.8
Brownies	36	1.7
Fudge	14	0.3
Pizza	21	0.3
Yogurt	12	0

A. Work with the carbohydrate data for both groups of foods.

i) Create a double stem and leaf plot comparing the fruit and nut data with the snack food data. State at least two observations about the data.

ii) Create two box plots using the same scale, one for the fruit and nut data, and one for the snack food data. State at least two observations about the data.

iii) Create two histograms using the same scale you used for the box plots, one for the fruit and nut data, and one for the snack food data. State at least two observations about the data.

iv) What can you conclude about the health benefits of eating fruits and nuts compared to eating snack foods with respect to carbohydrate content?

B. Repeat **part A** using the dietary fibre data.

C. What advice would you give a friend about his or her consumption of fruits and nuts compared to snack foods? Support your advice with evidence from the data.

UNIT 6 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-F4 Data Analysis: distribution of data **10-F5** Displaying Data: construct and interpret	1 h	• Grid paper (BLM) • Rulers

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of some of the outcomes in this unit. You may wish to use only one of **Part A or B**. It could supplement the unit test. It could also be used as enrichment material for some students. You can assess performance on the task using the rubric below. Make sure students have reviewed the rubric before beginning work on the task.

Sample Solution

A. i)

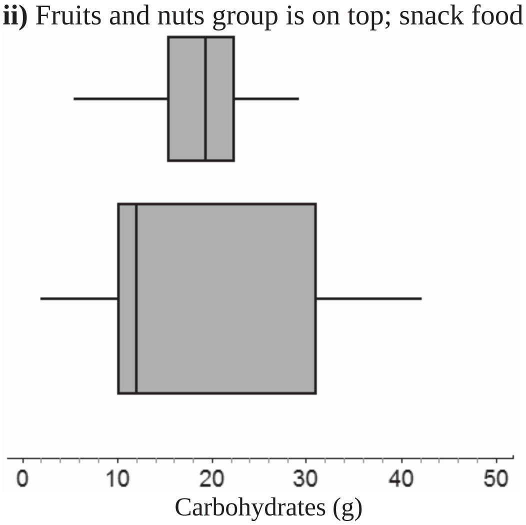
Carbohydrates in

Fruits and Nuts		Snack Foods
Leaves	Stems	Leaves
7 5	0	2 4
9 7 7 6 6 5 4 2 0	1	0 0 0 0 0 1 1 1 2 4 6
9 8 7 4 4 2 2 1 1 0	2	1 3
	3	1 5 6 6
	4	2 2

Sample response:

- Some of the snack foods ($\frac{6}{21}$) contain more carbohydrates than any of the fruits and nuts.
- There were about equal numbers from both food groups (13 vs. 11) that contained between 0 g and 20 g of carbohydrates.

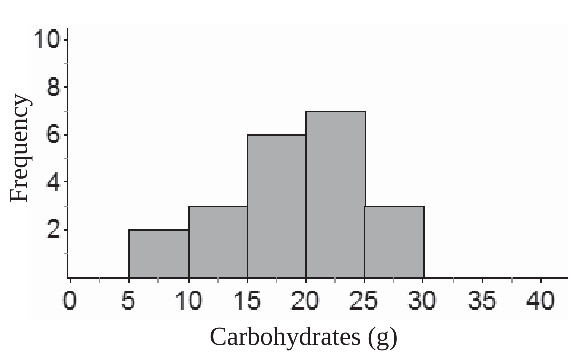
ii) Fruits and nuts group is on top; snack food group is on the bottom



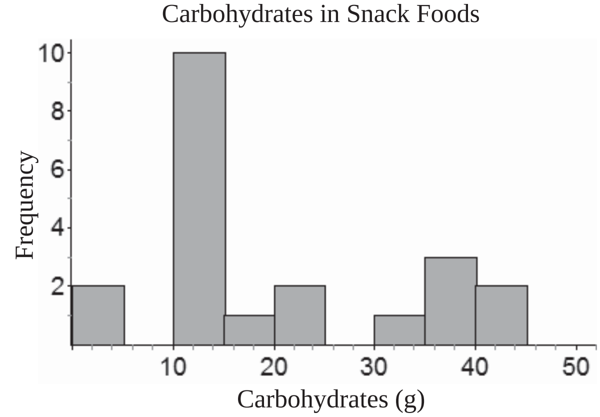
Sample response:
- The data values for snack foods are much more variable than the values for fruits and nuts. The snack food data set has a greater range.
- Generally, fruits and nuts have more carbohydrates than snack foods, but there are some snack foods that have a lot more carbohydrates than fruits and nuts.

iii) *Sample response*:

Carbohydrates in Fruits and Nuts



Carbohydrates in Snack Foods



Sample Solution [Continued]

Sample response:

- 6 out of 21, or 29% of the snack foods have more than 30 g of carbohydrates. 100% of the fruits and nuts have less than 30 g of carbohydrates.
- No fruits and nuts have as many grams of carbohydrates as some snack foods do.

iv) *Sample response*:

Since a healthy diet consists of foods that tend to be lower in carbohydrates, there are many snack foods that you should avoid because they are very high in carbohydrates. All the fruits and nuts and many of the snack foods are probably healthy choices.

B. i) Dietary Fibre in

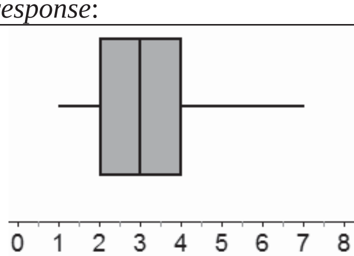
Fruits and Nuts / Snack Foods

Leaves	Stems	Leaves
	0	0 0 0 1 2 2 3 3 7 8 8
0 0 0 0	1	0 0 7 7
0 0 0 0	2	0 0 0 0 0
0 0 0 0	3	
0 0 0 0	4	0
0 0 0	5	
0	6	
0	7	

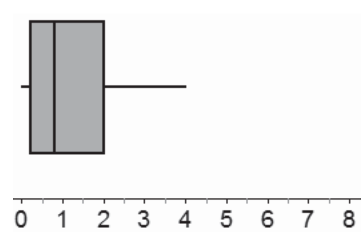
Sample response:

- Fruits and nuts are generally higher in dietary fibre compared to many snack foods.
- The dietary fibre for snack foods is 2 g or less for almost all snack foods, but it is likely to be more than 2 g for fruits and nuts.

ii) *Sample response*:



Dietary fibre (g) in fruits and nuts

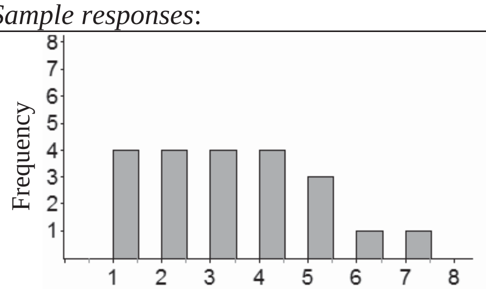


Dietary fibre (g) in snack foods

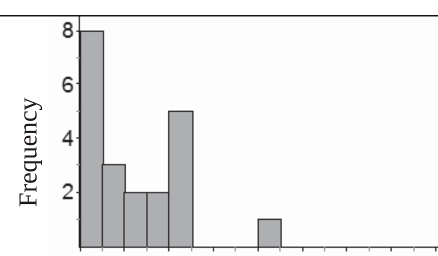
Sample response

- For the data on fruits and nuts, the median amount of dietary fibre is 3 g/serving. Over 40% of the fruits and nuts contain at least 4 g of fibre.
- For the snack food data, the median amount of dietary fibre is 0.8 g/serving. About 75% of the snack foods contain less than 2 g of fibre.

iii) *Sample responses*:



Dietary fibre (g) in fruits and nuts



Dietary fibre (g) in snack foods

Sample response:

- There is a broader range of amounts of dietary fibre in fruits and nuts than in snacks.
- Most snack foods contain very little dietary fibre.

iv) The median amount of dietary fibre per serving is much higher in the fruits and nuts group compared with the snack food group. Most of the snack foods contain very little dietary fibre. Since snack foods are lower in dietary fibre compared to fruit and nuts, it is possible that they are not as healthy choices as fruits and nuts.

C. All the graphs showed that there were some snack foods that were high in carbohydrates while none of the fruits and nuts were high in carbohydrates. All the graphs showed that most of the fruits and nuts were higher in dietary fibre than most of the snack foods. Since a healthy diet consists of foods that tend to be low in carbohydrates and high in dietary fibre, I would recommend that my friend eat more fruits and nuts than snack foods on a daily basis and avoid the snack foods that are high in carbohydrates or eat them in small quantities.

UNIT 6 Performance Task Assessment Rubric

	Level 4	**Level 3**	**Level 2**	**Level 1**
Creating the Appropriate Graphs for Data Involving One Variable	Organizes and displays the data in an effective manner, appropriate to the type of graph asked for.	Organizes and displays the data with considerable success, appropriate to the type of graph asked for.	Organizes and displays the data with some success, appropriate to the type of graph asked for.	Has difficulty organizing and displaying the data, and using the appropriate type of graph.
Making Observations from the Graphs	Provides detailed, logical and accurate observations.	Provides logical and accurate observations with considerable detail.	Provides some observations with little detail.	Provides limited or inaccurate observations
Making Conclusions	- Provides thorough, clear, and insightful explanations/ justification using a range of words, pictures, symbols, and/or numbers when drawing conclusions. - Uses graphs effectively to interpret similarities and differences.	- Provides complete, clear, and logical explanations/ justification using appropriate words, pictures, symbols, and/or numbers when drawing conclusions. - Uses graphs to interpret similarities and differences with considerable success.	- Provides partial explanations/ justification using simple words, pictures, symbols, and/or numbers when drawing conclusions. - Uses graphs to interpret similarities and differences with some success.	- Provides limited or inaccurate explanations/ justification using minimal words, pictures, symbols, and/or numbers when drawing conclusions. - Uses graphs to interpret similarities and differences with limited success.

UNIT 6 Blackline Master 1

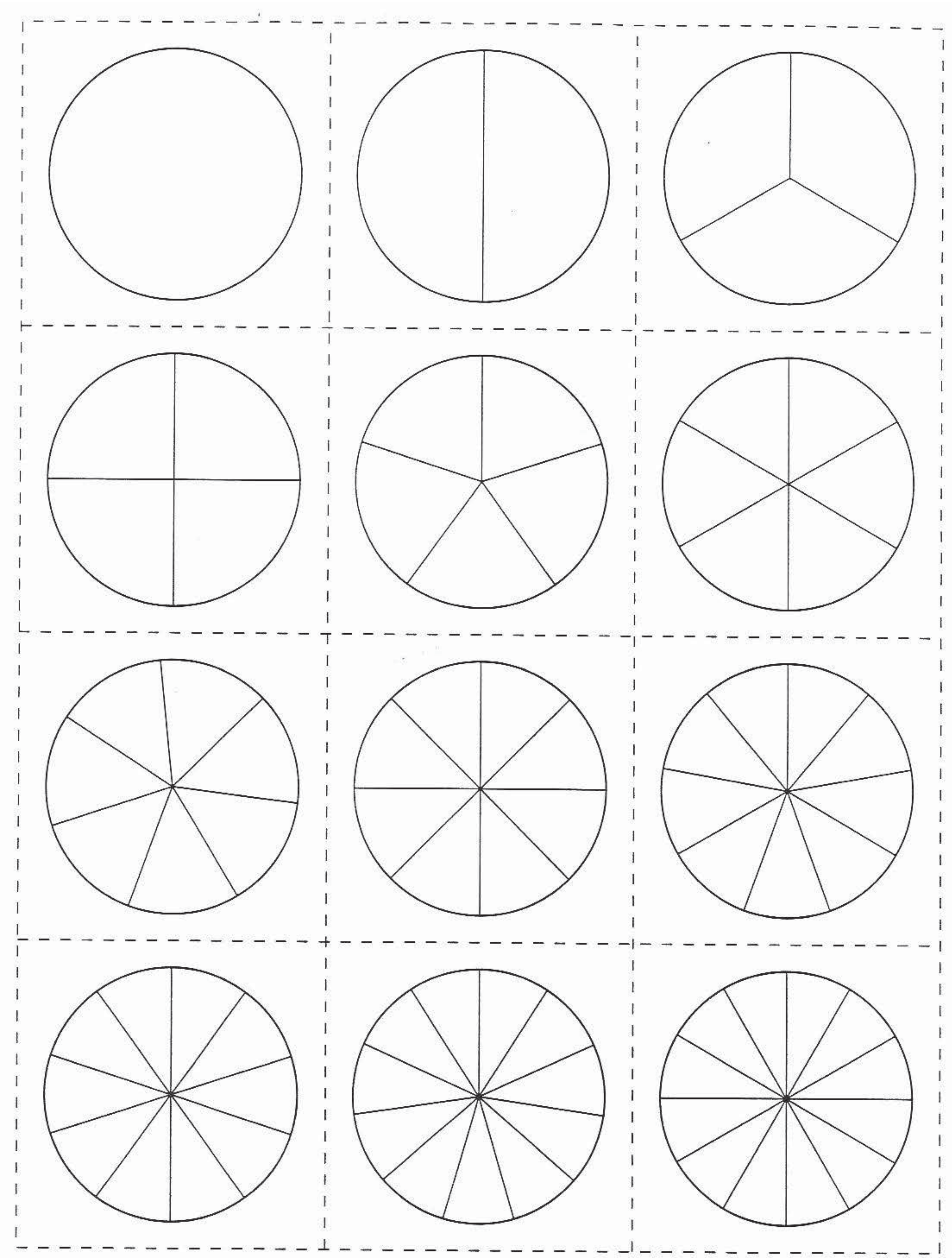
Grid Paper (0.5 cm by 0.5 cm)

UNIT 6 Blackline Master 2

Grid Paper (1 cm by 1 cm)

UNIT 6 Blackline Master 3

Fraction Circle Spinners



UNIT 7 TRIGONOMETRY

UNIT 7 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 227** **TG p. 243**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Grid paper • Rulers • Protractors • Compasses • Calculators	All questions
Chapter 1 Defining Trigonometric Ratios				
7.1.1 Using Similarity Properties to Solve Problems **SB p. 229** **TG p. 245**	**10-D6 Similar Triangles and Right Triangles: solve problems** • solve problems using the proportionality relationship among sides in similar triangles • apply the Pythagorean theorem in appropriate situations	1 h to 2 h	• Calculators	Q1, 3, 8, and 9
7.1.2 EXPLORE: Special Ratios in Similar Triangles (Optional) **SB p. 235** **TG p. 247**	**10-D5 Similar Triangles: apply properties** • apply side and angle relationships when developing the primary trig ratios **10-D7 Trigonometric Functions: relate to ratios in similar right triangles** • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles	1 h	• Rulers • Protractors • Calculators	Observe and assess questions
CONNECTIONS: Using a Clinometer **SB p. 237** **TG p. 249**	Explore a real world application of similar triangles	2 h	• Drinking straws • Grid paper • Cardboard • Glue • String • Weight	N/A
7.1.3 The Sine, Cosine, and Tangent Ratios **SB p. 238** **TG p. 250**	**10-D5 Similar Triangles: apply properties** • apply side and angle relationships when developing the primary trig ratios **10-D7 Trigonometric Functions: relate to ratios in similar right triangles** • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles [Cont'd]	2 h	• Calculators	Q1, 2, 5, and 7

UNIT 7 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
7.1.3 The Sine, Cosine, and Tangent Ratios [Cont'd]	**10-D11 Trigonometric Values: use calculators** • use calculators to determine the trig ratios sin θ, cos θ, and tan θ			
7.1.4 Trigonometric Identities **SB p. 243** **TG p. 253**	**10-D7 Trigonometric Functions: relate to ratios in similar right triangles** • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles • relate reciprocal ratios to primary trig ratios **10-D9 Trigonometric Identities** • understand what identities are • test statements to see if they are identities • understand why $\sin x = \cos (90 - x)$ • understand why $\sin^2 x + \cos^2 x = 1$ **10-D10 Trigonometric Values: special angles** • use the Pythagorean theorem to determine exact values for the sine, cosine, and tangent of 30°, 60°, and 45° angles	1 h	• Calculators	Q1, 2, 4, 6 and 9
Chapter 2 Applying Trigonometric Ratios				
7.2.1 Calculating Side Lengths and Angles **SB p. 249** **TG p. 256**	**10-D11 Trigonometric Values: use calculators** • use calculators to determine the trig ratios sin θ, cos θ, and tan θ **10-D13 Trigonometric Ratios: solve problems** • calculate side lengths and angles using trig ratios (use of calculators is required)	1 h to 2 h	• Calculators • Grid paper	Q1, 3, 4, and 9
7.2.2 Angles of Elevation and Angles of Depression **SB p. 252** **TG p. 259**	**10-D12 Trigonometric Values: right triangles — apply to solve problems** • explore angles of elevation (measured from the horizon up) and angles of depression (measured from the horizon down) in real world settings **10-D13 Trigonometric Ratios: solve problems** • calculate side lengths and angles using trig ratios (use of calculators is required)	1 h	• Calculators	Q1, 6, and 7

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
7.2.3 Areas of Polygons **SB p. 255** **TG p. 262**	**10-D12 Trigonometric Values: right triangles — apply to solve problems** • find areas of polygons using right triangle trigonometry **10-D5 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • apply formulas for area in a variety of contexts • develop non-routine formulas to determine area • understand that areas of regular polygons can be determined by dividing the area into familiar shapes	1 h	• Calculators	Q1, 2, and 4
GAME: Race to Five **SB p. 259** **TG p. 264**	Apply and practise trig ratio skills in a game situation	30 min	• Dice • Calculators	N/A
7.2.4 Vectors and Bearings **SB p. 260** **TG p. 265**	**10-D14 Vectors and Bearings: solve problems** • solve bearing and vector problems using the Pythagorean theorem and/or trigonometric ratios	2 h	• Rulers • Protractors • Calculators	Q1, 2, and 4
CONNECTIONS: Relating Trigonometric Ratios to Circles **SB p. 265** **TG p. 268**	Explore a mathematical application of trig ratios	20 min to 30 min	• Rulers • Calculators • Protractors	N/A
UNIT 7 Revision **SB p. 266** **TG p. 269**	Review the concepts and skills in the unit	1 h	• Rulers • Compasses (optional) • Protractors • Calculators	All questions
UNIT 7 Test **TG p. 270**	Assess the concepts and skills in the unit	1 h	• Rulers • Compasses (optional) • Protractors • Calculators	All questions
UNIT 7 Performance Task **TG p. 272**	Assess concepts and skills in the unit	1 h	• Calculators • Protractors • Rulers	Rubric provided

Math Background

• Trigonometry is a branch of geometry that focuses on computation. Students use some measurements of triangles to deduce other measurements. Trigonometric applications require an understanding of the concept of similarity and its application, as well as an ability to use the Pythagorean theorem.

• A review of the relationships among similar triangles and applications of those relationships is developed in the **Getting Started**, prior to the introduction of the trigonometric ratios.

• Through the **Explore** feature, students can observe that in a right triangle with a given acute angle, the ratios of the lengths of each leg to the hypotenuse and the ratio of the two legs are predictable. This exploration provides an informal introduction to core principles associated with trigonometry involving right angled triangles. (Explore **lesson 7.1.2** is optional, so formal trigonometric language is not used.) The ratios are formally defined in **lesson 7.1.3** as the sine, cosine, and tangent of the angle. The reciprocal ratios, secant, cosecant, and cotangent are then introduced in **lesson 7.1.4**. The use of the sine law in non-right triangles follows in Class XI. The sines, cosines, and tangents of angles greater than 90° are also discussed in future work.

• Students apply trigonometric or trig ratios in a variety of settings including the determination of areas of polygons as well as work with vectors and bearings.

• As students proceed through this unit, they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- Students use problem solving in **question 8** of **lesson 7.2.1**, where develop a strategy to solve a problem. They use the strategy of drawing a picture to help them solve problems in **question 9** of **lesson 7.2.1** and **questions 1 and 6** of **lesson 7.2.2.**

- They use communication frequently as they explain their thinking, for example, **question 9** of **lesson 7.1.1** and **question 7** of **lesson 7.1.3**. You might notice that the last question in most lessons requires an element of communication.

- They make connections in each **Try This** problem, where the new learning is connected to a problem they can already solve. There are also specific connections made. For example, **questions 4** of **lesson 7.1.4** help students connect the value of tan 60° to what they know about equilateral or right angled triangles by drawing a picture. There are real-world connections to the slopes of ramps and ladders in **questions 4 and 9** of **lesson 7.2.1** and connections to algebra in **question 7** of the same lesson.

- Students use reasoning in answering questions such as **question 8** of **lesson 7.1.4,** where they examine the truth of certain statements.

- They represent trigonometric ratios visually in the final **Connections** feature of the chapter and they use visualization as they estimate angle measures or trig ratios in **question 5** of **lesson 7.1.3** and **question 1** of **lesson 7.2.4**. Students also use visualization in the **Try This** in **lesson 7.2.3**.

Rationale for Teaching Approach

• The unit is divided into two chapters. **Chapter 1** allows students to recall their work with similarity of shapes, particularly triangles as an introduction to the trigonometric ratios. Trigonometric ratios are then developed and introduced to students in **Chapter 2**. **Chapter 2** then focuses on applying the ratios to solve a variety of problems.

• Students observe some relationships between trigonometric quantities in the form of trig identities. Formally, they are introduced to $\cos x = \sin (90 - x)$ and $\cos^2 x + \sin^2 x = 1$; incidentally they work with other identities such as $\sin x = (\tan x)(\cos x)$.

• There is one **Explore** feature, which introduces the trigonometric ratios informally as discussed earlier.

• There are two **Connections** features. The first one focuses attention on a device used as an interesting real-world application of trigonometry. The second one allows students to see the connection between the geometric and the numerical meanings of the sine, cosine, and tangent ratios.

• The **Game** encourages students to estimate trigonometric ratios.

Technology in This Unit

It is expected that students will have access to a calculator for determining the sine, cosine, or tangent given an angle or the angle given one of the ratios. A trig ratio table is provided at the end of the student book unit in case a calculator is not available.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Similar Triangles: understand and apply properties **9** Similar Triangles: apply properties **8** Proportion: solve indirect measurement problems **8** Angle Pair Relationships: parallel and non-parallel lines **6** Area (of a triangle): relate to area of a parallelogram **7** Angles: estimate and measure using a protractor **7** Angles: sum **8** Pythagorean Relationship: application	Students will need to recall what they know about similarity as well as some skills in solving proportions, using the Pythagorean theorem, angle relationships, and area formulas.

Pacing	Materials	Prerequisites
1 h	• Grid paper (BLM in Unit 6) • Rulers • Compasses • Protractors • Calculators	• definition of similar triangles • solving proportions • facility with a protractor, ruler, and compass for geometric constructions and measurements • properties of triangles (angles sum to 180º) • formula for area of triangle • familiarity with the properties of parallel lines and transversals

Main Points to be Raised

• If one triangle is similar to another one, the side lengths of the first triangle are all multiplied by the same scale factor to determine the side lengths of the second triangle.

• The sum of the angles in a triangle is 180°.

• The area of a triangle is half the product of the base and height.

• The third side of a right triangle can be calculated using the lengths of the other two sides.

Use What You Know — Introducing the Unit

• You may need to remind students that when triangles are similar, the angles are the same and the proportions of the side lengths are identical. In other words, all sides are multiplied by the same scale factor.

• In this particular case, you may need to draw students' attention to the fact that the leg lengths need to be integers or the vertices will not be at intersection points on the grid.

Observe students as they work. You might ask:

• *How do you know the base cannot be 5 units long?* (The base is 3 units long now. The base of the similar triangle has to be a multiple of 3 or a factor of 3 units long. Otherwise, the corners will not touch the intersection points. 5 is neither a multiple nor a factor of 3.)

• *Why is there no smaller similar triangle that fits on the grid?* (The base would have to be 1 unit long since 1 is the only other factor of 3. If the base were 1, the height would have to be 0.67 so the proportions within the triangles are the same. In that case, the top vertex would not be on a grid point.)

• *If you turn your triangle 90°, will it still be similar?* (Yes. All that matters are the angle sizes and side lengths, not the position of the triangle.)

Skills You Will Need

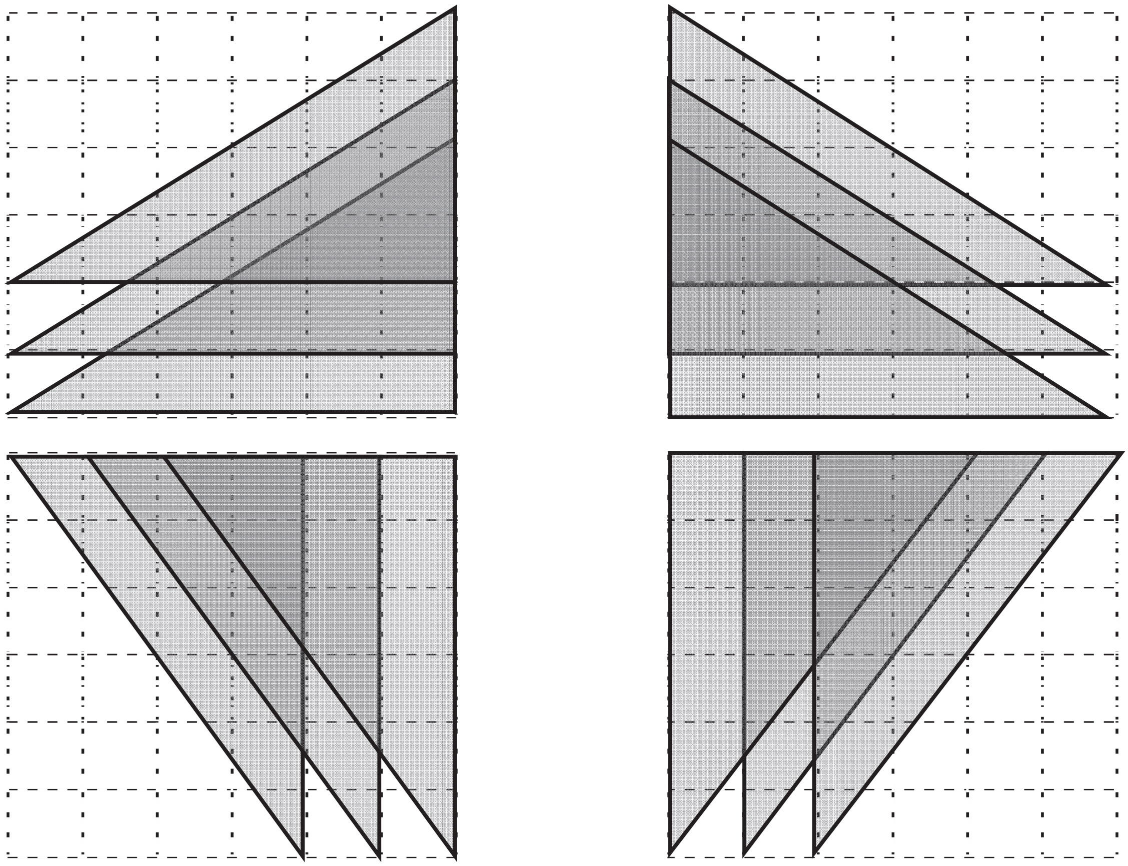
• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually.

Answers

In this and every lesson in this unit, there may be discrepancies between student responses and answers given in the answer guide due to rounding issues. Point out to students that any measurement is only an estimate, and thus any calculations using measurements are approximations. If students' answers are close to the answers in the guide and their work resembles the work shown in the answer guide, then they should be satisfied with their accuracy.

B.

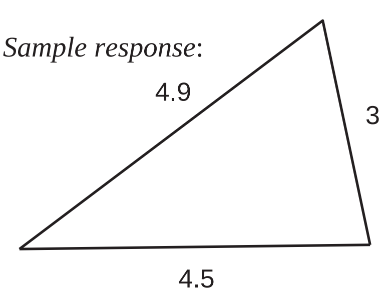


C. Each triangle is a right triangle with corresponding sides in the same proportion as the original triangle.
D. Because one side has to be 6 units long and that goes all across or all up and down the 6-by-6 grid. Every such possibility was shown.

1. a) 7.5 **b)** 22.5 **c)** 2.333 **d)** 1.1

2. a) 60° **b)** 120°

3. a) 4.15 **b)** *Sample response*:



4. a) 10 cm **b)** 4.36 cm

5. a) 1140 mm^2 or 11.4 cm^2 **b)** 5.6 cm^2

6. a and c, b and d; they are corresponding angles on parallel lines

Supporting Students

Struggling students

If students are not comfortable with the concept of similarity, show them one similar triangle with legs of lengths 6 and 4. Ask them what the side lengths of the first triangle were multiplied by to get the new triangle. Have them measure the angles to see that they are equal to the original angles.

Chapter 1 Trigonometric Ratios

7.1.1 Using Similarity Properties to Solve Problems

Curriculum Outcomes	Outcome relevance
10-D6 Similar Triangles and Right Triangles: solve problems • solve problems using the proportionality relationship among sides in similar triangles • apply the Pythagorean theorem in appropriate situations	Students can see the importance of ratios for solving measurement problems. This lesson is foundational in terms of laying groundwork necessary for the subsequent development of work with trigonometric ratios featured in the unit.

Pacing	Materials	Prerequisites
1-2 h	• Calculators	• solving proportions • Pythagorean theorem • dilatations

Main Points to be Raised

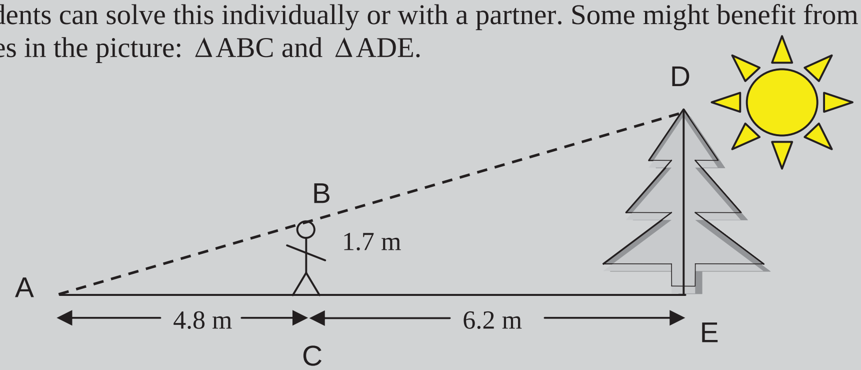
• You can set up and solve proportions involving the ratios of side lengths of similar triangles to calculate unknown lengths.

• Sometimes you need to use the Pythagorean theorem to set up and use appropriate proportions to solve for missing side lengths in similar right triangles.

• Sometimes the triangles are not provided, and the problem solver needs to use some ingenuity to create similar triangles that will help him or her solve a measurement problem.

Try This — Introducing the Lesson

A. Students can solve this individually or with a partner. Some might benefit from labelling the two nested triangles in the picture: ΔABC and ΔADE.



Observe while students work. You might ask:

• *What is the length of AE?* (4.8 + 6.2 = 11 m)

• *Why do you need to know the length of AE?* (to set up the proportion $\frac{1.7}{4.8} = \frac{DE}{AE}$)

• *Why are the triangles similar?* (Because they share one angle and each has a 90° angle, so the other corresponding angles must also be equal.)

The Exposition — Presenting the Main Ideas

• Work through the exposition with the students. Answer any questions they may have.

• Students could also find TU by solving the proportion $\frac{GA}{XU} = \frac{BA}{TU}$. Then they could calculate *s* by using the Pythagorean theorem.

• On the board, draw a figure like the one of the river shown at the end of the exposition. Help students see not only that the triangles are similar, but why it was important to place B in that place in order to create a right triangle. Make sure students understand that the 10 paces and 1 pace were arbitrary; any values could have been used. Whatever values are used, the person has to walk far enough from D to E to be able to sight both point C and point A in one line. Otherwise the similar triangles cannot be formed.

Revisiting the Try This

B. Students are encouraged to see how the shadow problem in **part A** provided an opportunity to use what was learned in the exposition.

Using the Examples

• Write the two problems used in **examples 1 and 2** on the board. Invite students to try them with their books closed. Once they have completed their own solutions, ask students to compare their solutions with the solutions shown in the worked examples.

• Provide time for students to ask questions if anything in the examples is not clear.

Practising and Applying

Teaching points and tips

Q 1c: Students can be reminded about the relationship between angles formed by parallel lines cut by a transversal. Since these angles are equal, the triangles are similar by AAA.

Q 4: Students may need to be reminded that the angle of incidence equals the angle of reflection, although the marks in the diagram should make that clear.

Q 7: Some students may find this problem difficult. They have to realize that values that are factors of 24 need to be scaled up appropriately to solve the problem.

Q 9: This closure question will allow students to synthesize what they have learned about applying similarity.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can apply similarity to calculate a missing side length in a non-contextual situation
Question 3	to see if students can apply similarity to solve a measurement problem when the triangles are not overlapping and in different orientations
Question 8	to see if students can apply similarity to solve a measurement problem when the triangles are nested
Question 9	to assess students' ability to communicate about using similarity to solve problems

Answers

A. 3.9 m

B. i) One is the triangle with vertices at the top of the tree, at the base of the tree, and at the end of the shadow. The other is the triangle with vertices at the head of the person, at the foot of the person, and at the end of the shadow.

ii) $\frac{h}{6.2+4.8} = \frac{1.7}{4.8}$

1. a) 3.6 units	**b)** 3.5 units	**c)** 15 cm	**8.** 166.67 m
2. 1.67 units	**3.** 43.2 m	**4.** 5.35 units	**9.** Since you know that the sides of similar triangles are proportional, if you know the dimensions of one triangle that might be easier to measure, it can give you information about the other triangle that is harder to measure.
5. 79.69 m	**6.** 8.63 m		
7. 24-32-40, 18-24-30, 10-24-26			

Supporting Students

Struggling students

You might encourage students to colour code matching side lengths to help them set up proportions correctly. Students could also sketch triangles such as the ones in **example 2** so that they are in the same orientation.

Enrichment

The use of a mirror as an indirect measuring tool, as depicted in **question 4**, would be an interesting approach for students to try out.

7.1.2 EXPLORE: Special Ratios in Similar Triangles

Curriculum Outcomes	Lesson relevance
10-D5 Similar Triangles: apply properties • apply side and angle relationships when developing the primary trig ratios **10-D7 Trigonometric Functions: relate to ratios in similar right triangles** • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles	Although this lesson is optional, before students work formally with the trigonometric ratios, it is helpful if they explore them informally. This work introduces the concept that it is reasonable to use the same trigonometric ratio no matter what the side lengths of the triangle, so long as the angle is known. (The formal language of opposite and adjacent side is introduced in the next lesson.)

Pacing	Materials	Prerequisites
1 h	• Rulers • Calculators • Protractors	• facility with a protractor and ruler for geometric constructions and measurements

Main Points to be Raised

• For any given acute angle in a right triangle, the following ratios are fixed:

- the side opposite the acute angle (vertical) to the hypotenuse

- the side adjacent the acute angle (horizontal) to the hypotenuse

- the side opposite the acute angle to the side adjacent to the acute angle

• As an angle increases from 0° to 90°, the ratio of the side opposite the acute angle to the hypotenuse increases. The ratio of the side adjacent the acute angle to the hypotenuse decreases. Hence, the ratio of the side opposite the acute angle to the side adjacent the acute angle also increases.

• The values of the ratio of opposite to hypotenuse of certain angles are identical to the values of the ratio of adjacent to hypotenuse of other angles. These ratios are equal in an isosceles right triangle.

Exploration

• Ask students to complete **parts A to D** with a partner or in a small group. Students should measure side lengths to the nearest millimetre (tenth of a centimetre) and use their calculators to calculate the ratios.

• Students can then work on **parts E** and **F.**

Observe while students work. You might ask:

• *Why might the values in that column have been slightly different? Do you think they really are different?* (They are close, so I think they are not really different. It is hard to measure precisely to the tenths place.)

• *Which side lengths increase as the angle increases in the pictures in **part H**? Which side lengths decrease?* (In the picture on the left, the horizontal side decreases and the vertical side increases in length as the angle increases, and the hypotenuse stays the same. In the picture on the right, the hypotenuse and vertical side increase in length as the angle increases, and the horizontal side stays the same length.)

• *What happens to the* $\frac{\text{horizontal}}{\text{hypotenuse}}$ *ratio as the angle increases? How does the picture on the left in **part H** show this?* (The ratio decreased since the horizontal side got shorter from the shortest to the tallest triangle while the hypotenuse did not change.)

Observe and Assess

As students are working, notice:

• Are they measuring precisely enough to detect differences in the ratios?

• Do some students realize that they do not need to calculate the values for 60° once they have found the measures for 30°?

Share and Reflect

Ask students what they notice about the ratios. Ask whether they measured all three triangles for each angle, or whether they realized that the ratios would have to be the same and only measured one triangle for each angle.

Answers

A, B, and C.

	vertical	horizontal	hypotenuse	$\frac{\text{vertical}}{\text{hypotenuse}}$	$\frac{\text{horizontal}}{\text{hypotenuse}}$	$\frac{\text{vertical}}{\text{horizontal}}$
ΔABG	1.9	1.9	2.7	0.704	0.704	1.00
ΔACF	4.1	4.1	5.8	0.707	0.707	1.00
ΔADE	5.1	5.1	7.2	0.708	0.708	1.00

D. The values in each of the last three columns are almost equal. (Note: The differences in values are due to rounding measurements. The values in each of the last three columns in the middle row are the most accurate.)

E. and F.
Sample response:
Note: There might be small errors in the hundredths place due to rounding.

Angle	30°	45°	60°
$\frac{\text{vertical}}{\text{hypotenuse}}$	0.500	0.707	0.866
$\frac{\text{horizontal}}{\text{hypotenuse}}$	0.866	0.707	0.500
$\frac{\text{vertical}}{\text{horizontal}}$	0.577	1.00	1.732

G. The triangles in each group of three are similar so the side ratios should be equal.

H. i) The triangles all have the same hypotenuse. As the angle increases, the vertical side gets bigger and the horizontal side gets smaller, so the $\frac{\text{vertical}}{\text{hypotenuse}}$ ratio would be greater and the $\frac{\text{horizontal}}{\text{hypotenuse}}$ ratio would be less.

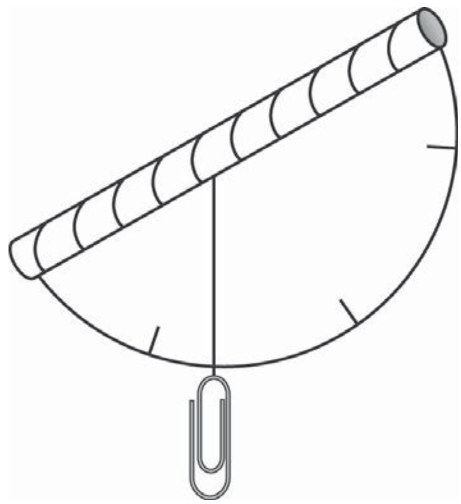
ii) The triangles are on the same base, so the horizontal value does not change. But the vertical value gets greater as the angle increases, so the $\frac{\text{vertical}}{\text{horizontal}}$ ratio would be greater.

I. The numbers in the top two rows are the same, only in reverse order.

CONNECTIONS: Using a Clinometer

There are several simple but interesting devices students can learn about for measuring indirectly. These include the clinometer. The clinometer featured here is made of simple materials and allows students to measure tall items indirectly. Students might try it out to measure a nearby building, tree, or flagpole.

An Internet search can provide information about alternate ways to create clinometers using a protractor and a tape measure.



Answers

1. Both are right triangles and they share a common angle at D, so they are similar (AAA).

2. $\frac{RO}{DO} = \frac{AX}{DX}$; $\frac{RO}{6} = \frac{9}{12}$; RO = 4.5 m; if RO is 4.5 m, then the tree is 4.5 + 1.8 (the height of the girl) = 6.3 m.

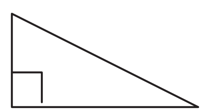
7.1.3 The Sine, Cosine, and Tangent Ratios

Curriculum Outcomes	Outcome relevance
10-D5 Similar Triangles: apply properties • apply side and angle relationships when developing the primary trig ratios **10-D7 Trigonometric Functions: relate to ratios in similar right triangles** • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles **10-D11 Trigonometric Values: use calculators** • use calculators to determine the trig ratios sin θ, cos θ, and tan θ	Students need to familiarize themselves with the trigonometric functions in right triangle situations so that they can later apply them in situations involving other types of triangles.

Pacing	Materials	Prerequisites
2 h	• Calculators	• properties of triangles (angles sum to 180º)

Main Points to be Raised

• The sides of a right triangle can be described with respect to each of the acute angles as opposite, adjacent, and hypotenuse.



The adjacent side is the leg that forms one of the arms of ∠A and the opposite side is the side that is not one of the arms of ∠A.

• The *trigonometric ratios* are defined in terms of the three sides of the right triangle:

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan x = \frac{\text{opposite}}{\text{adjacent}}$$

A mnemonic that some use to memorize these ratios is SOHCAHTOA, where SOH reminds students that sine is based on opposite and hypotenuse, CAH reminds students that cosine is based on adjacent and hypotenuse, and TOA reminds students that tangent is based on opposite and adjacent.

• We sometimes use the short form *trig ratios* to describe the trigonometric ratios.

• The size of the triangle is irrelevant for defining the trigonometric ratios since all right triangles with the same angles are similar. The trig ratios describe ratios of side lengths, which are the same for all of these triangles.

• Because the trigonometric ratios are associated with a particular angle, the trig ratios for the two acute angles in the triangle are different (other than in an isosceles triangle).

• The values of sine and cosine range from 0 to 1 for angles between 0° and 90°. The values of the sine and tangent increase as the angle increases and the value of the cosine decreases as the angle increases.

• Students can use their calculators to determine the trig ratios for various angles. They can also use their calculators to determine the angle for a given ratio. These angle values are sometimes called *arc sine*, *arc cosine*, and *arc tan* or inverse sine, inverse cosine and inverse tangent, and can be written as $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$.

Try This — Introducing the Lesson

A. Students can use the Pythagorean theorem to solve this problem. In each case the value of the length of the hypotenuse will be $s \times \sqrt{2}$**,** where s is the length of each leg. If the legs are each one unit long, the requested ratio will be $\frac{1}{\sqrt{2}}$, which is about 0.7. Alternatively, they can measure all the side lengths carefully with a ruler and calculate the ratio.

Observe while students work. You might ask:

• *Why did you draw a half-square?* (The two legs of the triangle have to be equal and that happens in a square.)

• *Why are you using the Pythagorean theorem?* (The length of the hypotenuse is the square root of the sum of the squares of the legs.)

The Exposition — Presenting the Main Ideas

• Draw a right triangle on the board to introduce the terms *adjacent side* and *opposite side* and the trig ratios: *sine*, *cosine*, and *tangent*. Help students see that the ratios are attached to a particular angle and that if a different angle is selected, the ratios are different.

• Acquaint students with the use of the calculator, first to determine the trig ratios for a certain angle and then to determine the angle for a given ratio using the shift button with the sin, cos, or tan key.

• Draw a right triangle with one small angle, e.g., 10°. Have students observe that the sine would be close to 0 since the opposite side is very small, but the cosine would be close to 1 since the adjacent side is almost as long as the hypotenuse. Then have students note that the sine of the 80° angle in the triangle is fairly large, while its cosine is fairly small.

• Make sure students understand that even if the triangle is rotated or flipped, the opposite side and the adjacent side for a particular angle do not change.

• Draw students' attention to the trig table at the end of the book. They can use this or their calculator to calculate trig values.

• To break the lesson into two days, you can introduce the lesson and then work on the examples in the textbook in groups of four on the first day. On the second day, you can ask students to read the exposition before working on the problems at the end of the lesson.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between the ratios from **part A** and the sine and cosine ratios, which, in the case of 45°, are equal.

Using the Examples

• The examples provide experience with estimating trigonometric ratios, determining an angle in a triangle when either trigonometric ratios or side lengths are given, and relating trig ratios.

• In groups of three, assign each student one example to read. Each student can then explain his or her example to the other students in the group.

• The side measures shown in the diagrams for the examples have been reduced or enlarged to fit on the page of the student book. The angles are correct.

Practising and Applying

Teaching points and tips

Q 1: Students need to apply the Pythagorean theorem to calculate two of the three ratios in each triangle. Alternatively, they can use their knowledge of Pythagorean triples.

Q 2: Allow for some latitude in student estimates. Estimates within a 20° range are reasonable.

Q 3: Encourage students to use their calculators for these questions, but to estimate first.

Q 5: Students need to use their calculators or estimate the size of one of the angles in the triangle and use the trig table to sketch the triangles for **parts a) and b)**. But for **part c)**, students only need to sketch a triangle whose height is 2.4 times its base; they do not need to calculate an angle size.

Q 6: Students need to have internalized the notion that smaller acute angles have smaller sines to answer **part a)**. For **part b)**, they need to use two examples to see that the relative sizes of the angles are not known from the given information.

Common errors

Some students mix up which side is opposite and which side is adjacent when triangles are not in standard position. Emphasize the fact that the adjacent side is actually an arm of the angle and the opposite side is not.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can use provided measurements to calculate sine, cosine, and tangent ratios
Question 2	to see if students can apply the Pythagorean theorem and then calculate the trig ratios of lengths in a triangle: sines, cosines, and tangents
Question 5	to see if students can apply what they've learned about the meanings of the ratios
Question 7	to see if students have made the connection between the concept of similarity to the concept of trig ratios

Answers

Normally, three significant digits are used to describe a trigonometric ratio. Actual measurements are provided to the level of the least precise measurement given in the problem.

A. i) The ratio is about 0.7.

ii) In a 4-4-5.66 triangle, $\frac{4}{5.66}$ = 0.71. Since all triangles with one 45° angle and a right angle are similar (AAA), the sides are proportional and it would be the same ratio in another similar triangle.

B. either the sine or cosine of 45°

1. a) sin A = $\frac{3}{5}$ = 0.60; cos A = $\frac{4}{5}$ = 0.80; tan A = $\frac{3}{4}$ = 0.75

b) sin A = $\frac{8}{17}$ = 0.47; cos A = $\frac{15}{17}$ = 0.88; tan A = $\frac{8}{15}$ = 0.53

2. a) sin A = 0.34; cos A = 0.94; tan A = 0.36

b) sin A = 0.98; cos A = 0.21; tan A = 4.72

3. a) sin 35° = 0.574; cos 35° = 0.819; tan 35° = 0.700

b) sin 55° = 0.819; cos 55° = 0.574; tan 55° = 1.429

c) sin 12° = 0.208; cos 12° = 0.978; tan 12° = 0.213

d) sin 80° = 0.984; cos 80° = 0.174; tan 80° = 5.671

4. a) 58.2° **b)** 76.1° **c)** 11.3°

5. *Sample responses*:
a) a right triangle with height of 1.4 units and hypotenuse of 2 units

b) a right triangle with a 60° angle

c) a right triangle with a base of 2 units and a height of 4.8 units

6. a) $\angle A > \angle B$; *Sample response*: This is because the values of the opposite sides increase as the angle increases.
b) No; *Sample response*: sin 75° = 0.966 and cos 85° = 0.087, and 75° < 85°. But sin 75° = 0.966 and cos 60° = 0.5 and 75° > 60°.

7. The trig ratios work for all right triangles with a particular pair of acute angles because all right triangles with those angles are similar.

Supporting Students

Struggling students

Some students might require additional experience calculating sines and cosines for drawn right triangles. By using triangles with different angles, students will get a better feel for estimates of values for these ratios.

7.1.4 Trigonometric Identities

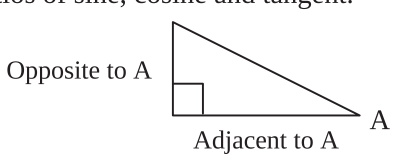
Curriculum Outcomes	Outcome relevance
10-D7 Trigonometric Functions: relate to ratios in similar right triangles • understand that primary trig ratios are equivalent for the equal angles in similar right triangles • investigate the three primary ratios between the lengths of pairs of sides in right angle triangles • relate reciprocal ratios to primary trig ratios **10-D9 Trigonometric Identities** • understand what identities are • test statements to see if they are identities • understand why $\sin x = \cos (90 - x)$ • understand why $\sin^2 x + \cos^2 x = 1$ **10-D10 Trigonometric Values: special angles** • use the Pythagorean theorem to determine exact values for the sine, cosine, and tangent of 30°, 60°, and 45° angles	Students begin to see how deductive processes allow them to use information about one trig ratio to determine others.

Pacing	Materials	Prerequisites
1 h	• Calculators	• algebraic skills for working with ratio combinations • Pythagorean theorem

Main Points to be Raised

• There are trigonometric relationships, called *identities* that are true for all values of an angle for which the ratios are defined. These include the relationship between the sine of one angle and the cosine of its complement, and the fact that $\sin^2 x + \cos^2 x = 1$.

• The reciprocal trig ratios are based on the primary trig ratios of sine, cosine and tangent.



$$\sec x = \frac{1}{\cos x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc x = \frac{1}{\sin x} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cot x = \frac{1}{\tan x} = \frac{\text{adjacent}}{\text{opposite}}$$

• The values of secant and cosecant are always 1 or more except when they are not defined. Like tangent, the values of cotangent can be greater than, less than, or equal to 1.

• Knowing any one of the trig ratios for an acute angle allows you to calculate the other five trig ratios.

• There are certain trig ratios, those related to 0°, 90°, 30°, 60°, and 45° that can be calculated exactly based on geometric properties. Students should be able to reconstruct those values. In particular they should realize that for 45°, the sine and cosine are equal and the tangent is 1, that sin 30° = cos 60° = 0.5 and that cos 0° = sin 90° = 1.

Try This—Introducing the Lesson

A. Some students will realize they could have drawn two identical triangles, since the sine of one of the acute angles is the same as the cosine of the others. Others will use their calculators or a trig table to help them create the triangles.

Observe while students work. You might ask:

• *What is the sine of the base angle? How do you know?* (For my triangle, it is $\frac{1}{3}$. I know that since I made the opposite side 1 and the hypotenuse 3.)

• *Why is the cosine of the base angle in the other triangle the same?* (Because now the adjacent side is 1, but the hypotenuse is still 3.)

The Exposition — Presenting the Main Ideas

• Draw a right triangle with sides 3, 4, and 5 on the board and review with students the definitions of sine, cosine and tangent for the base angle

• Point to the angle for which the cosine is $\frac{3}{5}$. Ask students to point out an angle for which the sine is that same value. Make sure students realize that the angles that are complements in a triangle have opposite sines and cosines.

• Draw student attention to the three sides of the triangle and ask them to indicate how the Pythagorean theorem applies. Divide both sides of the equation by the square of the hypotenuse and ask students to replace the values of $\frac{3}{5}$ and $\frac{4}{5}$ with expressions involving trigonometric ratios. Help them see why $\sin^2 x + \cos^2 x = 1$.

Inform students that there are three other trig ratios that are defined as the reciprocals of the three they know. Ask them to describe those ratios in terms of the opposite side, adjacent side, and hypotenuse. Introduce the terms secant, cosecant, and cotangent, with the appropriate abbreviations. Point out how knowing one of the reciprocal ratios automatically tells the value of one of the primary ratios, and vice versa.

• Have students read through the initial part of the exposition to confirm what was just modelled.

• Finally work through the section about the trig ratios for the special angles described on the third page of the exposition with the students.

Revisiting the Try This

B. This question allows an opportunity to make a formal connection between the identity relating sines and cosines of complementary angles with their informal exploration of this idea in **part A.**

Using the Examples

• The examples highlight the relationships between the trig ratios.

• Assign each student one example to read. Each student can then explain both solutions of his or her example to the other student.

Practising and Applying

Teaching points and tips

Q 1: Students do not need to measure. They can use the provided values and the Pythagorean theorem.

Q 2: Students might use different relationships for different ratios. For example, to calculate sine when cosine is known, they might use the fact that $\sin^2 x + \cos^2 x = 1$, but to calculate secant, they could use a definition.

Q 4: Students might draw an equilateral triangle and divide it into congruent right triangles.

Q 8: Students can approach **parts a) and b)** through visualization. They need to use algebraic skills to explain their thinking in **part d)**. **Part e)** can be solved using a counterexample.

Q 9: Students need to use a variety of relationships between trig ratios to answer this question.

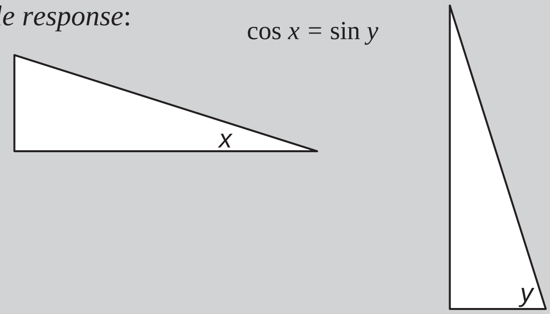
Suggested assessment questions from Practising and Applying

Question 1	to see if students can use provided measurements to calculate reciprocal trig ratios
Question 2	to see if students recognize and apply the interrelationships among the trig ratios
Question 4	to see if students can draw a picture to explain the value of a given trigonometric ratio
Question 6	to see if students can determine values using the relationship between complementary angles as well as inverse trig functions
Question 8	to see if students can reason both algebraically and geometrically about properties of the trig ratios

Answers *Note that three significant digits are used to describe most trigonometric ratios.*

A. *Sample response*:

cos x = sin y



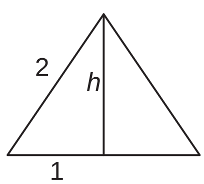
B. cos x = sin $(90 – x)$

1. a) sec x = 1.152; csc x = 2.016; cot x = 1.75
b) sec x = 1.033; csc x = 4; cot x = 3.873

2. a) cos x = 0.6; tan x = 1.333; cot x = 0.75
sec x = 1.667; csc x = 1.25
b) sin x = 0.917; tan x = 2.29; cot x = 0.436
sec x = 2.5; csc x = 1.091
c) sin x = 0.707; cos x = 0.707; cot x = 1.0
sec x = 1.414; csc x = 1.414
d) sin x = 0.980; cos x = 0.2; tan x = 4.899
cot x = 0.204; csc x = 1.021

3. a) 30° **b)** 35.5° **c)** 72.9°

4.



$h^2 = 2^2 – 1^2 = 4 – 1 = 3$, so $h = \sqrt{3}$
tan 60° = $h \div 1 = \sqrt{3} \div 1 = \sqrt{3}$

5. a) x is greater; if sec x > sec y, then cos x < cos y and that means x is a bigger angle
b) y is greater; if csc x > csc y, then sin x < sin y and $x < y$
c) x is greater; if cot x < cot y, then tan x > tan y and since the numerator is the opposite side, x is the greater angle

6. a) 53° **b)** 48° **c)** 53° **d)** 37° **e)** 39°

7. Since sin 45° = 0.707, an angle with a sine of 0.7 must be close to 45°.

8. a) True; if x < 45°, its opposite side is shorter relative to its adjacent side so the cosine, $\frac{\text{adjacent}}{\text{hypotenuse}}$, will be greater than the sine, $\frac{\text{opposite}}{\text{hypotenuse}}$.
b) False; if x < 45°, its opposite side is shorter relative to its adjacent side and the hypotenuse so the cosine, $\frac{\text{adjacent}}{\text{hypotenuse}}$, will be greater than the tangent.

c) True; sin x = (tan x)(cos x) because
tan = $\frac{\text{opposite}}{\text{adjacent}}$ and cos = $\frac{\text{adjacent}}{\text{hypotenuse}}$, so
$\frac{\text{opposite}}{\text{adjacent}} \times \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}$, which is sin.
d) False; I tried it with x = 30° and it did not work since cos 30° = 0.866, but sin 30° × tan 30° = 0.5 × 0.577 = 0.289.

9. *Sample responses*:
- The cotangent is just the reciprocal of the tangent, so cot x = 2.
- If tan x is 0.5 or $\frac{1}{2}$, I know that $\frac{\sin x}{\cos x} = \frac{1}{2}$, so cos x = 2 sin x.

Since $\sin^2 x + \cos^2 x = 1$, then $\sin^2 x + 4\sin^2 x = 1$.

Since $5\sin^2 x = 1$, $\sin x = \sqrt{0.2} = 0.447$.
- That means cos x = 2(0.447) = 0.894.
- If sin x = 0.447, then csc x = 2.236.
- If cos x = 0.894, then sec x = 1.118.

Supporting Students

Struggling students

You may choose not to assign **question 7 or 8**, which focus on reasoning. For **question 9**, you might suggest a sequencing of calculations, first getting the cotangent, then the sine or cosine, then the other and then the two other reciprocal ratios. You might have to help students go from the tangent ratio to the calculation of sine or cosine. You can further support students by encouraging them to use trig tables so they have reasonable estimates for trig ratios to compare to.

7.2.1 Calculating Side Lengths and Angles

Curriculum Outcomes	Outcome relevance
10-D11 Trigonometric Values: use calculators • use calculators to determine the trig ratios sin θ, cos θ, and tan θ **10-D13 Trigonometric Ratios: solve problems** • calculate side lengths and angles using trig ratios (use of calculators is required)	Students should become comfortable using trigonometric ratios to calculate unknown side or angle measures in fairly simple situations before they encounter more complex situations.

Pacing	Materials	Prerequisites
1-2 h	• Calculators • Grid paper (BLM in Unit 6)	• Pythagorean theorem • graphing linear relationships on a Cartesian coordinate system

Main Points to be Raised

• To determine a missing triangle measure using trig ratios, it is important to first determine which ratio would be appropriate to use.

• Usually, more than one ratio can be used to calculate an unknown measure.

• Sometimes the Pythagorean theorem must be applied to determine a measure before using a trig ratio.

• Though right angles are not always indicated explicitly, they can be assumed for certain situations (e.g., between the ground and a wall in **example 1**).

Try This — Introducing the Lesson

A. Suggest that students solve this problem independently and then talk over their solutions with a classmate. Most students will recognize quite readily that they can use sin 50° to determine *x*, but some may not realize that in this situation it makes sense to use cos 40°, and not cos 50°.

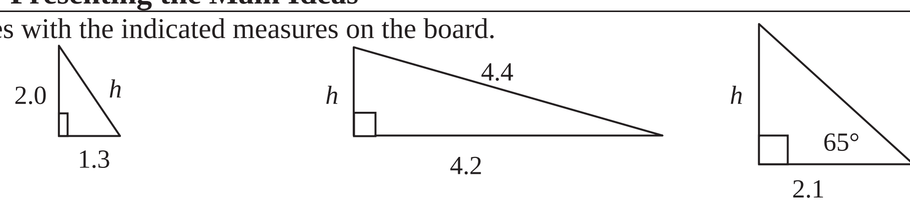
Observe while students work. You might ask:

• *What ratio describes the sine of 50°?* ($x \div 1.2$)

• *What is the other angle in the triangle? How do you know?* (40°; it is 90° – 50°)

The Exposition — Presenting the Main Ideas

• Draw these triangles with the indicated measures on the board.



• Ask students:

How could you use the Pythagorean theorem to calculate length h in the first triangle? ($1.3^2 + 2.0^2 = h^2$)

How do you know the tangent of one of the angles in the first triangle is 2.0 ÷ 1.3? Which angle? What is its value? (2.0 and 1.3 are the legs of a right triangle; the angle in bottom right corner has that tangent since 2.0 is opposite and 1.3 is adjacent; the angle is 57°.)

How could you calculate the value of h in the middle triangle using the Pythagorean theorem? ($4.4^2 – 4.2^2 = h^2$)

How could you calculate the value of h in the middle triangle using a trig ratio? (first find angle with cosine (4.2 ÷ 4.4), which is 17.3°; then $h = 4.2 \times \tan 17.3°$)

Why can you NOT use the Pythagorean theorem to calculate the value of h in the last triangle? (You only know one side and not two.)

Why would you use the tangent ratio instead of the cosine or sine ratio to calculate h in the third triangle? (I know that $\tan 65° = h \div 2.1$, so I can multiply $2.1 \times \tan 65°$ to calculate *h*.)

• Students can refer back to the exposition for support as they work through the rest of the lesson.

Revisiting the Try This

B. No; because only the hypotenuse is given in the problem, the student must calculate either the sine or cosine value first. Once another side length is known, the tangent function can be used.

Using the Examples

• For **example 2**, draw a pair of axes on the board. Remind students how to graph an equation like $y = 2x$. Have students observe the angle at which the line meets the x-axis and estimate it. Now draw a triangle connecting (0,0), (1, 0), and (1, 2). Ask students how they could use this triangle to calculate the angle. Once students make some suggestions, have them read the explanation.

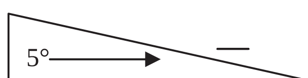
• Students can use **example 1** for reference as they work through the **Practising and Applying** exercises.

Practising and Applying

Teaching points and tips

Q 1a: Students can use what they know about angles in a triangle to calculate z without using trig ratios. They can calculate x using the cosine of 18° or sine of 72° or y using a tangent ratio.

Q 4: Students will benefit from making a quick sketch:



This will help them see how $\frac{1}{7}$ relates to the tangent of 5°.

Q 6: Students can calculate the base using the cosine of 37° and the height using either sin 37°, tan 37°, or the Pythagorean theorem.

Q 7: Students should see that lines of the same slope meet the axis at the same angle.

Q 8: Students might use the information from **example 2** to recognize that the line $y = 2x - 1$ meets the x-axis at an angle of 63.4°. Since $y = x$ meets the axis at a 45° angle, these values can be subtracted to calculate the angle of intersection.

Q 9: Students must solve two problems here: one where the angle is 60° and one where it is 75°.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can directly apply the definitions of the trigonometric ratios
Question 3	to see if students can solve for a missing angle in a triangle
Question 4	to see if students can apply their knowledge and reasoning skills in a practical situation
Question 9	to see if students can solve a more complex problem using trigonometric ratios

Answers

A. i) $\sin 50° = \frac{x}{1.2}$, so $x = 1.2 \sin 50° = 0.92$

ii) The other angle is 40° since 90° – 50° = 40°, so $\cos 40° = \frac{x}{1.2}$ and $x = 1.2 \cos 40° = 0.92$.

B. No. To calculate the tangent when you know the angle, you need at least one of the opposite or adjacent side lengths. Otherwise you end up with an equation with two unknowns, which you cannot solve: $\tan 50° = \frac{x}{?}$ or $\tan 40° = \frac{?}{x}$.

1. a) $z = 18°$; $y = 3.25$; $x = 10.51$	**4.** No. It would be closer to 8° when the base is 7. To find the base angle a, use $\tan a = \frac{1}{7}$; $a = 8.13$.
b) $y = 5.7°$; $z = 84.3°$; $x = 9.95$	
2. 3.09 m	**5.** About 9.1°
3. 65.2°	**6.** 24.03 square units

Answers [Continued]

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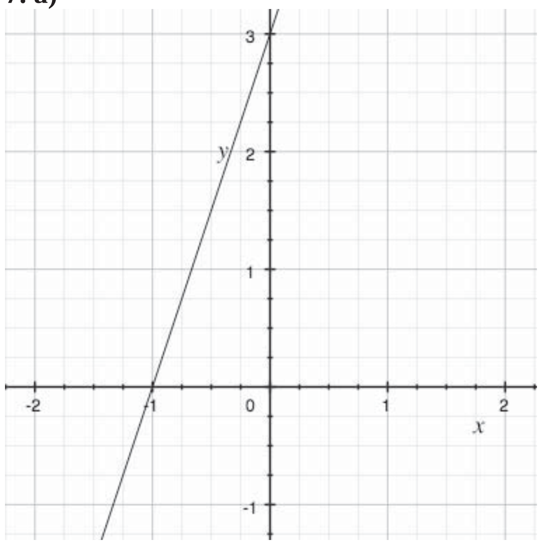
<td>7. a)


b) 71.6° c) 71.6° since the lines are parallel</td>
<td>8. 18.4°

9. a) Minimum 2.07 m and maximum 4 m

b) Closer at 75°

10. You only need one angle and one side length. If you know one acute angle, you can subtract it from 90° to find the other acute angle. If you know one acute angle and one side length, you can use sin or cos to find the other side length. Then you can use the Pythagorean theorem to find the third side length.
Or, you only need two side lengths. You can use the Pythagorean theorem to find the third side. You could use arc sin, arc cos, or arc tan to find the angles.</td>
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Supporting Students

Struggling students

Provide more examples like those in **question 1** and then more like those in **question 2** to replace some of the more complex questions, such as **questions 4~~,~~ 7, 8**, **and 9**.

7.2.2 Angles of Elevation and Angles of Depression

Curriculum Outcomes	Outcome relevance
10-D12 Trigonometric Values: right triangles — apply to solve problems • explore angles of elevation (measured from the horizon up) and angles of depression (measured from the horizon down) in real world settings **10-D13 Trigonometric Ratios: solve problems** • calculate side lengths and angles using trig ratios (use of calculators is required)	Many problems requiring the use of trigonometry involve what are called angles of elevation and angles of depression. These types of problems should be introduced separately.

Pacing	Materials	Prerequisites
1 h	• Calculators	• calculating sin, cos, tan, arc sin, arc cos, and arc tan with a calculator

Main Points to be Raised

• Angles of elevation and angles of depression are both angles with the horizontal.

• Angles of elevation are used when looking up and angles of depression when looking down.

• Problems involving angles of elevation sometimes require consideration of the height of a person below eye level.

Try This — Introducing the Lesson

A. Students can work alone or with a partner.

Observe while students work. You might ask:

• *Why did he multiply instead of divide?* (tan 25° = height of tree ÷ 6.2, so you have to multiply both sides of the equation by 6.2)

• *Why did you decide to use the tangent instead of the sine?* (For sine, you need to know the opposite side or the hypotenuse and I did not know either of those.)

• *Why does calculating the height of the triangle not completely solve the problem?* (Part of the tree height is below the triangle.)

The Exposition — Presenting the Main Ideas

• Ask students to look at the two pictures in the exposition. Discuss why one might be called an angle of elevation and the other an angle of depression. Draw attention to the fact that the angle is always with the horizontal.

• Ask students to explain why the height of the building involves both the height of the triangle and the additional height below it.

Revisiting the Try This

B. This question allows students to relate their solution to **part A** to the new terminology they have just learned.

Using the Examples

• Ask students to read through **examples 1 and 2**. Answer any questions students have about those examples. Point out how the answer in **example 2** would have changed if 347 x tan 3˚ was calculated without rounding.

• Draw the picture for **example 3** on the board. Help students see that the height of the triangle can be described using information from the triangle on the left as well as using information from the triangle on the right. Ask students to generate calculations involving both base angles to describe that height.

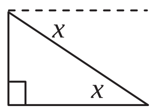
• Then work through **example 3** with the students. Make sure they understand why the 1.4 was subtracted from 100 and why tangent ratios were used (since information about opposite and adjacent sides was known).

Practising and Applying

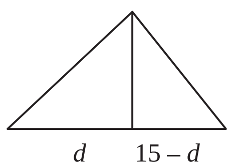
Teaching points and tips

Q 1: Students need to remember to subtract 1.5 from 18.6 to determine the length of the opposite side of the triangle.

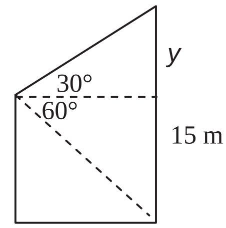
Q 2 and 5: Some students may not realize that the value of the angle of depression is the same as the angle at the base of the triangle.



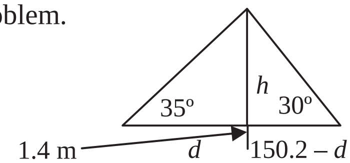
Q 4: To solve the problem, students need to draw a sketch and use a variable to describe the total base of 15in terms of two parts, e.g., d and $15 - d$. They can then express the height of the triangle as either the product of d and tan 22° or the product of (15 – d) and tan 31°. These values can be set equal and the equation solved to determine d. Since $h = d \tan 22°$, h can then be calculated.



Q 5: A diagram would be helpful for the student to approach this problem.

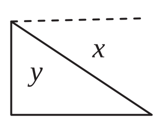


Q 6: Again, a diagram is helpful in solving the problem.



Common Errors

Students often mistake which angle they must determine when calculating an angle of depression. For example, to determine the angle of depression in the situation below, they often use angle y instead of angle x.



Watch for this and keep reminding students to draw the horizontal to help them see which angle is which.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can solve a standard problem involving angle of elevations
Question 6	to see if students can solve a more complex problem involving angle of depressions
Question 7	to see if students can visualize and discuss situations involving different angles of depression

Answers

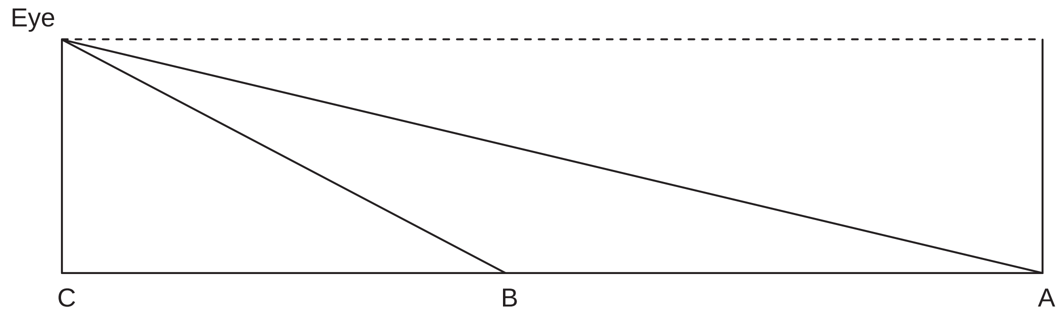
A. i) $\tan 25° = \frac{\text{opposite side}}{6.2}$, so opposite side = $6.2 \times \tan 25°$

ii) He forgot to add his height below his eyes since that is also part of the tree's height.

B. It is an angle of elevation since he is looking up to the top of the tree.

1. 31.9°	**4.** 3.62 m
2. 12.1°	**5.** 20 m
3. 45.1 m	**6.** 48.94 m

7. The angle of depression increases as you get closer to an object since the opposite side stays the same and the adjacent side gets smaller. If the object were directly below (C in the diagram), the angle of depression would be 90°.



Supporting Students

Struggling students

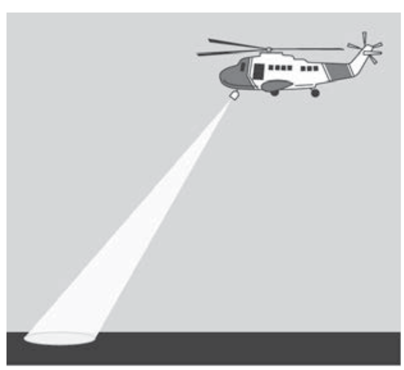
Add questions similar to **questions 1 and 2** to replace **questions 4 to 6** for struggling students.

Enrichment

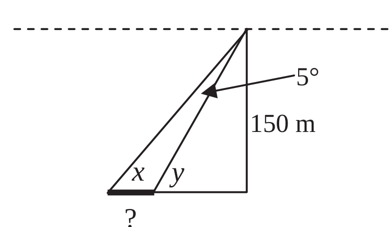
• Students can create problems similar to **questions 4 and 6** on their own. They can trade problems with a classmate to solve.

• You can also assign this problem. The solution is indicated below it.

Problem: A searchlight is attached to the front of a helicopter. The helicopter is flying 150 m above ground and the centre of the beam is aimed at an angle of 70° from the horizontal. The beam spreads out at an angle of 5°. How wide a distance does the beam cover?



Solution: Students need to use both a 72.5° and a 67.5° angle of depression. Some students might use 70° and 75° or 65° and 70° instead, but the wording of the problem suggests that 70° is the "centre" of the angle of depression. Since the beams from the helicopter can be thought of as the transversals of two parallel lines (the ground and the horizontal), angles *x* and *y* are 67.5° and 72.5° respectively. The length required is the difference between the length of the adjacent side to the base 67.5° angle in the triangle with an angle of depression of 67.5° and the length of the adjacent side to the base 72.5° angle in the triangle with an angle of depression of 72.5°.



7.2.3 Areas of Polygons

Curriculum Outcomes	Outcome relevance
10-D12 Trigonometric Values: right triangles — apply to solve problems • find areas of polygons using right triangle trigonometry **10-D5 Area, Perimeter, Surface Area, Capacity, and Volume: determine** • apply formulas for area in a variety of contexts • develop non-routine formulas to determine area • understand that areas of regular polygons can be determined by dividing the area into familiar shapes	Sometimes the familiar formulas for areas of polygons cannot be directly applied since side measurements are not known. The use of trig ratios can allow students to work around those difficulties.

Pacing	Materials	Prerequisites
1 h	• Calculators	• area formulas for triangles and parallelograms • properties of circles (360°) • definition of rhombus • names of polygons (octagon, decagon, etc.)

Main Points to be Raised

• A formula for the area of a parallelogram that is based on the formula $A = bh$ is $A = ab \sin x$, where a and b are adjacent sides of the parallelogram and x is the acute angle between them. The formula could also be written as $b \times a \sin x$, where $a \sin x$ is equal to the height, h.

• A formula for the area of a triangle that is based on the formula $A = bh \div 2$ is $A = ab \sin x \div 2$ where a and b are adjacent sides of the triangle and x is the acute angle between them.

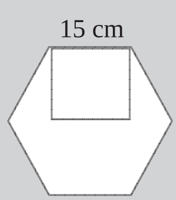
• Every regular polygon can be divided into isosceles triangles. The polygon's area is the sum of the areas of the triangles. The vertex angle of each isosceles triangle is $360 \div n$ where n is the number of sides of the polygon.

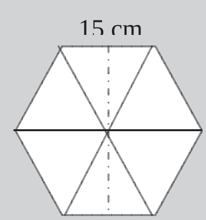
Try This — Introducing the Lesson

A. Students should work with a partner.

Observe while students work. You might ask:

• *How do you know the square must be more than 15 cm on a side?* (A 15 cm square would not be big enough.)





• *How do you know the side length is less than 30 cm?* (The dashed line that is the height of the triangle is shorter than the 15 cm diagonal side of each triangle. So the dashed line that is the height of the hexagon is shorter than 30 cm.)

The Exposition — Presenting the Main Ideas

• Draw a parallelogram on the board. Label the side lengths and one of the acute angles with appropriate measurements. Mark the height with *h*, but do not indicate its length. Ask students to suggest how to calculate the area of the parallelogram. If they say they need the height measurement, ask them how to find it from the information that is given. Work through the problem to see that the height can be calculated using a trigonometric ratio involving the angle.

• Draw an acute triangle as half of a parallelogram. Label two adjacent side lengths and the angle between them. Ask students how they might calculate the area using the given information.

• Draw a regular hexagon on the board and divide it into the six triangles that make it up (as shown in the student book). Ask students to determine the measure of each angle in each triangle. Ask pairs of students to figure out how to calculate the area of the whole hexagon using the triangles. As a group, share ideas.

• Draw a regular octagon and divide it into eight isosceles triangles. Ask students to calculate the area of the octagon.

• Suggest that students read through the exposition for further explanation.

Revisiting the Try This

B. Students can either use the method shown for the regular hexagon or adapt the one shown for the octagon to determine the area of the hexagon in **part A.**

Using the Examples

Write the two example problems on the board while students' books are closed. Ask each student to choose one of the example problems to solve. They can then compare their answer to the one in **example 1 or 2**, as appropriate.

Practising and Applying

Teaching points and tips

Q 1: If required, remind students that a rhombus is a parallelogram with four equal sides.

Q 4: Students will need to calculate the side lengths and angle measures for the triangles that make up the pentagon, hexagon, and decagon before they can proceed:

= Pentagon: each base has a length of 12 cm and angles of 54°

- Hexagon: each base has a length of 10 cm and angles of 60°

- Decagon: each base has a length of 6 cm and angles of 72°

In later classes, students will learn that a circle is the shape with maximum area for a given perimeter, so it is reasonable that shapes with more sides (that better approximate a circle) have greater areas.

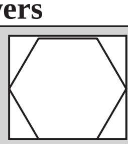
Q 5: Students need to realize that the height of the triangle can be expressed in terms of trig ratios involving both of the base angles.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can use a provided formula for area
Question 2	to see if students can calculate the area of a given composite shape
Question 4	to see if students can apply the methods learned in a problem solving situation

Answers

A. i)



A. ii) *Sample response*:
The side of the square is less than 30 cm. More than 75% of the square appears to be in the hexagon. I would estimate 75% of 30 x 30 would be close. My estimate is 675 cm^2

B. i) $s = 15$, so $A = 585$ cm^2

ii) Yes, the estimate was reasonable, though it was a bit larger than the actual area.

Answers [Continued]

1. 346.41 cm^2 **2.** 26.57 square units **3.** 769.42 cm^2 **4. a)** $A_{square} = 225$ cm^2 $A_{pentagon} = 247.75$ cm^2 $A_{hexagon} = 259.81$ cm^2 $A_{decagon} = 276.99$ cm^2	**4. b)** The area gets bigger as more sides are added even though the perimeter stays the same. **c)** A circle is like a polygon with many, many sides. **5.** $A = (b_1 + b_2)(b_1 \tan X) \div 2$ or $(b_1 + b_2)(b_2 \tan Z) \div 2$ **6.** The lengths of two sides (*a* and *b*) and the angle between them, *x*, since the formula is $A = \frac{1}{2}ab \sin x$.

Supporting Students

Struggling students

Some students may need additional work with describing the angles of the triangles that make up regular polygon shapes. Other students may benefit if the standard formulas for area of a triangle and parallelogram are displayed for reference.

Enrichment

Students can also be asked to find the side lengths for a shape with a particular area that includes a particular angle, e.g., a parallelogram with an area of 40 square units and one angle of 20°.

GAME: Race to Five

This game provides an opportunity for students to begin to associate the approximate values of sines, cosines, and tangents with the appropriate angles.

7.2.4 Vectors and Bearings

Curriculum Outcomes	Outcome relevance
10-D14 Vectors and Bearings: solve problems • solve bearing and vector problems using the Pythagorean theorem and/or trigonometric ratios	The use of vectors and bearings has real-world value for students and is an appropriate application of right triangle trigonometry. Work with vectors will also support later work in physics.

Pacing	Materials	Prerequisites
2 h	• Rulers • Protractors • Calculators	• Pythagorean theorem • compass directions • accurate angle measurement

Main Points to be Raised

- The length of a vector is proportional to the distance it represents.
- A bearing is a clockwise measure from 0° North.
- The direction of a vector is determined by the bearing it represents.
- A single vector can be used to replace a pair or groups of three vectors by joining the initial point of the first vector to the final point of the final vector. This will later be called a vector sum.
- A vector is associated with a distance and a direction.

Try This — Introducing the Lesson

A. Students should work with a partner and draw a sketch to model the situation presented in the problem.

Observe while students work. You might ask:

- *How do you know that is a right triangle?* (East is a 90° turn from south.)
- *How do you know the triangle is isosceles?* (It is the same distance south as east.)
- *How will you calculate that length?* (I will use the Pythagorean theorem; the legs are both 2.)

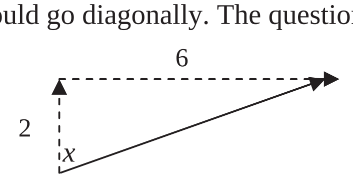
Some students might recognize that these directions would lead to different results at the North Pole.

The Exposition — Presenting the Main Ideas

- Draw a circle on the board, marking on N, S, E, and W directions. Draw the centre of the circle and radii to N, then E, then S, then W. Ask students to name each angle if one arm of the angle is always the arm pointing to the north and the other arm is each of the other directions (E, S, and W). Mark the values 000°, 090°, 180°, and 270° respectively.
- Ask students to find the angles describing NE (045°) and SE (135°). Mention an angle and ask students where the bearing would point, e.g., 040° would be a bit closer to north than northeast or 200° would be just slightly to the west of south.
- Then draw a vector like this one:

- Indicate that the vector represents a distance. For example, it could be 2 km. Ask what the bearing is (000°).
- Add the second vector.

- Ask why the new vector might represent a distance of 6 km (it is three times the length of the first vector). Ask what the bearing is (090°). Point out that if someone started at the same original position and wanted to end up in the final position, he or she could go diagonally. The question is how far the distance would be and what angle should be used.



The Exposition — Presenting the Main Ideas [Continued]

• Ask students how to determine the distance (using the Pythagorean theorem with sides of 2 and 6).

• Point out that to calculate the bearing, the value of $\angle x$ must be determined. Ask students what trig ratio can be used to determine the value of the angle (tangent since tan $x = 3$). (Since $x = 72°$ is the angle with that tangent, the bearing is 072°.)

• Suggest that students read the last part of the exposition. Respond to any questions they might have. Make sure they understand why the 63° was added to 180° to find the bearing.

Revisiting the Try This

B. Students should realize that they are working with a tangent of 1 (or an angle of 45°) in the problem solved in **part A,** but that the 45° needs to be added to 90° to describe the bearing.

Using the Examples

Work through both examples with the students. Note that only simple relationships are shown when more than two vectors are used.

Practising and Applying

Teaching points and tips

Q 1: Accept estimates within a reasonable range.

Q 3: These values were specially selected so that students could see that going backwards means adding or subtracting 180° from a bearing. For example, going at a bearing of 225° first and then switching to a bearing of 45° involves going backwards on the same path.

Q 4: The students can use:

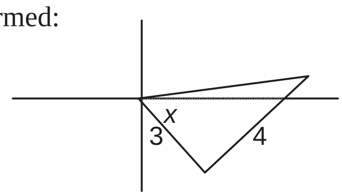
a) $\tan^{-1}$ (3.5 ÷ 2.0) to determine the bearing and the Pythagorean theorem to calculate the distance

b) 360 – $\tan^{-1}$ (1.7 ÷ 1.4) to determine the bearing and the Pythagorean theorem to calculate the distance

c) 90 + $\tan^{-1}$ (3.5 ÷ 2.5) to determine the bearing and the Pythagorean theorem to calculate the distance

For **part d),** the students need to realize that the last segment at a bearing of 045° is just a continuation of the vector that describes the combination of the first two segments.

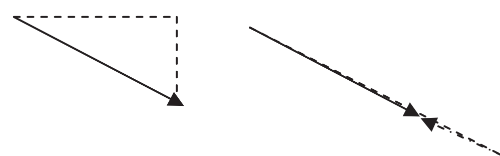
Q 5: The values in the problem were set up so that a right triangle would be formed, even though the vectors were not horizontal or vertical. In that way students could easily determine the angles in the triangle formed:



tan $x = 1.333$, so $x = 53.1°$

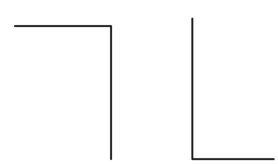
Students can subtract from 135° to calculate the bearing and use the Pythagorean theorem to calculate the distance.

Q 6: Students should draw the 10 km vector at a bearing of 120° and base their solutions on that drawing. For example, the first picture is based on a right triangle with an initial bearing of 090°, creating a 30°-60°-90° triangle by making the second distance 5 km and using the Pythagorean theorem to calculate the first distance. The second diagram is based on going too far and then reversing the direction (using 120° + 180°).



Another option is to use the same bearing but to do it in two steps.

Q 7: Students should draw both possible pictures:



They can see that the lengths are the same and so are the bearings.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can visually estimate bearings for given vectors
Question 2	to see if students understands what a bearing is
Question 4	to see if students can apply trigonometric ratios to solve simple vector and bearing problems

Answers

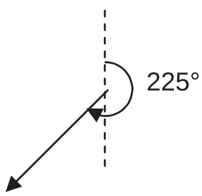
A. I am about 2.8 km from where I started. I drew a right triangle with two legs of 2 and calculated the hypotenuse using the Pythagorean theorem.

B. 2.8 km at a bearing of 135°

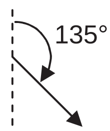
1. *Sample responses* (values within 20° are reasonable):
a) 045° **b)** 330° **c)** 150°

2. *Sample responses* (as the distances may vary):

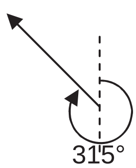
a)



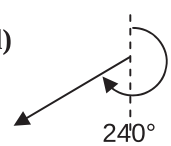
b)



c)



d)



3. a) i) 1 m at a bearing of 225°
ii) 0 m (no displacement)
iii) 1 m at a bearing of 050°

b) The directions were directly opposite each other.

4. a) 4.03 km at a bearing of about 060°

b) 2.2 km at a bearing of about 310°

c) 4.3 km at a bearing of about 145°

d) 3.12 km at a bearing of 045°

5. 5 km at a bearing of about 082°

6. *Sample response*:
5 km at a bearing of 120° and another 5 km at the same bearing; 12 km at a bearing of 120° and 2 km at a bearing of 300°; 8.7 km at a bearing of 90° and 5 km at a bearing of 180°

7. a) No; Both times the vector is the hypotenuse of a triangle with legs of 4 and 2 and so the Pythagorean theorem would give the same value both times (about 4.5 km).

b) No; Both times the bearing is 153°.

Supporting Students

Struggling students

As long as students are able to complete **questions 1, 2, and 4**, you may choose not to have them complete the other questions.

Enrichment

• Students are encouraged to discover multiple possible solutions for **question 6** and compare them with those found by others.

• Students can try to work with more complicated vector trips, e.g., using a single vector to describe a trip made up of these parts: 1.0 km at a bearing of 0°, followed by 3.0 km at a bearing of 90°, followed by 1.4 km at a bearing of 180°.

CONNECTIONS: Relating Trigonometric Ratios to Circles

This **Connection** exposes students to an interpretation of sine, cosine, and tangent that also applies to obtuse and reflex angles. It also introduces them to a physical interpretation of tangent and makes a connection to the geometric concept of the tangent to a circle.

Answers

1 a) 0.64 cm **b)** 0.77 cm **c)** 0.84 cm

2. The values in **question 1** are the sine, cosine, and tangent of 40°.

3. Using ASA, $\angle$CAE = $\angle$FEA, AE = AE, and $\angle$AEC = $\angle$FAE

4. *Sample response*:
$\angle$FEA = 40° because FE is parallel to AC
$\angle$AEC = 50° since 180° – (90° – 40°)
$\angle$CED = 40° since 90° – 50°
$\angle$ EFA = $\angle$ECD = 90°; $\angle$FAE = $\angle$CDE = 90°; $\angle$AEF = $\angle$DEC = 40°
Using AAA, ΔEFA is similar to ΔECD.
(One could also show similarity by measuring lengths and using corresponding proportional sides.)

5. $\frac{ED}{EA} = \frac{EC}{EF}$
I found out in **question 2** that AC = cos 40° and in **question 3** that EF = AC so EF = cos 40°.
I found out in **question 2** that EC = sin 40°. So $\frac{ED}{1} = \frac{\sin 40^\circ}{\cos 40^\circ}$, but $\frac{\sin 40^\circ}{\cos 40^\circ}$ = tan 40° so ED = tan 40°.

UNIT 7 Revision

Pacing	Materials
1 h	• Rulers • Protractors • Calculators

Question(s)	Related Lesson
1 – 4	Lesson 7.1.1
5	Lesson 7.1.3
6, 7	Lesson 7.1.4
8, 9, 10, 11	Lesson 7.2.1
12, 13	Lesson 7.2.2
14	Lesson 7.2.3
15, 16	Lesson 7.2.4

Revision Tips

Q 2b: Some students may not see that there are two triangles, since one is nested inside the other.

Q 8: Some students might calculate the various values that are mentioned, but others might use reasoning based on the angle size.

Q 8: The solution is easiest if students use the sine ratio.

Q 15: Accept a reasonable estimate, perhaps within a range of about 20°.

Answers

1. The pair in A are similar since the sides are in the same proportion (scale factor of 1.4).

2. a) 1.8 units **b)** 6.67 units

3. 12 m **4.** 25 m

5. a) sin A = 0.467, cos A = 0.9, tan A = 0.519;
$\angle$A = 27°

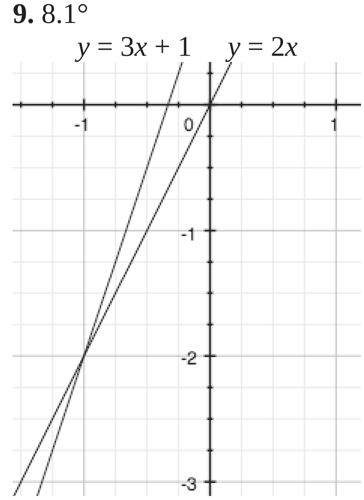
b) sin A = 0.514, cos A = 0.857, tan A = 0.60,
$\angle$A = 31°

6. For **part a**, sec A < cot A < csc A
For **part b**, sec A < cot A < csc A
In both cases, the order is the same; because the sizes are in reverse order to the sizes of the reciprocal primary ratios.

7. a) 78.5° **b)** 11.5° **c)** 56.3° **d)** 5°

8. 4.97 m

9. 8.1°



10. 29° **11.** 24.4° **12.** 103.92 m

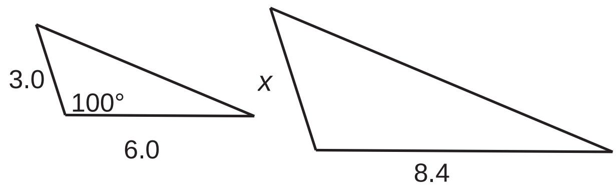
13. 12.05 m **14.** 2472.73 cm^2

15. *Sample responses* (values within 20° are reasonable):
a) 150° **b)** 65°

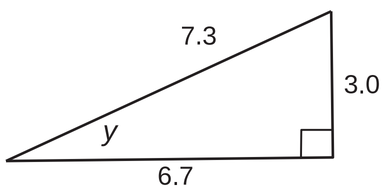
16. 7.8 km at a bearing of 140°

UNIT 7 Trigonometry Test

1. These two triangles are similar.
What is the value of *x*?



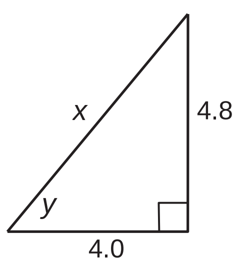
2. a) What are the sine, cosine, and tangent of *y?*



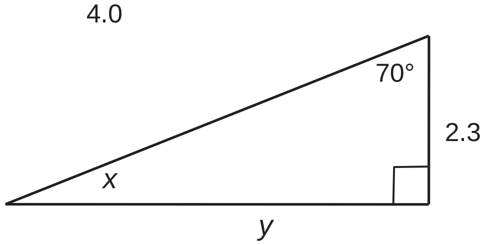
b) What is the measure of angle *y?*

3. Determine the values of *x* and *y* for each triangle.

a)



b)



4. An angle has a sine of 0.5. Calculate the values of each of the other 5 trig ratios.

5. Explain why the cosecant of 45° has to be about 1.414.

6. Fill in each blank with an acute angle.

a) cos ____ = 0.45

b) sin ____ = 0.15

c) tan ____ = 4.1

d) cos ____ = sin 37°

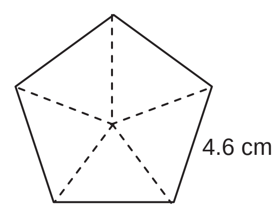
7. Why does the sine of an acute angle have to be less than 1?

8. A 6.0 m ladder is resting against a wall, making an angle of 50° with the wall. How high up the wall does the ladder go?

9. What is an angle of depression?

10. Lemo is looking up at a tree that is 500 m away. The angle of elevation is 4°. Lemo's eyes are about 1.5 m above ground. How high is the tree?

11. Calculate the area of this regular pentagon



12. Draw a vector to represent each bearing.

a) 145°

b) 300°

13. Draw a single vector to represent the entire trip made up of these two segments.

Part 1: 3.2 km at a bearing of 020°

Part 2: 4.6 km at a bearing of 110°

UNIT 7 Test

Pacing	Materials
1 h	• Rulers • Protractors • Calculators

Question(s)	Related Lesson
1	Lesson 7.1.1
2, 3, 7	Lesson 7.1.3
4, 5, 6	Lesson 7.1.4
8	Lesson 7.2.1
9, 10	Lesson 7.2.2
11	Lesson 7.2.3
12, 13	Lesson 7.2.4

Select questions to assign depending on the time available.

Answers

1. 4.2

2. a) $\sin y = \frac{3.0}{7.3} = 0.411$

$\cos y = \frac{6.7}{7.3} = 0.918$

$\tan y = \frac{3.0}{6.7} = 0.448$

b) $y = 24.1°$

3. a) $x = 6.2$; $y = 50.2°$ **b)** $x = 20°$; $y = 6.3$

4. $\cos x = 0.866$
$\tan x = 0.577$
$\cot x = 1.732$
$\sec x = 1.155$
$\csc x = 2$

5. csc *x* is the reciprocal of sin *x*. The ratio of the sides in an isosceles right triangle is 1-1-root 2 and so the value of csc x is root 2. This is about 1.414.

6. a) 63.3° **b)** 8.6°

c) 76.3° **d)** 53°

7. $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$; Since the hypotenuse is the longest side in the triangle, the numerator is less than the denominator and the fraction is less than 1.

8. 3.9 m

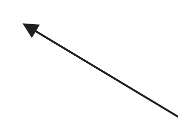
9. An angle of depression is the angle formed by the horizontal and your line of sight with an object when you look down.

10. 36.5 m

11. 36.4 cm^2

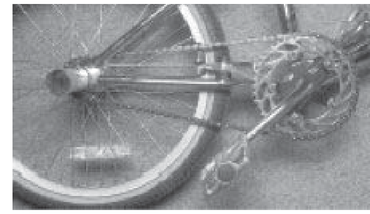
12. a)

b)



13. 5.6 km at a bearing of 075°

UNIT 7 Performance Task — Trigonometry and Bike Chains

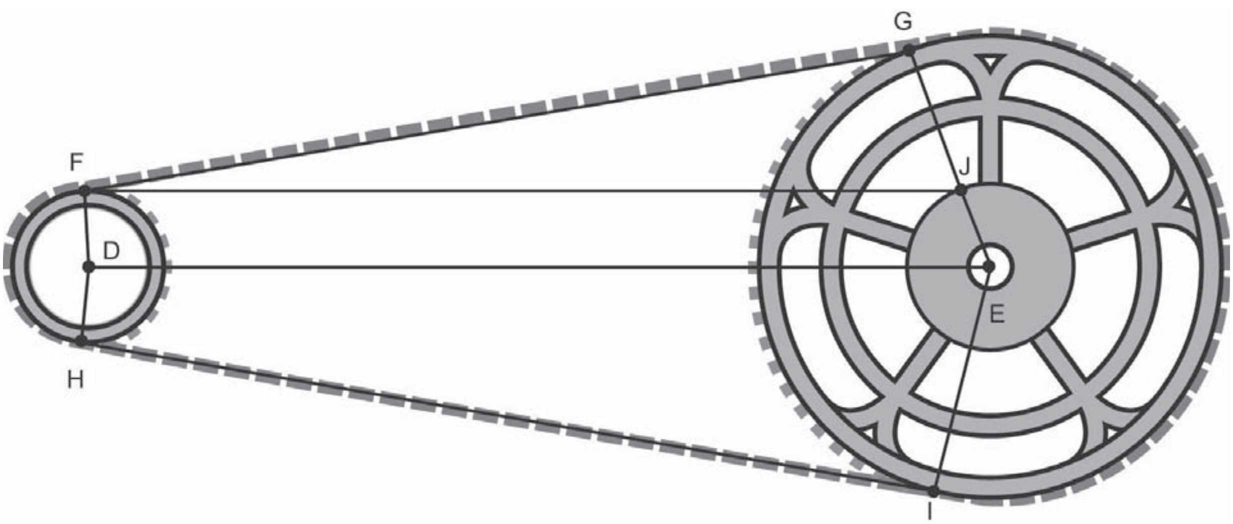


Look at the photo of a chain on a bicycle. Your task is to figure out, to the nearest millimetre, how long the chain needs to be.

The diameter of the larger sprocket wheel is 176 mm and the diameter of the smaller sprocket wheel is 90 mm. The centres of the two sprockets are 420 mm apart.

Assume that the straight lengths of chain meet the sprocket wheels at 90° angles.

This diagram below is a model of the chain connecting the two sprocket wheels.



Use trigonometry to determine the length of the chain.

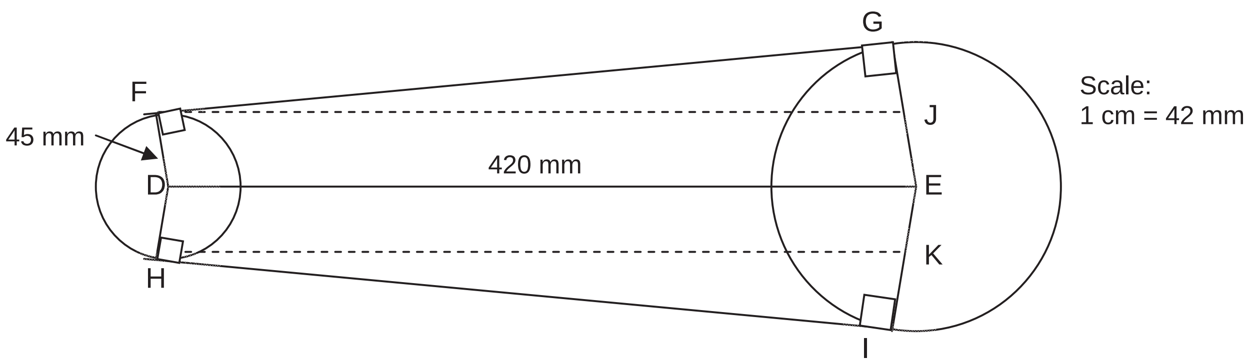
UNIT 7 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
10-D7 Trigonometric Functions: relate to ratios in similar right triangles **10-D11** Trigonometric Values: use calculators **10-D13** Trigonometric Ratios: solve problems	1 h	• Calculators • Protractors

How to Use This Performance Task

You might use this task as a rich problem to assess student understanding of a number of outcomes in this unit. It could replace or supplement the unit test. It could also be used, if you wish, as enrichment material for some students. You can assess performance on the task using the rubric on the next page. Make sure students have reviewed the rubric before beginning work on the task.

If students do not seem to know where to start, you might provide this picture with additional lines drawn on it. Suggest that they consider right triangles ΔFGJ and ΔHIK to help them solve the problem.



Sample Solution (with lines FJ and HK added):

GE = 88 mm since the diameter of the larger circle is 176 mm.

I do not know the length of GF, but I can figure out the lengths of GJ and FJ, so I will use cosine.

$\cos \angle GJF = \frac{GJ}{FJ}$

Since DF = EJ = 45 mm, GJ = GE – DF = 88 – 45 = 43 mm.

FJ = DE since the shape FJED is a parallelogram and these are opposite sides.

$\cos \angle GJF = \frac{43}{420} = 0.102$

$\angle$GJF is about 84°.

That means FG = 420 × sin 84° = 420 × 0.995 = 417.7 mm.

ΔFGJ is congruent to ΔHIK, so HI is also 417.7 mm.

I need to calculate the amount of chain around the right part of the big wheel and the left part of the small wheel.

FJ is parallel to ED, so $\angle$GED = $\angle$GJF. That is 84°.

Also HK is parallel to ED, so $\angle$ DEI = 84°.

If you put them together, $\angle$ GEI = 168°. That means $\frac{168}{360}$ of the big wheel is on the inside and does not have chain around it. But $\frac{168}{360}$ of the small wheel does have chain around it since the circle arrangements are similar.

It also means that $\frac{192}{360}$ of the big wheel has chain around it.

I can find the circumference of both wheels, take the correct proportions of them, and then add that amount of chain to the total.

$C_{\text{big circle}} = \pi \times 176 \approx 552.9$ mm; $\frac{192}{360} \times 552.9 \approx 294.9$ mm

$C_{\text{small circle}} = \pi \times 90 \approx 282.7$ mm; $\frac{168}{360} \times 282.7 \approx 131.9$ mm

The total chain length is 2 × 417.7 + 294.9 + 131.9 = 1262.2 mm.

Unit 7 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Problem solving	Independently solves the problem accurately by: • creating appropriate triangles • using appropriate trig ratios to calculate the straight lengths of chain • correctly using the circumference formula for a circle • using the appropriate fractions of the circumference to calculate the curved lengths of chain	Using the suggestion of additional lines to draw on the diagram, solves the problem accurately by: • creating appropriate triangles • using appropriate trig ratios to calculate the straight lengths of chain • correctly using the circumference formula for a circle • using reasonable estimates for fractions of the circumference to calculate the curved lengths of chain	Using the suggestion of additional lines to draw on the diagram, solves the problem accurately by: • creating appropriate triangles • using appropriate trig ratios to calculate the straight lengths of chain • recognizing the need to calculate the curved lengths of chain, somehow using the circumference of the circle	Using the suggestion of additional lines to draw on the diagram, does not calculate the lengths of even the straight lengths of chain
Communication	Communicates the steps of the solution completely and clearly	Communicates most steps of the solution reasonably clearly	Communicates some steps of the solution reasonably clearly	Does not communicate thinking clearly

UNIT 8 GEOMETRY

UNIT 8 PLANNING CHART

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Getting Started **SB p. 269** **TG p. 278**	Review prerequisite concepts, skills, and terminology and pre-assessment	1 h	• Rulers • Protractors • Compasses	All questions
Chapter 1 Symmetry and Reasoning				
8.1.1 2-D and 3-D Reflectional Symmetry **SB p. 270** **TG p. 280**	**10-E3 2-D and 3-D Shapes: explore properties and test conjectures** • compare 2-D and 3-D mirror symmetry	2 h	• Cubes (net) (optional) • Piece of paper (rectangular) • Models of several prisms (nets, clay models and knife to cut them, or models made of cubes) and a tetrahedron (net or clay model) (optional)	Q2, 3, 4, and 9
8.1.2 2-D and 3-D Rotational Symmetry **SB p. 275** **TG p. 284**	**10-E3 2-D and 3-D Shapes: explore properties and test conjectures** • compare 2-D and 3-D rotational symmetry • compare mirror and rotational symmetry	2 h	• Square, rectangle, and cube (net) • Regular polygons and other shapes (optional) • 3-D prisms (actual or made from nets) (optional) • Straws or long sticks (optional)	Q2, 4, 7, and 10
8.1.3 Reasoning **SB p. 279** **TG p. 287**	**9-E1 Congruent Triangles and Angle Properties: informal deductions** • distinguish between inductive and deductive reasoning using both mathematical and non-mathematical reasoning **10-E2 Geometric Reasoning: inductive and deductive** • use inductive and deductive reasoning in situations such as generalizing relationships and proving theorems	2 h	• Protractors • 3-D shapes (nets) (optional)	Q1, 3, 7, and 8
Chapter 2 Constructions				
8.2.1 EXPLORE: Rigidity (Optional) **SB p. 284** **TG p. 292**	• Consider one of the physical properties of triangles (rigidity) that makes triangles unique	45 min	• Sticks of varying lengths • Tape (or another material to attach sticks, e.g., putty)	Observe and Assess questions

UNIT 8 PLANNING CHART [Continued]

	Outcomes or Purpose	Suggested Pacing	Materials	Suggested Assessment
Chapter 2 Constructions [Continued]				
8.2.2 Perpendiculars and Bisectors **SB p. 285** **TG p. 294**	**10-E4 Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors)** • consider the concepts of perpendicular and angle bisectors • locate incentres and circumcentres and construct incircles and circumcircles using perpendicular and angle bisector constructions	2 h	• Rulers • Compasses • Protractors	Q1, 3, 5, and 6
8.2.3 Medians and Altitudes **SB p. 291** **TG p. 300**	**10-E4 Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors)** • consider the concepts of medians and altitudes of triangles • locate centroids (centres of gravity) and orthocentres using median and altitude constructions	2 h	• Rulers • Compasses • Protractors • Paper triangle (optional)	Q1, 4, 6, and 9
CONNECTIONS: Paper Folding Constructions **SB p. 295** **TG p. 304**	Explore constructions other than those involving ruler and compass	1 h	• Blank paper	N/A
GAME: Balancing Triangles **SB p. 295** **TG p. 304**	Apply some of the unit concepts (centroid constructions and centroid as centre of gravity)	15 min	• Stiff paper • Scissors • Rulers • Compasses • Pencil	N/A
UNIT 8 Revision **SB p. 296** **TG p. 305**	Review the concepts and skills in the unit	2 h	• Rulers • Compasses • Protractors	All questions
UNIT 8 Test **TG p. 308**	Assess the concepts and skills in the unit	1 h	• Rulers • Compasses • Protractors	All questions
UNIT 8 Performance Task **TG p. 312**	Assess concepts and skills in the unit	1 h	• Rulers • Compasses • Protractors	Rubric provided
UNIT 8 Blackline Masters **TG p. 315**	BLM 1 Isometric Dot Paper BLM 2 Net of Cube BLM 3 Net of Rectangle-based Prism BLM 4 Net of Right Triangle-based Prism BLM 5 Net of Regular Hexagon-based Prism BLM 6 Net of Regular Pentagon-based Prism BLM 7 Net of Regular Octagon-based Prism BLM 8 Net of Square-based Pyramid BLM 9 Net of Tetrahedron BLM 10 Net of Cylinder BLM 11 Net of Cone BLM 12 Regular Polygons and Other Shapes			

Math Background

• In this unit, students extend their knowledge of reflectional and rotational symmetry to three dimensions.

• They also more formally consider the role of both inductive and deductive reasoning in mathematics.

• Students learn the rationale and procedures for performing a variety of geometric constructions, including angle and perpendicular bisectors, medians and altitudes of triangles, and circumcentres, incentres, circumcircles, and incircles of triangles.

• As students work through this unit they will use a variety of mathematical processes, including problem solving, communication, reasoning, representation, visualization, and making connections.

For example:

- They use problem solving in **question 7** of **lesson 8.1.2**, where they analyze a hexagonal prism to consider all possible axes of turn symmetry. Another example of problem solving is found in **question 5** of **lesson 8.2.2**, where students decide how to construct triangles to meet given conditions.

- They use communication frequently to explain their thinking, for example, in **question 3** of **lesson 8.1.2**, where they explain why a triangle cannot have turn symmetry of order 2, and in **question 9** of **lesson 8.2.3**, where they explain how the construction of a median relates to the notion of centre of gravity.

- They use reasoning in answering questions such as **questions 6 and 9** of **lesson 8.1.1**, where they reason about possible lines of symmetry for a triangle and generalize what they have learned about symmetry in polygons to other 2-D and 3-D shapes. **Lesson 8.1.3** is devoted completely to reasoning, both deductive and inductive. Students also use reasoning extensively in **lesson 8.2.2**, for example, in **questions 2b, 4, and 6** where they use inductive reasoning.

- They use visualization in **lessons 8.1.1 and 8.1.2** when they visualize planes and axes of symmetry for 3-D shapes, and lines and points of symmetry for 2-D shapes. They also use visualization in **lesson 8.2.3** where they visualize how a third altitude or third median intersects the other two, as well as visualizing how a median divides a triangle into two equal halves.

- Students make connections, for example, in **lessons 8.1.1 and 8.1.2,** where they relate symmetry in two and three dimensions. There is a connection to trigonometry in **question 8** of **lesson 8.2.2.** The **Connections** feature extends the concept of constructions beyond straight-edge and compass to other forms, in particular, paper folding constructions.

Rationale for Teaching Approach

• This unit is divided into two chapters.

- **Chapter 1** deals with the extension of existing knowledge about reflectional and rotational symmetry of shapes and objects before introducing the formal language of inductive and deductive reasoning processes.

- **Chapter 2** helps students to apply knowledge they already have about polygons to create new shapes that are based on features of those polygons. Various constructions are featured in the development of more advanced geometric concepts such as altitudes, medians, centroids, circumcentres and orthocentres.

• The optional **Explore** feature in **Chapter 2** introduces the concept of rigidity and how it applies to constructions.

• The **Connections** feature explores centres of gravity and allows students to apply what they have learned about constructions to a real-world physical situation.

Technology in This Unit

No technology is required in this unit, other than the use of ruler, straight-edge, compass, and protractor. However, students with access to a computer program like Geometer's Sketchpad can explore many ideas presented in the unit.

Getting Started

Curriculum Outcomes	Outcome relevance
9 Congruent Triangles: properties and minimum and sufficient conditions **7** Angles: estimate and measure using a protractor **6** Rotational symmetry: properties	Students will experience more success in this unit if they review what they already know about conditions for congruence of triangles, rotational symmetry, and construction of angles and triangles.

Pacing	Materials	Prerequisites
1 h	• Rulers • Protractors • Compasses	• 2-D reflectional symmetry • 2-D rotational symmetry • coordinates • vocabulary for 3-D objects • use of protractors and compasses to construct angles, segments, and triangles • conditions for congruence of triangles

Main Points to be Raised

• You can determine a line of symmetry by folding a 2-D shape. You can also use a ruler; matching points on either side of the line of symmetry should be equally far from the line. Another way to test for reflectional symmetry is to use coordinates; if the line of symmetry is the *y*-axis, there should be matching points (x, y) and ($–x$, y) on or in the shape.

• Two polygons are congruent if all their sides and angles match.

• A shape has rotational symmetry if, when it is turned, it can fit into an outline of itself in less than a full turn.

• You can create a shape with rotational symmetry by taking a polygon, turning it a half turn about the midpoint of one edge, and then putting the two pieces together to make one shape.

Use What You Know — Introducing the Unit

• Students can try this activity alone or with a partner.

• Observe students as they work. You might ask:

- *Why is it convenient to be able to use a ruler to decide whether a shape on a printed page is symmetrical?* (Because the page does not have a coordinate system on it and because I cannot cut out the shape to fold it.)

- *Why is it not necessary to test every angle in the two trapezoids to see if they match?* (I know that the sum of the four angles in each shape is 360°, so I need to check only three of the angles.)

- *Why did you put the trapezoids together that way to create the shape with rotational symmetry?* (I know that if I do it that way, there is a horizontal line of symmetry and a vertical line of symmetry. Because of this, there is automatically rotational symmetry.)

Skills You Will Need

• To ensure students have the required skills for this unit, assign these questions.

• Students can work individually.

Common Errors

Some students will think you have to measure all angles to test to see whether the trapezoids are congruent. This is not incorrect, but it is simply not as efficient as it could be. Other students will suggest you have to measure only two pairs of angles. They assume that the trapezoid is isosceles. Without stating that assumption, this is insufficient information.

Answers

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A. *Sample response*:
i) Trace the trapezoid and cut it out. Fold it so that the top vertices overlap and the bottom vertices overlap—if the halves match exactly, the trapezoid is symmetrical.

ii) Measure to locate the midpoints of both the top and bottom sides. Draw a line from the middle of the top side length to the middle of the bottom side length. Measure the resulting halves to see they are congruent. If they are, the trapezoid is symmetrical.

iii) Trace the trapezoid onto a coordinate grid so that the *y*-axis is the line that I think is the line of symmetry. If the coordinates of the vertices on the right match the coordinates of the vertices on the left, except for having opposite signs, the shape is symmetrical.

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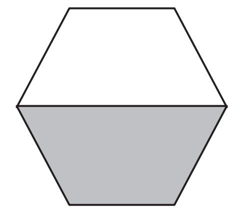
B. Measure the four side lengths and three angles of each.

C. *Sample response*:

I know it has rotational symmetry because if I turn it around its centre, it fits onto itself at least twice during a complete rotation. It actually fits six times.

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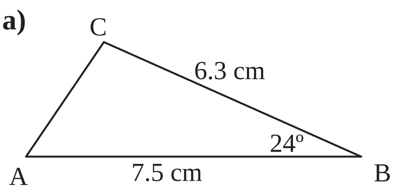
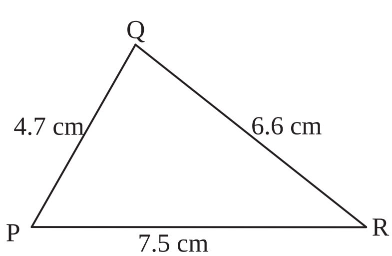
1. a) Pentagon-based prism

b) Octagon-based pyramid

c) Cone

d) Cylinder

e) Sphere

2. a)

b)


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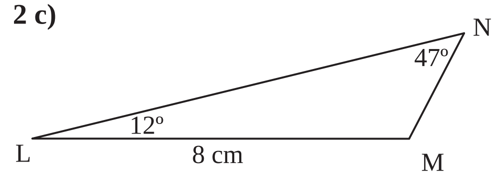
2 c)

3. a) Three corresponding sides the same in both triangles (SSS), two corresponding sides and the angle between them the same in both triangles (SAS), or two corresponding angles and one corresponding side the same in both triangles (ASA).

b) *Sample response*:
Knowing VZ = VX would show congruence using SAS; or knowing ∠ Y = ∠ W would show congruence using ASA; or knowing ∠ Z = ∠ X would show congruence using AAS.

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Supporting Students

Struggling students

If students are struggling with drawing triangles in **question 2**, suggest the sequence they should use for each. For example, in **question 2 a)**, you might ask them why it makes sense to draw ∠ B before drawing BC.

Chapter 1 Symmetry and Reasoning

8.1.1 2-D and 3-D Reflectional Symmetry

Curriculum Outcomes	Outcome relevance
10-E3 2-D and 3-D Shapes: explore properties and test conjectures • compare 2-D and 3-D mirror symmetry	Symmetry is a mathematical topic that is particularly relevant to our everyday lives. It helps us understand many of the natural and human-made objects around us. Work on this important topic is a good way to develop students' mathematical reasoning.

Pacing	Materials	Prerequisites
2 h	• Cubes (BLM net) (optional) • Piece of paper (rectangular) • Models of several prisms (BLM nets, clay models and knife to cut them, or models made of cubes) and a tetrahedron (BLM net or clay model) (optional)	• terms: *perpendicular, bisect, plane, regular* • names of various prisms

Main Points to be Raised

• Reflectional symmetry is sometimes called mirror symmetry.

• A mirror line is also called a line of reflection or a line of symmetry. It has the property that one half of the shape would fall exactly on top of the other half if the shape were folded on the line. Sometimes there are many lines of symmetry.

• A segment joining any point on one side of a mirror line with its mirror image on the other side is bisected by and is perpendicular to the mirror line.

• A mirror line, or line of symmetry, in 2-D is equivalent to a plane of symmetry in 3-D.

• A line segment joining any two matching points on either side of a plane of symmetry is bisected by the plane and is perpendicular to it.

• Regular right prisms always have more than one plane of symmetry.

• The planes of symmetry of a prism are extensions of lines of symmetry of the faces of the prism.

Try This — Introducing the Lesson

A. and B. Although students can simply use the picture in the book, it might be helpful if small groups of students had actual models of these structures to hold and view.

Observe while students work. You might ask:

• *How are the two structures the same? How are they different?* (Same: both use 15 cubes, both have flat parts and tall parts, both look the same on the left. Different: the structure on the right has a part four cubes high and the one on the left does not; the structure on the right does not have a cube in the front on the right side.)

• *Does either of these shapes look like a building to you?* (Not one in my village, but I have seen pictures of buildings in other countries somewhat like these.)

• *Why does it make sense that the structure on the left has seven cubes on its left side and seven on its right side?* (Since the two sides match, there must be the same number of cubes on either side.)

The Exposition — Presenting the Main Ideas

• Hold up a piece of paper. Ask students where to fold it to find the mirror line (also called the line of reflection or line of symmetry). Follow their suggestions. Draw a point, its opposite point and the line segment connecting them. Demonstrate that the segment is bisected by the line of symmetry and that the two lines are perpendicular. Ask if there is another line of symmetry for the rectangle and then show it.

• Relate back to the first activity in the unit. Show how a line of symmetry could be constructed by folding, by using a mirror, or by measuring. Two points that are matching could be connected by a segment. The midpoint could be determined and a line drawn at right angles to it to locate the line of symmetry.

• Hold up a prism. Ask students how they might cut it into two mirror opposite halves. Discuss suggestions. If you have a clay model, cut it with a knife. If you have a model made out of cubes, dismantle it to show the two halves. If you are not able to divide your model, ask students to visualize the two halves of the prism.

• Remind students that the word *plane* means a flat surface.

• Show students a plane of symmetry. Point out how the plane passes through lines of symmetry of the faces.

• Ask students to read through **pages 270 and 271**.

• Ask students if they understand why the pentagonal prism has six, and not five, planes of symmetry. Depending on student readiness, you might help them notice that the seven faces (two pentagons and five rectangles) have a total of 2 × 5(from the pentagons) + 5 × 2(from the rectangles) = 20 lines of symmetry. Notice that five of the planes of symmetry pass through three of those lines (two on the pentagon bases and one on a rectangle) and one plane of symmetry passes through five lines of symmetry of the rectangles. That accounts for all 20 lines. Planes of symmetry can also pass through edges, and in this case they do.

• Mention that all the prisms and pyramids we use in this unit will be right prisms or pyramids. For prisms, this means that one base is directly above the other and not shifted over. For pyramids, it means the apex of the pyramid is directly over the centre of the base.

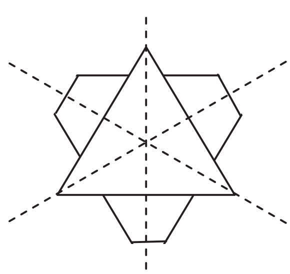
Revisiting the Try This

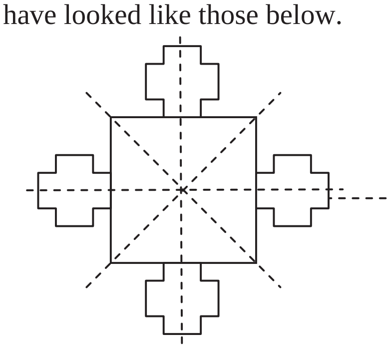
C. These questions give the students an opportunity to use the new terminology (*plane of symmetry*) in relation to their thinking in **parts A and B.**

Using the Examples

• Assign pairs of students to read through the examples. One person can focus on **example 1** and the other person can focus on **example 2**. They should then explain to each other what they learned.

• Make sure that students understand that different shapes could have been added to the sides of the triangles or squares in **example 1**. For example, they might have looked like those below.





• Encourage students to use the examples as a reference when working through **Practising and Applying**.

Practising and Applying

Teaching points and tips

Q 1: Encourage students not to stop with the first plane. If cubes are not available, they may choose to draw the structure on the blackline master of isometric grid paper.

Q 2: The drawings do not have to be exact for the purpose of this question.

Q 3: Some students might need a reminder that the plane can go through an edge of the object. It does not go only through lines of symmetry of the faces. Encourage students to visualize where the planes would be and to think of all the lines of symmetry any plane would go through.

Q 4: Students will need to remember to consider the rectangle faces and not just the bases. They can use the blackline master of the regular polygons.

Q 5: Suggest that students look at **example 1** to get ideas for this question.

Q 6: Students will need to recall that every triangle is equilateral, isosceles, or scalene.

Q 8: It would be helpful to have a model of a tetrahedron available.

Common Errors

Many students will forget to take into account the rectangle faces when considering the planes of symmetry of a prism. It will be helpful if you have prisms available for students to handle or view.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can locate lines of symmetry for 2-D shapes
Question 3	to see if students can locate planes of symmetry for 3-D shapes
Questions 4 and 9	to see if students can use prior experiences and reasoning to make a generalization

Answers

A. *Sample response*:
The first shape looks the same on its right as on its left, and the other one does not.

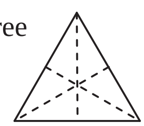
B. *Sample response*:
People tend to like symmetric designs and asymmetric designs might put more stress on one side of the building than on the other.

C. i) The structure on the left.

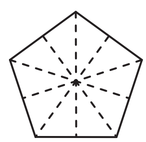
ii) One; a vertical plane through the centre of the middle cube.

1. Four

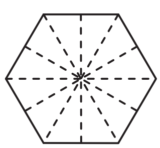
2. a) Three



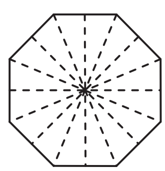
b) Five



c) Six



d) Eight

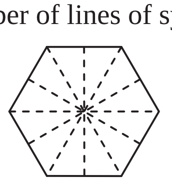


3. a) Four **b)** Six

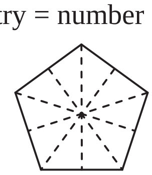
c) Seven **d)** Nine

4. a) The number of lines of symmetry is the same as number of sides. If the polygon has an even number of sides, n, there are $\frac{n}{2}$ lines connecting pairs of vertices and $\frac{n}{2}$ lines connecting midpoints of sides to midpoints of opposite sides, and $\frac{n}{2} + \frac{n}{2} = n$. If the polygon has an odd number of sides, n, there are n lines connecting vertices to the midpoints of the opposite sides.

Number of lines of symmetry = number of sides



Even

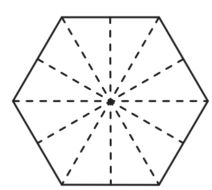


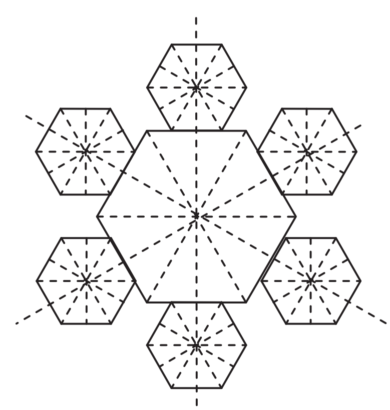
Odd

4. b) Each of these lines of symmetry results in a plane of symmetry if it is extended. There is also one more plane of symmetry that extends from the lines of symmetry that cut the lateral rectangular faces in half. This means there is one more plane of symmetry for a prism than the number of lines of symmetry in the regular polygon base.

c) Each of these lines of symmetry results in a plane of symmetry if it is extended so the number of planes of symmetry for a pyramid is the same as the number of lines of symmetry in the regular polygon base.

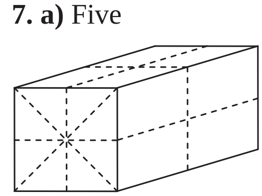
5. *Sample response*:



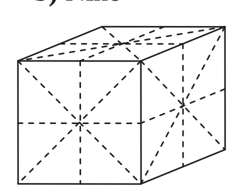


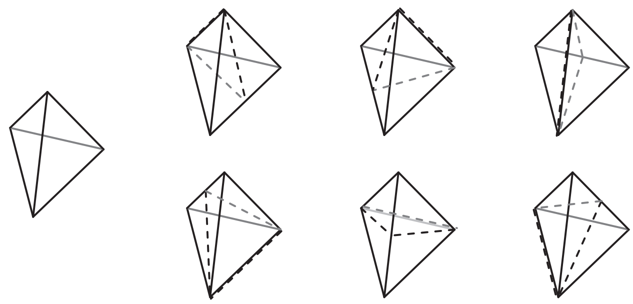
6. No. Scalene triangles do not have any, isosceles triangles have only one, and equilateral triangles have three. There are no other kinds of triangles.

7. a) Five



b) Nine



<table>
<tr>
<td>8. Six planes of symmetry; Sample response:
Each plane passes through an edge of the tetrahedron and the midpoint of the of the opposite edge.
</td>
<td>9. a) A circle has an infinite number of lines of symmetry.

b) A cylinder has an infinite number of planes of symmetry on the base corresponding to the infinite number of lines of symmetry in the circle base.
It also has one more plane that cuts across the lateral curved surface

c) A cone has an infinite number of planes of symmetry on the base corresponding to the infinite number of lines of symmetry in the circle base.</td>
</tr>
</table>

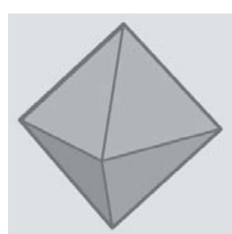
Supporting Students

Struggling students

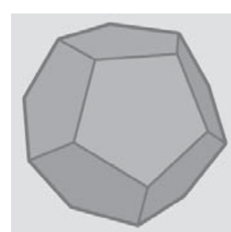
Many students with poor visualization skills will struggle, particularly with the 3-D symmetry. Some will benefit from having models to work with. Others who think more analytically might be benefit from counting to determine numbers of planes of symmetry; they can count how many total lines of symmetry can be found on the faces of the shapes and use that information to help them.

Enrichment

Students might be interested in investigating planes of symmetry for more complicated shapes than prisms or pyramids. For example, they might consider octahedrons or dodecahedrons.



Octahedron



Dodecahedron

8.1.2 2-D and 3-D Rotational Symmetry

Curriculum Outcomes	Outcome relevance
10-E3 2-D and 3-D Shapes: explore properties and test conjectures • compare 2-D and 3-D rotational symmetry • compare mirror and rotational symmetry	Symmetry is a mathematical topic that is particularly relevant to our everyday lives. It helps us understand many of the natural and human-made objects around us. Work on this important topic is a good way to develop students' mathematical reasoning.

Pacing	Materials	Prerequisites
2 h	• Square, rectangle, and cube (BLM net) • Regular polygons and other shapes (BLM) (optional) • 3-D prisms (actual or made from BLM nets) (optional) • Straws or long sticks (optional)	• performing a rotation in a plane

Main Points to be Raised

• Turn symmetry is also called rotational symmetry.

• A shape has rotational symmetry if, when it is turned around a centre of rotation (or turn centre), it fits into its own outline again before completing a full turn.

• The order of turn symmetry tells the number of times the shape fits into its own outline within a full turn. A shape with no rotational symmetry has turn symmetry of order 1.

• Rotational symmetry can refer to 2-D shapes or to 3-D objects. In the case of 2-D shapes, the turn is around a point called the centre of rotation. In the case of 3-D objects, the turn is around a line called the axis of rotation.

• An axis of rotation always connects two of the following: a vertex, the centre of a face, or the midpoint of an edge.

Try This — Introducing the Lesson

A. Allow students to try this alone or with a partner. Observe while students work. You might ask:

• *Can the small block fit if you turn it upside down? Will it look right?* (Yes, it will look right since the cross looks the same upside down and it will still be in the front.)

• *Can the small block fit if you turn it backwards? Will it look right?* (It will fit but it will not look right. You will not be able to see the design as the instructions say you have to.)

• *Can the small block fit if you turn it a quarter turn to the left? Will it look right?* (Yes, it will look right since the cross looks the same if you turn it a quarter turn and it will still be in the front.)

The Exposition — Presenting the Main Ideas

• Hold up a square shape, for example, a square piece of paper. Have a student trace the square on the board as you hold it. Make a mark on the top edge only.

• Place your finger on the centre of the square and turn it a quarter turn. Have students notice that the square still fits into the outline. Explain that you held the shape at the turn centre and that because the shape fit back into its outline without going all the way around, it has turn or rotational symmetry.

• Ask students to predict how many ways the square can fit into its outline before you turn it fully around (four times). Demonstrate that this is the case. Tell them that because the shape fits four times, it has turn symmetry of order 4. Have them see that if you place your finger on the shape at a different point, the shape will not turn to fit into its outline, but this does not matter. As long as there is some point where it can be turned to fit into its outline in less than a full turn, the shape has turn symmetry.

• Now hold up a rectangle that is not square. Ask students to predict whether it has turn symmetry and what the order would be (Yes, it has turn symmetry of order 2.).

• Discuss why the order is no longer 4—there are two pairs of different equal sides and not four equal sides, so quarter turns will not result in a fit within the outline.

• Now hold up a cube and ask students what they think rotational symmetry might mean in relation to a cube.

• Demonstrate by placing a stick through the centre of two opposite faces. Mark one of the other faces. Turn the cube a half turn around, keeping the stick in place. Have students notice that what they see looks exactly the same as what they saw before the turn, but the marked face has moved.

• Have students read through the exposition. See if there are questions. You may wish to demonstrate 3-D rotational symmetry with a rectangular prism as well.

Revisiting the Try This

B. This question will reinforce the connection between the **Try This** and the main ideas presented in the exposition.

Using the Examples

• Pose the question in **example 1** on the board. Have students work in pairs to try to solve the problem. Then suggest they read **example 1** in the text. Make sure students understand that a parallelogram has turn symmetry but not reflectional symmetry.

• Next, pose the question in **example 2**. Hold up the shape or, if possible, provide prisms for small groups of students to work with. After they arrive at their own solutions, suggest that they read **example 2** in the text.

Practising and Applying

Teaching points and tips

Q 2: Ask students why the answer would be different if the polygons were not regular.

Q 3: You may have to remind students of the types of triangles they might consider: equilateral, isosceles, and scalene.

Q 5 to Q 10: If possible, provide actual shapes to pairs or small groups of students.

Q 5: Remind students that the axes of rotation always connect two of: vertices, centres of faces, or midpoints of edges and that the connected points are normally "opposite" each other.

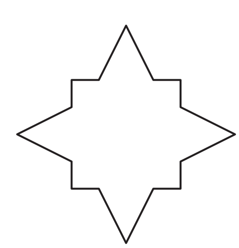
Common Errors

Many students have difficulty visualizing 3-D symmetry. They often confuse reflectional and rotational symmetry. Keep reminding students of the difference between these.

Suggested assessment questions from Practising and Applying

Question 2	to see if students can generalize about the rotational symmetry of regular polygons
Question 4	to see if students can create shapes with a particular order of turn symmetry
Question 7	to see if students can determine the axes of rotation for 3-D shapes
Question 10	to see if students can recognize and explain the underlying principles of 3-D rotational symmetry

Answers

A. Four **B. i)** Rotational symmetry is about turning something so it looks the same and that is the idea you need to fit the small cube in the hole in more than one way.	**B. ii)** Four; it looks the same, or fits the square hole four times during a complete rotation.
1. a) 3 **b)** 5 **c)** 6 **d)** 8 **2.** The order of turn symmetry is the same as the number of sides. You can divide any regular polygon with n sides into n congruent triangles. If you were to focus on one of the triangles while turning the polygon, that triangle would match each other triangle n times. **3.** No. Isosceles triangles and scalene triangles have order 1 and equilateral triangles have order 3. There are no other kinds of triangles.	**4.** *Sample response*: 

Answers [Continued]

5. a) Opposite vertices, midpoints of edges to midpoints of diagonally opposite edges, and centres of faces to centres of opposite faces.

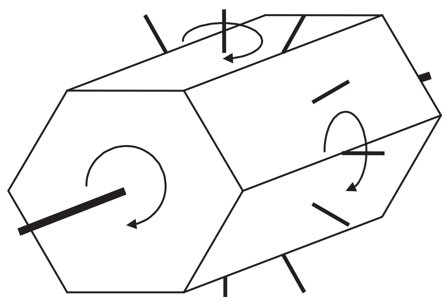
b) Thirteen

6. Order of 8 around an axis passing through the centre of the base and the apex of the pyramid.

7. a) Seven

b)

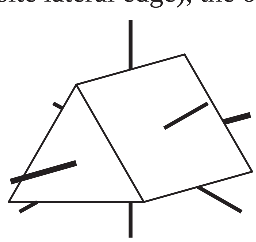
- Using the axis that passes through the centres of the bases, the order of turn symmetry is 6.
- Using any of the other axes (one passes through the centres of opposite lateral faces and the others pass through the midpoints of opposite lateral edges), the order of turn symmetry is 2.



8. a) Four

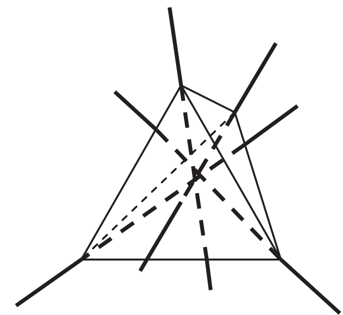
b)

- Using the axis that passes through the centres of the bases, the order of turn symmetry is 3.
- Using any of the other axes (each of the other axes passes through the centre of a lateral face and the midpoint of the opposite lateral edge), the order of turn symmetry is 2.

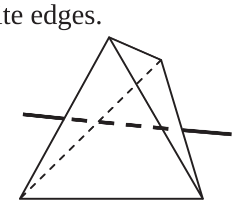


9. a) Seven

b) Four axes of rotation (each passes through a vertex and the centre of the opposite face) of order 3.



Three axes of rotation of order 2 (each axis of rotation passes through the midpoint of an edge and the midpoint of the opposite edge). One is shown below, but there are two others since there are three pairs of opposite edges.



10. Both have an infinite number of axes of rotation:

- For a cylinder, you can visualize an axis that passes through the centres of the bases, but you can also visualize an infinite number of axes that pass from any point on the curved lateral surface at the "equator" to a point on the opposite side.
- For a sphere, you can visualize an infinite number of axes that pass through the centre of the sphere from any point on the surface to a point on the opposite surface.

Supporting Students

Struggling students

• Struggling students will benefit greatly from working with actual shapes, especially if they are provided with sticks or similar objects to serve as axes of rotation. They may have more difficulty coming to generalizations, as is required in **questions 2, 3, and 10.** Encourage them to work with a partner for those questions.

• Help students use their models by showing them how to mark a face to keep track of where it goes.

Enrichment

Some students might come to generalizations about the order of rotational symmetry for all regular prisms, regular pyramids, and the other Platonic solids (octahedron, dodecahedron, and icosahedron).

8.1.3 Reasoning

Curriculum Outcomes	Outcome relevance
9-E1 Congruent Triangles and Angle Properties: informal deductions • distinguish between inductive and deductive reasoning using both mathematical and non-mathematical reasoning **10-E2 Geometric Reasoning: inductive and deductive** • use inductive and deductive reasoning in situations such as generalizing relationships and proving theorems	Reasoning is at the heart of mathematics. Recognizing good reasoning, whether inductive or deductive, is an important skill for students to develop.

Pacing	Materials	Prerequisites
2 h	• Protractors • 3-D shapes (BLM nets) (optional)	• concepts of rotational and reflectional symmetry • concept of supplementary angles • sum of angles in a triangle • familiarity with the terms *scalene, pyramid, prism, right triangle*

Main Points to be Raised

• Inductive reasoning is reasoning based on examples that suggest something might be true more generally. For example, if you measure the angle sums in several quadrilaterals and the total is always 360°, you might reason, inductively, that this will be true for all quadrilaterals.

• Deductive reasoning is based on known or assumed principles. For example, if you know (although your knowledge may be informal) that the sum of the angles in a triangle is 180°, you can reason, deductively, that the sum of the angles in any quadrilateral is 360°. This is because you can analyse that every quadrilateral can be broken into two triangles.

• When you believe something to be true, but you are not sure, your belief is a hypothesis or a conjecture. If you can use deductive reasoning to prove that it must be true, the conjecture becomes a theorem.

• If you use inductive reasoning to come to a conclusion and you check your conclusion using yet another situation, you are verifying your conjecture. If you find a situation that shows your conjecture was incorrect, you have found a counterexample.

• To disprove a conjecture, you need only one counterexample. To prove a conjecture is true, you have to either examine every possible case or use deductive reasoning.

Try This — Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. Observe while students work. You might ask:

• *Why might you say that Buthri is using personal experience or examples?* (Buthri is suggesting that this is something she has seen many times and that is why she thinks it is true.)

• *Why might you suggest that Mindu is using a scientific fact?* (Mindu is talking about what he has learned about bodies; it is not based his own observation of bodies.)

• *How could Mindu have learned the fact that he used?* (Maybe by reading a science book.)

The Exposition — Presenting the Main Ideas

• Have students read through the exposition.

• Go over the example about the polygon with them on the board to make sure they understand it.

Revisiting the Try This

C. This will reinforce the connection between the **Try This** and the main ideas presented in the exposition.

Using the Examples

• Pose the parts of **example 1** aloud, one at a time, with the students. For each statement, ask them to tell whether they think they are seeing inductive or deductive reasoning and why. They can check their answers by reading **example 1.**

• Ask students whether they think there are shapes that have one kind of symmetry (reflectional or rotational) and not the other. Encourage students to provide examples (e.g., a parallelogram has rotational, but not reflectional, symmetry or a tall isosceles triangle has reflectional, but not rotational, symmetry). Then ask them to read through **example 2**. Point out how, if we see something a few times, we may think it is always true, but if we can find one example where it is not true, we know that our conjecture was not true.

• Ask pairs of students to read through **examples 3 and 4** and discuss them together. Answer any questions they might have.

• Make sure students understand that deductive reasoning requires them to prove that something is true in an arbitrary case. For example, if it is a fact about all triangles, it has to be true for right, acute, and obtuse triangles and for equilateral, isosceles, and scalene triangles. If it is a fact about all regular polygons, it has to be true for a polygon of *n* sides, even if you do not know the value of *n*. This requires you to analyse what makes a shape a polygon.

• Finally, have students work in pairs to provide examples, different from those already presented in the text, of each of these: inductive reasoning, deductive reasoning, a conjecture, a theorem, and a counterexample. Ask different pairs of students to share these examples.

• It is not required, or expected, that students will use traditional two-column proofs (one column labelled "facts" and the other column labelled "reasons") to demonstrate their deductive reasoning. What is expected is modelled in the text. These are sometimes called paragraph proofs.

Practising and Applying

Teaching points and tips

Q 2: If necessary, provide the information in the chart. First, though, remind students to think back to their work in **lesson 8.1.1**. Remind students that their work in the previous lesson was completely based on inductive reasoning. For **part d)**, students should understand that they have to analyse the components of a prism with an arbitrary base, e.g., a base with *n* sides.

Q 3: You may need to check that students understand the question, e.g., that a prism with, say, a square base has one more face than a pyramid with that same base. Again, students must assume the shape has an arbitrary *n*-sided base.

Q 5: Make sure students do not test this conjecture using only rectangles.

Q 6: Students should actually draw shapes and measure the angles with a protractor.

Common Errors

• Some students will find it difficult to distinguish inductive from deductive reasoning because they view the principle from which they are deducing as an example. You may have to remind them that mathematicians have chosen to simply accept certain "truths" from which other information can be deduced (these are called axioms). An example is the principle of commutativity for adding two numbers.

• Many students will have difficulty stating their assumptions clearly enough that other information can be deduced. You might suggest they use a format such as the one shown to the right. However, students still need to be able to justify their assumptions.

I am assuming these facts:
•
•
•

Suggested assessment questions from Practising and Applying

Question 1	to see if students can distinguish between inductive and deductive reasoning
Question 3	to see if students can use deductive reasoning to learn about geometric shapes
Question 7	to see if students can use a counterexample to disprove a conjecture
Question 8	to see if students understand and can communicate about the fundamental difference between inductive and deductive reasoning

Answers

A. i) She is using examples of people she knows who have given birth.

ii) He is using the knowledge that women have different organs than men, making it possible for woman to give birth and impossible for men to give birth.

B. *Sample response*:
I know the sun will rise tomorrow. I am convinced this is true because I have noticed the sun every morning.

C. i) Inductive; she based her conclusion on her experience, or examples.

ii) Deductive; he based his conclusion on his knowledge of human anatomy.

1. a) Deductive **b)** Inductive

c) Inductive **d)** Deductive

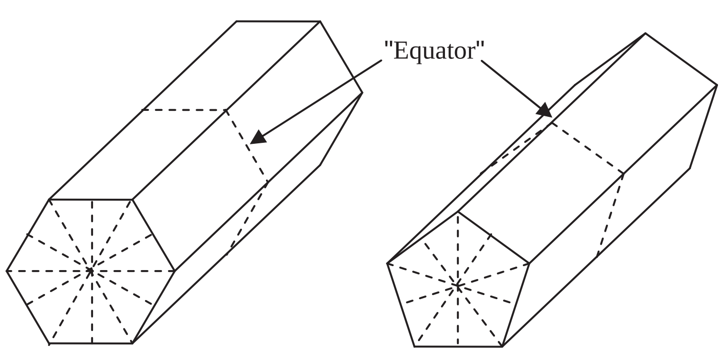
2. a)

Number of sides in the polygon base	Number of planes of symmetry
3	4
4	5
5	6
6	7

b) *Sample response*:
The number of planes of symmetry of a regular polygon-based prism is always one more than the number of sides in the polygon base.

c) I used the examples in the chart to come up with the conjecture.

d) If a prism has an n-sided base, there are n lines of reflection in the base. If each line is extended into a plane of reflection, there are n planes of symmetry. In addition, there is one more plane that cuts through the "equator" of the prism.



2. d) [Cont'd] So, if there are n sides on the base of a regular polygon-based prism, there are $n + 1$ planes of symmetry.

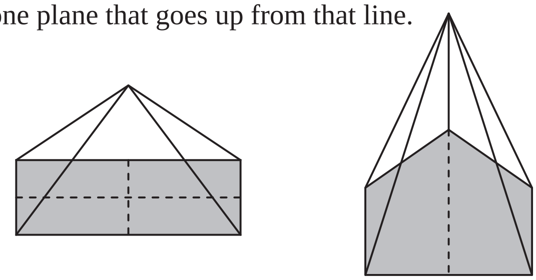
3. a) I checked using some prisms and pyramids and found that a pyramid always has one face fewer than a prism with the same base.

Number of sides in the base	Number of faces of prism	Number of faces of pyramid
3	5	4
4	6	5
5	7	6
6	8	7

b)
- For a prism, if the base has n sides, then there are n edges joining the vertices of the top base with the vertices of the bottom base. This results in n lateral rectangle faces along with the two faces that are bases. So, the total number of faces for a prism is $n + 2$.
- For a pyramid, if the base has n edges, then there are n edges joining the vertices of the base to the apex. This results in n lateral triangle faces along with the one face that is the base. So, the total number of faces for a pyramid is $n + 1$.
- Since $(n + 2) – (n + 1) = 1$, a pyramid always has one face fewer than a prism with the same base.

Answers [Continued]

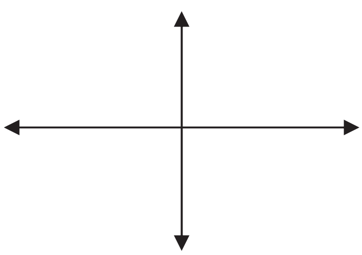
4. a) I used bases that were a rectangle (but not a square) and a pentagon. The pyramid with the rectangle base has 2 planes of symmetry that come up from the two lines of symmetry on the base. The same is true for the pentagonal base — there is one line of symmetry and one plane that goes up from that line.



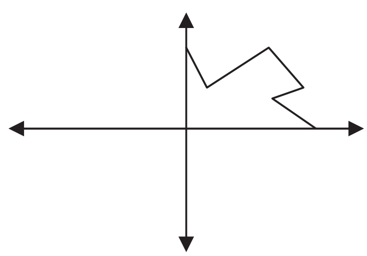
4. b) I realized that if I cut horizontally across a pyramid anywhere along the height, I would not get a plane of symmetry since the bottom would be much wider than the top. So, I knew each of the planes had to go through the top vertex and the base, cutting the base into congruent mirror halves. That means each plane cuts the base through one of the lines of reflection. So, a pyramid always has the same number of planes of symmetry as the number of lines of symmetry on the base.

5. I tried it with several shapes and it looked like it might be true.

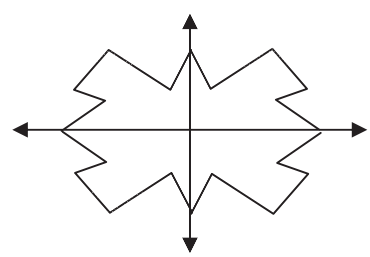
I imagined the two lines of reflection without describing exactly what the shape looks like. I noticed it looked like a coordinate system so I used coordinates.



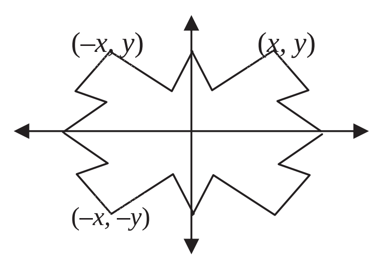
Then, I imagined an irregular design in the top right quadrant.



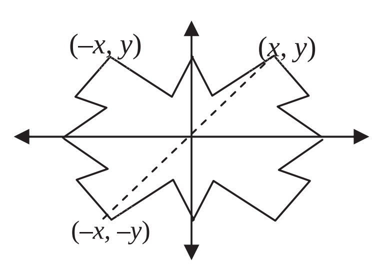
Then I looked at what the full shape would look like.



If I label a point in the top right quadrant (x, y), then I know there is a corresponding point in the bottom left quadrant that is ($-x$, $-y$). This is because the reflection in the vertical line moved it to ($-x$, y) and the reflection in the horizontal axis moved it to ($-x$,$-y$).



I can draw a line connecting the points (x, y), and ($-x$,$-y$) and it goes through (0, 0). That means if I do a half turn with a turn centre of (0, 0), each point (x, y) ends up as ($-x$,$-y$) and so the shape will have to fit into its outline.

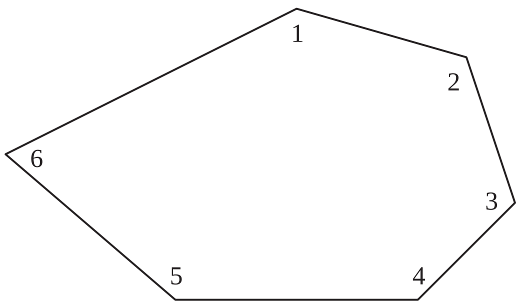


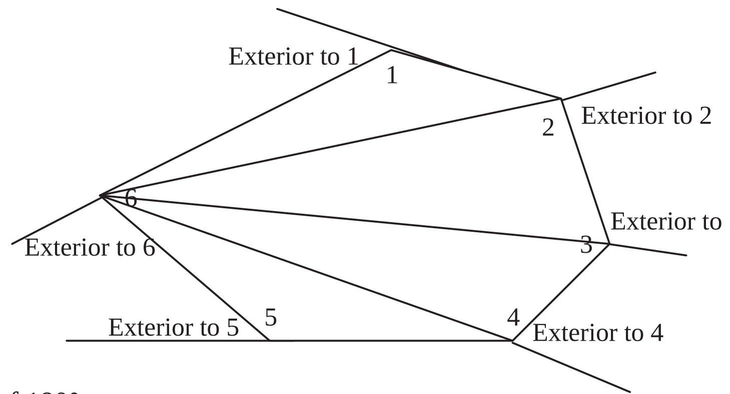
6. a) The sum is 360° each time.

b) My conjecture is that the sum of the exterior angles of any polygon is 360°.

6. c) Suppose the shape is a polygon with *n* sides and *n* vertices. I can mark the angles 1, 2, 3, …, *n*.

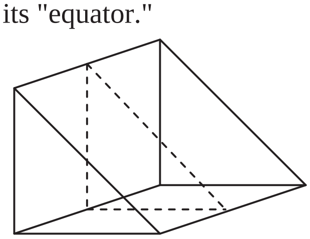
If there are *n* vertices, the shape can be divided into *n* – 2 separate triangles. This is because you can connect any vertex to all *n* – 1 other vertices to create *n* – 2 triangles.





Each triangle has an angle sum of 180°.
The total angle measure in *n* – 2 triangles is (*n* – 2) × 180.
Each exterior angle is 180° – the interior angle.
So the sum of the exterior angles is (180 – angle 1) + (180 – angle 2) + … + (180 – angle *n*).
This is 180*n* – (angle 1 + angle 2 + …. + angle *n*) = 180 *n* – 180(*n* – 2) = 360.

7. There may not be any planes of symmetry travelling through the bases because the bases are not symmetrical, but any prism always has a plane of symmetry through its "equator."



8. With deductive reasoning, you can be sure that it is true. With inductive reasoning, you must depend on examples, and there may always be a counterexample that you have not yet discovered.

Supporting Students

Struggling students

• Deductive reasoning is difficult for many students. Each problem is new and presents new challenges. For the first few exercises, you might suggest the steps students might follow and have them simply provide the reasons before they try one on their own.

• You might have students work first exclusively on the 2-D problems (**questions 5 and 6**) before introducing the 3-D problems.

• You might also start students with simpler proofs, e.g., proving that the sum of the angles in a square is 360°, proving that any rectangle has rotational symmetry, proving that any point on the perpendicular bisector of a segment is equally far from the end points of the segment, or proving that any isosceles triangle has reflectional symmetry.

Enrichment

Students might be interested in learning about *Euclid's Elements* and how geometry was developed deductively based on his stated axioms. Others might be interested in investigating deductions in geometries with other axiomatic systems, either spherical geometry or hyperbolic geometry.

Chapter 2 Constructions

8.2.1 EXPLORE: Rigidity

Purpose	Lesson relevance
• Consider a physical property of triangles (rigidity) that makes triangles unique	This optional lesson allows students to experiment physically with the concept of rigidity. The rigidity of triangles has many real-world applications.

Pacing	Materials	Prerequisites
45 min	• Sticks of varying lengths • Tape (or another material to attach sticks, e.g., putty)	• meaning of the terms *rhombus, pentagon,* and *right isosceles triangle*

Main Points to be Raised

• Triangles are the only 2-D polygons that are rigid. All triangles are rigid.

• Any 3-D shape with exclusively triangular faces is rigid since each face is rigid.

Note: The word "Constructions" has been chosen for the title of this chapter. This term often evokes the idea of ruler and compass constructions, and many of the constructions in this chapter will be of that sort. However, the term "construction" is also used in a broader sense to mean building or creating shapes. In this lesson, students construct 2-D and 3-D shapes with sticks and tape. In the ***Connections*** *feature, they use paper folding constructions.*

Exploration

• Before inviting students to work on **parts A, B, and C**, show them how to push over the square to make a non-square rhombus and model the construction of the skeleton of the cube. (A square is a special kind of rhombus.)

Provide pencils or sticks to serve as edges of shapes, as well as some material for joining them.

• Ask students to work with a partner or in a small group. Observe while students work. You might ask:

- *I notice you made only isosceles triangles. How can you make a scalene triangle?* (I need to use sticks of three different lengths or use the tape to join the sticks to make different lengths.)

- *What shapes did you create by adding the diagonal to the square?* (I ended up with two triangles.)

- *Why do you think it was not rigid when you added just one diagonal to the pentagon? How many diagonals did you need to add?* (Because even though there was one triangle, there was also a trapezoid, which is not rigid. I needed two diagonals to make triangles, which are always rigid.)

- *How did you test its rigidity?* (I tried to push over one vertex to see if it moved.)

Observe and Assess

As students are working, notice:

• Are they creating the required shapes correctly?

• Are they testing appropriately for rigidity?

• Do they make reasonable predictions before they test their models?

• Do they distinguish correctly between inductive and deductive reasoning?

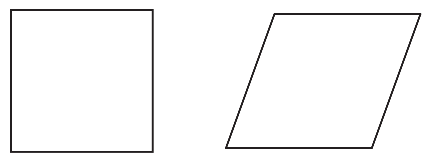
Share and Reflect

Ask students to share their responses. In particular, ask that students identify and describe the 3-D shapes they made. You might also ask:

• *If you make 2-D shapes out of stone, would you be able to test their rigidity?* (No. You cannot push a shape to test its rigidity if it is made out of stone; it is the stone that is rigid, not the shape.)

• *Why do you think someone building a structure might want to know which shapes are rigid?* (If you are building a structure, you do not want it to collapse.)

Tell students that the reason triangles are rigid is that given the three sides of a triangle, the angles formed by those given lengths are fixed. In other words, there is only one set of angles that go with any particular set of three sides of a triangle; there is no way you can create a triangle with different angles by pushing it. This is not the case with, for example, quadrilaterals. You can have two quadrilaterals with the same side lengths but with different angles, e.g., as below:



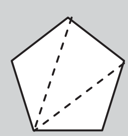
You may want to model why, for example, a triangular prism or a cube is not rigid. Students can imagine one face being attached to a wall and imagine how you could push the opposite face down and distort the original shape.

Answers

A. All triangles are rigid.

B. **i)** It makes the square a composite shape made up of two triangles, and all triangles are rigid. This is deductive reasoning because it is based on an assumed fact- that all triangles are rigid. But if I was not sure of the rigidity of triangles, it would still be inductive reasoning.

ii) I added two diagonals that both started at the same vertex and ended at adjacent vertices, dividing the regular pentagon it into three triangles. It worked because all triangles are rigid.



iii) I would add diagonals to divide each shape into triangles.

C. Any 3-D shape that has only triangle faces is rigid because each face is rigid.

D. i) They are the only rigid shape. They can be used to make other shapes rigid.

ii) Structures such as bridges and towers need to be stable and rigid.

Supporting Students

Enrichment

Some students might want to use the principle of rigidity of triangles to use sticks to design a bridge that will span a long distance and not buckle when there is weight applied to it.

8.2.2 Perpendiculars and Bisectors

Curriculum Outcomes	Outcome relevance
10-E4 Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors) • consider the concepts of perpendicular and angle bisectors • locate incentres and circumcentres and construct incircles and circumcircles using perpendicular and angle bisector constructions	Straight-edge and compass constructions are an important historical aspect of geometry. They focus students on the reasoning process that is integral to mathematics.

Pacing	Materials	Prerequisites
2 h	• Rulers • Compasses • Protractors	• conditions for congruence of two triangles • angle relationships when a transversal cuts parallel lines

Main Points to be Raised

• Angle bisectors, perpendicular bisectors, and a perpendicular to a line from a point can be constructed using just a straight-edge and compass. We usually use a ruler as the straight-edge, but the intent is to not use the measurements shown on the ruler. *Note: Students will have met these constructions in earlier classes, but they should be reviewed.*

• The circumcircle of a triangle passes through all three vertices. Its centre, called the circumcentre, is found at the intersection of the perpendicular bisectors of any pair of sides. This construction is based on the principle that the centre of a circle is equally distant from all points on the circle.

• The incircle of a triangle touches each side of the triangle at only one point. Its centre is called the incentre. The incentre is formed by the intersection of any two angle bisectors. This construction is based on the principle that any point on an angle bisector is equally far from the arms of the angle it bisects. This is based on principles related to the congruence of triangles.

Note: The next three points are optional, but might be valuable to raise:

• Angle bisection is based on the principle that triangles with matching sides (SSS) are congruent.

• The construction for a perpendicular bisector is based on the principle that the diagonals of a rhombus form a right angle. This is based on the fact that triangles are congruent with SSS or ASA and on the equality of alternate interior angles when parallel lines are cut by a transversal.

• The construction of a perpendicular to a line through a point is based on the principle that all points on a perpendicular bisector are equally far from the two ends of the segment it bisects. This, in turn, is based on the principle that SSA results in congruent triangles.

Try This — Introducing the Lesson

A. and B. Allow students to try these alone or with a partner. They can use rulers. For **part B**, make sure that they measure the distance from a point to the line on a perpendicular.

Observe while students work. You might ask:

• *How do you know your point is just as far from P as from Q?* (I measured with my ruler.)

• *What did you notice about all five points?* (They form a line.)

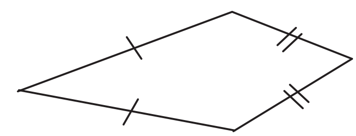
• *Why can the point not be closer to that line?* (The point has to be the same distance from both lines. If I move it there, it will be too close to one of the lines compared to the other.)

The Exposition — Presenting the Main Ideas

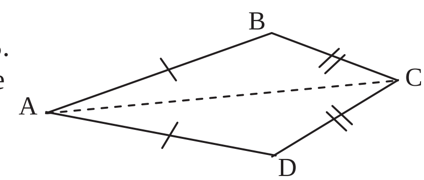
• There is a great deal of material in the exposition, although some of it is review. It is best to go through the material with the whole class. In this way, you can add the underlying explanations about how the constructions work if you feel your students are prepared to understand the additional background.

• Draw an angle on the board. Demonstrate the steps on **page 285** that show how to create an angle bisector. You may choose to clarify why the construction makes sense by going through the steps shown below.

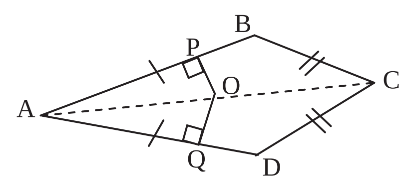
- By following the steps in the text, you have created this shape with equal sides indicated.



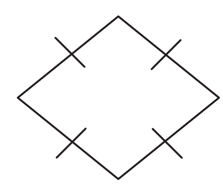
- By adding a line you can see that $\Delta ABC \cong \Delta ADC$ by SSS. That means the two angles created at A are equal and so the angle is bisected.

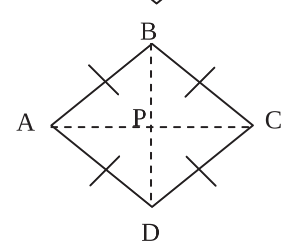


• Point out that any point on the angle bisector is equally distant from the two arms of $\angle A$ by drawing this diagram.
- The angles at P and Q have to be right angles since that is what is meant by distance to a line.
- $\Delta APO \cong \Delta AQO$ based on ASA.



• Draw a line segment on the board. Demonstrate the steps on **page 286** that show how to create a perpendicular bisector. If your students are ready, help them understand why the construction makes sense by going through the steps shown below.
- By following the steps in the text, you have created a rhombus. You know it is a rhombus because all its sides are equal. That means it is also a parallelogram.
- By adding the diagonals, you can see that $\angle PAB = \angle PCD$ and $\angle ABP = \angle CDP$ because they are alternate interior angles when parallel lines are cut by a transversal. This means $\Delta ABP \cong \Delta CDP$ based on ASA. Therefore AP = CP. Because of this, $\Delta ABP \cong \Delta CBP$ based on SSS. This means the two angles $\angle BPA$ and $\angle BPC$ must be equal. Since they total 180°, each must be 90°. You created a perpendicular bisector.





• Make sure students understand that any point on a perpendicular bisector is equally far from the two endpoints of the segment bisected. This is based on the congruence of ΔABP and ΔCBP because of SSA. [BP = BP, AB = CB, and there are two 90° angles at P.]

• Draw a line segment and a point off the segment. Demonstrate the steps on **page 286** that show how to create a perpendicular from a point to a line. If your students are ready, help them see that they have really used the previous construction for a perpendicular bisector to perform this construction. P has to be on the perpendicular bisector since it is equally distant from A as from B.

Revisiting the Try This

C. This question will reinforce the connection between the **Try This** and the main ideas presented in the exposition. You may want to let students know that some people call the type of problem in **Try This** a locus problem. In a locus problem, you are looking for all the positions that meet a certain condition.

Using the Examples

• Ask the students to close their books. Then present the problem in **example 1**. Let them try to solve the problem on their own. They can check their solution against the solution on **pages 287 and 288**. Some students may need you to lead them through the example again.

• **Examples 2 and 3** show students how to create circumcircles and incircles. Go through these examples with the students. It may be helpful to make sure students understand why each construction is valid.

• For the construction of the circumcircle, the circumcentre has to be equally distant from all three points or the circle will not go through all of them. Because all the points that are just as far from D as from E are on its perpendicular bisector, and all the points equally distant from E and from F are on its perpendicular bisector, the only possible position for the circumcentre is the intersection of any two perpendicular bisectors. It does not matter which two are chosen, since all three points are involved in any choice of two bisectors.

• The incentre has to be equally distant from each side of the triangle. Refer back to the last part of the explanation above for creating an angle bisector, if you have not already demonstrated it, to show why each point on an angle bisector is equally distant from both arms. By bisecting two angles, the point on the intersection of the bisectors is equally far from AB as from AC, and equally far from AC as from BC, so it is equally distant from all three sides. Students should understand that the reason for including Step 2 in the construction is to define the radius of the incircle so that it can be drawn.

Practising and Applying

Teaching points and tips

Q 1: Make sure students understand that they can use their rulers and protractors to create the triangle, but that they need to use only a straight-edge and compass to create the circumcircles. It is easiest if the sequence of constructions is AB followed by $\angle$ B and then BC for **part a)**, DE followed by $\angle$ E and then EF for **part b)**, and JK followed by $\angle$ K and then KH for **part c)**.

Q 2: Notice that it was important that different types of triangles were used in **question 1** in order for students to be able to answer this question.

Q 3: Ensure students know that they can use rulers and protractors to create the triangle, but should use only a straight-edges and compass to create the incircles.

Q 5: For **part a)**, refer students back to **example 1.** Students need to realize that the easiest way to construct a 45° angle is to create an isosceles right triangle. An easy way to construct a 30° triangle is to create an equilateral triangle and bisect one of the angles. These are the steps the students must follow:

For **part a)**:

- Draw line segment AC of any length, then construct the perpendicular bisector of AC. Construct a circle with centre at A and radius that is half AC.
- Construct a perpendicular to AC at A by extending AC an equal distance on the other side of A.
- Construct an angle bisector of the right angle formed by the perpendicular and AC at A.
- Vertex B is located where the circle's circumference intersects the angle bisector. The angle is half of a right angle, so it is 45°.

For **part b)**:

- Draw line segment AC of any length, then construct the perpendicular bisector of AC at D.
- Construct a circle with centre at A and radius AD, which is half of AC.
- Construct a circle with centre at D and radius AD.
- Name the intersection point of the two circles E.
- Connect A and E with line segment AE.
- $\angle$ DAE is 60° since ΔAED is equilateral since AE = AD and AD = DE. Bisect $\angle$ DAE to get a 30° angle. Locate B where the angle bisector intersects the circumference of the circle with centre at A.

Q 6b: For different types of quadrilaterals, students could consider squares, rectangles, parallelograms, trapezoids, kites, or arbitrary quadrilaterals.

Q 8: The proof need not be in formal two-column format.

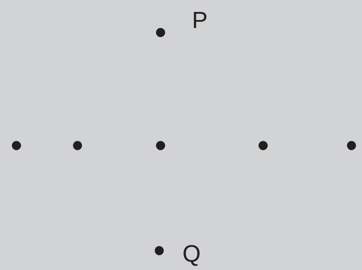
Suggested assessment questions from Practising and Applying

Question 1	to see if students can create a triangle and a circumcircle
Question 3	to see if students can create a triangle and an incircle
Question 5	to see if students can recognize triangle relationships and use that information to solve problems involving straight-edge and compass constructions
Question 6	to see if students can use inductive reasoning in a geometric setting to form and test a conjecture

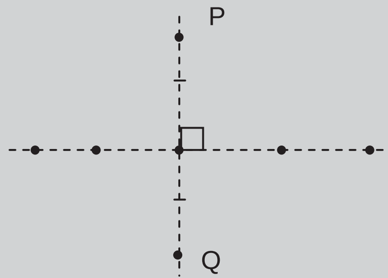
Answers

A. *Sample response*:

i)

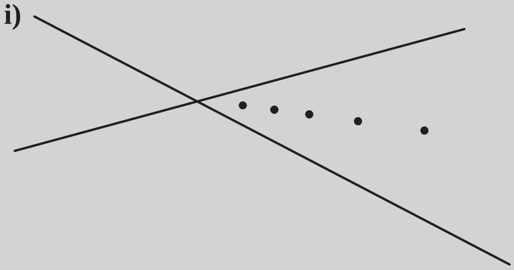


ii) I found the point halfway between P and Q by measuring. Then I used my ruler to find other points that worked.

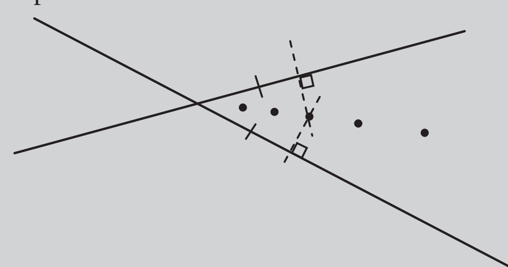


iii) I measured from each point to P and to Q to check.

B. i)



ii) I guessed that it would make sense to use points between the two lines. I tried a point that looked like it was equally far from both and checked. I adjusted the point when I had to.

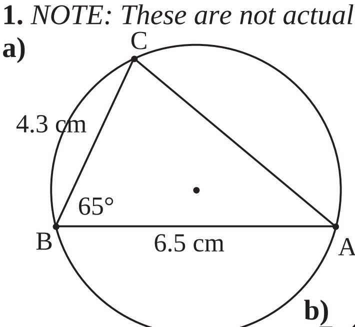


iii) I measured from each point to each line to check.

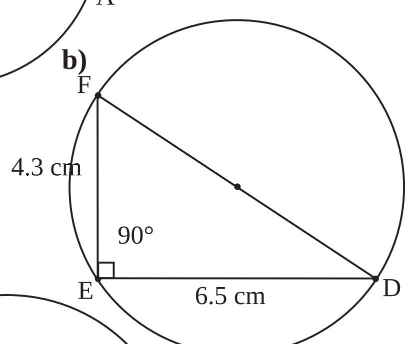
C. i) All five points were on the perpendicular bisector of PQ.
ii) All five points were on an angle bisector.

1. *NOTE: These are not actual size.*

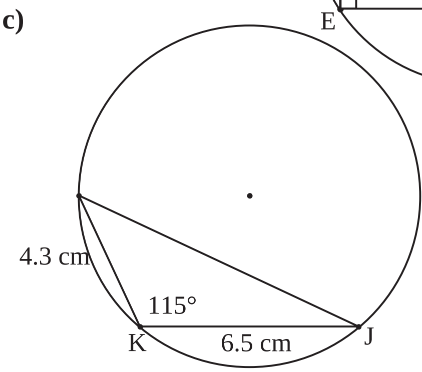
a)



b)



c)



2. a)
- In the acute triangle, the circumcentre is inside the triangle.
- In the right triangle, the circumcentre is on the hypotenuse.
- In the obtuse triangle, the circumcentre is outside the triangle.

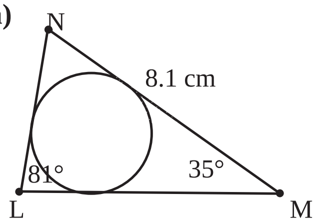
b) Obtuse triangles

c) *Sample response*:
I could verify it by induction, using many examples of different obtuse triangles.

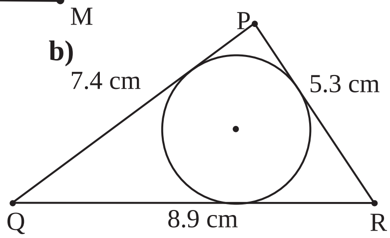
Answers [Continued]

3. NOTE: These are not actual size.

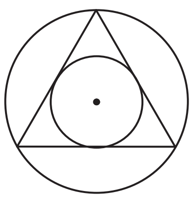
a)



b)

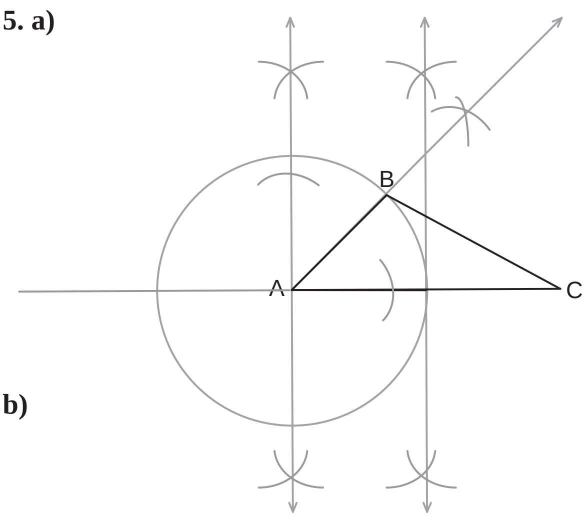


4. a)

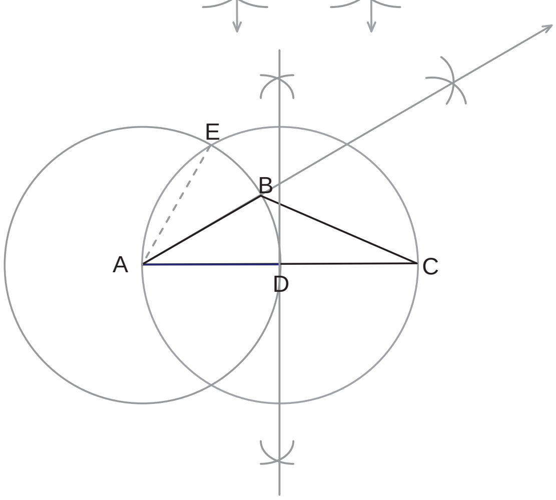


b) The circumcentre and incentre are the same point.

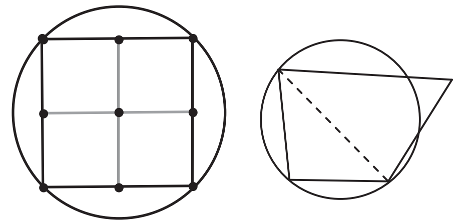
5. a)



b)

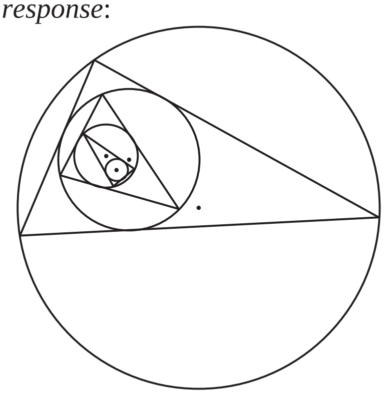


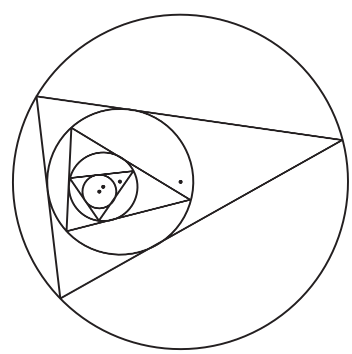
6. a) and b) *Sample response*:
I drew a circumcircle for three of the vertices and it went through the fourth one for a square. When I tried with a non-isosceles trapezoid, it did not work.



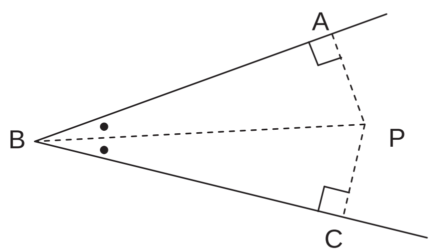
I conclude that not all quadrilaterals have circumcircles. One counterexample was enough to prove that this is true.

7. There is no pattern.
Sample response:





8. a) Construct perpendiculars from P to the arms of the angle to make points A and B. The angles at B are equal.

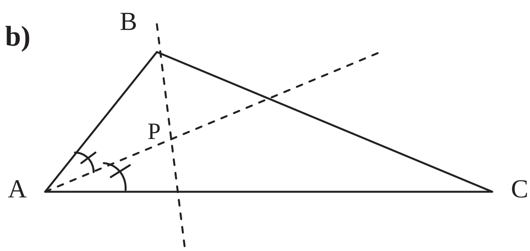


To prove that AP = CP:

Since, $\sin \angle B = \frac{AP}{BP}$ and $\sin \angle B = \frac{CP}{BP}$

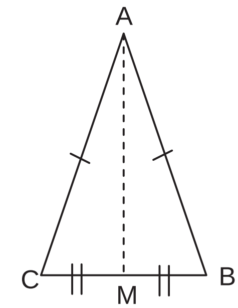
AP = BP × sin $\angle$ B and CP = BP × sin $\angle$ B.
Therefore AP = CP.

b)

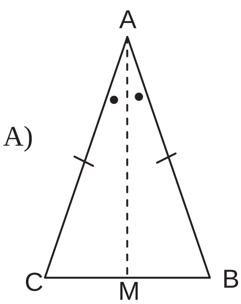


Since P is on the bisector of $\angle$ A, it is as far from AB as from AC. Since it is also on the bisector of $\angle$ B, it is as far from AB as from BC. That means P is equally far from all three sides, which makes it the incentre.

9. a) AM is the perpendicular bisector of BC.
ΔABM ≅ ΔACM because of SSS:
• CM = BM (M is the midpoint of CB because AM bisects CB)
• CA = BA (triangle is isosceles)
• MA = MA (shared side length)



AM is the angle bisector of $\angle$ A.
ΔABM ≅ ΔACM because of SAS:
• MA = MA (shared side length)
• CA = BA (triangle is isosceles)
• $\angle$ MAC = $\angle$ MAB (AM bisects $\angle$ A)



This proves that the line that is the perpendicular bisector must also be the angle bisector.

b) Equilateral triangles have this property for all three angles and opposite side lengths because the arms of the angle from each vertex are of equal length.

Supporting Students

Struggling students

• This is a difficult lesson. In each situation, the student must evaluate what is required, recall the necessary constructions, and perform the constructions correctly. You may wish to assign fewer problems for struggling students to try. You may suggest **questions 1, 3, 5, and 6.** They might also try **question 2** and then either **question 8** or **question 9**.

• Struggling students might focus on how to perform the constructions rather than on why the constructions work. However, it is important that they understand the principles behind at least some of the constructions.

Enrichment

Students might investigate a number of other possible ruler and compass constructions. For example, they might attempt constructions of some regular polygons, e.g., a regular hexagon or a regular pentagon.

8.2.3 Medians and Altitudes

Curriculum Outcomes	Outcome relevance
10-E4 Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors) • consider the concepts of medians and altitudes • locate centroids (centres of gravity) and orthocentres using median and altitude constructions	Straight-edge and compass constructions are an important historical aspect of geometry. They focus students on the reasoning process that is integral to mathematics.

Pacing	Materials	Prerequisites
2 h	• Rulers • Compasses • Protractors • Paper triangle (optional)	• formula for area of a triangle • constructions from **lesson 8.2.2**

Main Points to be Raised

• The median of a triangle is a line segment that joins a vertex to the midpoint of the opposite side. There are three medians in each triangle.

• The centroid, or centre of gravity, of a triangle is its balance point. It is located at the intersection of the medians of the triangle. Its location is fixed once the intersection of two medians has been found.

• To construct a median, you locate the midpoint of a side without measuring with a ruler. You can do that by constructing a perpendicular bisector of that side.

• The altitude of a triangle is the perpendicular from a vertex to the opposite side. It is sometimes inside the triangle and sometimes outside of it.

• The orthocentre of a triangle is the intersection of its altitudes. Its location can be found by determining the intersection of any two of the altitudes.

Try This — Introducing the Lesson

A. Allow students to try these alone or with a partner. Students can use rulers and compasses to draw their scale models and protractors to check the right angle for the height. Observe while students work. You might ask:

• *How did you make sure your triangles had the correct side lengths?* (I use 1 mm to represent 1 m to draw the scale model. After I drew the long side, I used a compass to help me make sure the other sides were the right lengths.)

• *How did you determine the area of* ΔBMC*?* (I drew a perpendicular from M to BC to be the height. I multiplied the height by 89 and divided by 2.)

• *Why might you have expected this result?* (After I did it, I realized I could have drawn the perpendicular from C down to AB. Then both triangles would have the same base and height, and therefore the same area.)

The Exposition — Presenting the Main Ideas

• Model for the students how to construct a median for a triangle. Make sure they understand that they are not to measure with their rulers to locate the midpoint of the opposite side, but, rather, they are to use the perpendicular bisector. You might draw a triangle on the board to show what to do.

• Next, model that if you construct two medians, they intersect at a point called the centroid. Show how the third median goes through exactly the same point. If you have a paper triangle, you might show how the triangle would balance on the point of a pencil if you placed it at that point, but not if you placed it at other points.

• Remind students of how to construct a perpendicular from a point to a line to model how to draw the altitudes of a triangle. Make sure students understand that for an obtuse triangle, an altitude might be outside of the triangle.

• Model the construction of two altitudes of a triangle, as shown on **page 292** of the text. Indicate or model that the third altitude will go through exactly the same point, called the orthocentre.

• Students might scan the ideas on **pages 291 and 292** to review what you modelled.

Revisiting the Try This

B. and C. These questions show students how they could have used the notions of median and altitude to complete **part A.** The questions also provide the opportunity to notice that the median of a triangle divides the triangle into two smaller triangles with the same area. This is further explored in **question 4** of the **Practising and Applying**.

Using the Examples

Write the problems from the two examples on the board. Have students work in pairs to try to solve the problems. They can then compare their solutions to the solutions on **pages 292 and 293**.

Practising and Applying

Teaching points and tips

Q 1: Students can use rulers and protractors to draw the triangle, but should use constructions to locate the centroids. The question is set up to show an acute, a right, and an obtuse triangle to prepare students to answer **question 2.**

Q 3: Students can use rulers and protractors to create the triangles, but should use straight-edge and compass constructions for the altitudes. There are three possible altitudes they can use. Answers might vary slightly, due to rounding.

Q 4, 5, and 7: Students should work on these on their own, but compare with a classmate's work.

Q 4: This question is important. It is designed to bring out the property that a median divides a triangle into two equal halves. This is fundamental to explaining why the centroid is a centre of gravity.

Q 6: This question is designed to bring out other reasons why equilateral triangles are special.

Q 7: The relationship in **part e)** is simply an interesting geometric fact.

Q 9: This question requires a lot of reasoning. Students might want to work in small groups or the question might be handled in a class discussion.

Common Errors

• Many students are limited in their notion of altitude. They are not comfortable with an altitude being outside of the triangle and they simply join a vertex to the midpoint of the opposite side. It is important to emphasize that it is possible for an altitude to be outside of a triangle.

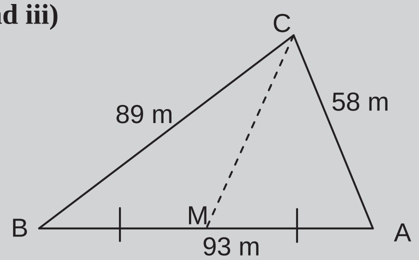
• It is important for students to be careful in their constructions. If they are not careful, some of the principles being explored will not be clear. For example, if medians are not drawn carefully, then the third median might not go through the intersection of the other two.

Suggested assessment questions from Practising and Applying

Question 1	to see if students can locate the centroid of a triangle
Question 4	to see if students can develop the reasoning as to why a median divides a triangle into shapes of equal area
Question 6	to see how well students use inductive reasoning to build a conjecture
Question 9	to see how well students use reasoning and communication to connect what they learned about medians to the notion of the centre of gravity

Answers

A. i) and iii)



ii) 2511 m^2 (2600 m^2, when rounded to the proper precision); base is 93 m, height is 54 m

A. iv) The area of each triangle is 1255 m^2 (1300 m^2 when rounded to the proper precision); base is 46.5 m, height is 54 m

v) The area of each triangle, ΔBMC and ΔAMC, is half the area of ΔABC.

B. The median from C to the midpoint of AB was used. I could have used the median from B to the midpoint of CA or the median from A to the midpoint of BC.

C. Altitude is 54 m

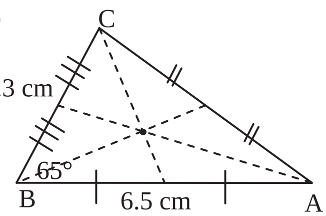
Answers [Continued]

1. *Notes:*

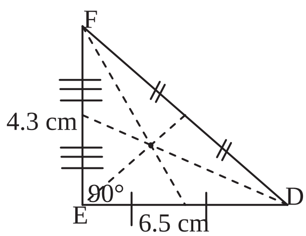
• *These shapes are not actual size and the third median is optional.*

• *For each diagram, the centroid is the intersection point of the medians indicated by a bold dot inside the triangle.*

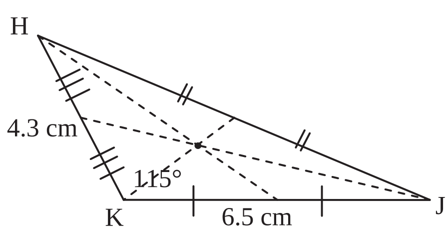
a)



b)

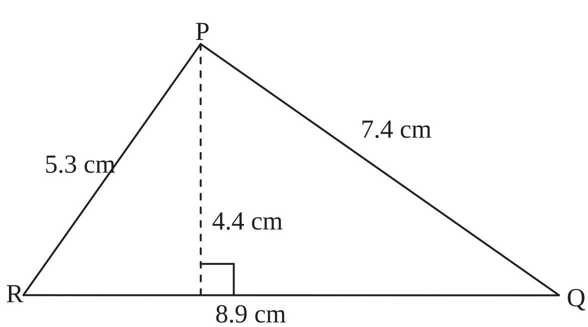


c)



2. No, it is impossible since the centroid is the intersection of the medians. A median is always inside the triangle because it travels from a vertex to the midpoint of its opposite side.

3. a) *Sample response*:

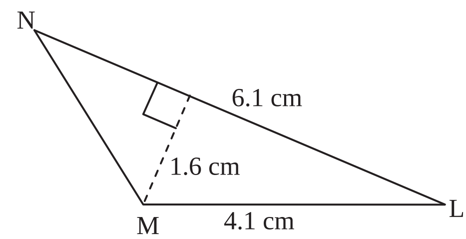


Using the altitude from P, $A = bh \div 2 = (8.9)(4.4) \div 2 = 19.58\text{ cm}^2 \approx 19.6\text{ cm}^2$

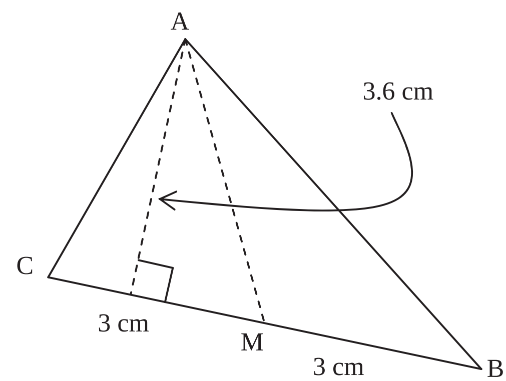
3. b) *Sample response*:

Using the altitude from M,

$A = bh \div 2 = (6.1)(1.6) \div 2 = 4.88\text{ cm}^2 \approx 4.9\text{ cm}^2$



4. a) *Sample response*:



b) ΔABC's altitude is also the altitude of ΔAMB and ΔAMC.

c) ΔAMB: $A = bh \div 2 = (3)(3.6) \div 2 = 5.4\text{ cm}^2$

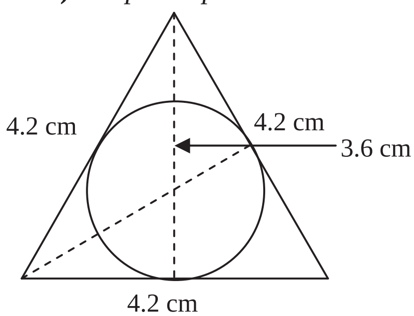
ΔAMC: $A = bh \div 2 = (3)(3.6) \div 2 = 5.4\text{ cm}^2$

The median divides the area of ΔABC in half.

d) The median of a triangle divided the area of the triangle in half regardless of the triangle's dimensions.

e) The median of a triangle divides the area of any triangle in half. I used inductive reasoning (through examples).

5. a) and b) *Sample response*:

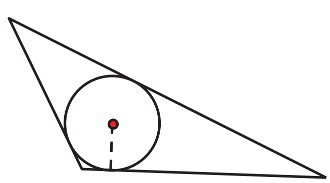


I estimate that the incircle covers more than half the triangle.

c) The area of the triangle is 7.6 cm^2

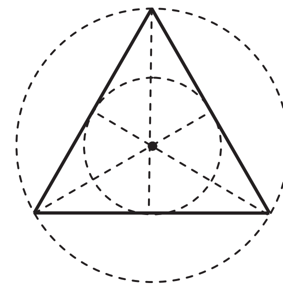
[$\frac{1}{2}$(4.2 cm)(3.6 cm) ≈ 7.56 cm^2 ≈ 7.6 cm^2].
The area of the circle is 4.5 cm^2 [π(1.2 cm)^2 ≈ 4.5 cm^2].
Therefore, the circle covers 4.5 ÷ 7.6 ≈ 59% of the triangle, so my estimate was reasonable.

d) *Sample response*:
These are the findings of my classmate.
Area of triangle is 7.85 cm^2 and area of incircle is 3.04 cm^2. The circle covers approximately 39% of the area of the triangle.



e) The percentage of the triangle's area covered by the incircle depends on the type of triangle. It can be more than half or less than half.

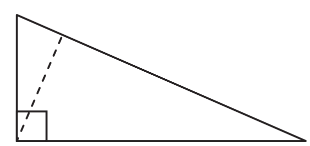
6. a), b), c) All four centres are in the same location in any equilateral triangle.



7. d) The ratio for PQ to PS and for PR to PT is the same, approximately 3:1.

e) No matter what the dimensions of the right triangle were, the ratio for PQ to PS and for PR to PT was always the same, approximately 3:1.

8. The two legs of the right triangle are also two of the altitudes.



9. Each median divides the area of the triangle in half. Therefore, the location where all the medians intersect has half the area on each side, no matter which way you are dividing the triangle in half.

Supporting Students

Struggling students

Remind struggling students to refer back to the previous lesson to recall how to create an incentre for a triangle (for **questions 5 and 6**). You might also choose to make copies of the shapes they must create in **questions 1, 3, 4, 5, 6, and 7** so that they can focus on the constructions.

Enrichment

Students might investigate the centres of gravity for other shapes, in particular, irregular shapes or different types of quadrilaterals.

CONNECTIONS: Paper Folding Constructions

Although the straight-edge and compass are the traditional construction tools in mathematics, mathematicians now explore many other tools. Some tools are technological and some are simple tools like a plastic mirror. This activity simply shows that constructions can be created using other tools, in this case paper folding.

Some students might be interested in learning more about origami, the Japanese art of paper folding.

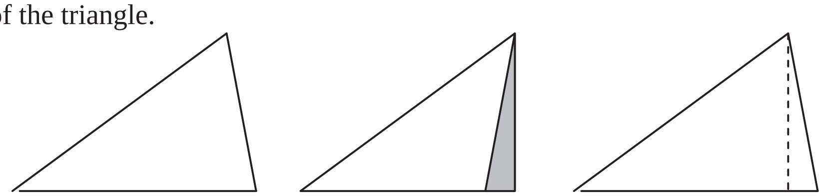
Answers

1. The circumcentre.

2. a) Fold the angle in half so the arms match and the fold goes through the vertex. The fold line is the angle bisector.

b) Fold to find the angle bisectors of two of the triangle's angles (using the method described in **part B i**). The incentre will be the intersection of the two fold lines.

c) Fold one side so the sides match and the fold also goes through the vertex opposite to it. The fold line is an altitude of the triangle.



d) Fold to find two altitudes of the triangle (using the method described in **part B iii**). The orthocentre will be the intersection of the two fold lines.

e) Fold the triangle so that two vertices match. The fold line is the midpoint of the side that connects those vertices. Unfold and fold again so that the new fold line travels through the midpoint of the side and the vertex opposite to it. The new fold line is a median of the triangle.

f) Fold to find two medians of the triangle (using the method described in **part B v**). The centroid will be the intersection of the two fold lines.

GAME: Balancing Triangles

This game is fairly quick. It provides practice with constructing a centroid. It reinforces that the centroid is indeed a centre of gravity.

UNIT 8 Revision

Pacing	Materials
2 h	• Rulers • Compasses • Protractors

Question(s)	Related Lesson(s)
1 – 3	Lesson 8.1.1
4 – 6	Lesson 8.1.2
7	Lessons 8.1.1 and 8.1.2
8, 9	Lesson 8.1.3
10	Lesson 8.2.1
11, 12	Lesson 8.2.2
13, 14	Lesson 8.2.3
15	Lessons 8.1.3 and 8.2.2
16	Lessons 8.1.3, 8.2.2, and 8.2.3

Revision Tips

Q 2: Remind students they might start with a rectangle and modify it. Other students might consider creating a shape in the first quadrant of a Cartesian plane. They can then reflect it in the *y*-axis and then reflect the combined shape in the *x*-axis to create the required shape.

Q 3: Students could consider prisms and pyramids to complete this question.

Q 8: Encourage students to consider some specific cases first.

Q 10: Omit this question if the **Explore lesson 8.2.1** was not done.

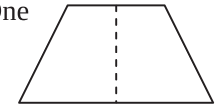
Q 11: Students will need to realize they should construct a 45° angle by creating an isosceles right triangle and then they can bisect it. The steps for **part a)** are shown here:

- Draw any line segment AC.
- Construct the perpendicular bisector of AC at D.
- Construct the perpendicular bisector of DC at E.
- Draw a circle with centre A and radius AE. Vertex B can be anywhere on the circumference of this circle and AB will be $\frac{3}{4}$ AC.
- Draw a perpendicular to AC at A. To do this you need to extend AC on the other side.
- Bisect the 90º angle formed at A to get a 45º angle and then bisect the 45º angle to get a 22.5º angle.
- Mark B where the second bisector intersects the circle.

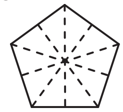
Q 13: Students will need to realize that $\angle$ M can be determined by subtracting the sum of the other two angles from 180°.

Answers

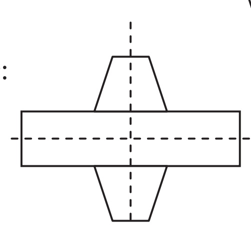
1. a) One



b) Five



2. *Sample response*:



3. *Sample response*:
A square based prism that is not a cube.

4. a) No turn symmetry, so the order of turn symmetry is 1.

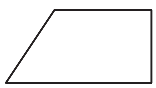
b) Turn symmetry of order 5 using the turn centre that is at the intersection of the five lines of reflection.

Answers [Continued]

5. a) The order of turn symmetry is 5 using the axis of turn symmetry that passes through the centre of the base and the apex of the pyramid.

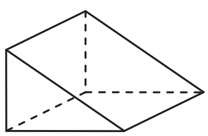
b) The order of turn symmetry is 5 using the axis of turn symmetry that passes through the centres of the two bases of the prism. The order of turn symmetry is 2 using any of the five axes of turn symmetry that pass through the centre of one of the lateral faces and the midpoint of the edge opposite to it.

6. a)



Non-isosceles trapezoid

b)



Scalene right triangle-based prism

7. a) A cone has an infinite number of planes of symmetry, each of which passes through the centre of the base circle and the apex of the cone.

b) A cone has an order of turn symmetry that is infinite, around the only axis of symmetry, which passes through the centre of the base circle and the apex of the cone.

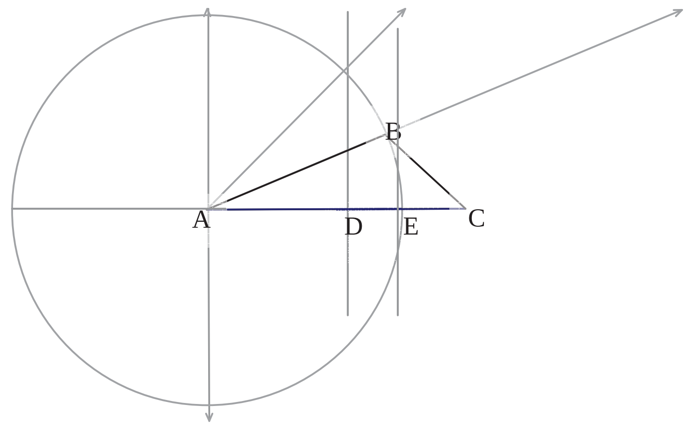
8. The number of edges of a pyramid is double the number of sides on the base.

Number of sides on the base	Number of edges on the pyramid
3	6
4	8
5	10
6	12

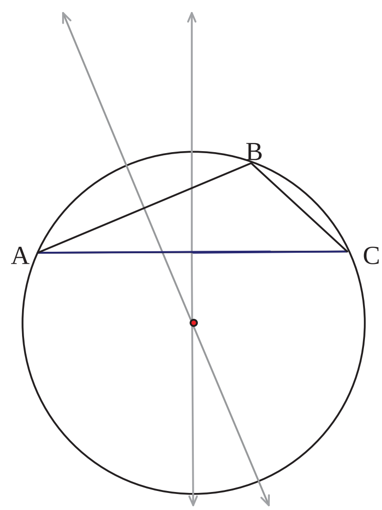
9. The order of turn symmetry is the same as the number of sides in the equilateral triangle. If you divide the triangle into three congruent triangles, by drawing lines from each vertex to the centre, and then turn the equilateral triangle around the centre, each small congruent triangles matches with the original triangle 3 times in a full rotation. This means the order of turn symmetry is 3, which is the same as the number of sides.

10. If she put a diagonal on every face, the prism would be rigid because each face would be made up of triangles.

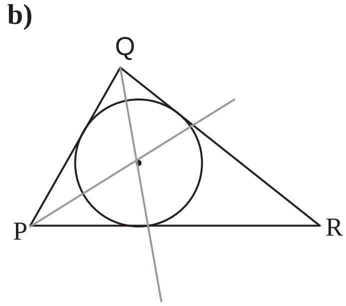
11.
a)



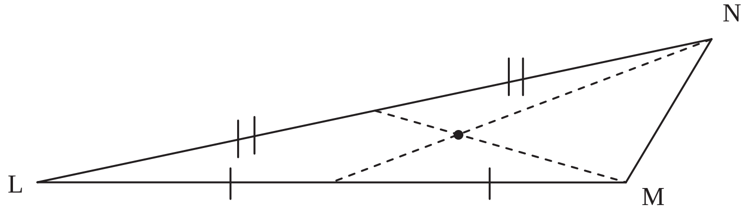
b)



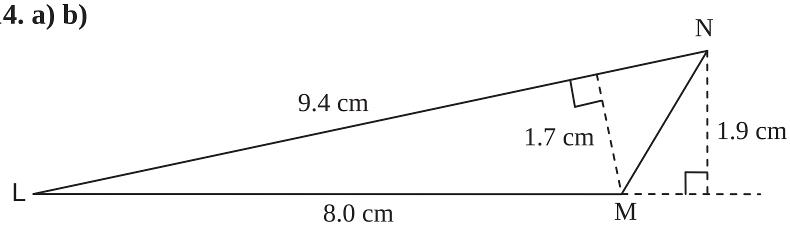
12. a) b)



13. a) b) c)
- Construct two medians. The intersection is the centroid:
- Construct perpendicular bisectors of two sides to find their midpoints.
- Draw the medians by joining each midpoint to the opposite vertex.



14. a) b)



Using the altitude from M, $A = bh \div 2 = (9.4)(1.7) \div 2 = 7.99 \approx 8.0\text{ cm}^2$
Using the altitude from N, $A = bh \div 2 = (8.0)(1.9) \div 2 = 7.6\text{ cm}^2$

b) The answers can be slightly different because of construction and measuring inaccuracies. If you could construct and measure perfectly, the answers would be the same.

15. A median divides a triangle into two triangles. Each of these has the same height. Each has a base that is half the base of the original triangle, so their bases are also equal. Base and height are the only two measurements needed to find the area of a triangle. Therefore, the two triangles have equal area.

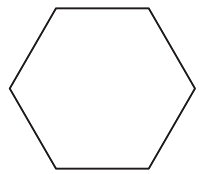
16. *Sample response*:
I could construct various kinds of isosceles triangles, including acute, right, and obtuse triangles, and then construct the three centres of each to see if they are collinear. This is inductive reasoning. (*Note: They are always collinear.*)

UNIT 8 Geometry Test

1. How many different lines of symmetry can be drawn in each shape? Sketch them.

a) rectangle

b) regular hexagon



2. How many planes of symmetry are there in a regular pentagonal pyramid? Sketch them.

3. Sketch a 2-D shape and a 3-D shape that have no reflectional symmetry.

4. Describe the rotational symmetry of each shape in **question 1**.

5. a) Describe the turn symmetry of a regular octagon-based pyramid.

a) Describe the turn symmetry of a regular octagon-based prism.

6. Describe the reflectional and rotational symmetry of a cylinder.

7. a) Write an example of inductive reasoning. The situation can be mathematical or not.

b) Write an example of deductive reasoning.

c) Explain the difference between these types of reasoning.

8. Use deductive reasoning to prove that an equilateral triangle has the same number of lines of symmetry as number of sides.

9. a) Draw ΔABC: AB = 2.8 cm, BC = 4.5 cm, and AC = 5.3 cm

b) Construct the circumcentre and the circumcircle of ΔABC.

10. a) Draw ΔPQR: PQ = 6.9 cm, QR = 8.3 cm, and $\angle Q = 62^\circ$

b) Explain how you could use constructions to locate the incentre of ΔPQR.

c) Construct the incentre and the incircle of ΔPQR.

11. a) Draw ΔLMN: LM = 9.7 cm, $\angle L = 32^\circ$, and $\angle M = 43^\circ$

b) Why does it make sense that the centroid of ΔLMN has to be inside the triangle?

c) Construct the centroid of ΔLMN.

12. a) Construct an altitude for ΔLMN from **question 11** and use it to measure the area of ΔLMN.

b) Construct a different altitude for ΔLMN and use it to measure the area of ΔLMN.

13. A conjecture is made that the location of a triangle's orthocentre can be inside, outside, or at a vertex, depending on the type of triangle. Explain how you could use inductive reasoning to prove or disprove the conjecture.

UNIT 8 Test

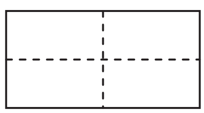
Pacing	Materials
1 h	• Rulers • Compasses • Protractors

Question(s)	Related Lesson(s)
1 – 3	Lesson 8.1.1
4 – 6	Lesson 8.1.2
7, 8	Lesson 8.1.3
9, 10	Lesson 8.2.2
11 – 13	Lesson 8.2.3

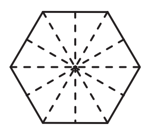
Select questions to assign according to the time available.

Answers

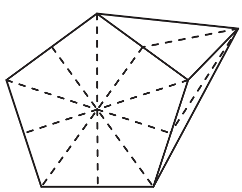
1. a) Two



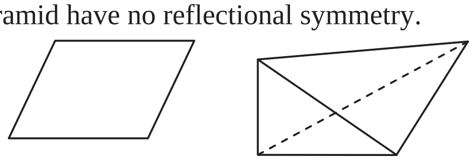
b) Six



2. Five planes of symmetry, as shown:



3. *Sample response*:
This parallelogram and scalene right triangle-based pyramid have no reflectional symmetry.



4. a) Rotational symmetry of order 2 using the centre that is at the intersection of the mirror lines.
b) Rotational symmetry of order 6 using the centre that is at the intersection of the mirror lines.

5. a) The order of turn symmetry is 8 using the pyramid's only axis of turn symmetry, which passes through the centre of the base and the apex of the pyramid.
b)
- The order of turn symmetry is 8 using the axis of turn symmetry that passes through the centre of both bases.
- The order of turn symmetry is 2 using any of the four axes of turn symmetry that pass through the centre of one of the lateral faces and the centre of the face that is opposite to it or any four axes of turn symmetry that pass through the midpoint of one of the lateral edges and the midpoint of the edge that is opposite to it.

6. A cylinder has an infinite number of planes of symmetry; each plane passes through the infinite number of reflection lines on both bases. It also has one plane of symmetry that is located parallel to the two base circles and halfway between them.
A cylinder has an order of turn symmetry that is infinite around the axis of rotation that passes through the centre of both bases. It also has turn symmetry of order 2 around any of an infinite number of axes that pass through the lateral surface of the cylinder and its centre at the "equator."

7. a) *Sample response*:
I know the rain will stop because it always does.

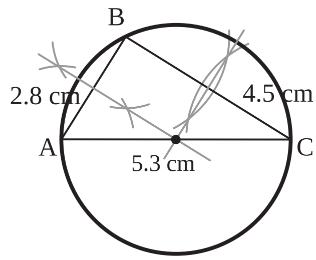
b) *Sample response*:
I know that this stove will not start burning because there is no kerosene.

c) With inductive reasoning you base your conclusion on your earlier experiences of the same phenomenon. With deductive reasoning you base your conclusion on information that is known or assumed to be true.

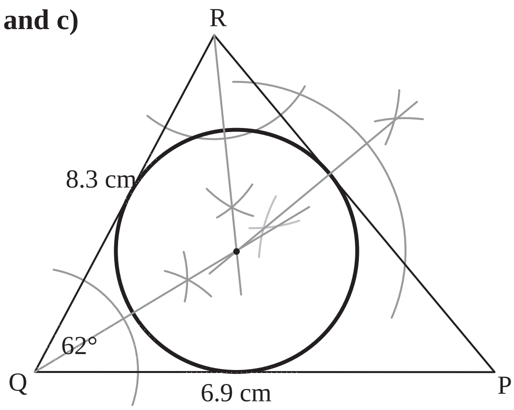
8. To show that equilateral triangles have three lines of reflection, construct a median that divides the triangle into two congruent right triangles (SSS). Since the two halves of the triangle are congruent and are mirror images, the median is a line of symmetry. Since you can repeat this with the other two medians, an equilateral triangle has three lines of symmetry.

Answers [Continued]

9. a) and b)



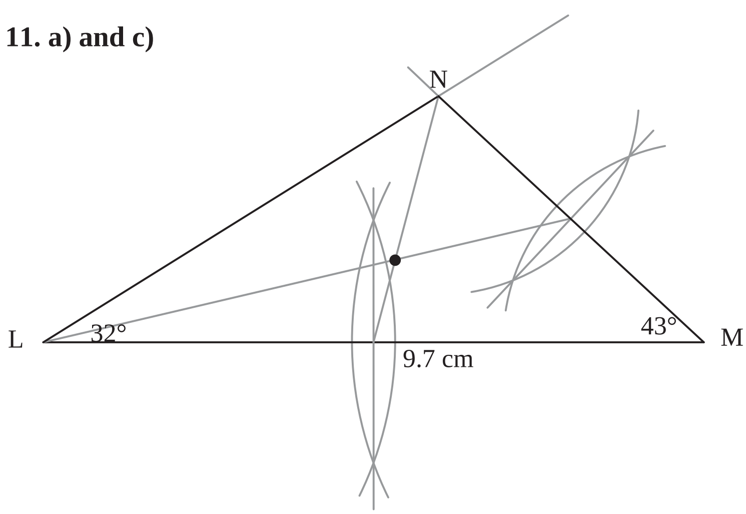
10. a) and c)



b)

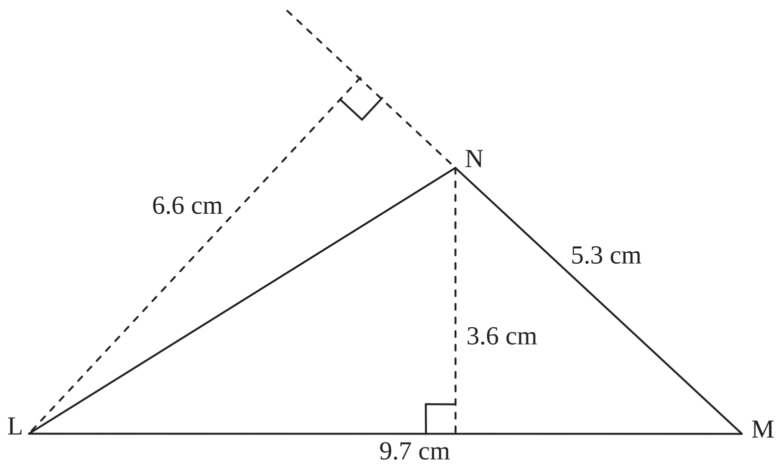
- Construct angle bisectors from two vertices.
- Construct a perpendicular from the intersection point of the angle bisectors to one of the sides.
- Construct a circle with centre at the intersection of the angle bisectors that passes through the intersection of the perpendicular and the side it was drawn to. That is the incircle. The incentre is the intersection of the angle bisectors.

11. a) and c)



b) *Sample response*:
Each median goes from a vertex to the midpoint of the opposite edge. The medians are contained by the triangle and must intersect at a point inside the triangle. This intersection point is the centroid.

12.



Using the altitude from N, $A = bh \div 2 = (9.7)(3.6) \div 2 = 17.46 \approx 17\ \text{cm}^2$.
Using the altitude from L, $A = bh \div 2 = (5.3)(6.6) \div 2 = 17.49 \approx 17\ \text{cm}^2$.

13. *Sample response*:
Draw various kinds of acute (equilateral, isosceles, and scalene), right (isosceles and scalene), and obtuse (isosceles and scalene) triangles, and construct two altitudes in each triangle in order to find each orthocentre.
Note that
- *the orthocentre of an acute triangle is inside the triangle*
- *the orthocentre of an obtuse triangle is outside the triangle*
- *the orthocentre of a right triangle is at the vertex that is the right angle*

UNIT 8 Performance Task — Exploring Triangle Properties

In this task, you will explore some properties of triangles.

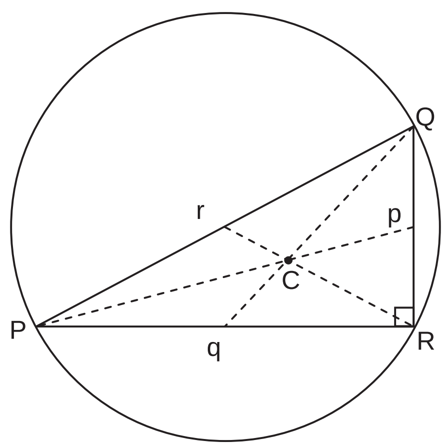
A. Construct any right triangle, ΔPQR, and perform the following constructions:

i) Construct the circumcentre and circumcircle.

ii) Find the midpoint of each side length and label it with the small letter of the vertex that is opposite to it. For example, if the vertex is P, label the opposite midpoint p.

iii) Construct all three medians, locate the centre of gravity, and label it C.

For example:



B. i) For your right triangle from **part A**, measure the hypotenuse and measure the distance from the triangle's centre of gravity to its circumcentre. What is the ratio of these distances?

ii) Repeat **part i)** for other right triangles until you can make a conjecture about the ratio of these distances in any right triangle.

C. Draw any triangle that is not a right triangle.

i) Make the constructions as described above in **part A**.

ii) Measure to find the ratio of the lengths of the two parts of any median (separated by the centre of gravity). For example, in the triangle above, you would find the ratio of RC to rC, PC to pC, and QC to qC.

iii) Repeat **parts i) and ii)** using different triangles until you can make a conjecture about the ratio of these distances in any triangle. Use examples of scalene, isosceles, and equilateral triangles.

D. Use your conjecture from **part C iii)** to prove deductively your conjecture from **part B ii)**.

UNIT 8 Performance Task

Curriculum Outcomes Assessed	Pacing	Materials
9-E1 Congruent Triangles and Angle Properties: informal deductions **10-E2** Geometric Reasoning: inductive and deductive **10-E4** Bisectors: examine intersection points (altitudes, medians, angle bisectors, and perpendicular bisectors)	1 h	• Rulers • Compasses • Protractors

How to Use This Performance Task

Before assigning the task, remind students of the valid conjectures they made and the sound reasoning they used in the **Practising and Applying** work throughout this unit. Do this to encourage them in their conjecturing and reasoning.

Then ask students to read the task and the assessment rubric. Ask if they have any questions before allowing them the time they need to perform the task.

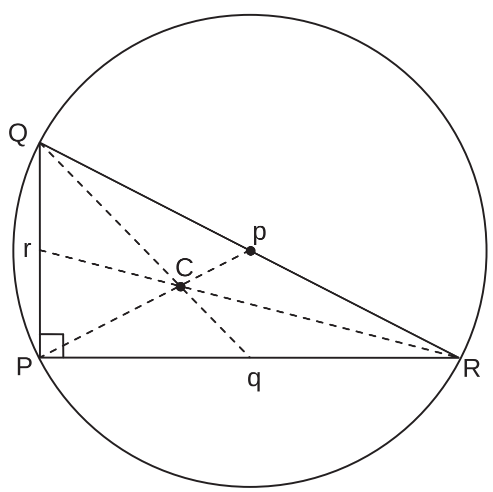
Notes:

• *In right triangles, the hypotenuse is six times as long as the distance from the circumcentre to the centre of gravity.*

• *In any triangle, medians are divided in the ratio of 2:1 by the centre of gravity.*

Sample Solution

A. Constructions are shown in the diagrams (students should show markings they used to construct each part). p is both the midpoint of QR and the circumcentre. This is directly related to the notion that a right angle subtends half a circle.



B. i) Cp = 1.1 cm and QR = 6.4 cm. The ratio is 6.4 ÷ 1.1 ≈ 6
ii) I conjecture that the ratio will always be 6:1.

C. i) PC ÷ Cp = 2.1 ÷ 1.1 ≈ 2, QC ÷ Cq = 2.7 ÷ 1.4 ≈ 2, and RC ÷ Cr = 3.9 ÷ 2.0 ≈ 2.
ii) I conjecture that the ratio will always be 2:1.

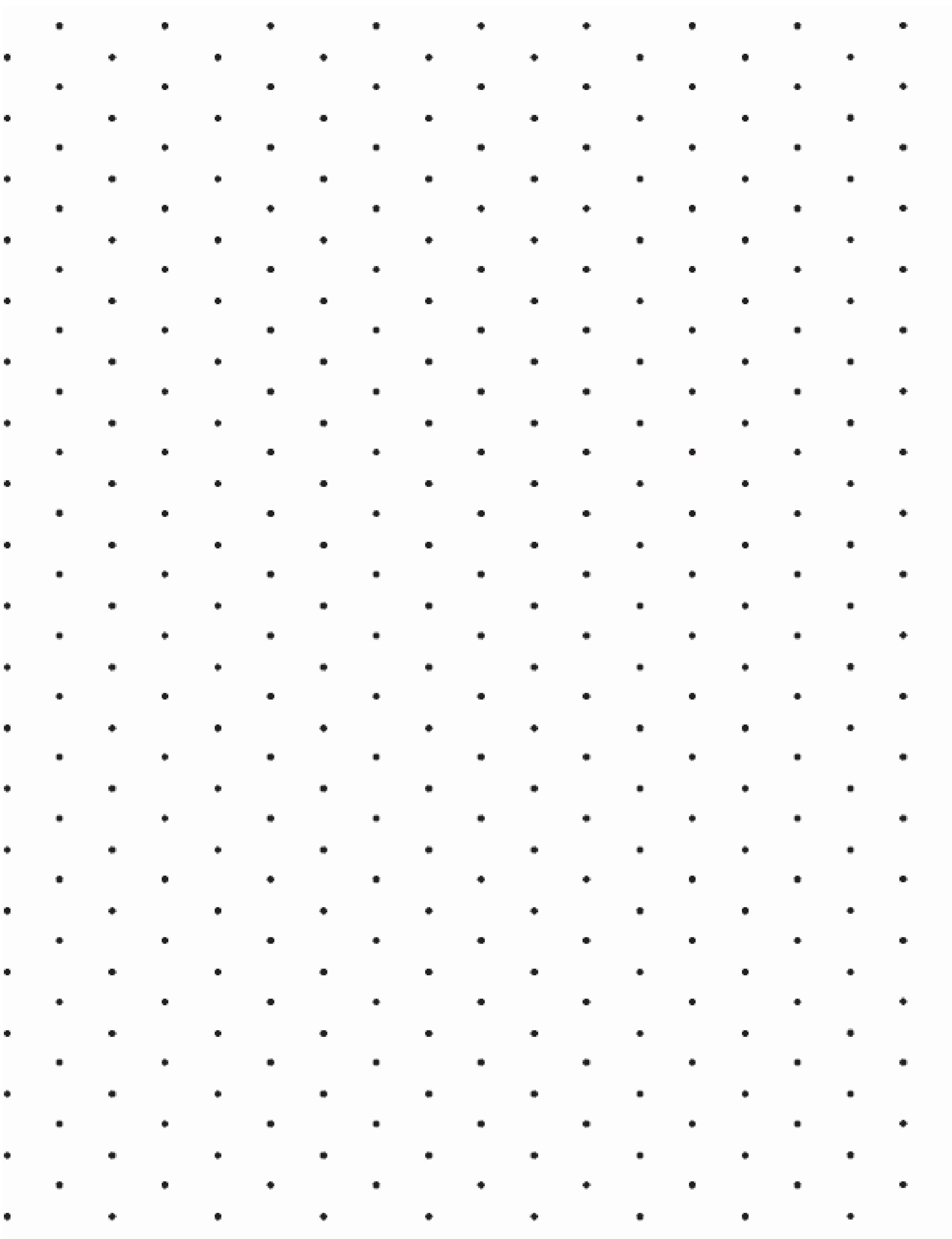
D. Qp = Rp = Pp because each is a radius of the circumcircle.
I assumed Cp × 2 = CP because of what I found in **part C**.
Therefore, since Pp = PC + Cp, Pp = Cp × 3.
QR = 2 × Rp, but Rp = Pp because they are both on the same circle centred at P.
So QR = 2 × Pp = 6 × Cp.

Unit 8 Performance Task Assessment Rubric

	Level 4	Level 3	Level 2	Level 1
Performing Constructions	Exceptionally accurate/careful constructions and measurements, marked clearly to show the methods of construction.	Reasonably accurate constructions and measurements, marked clearly to show the methods of construction.	Reasonably accurate constructions and measurements, marked clearly to show the methods of construction for some of the constructions, with signs of uncertainty in method.	Major errors in construction and/or measurement.
Reasoning Inductively and Deductively	Uses good inductive strategy to find conjectures and clear deductive reasoning to connect the findings from **parts B and C**.	Uses good inductive strategy to find conjectures and reasonable deductive reasoning to connect the findings from **parts B and C**.	Some trouble with inductive or deductive reasoning, but good attempts at reasoning.	Major flaws in inductive and deductive reasoning.

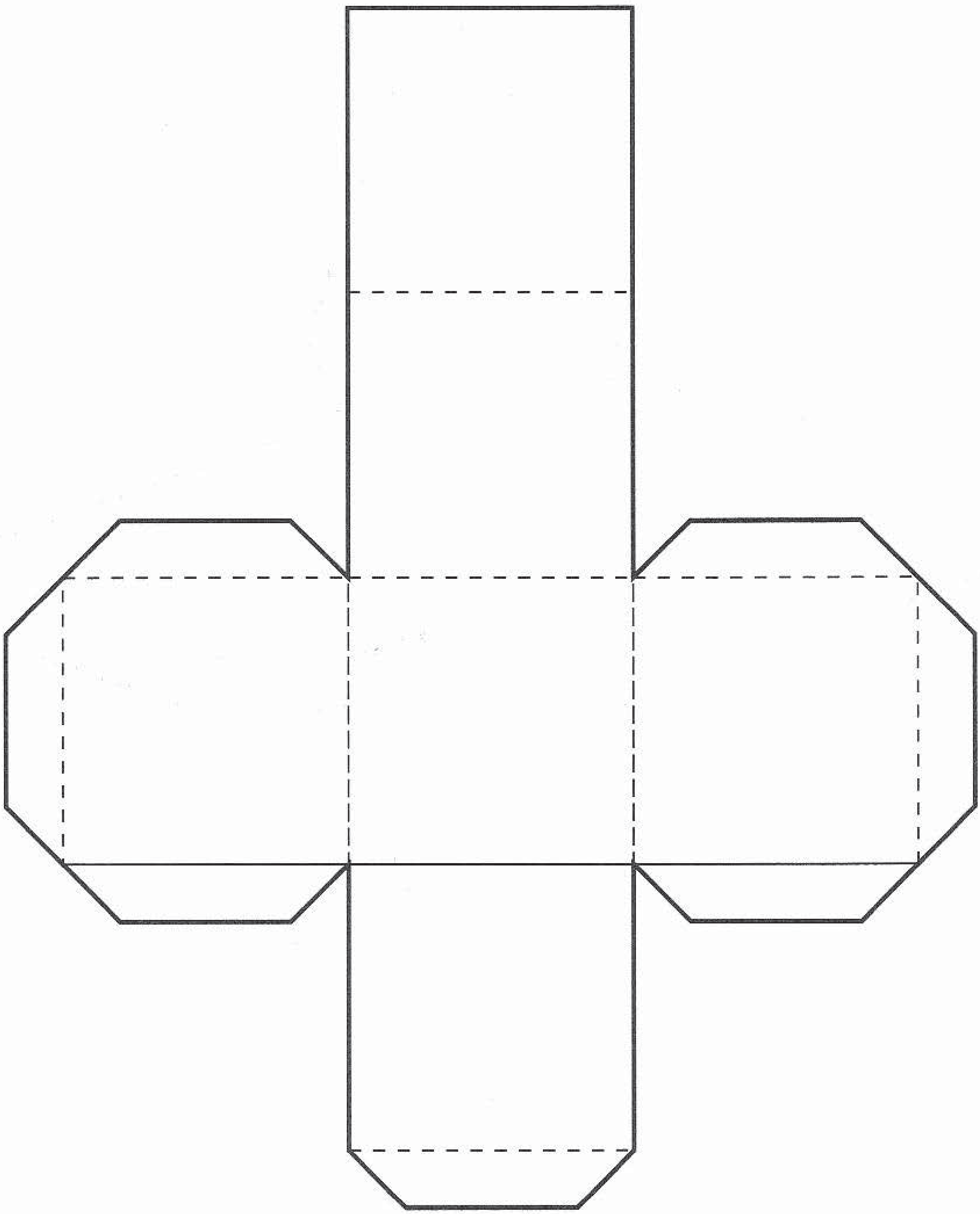
UNIT 8 Blackline Master 1

Isometric Dot Paper



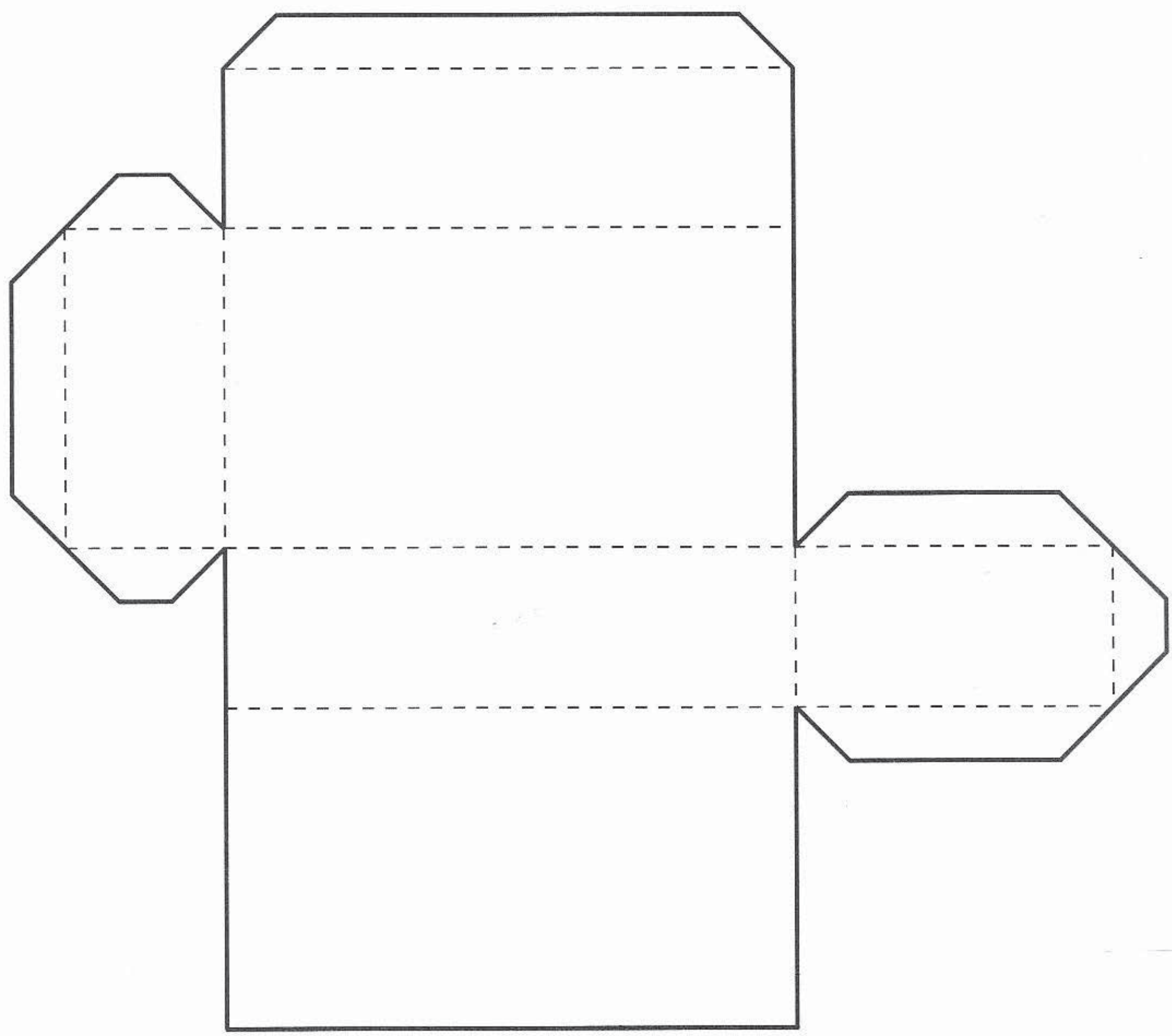
UNIT 8 Blackline Master 2

Net of Cube



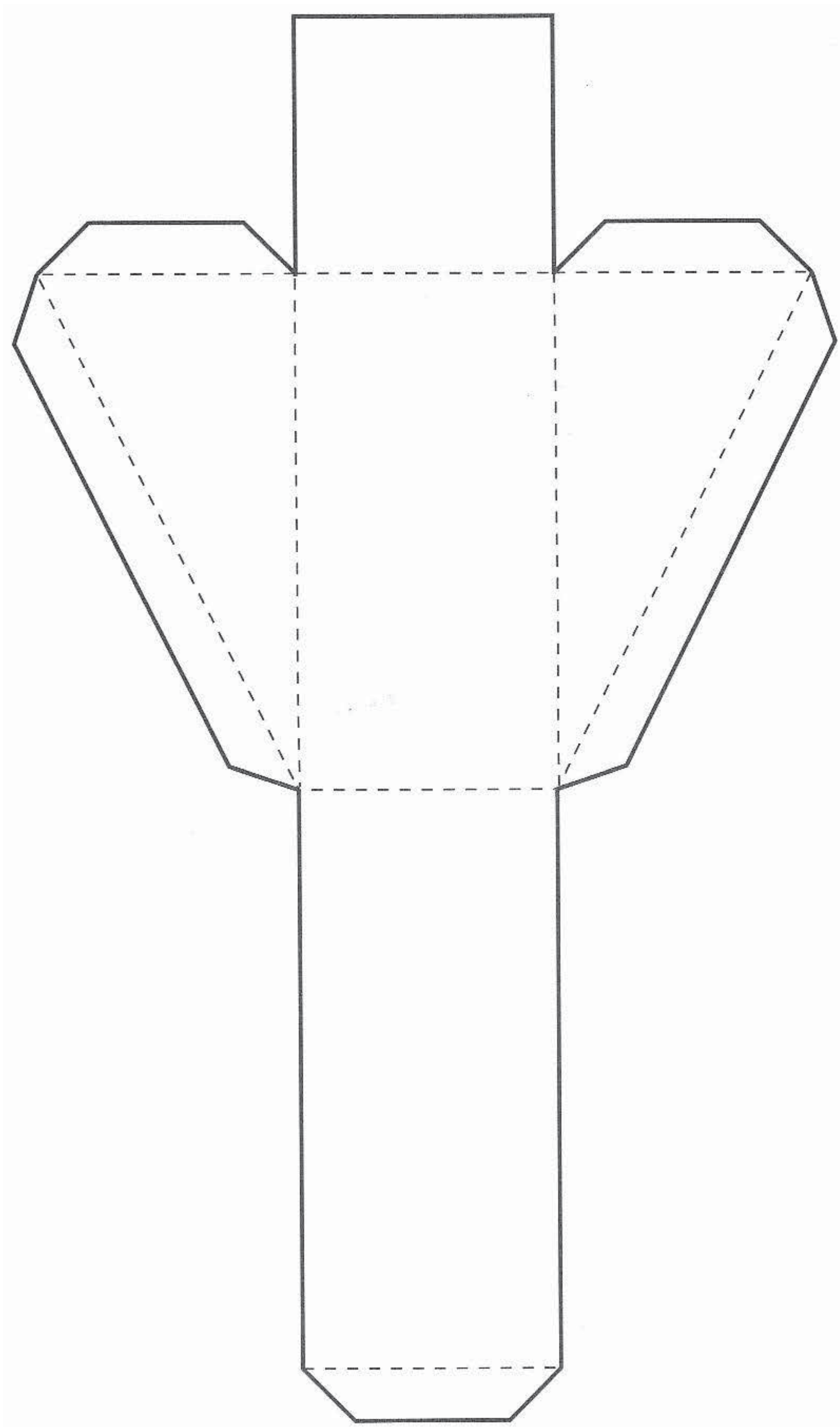
UNIT 8 Blackline Master 3

Net of Rectangle-based Prism



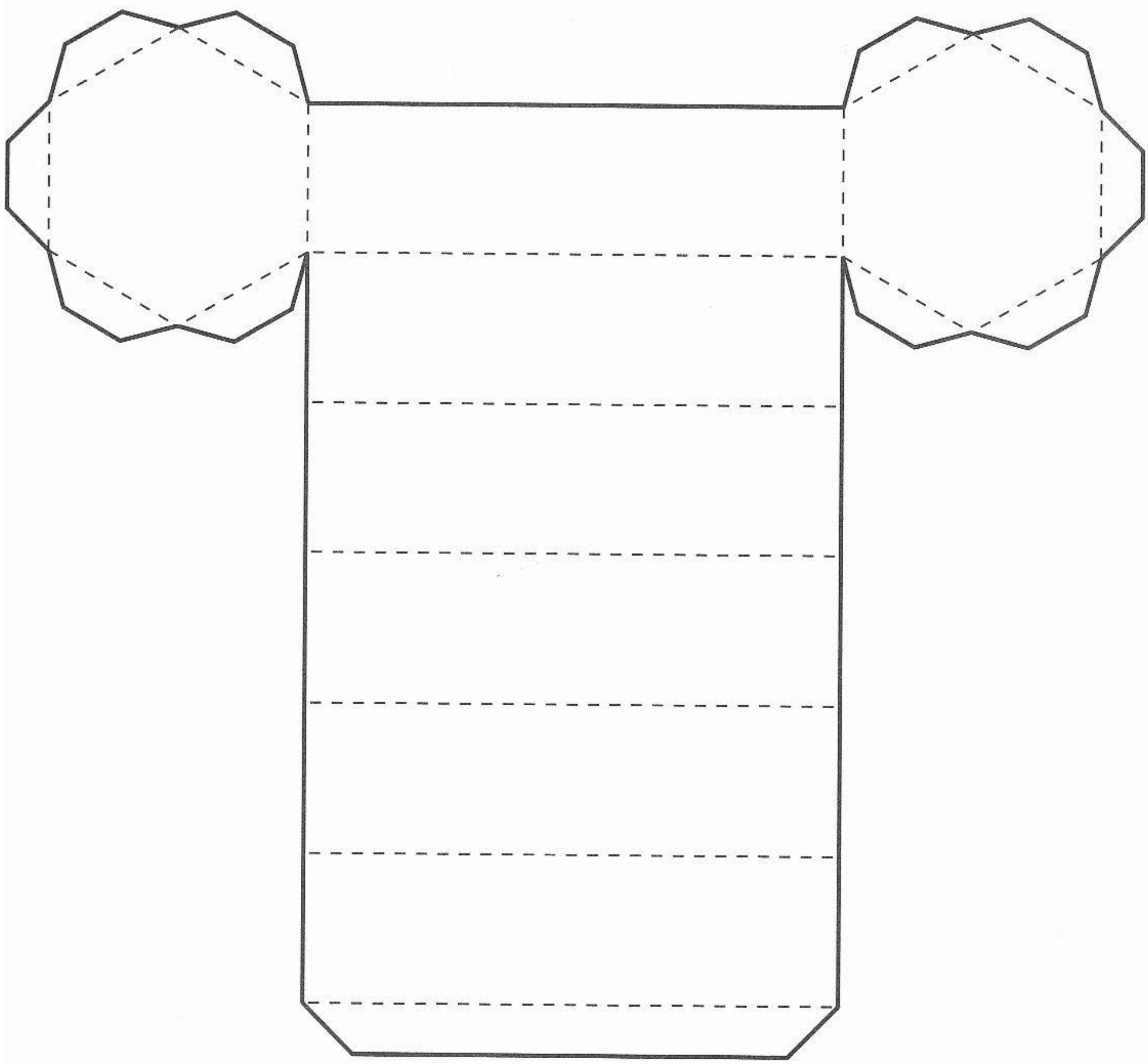
UNIT 8 Blackline Master 4

Net of Right Triangle-based Prism



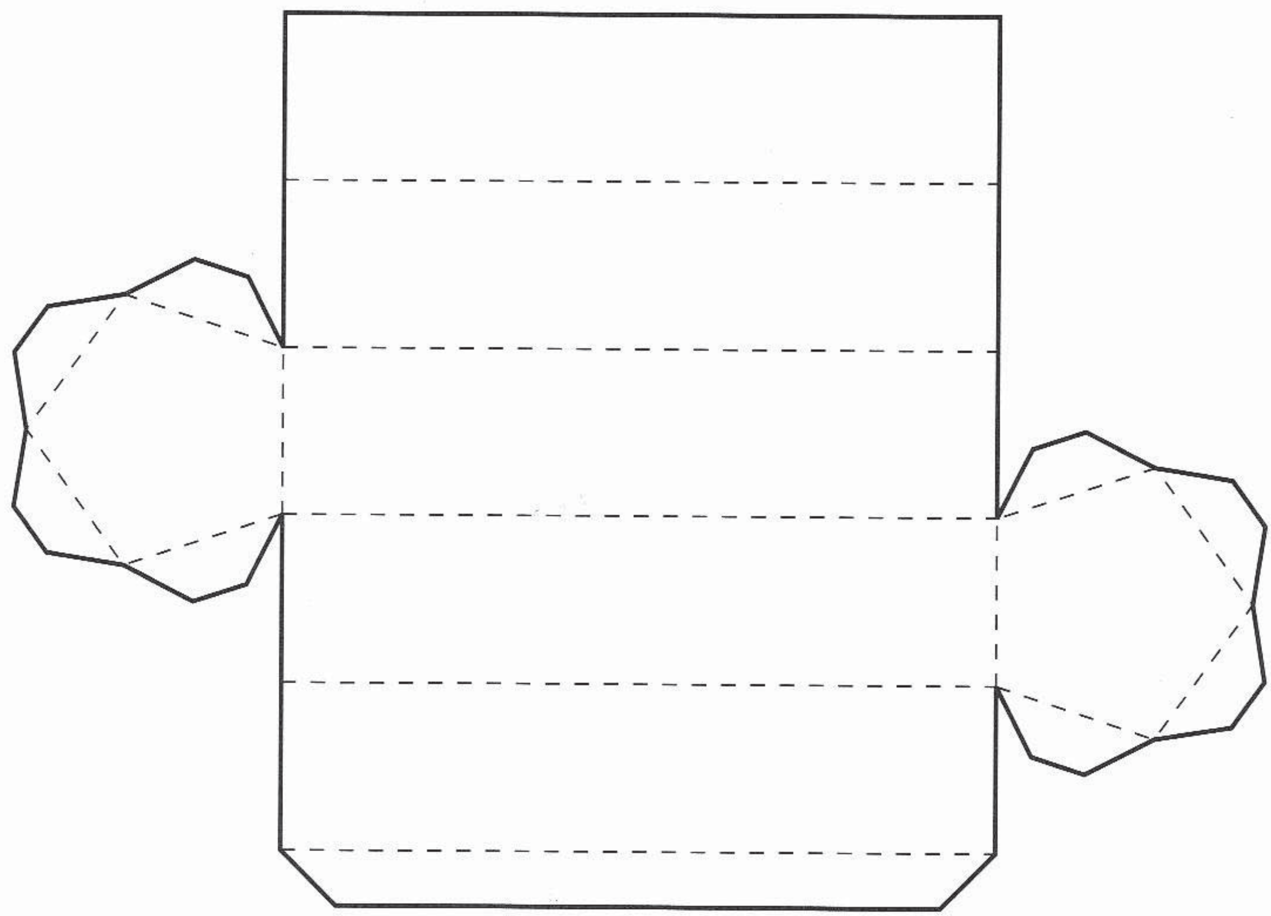
UNIT 8 Blackline Master 5

Net of Regular Hexagon-based Prism



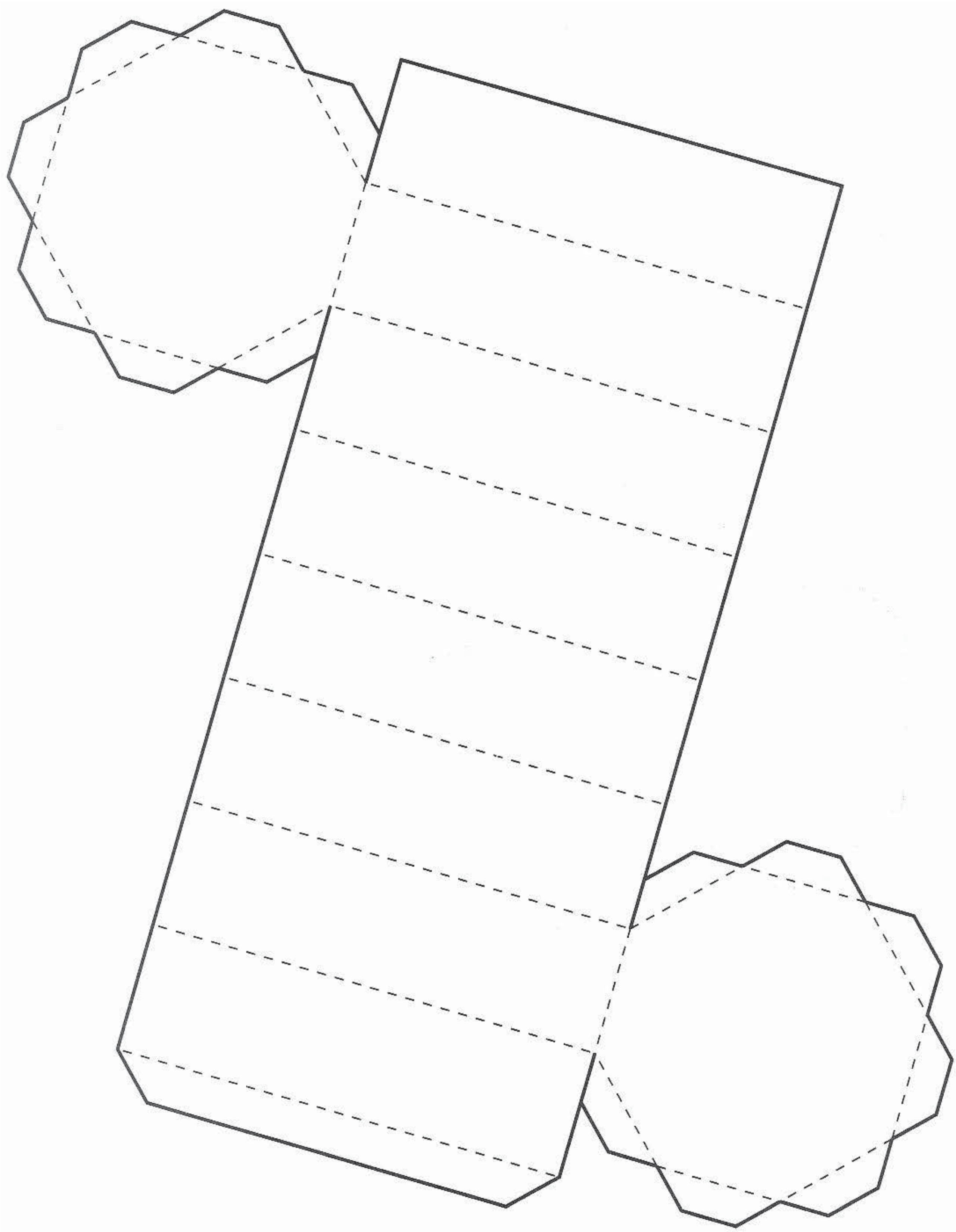
UNIT 8 Blackline Master 6

Net of Regular Pentagon-based Prism



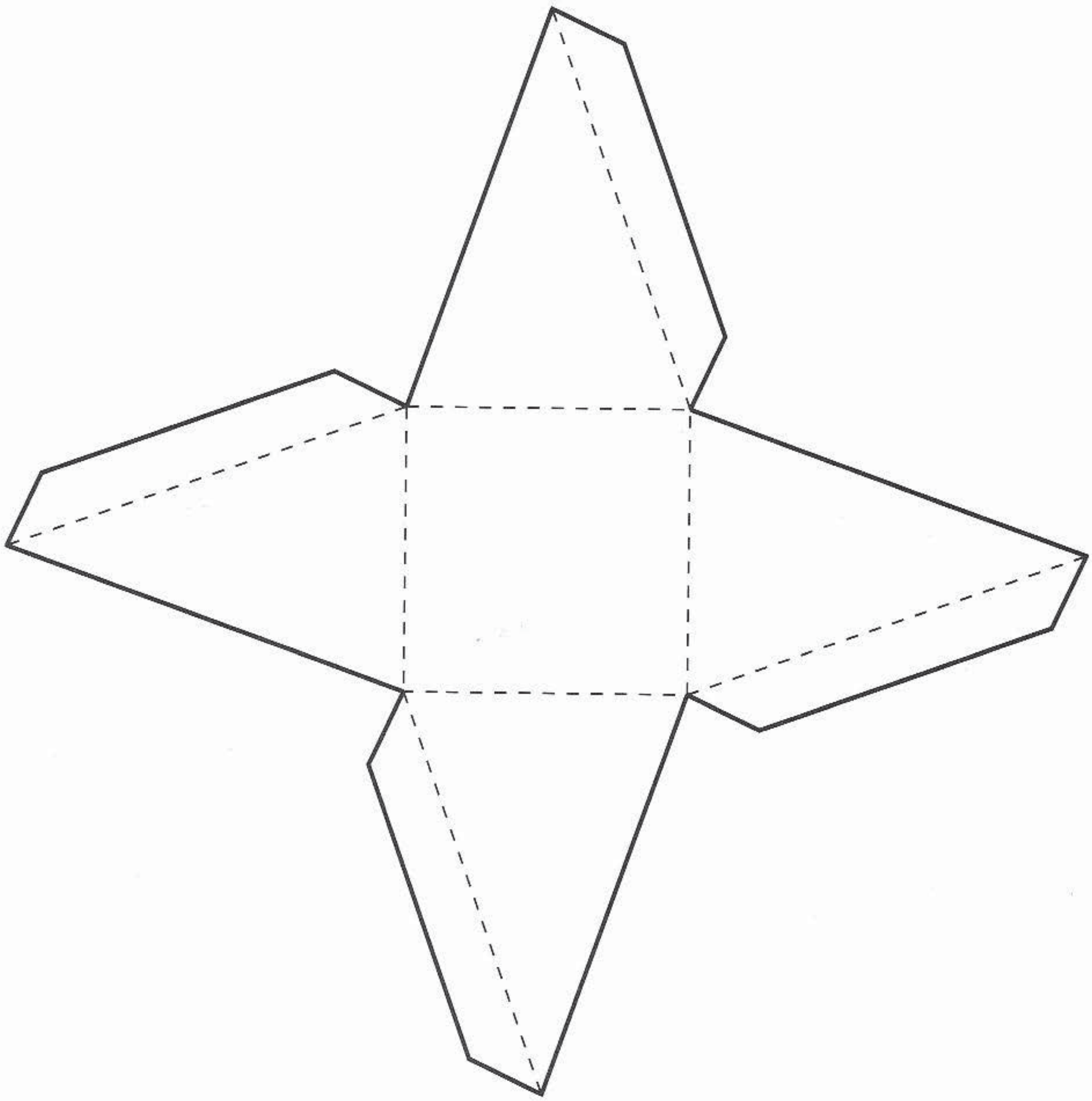
UNIT 8 Blackline Master 7

Net of Regular Octagon-based Prism



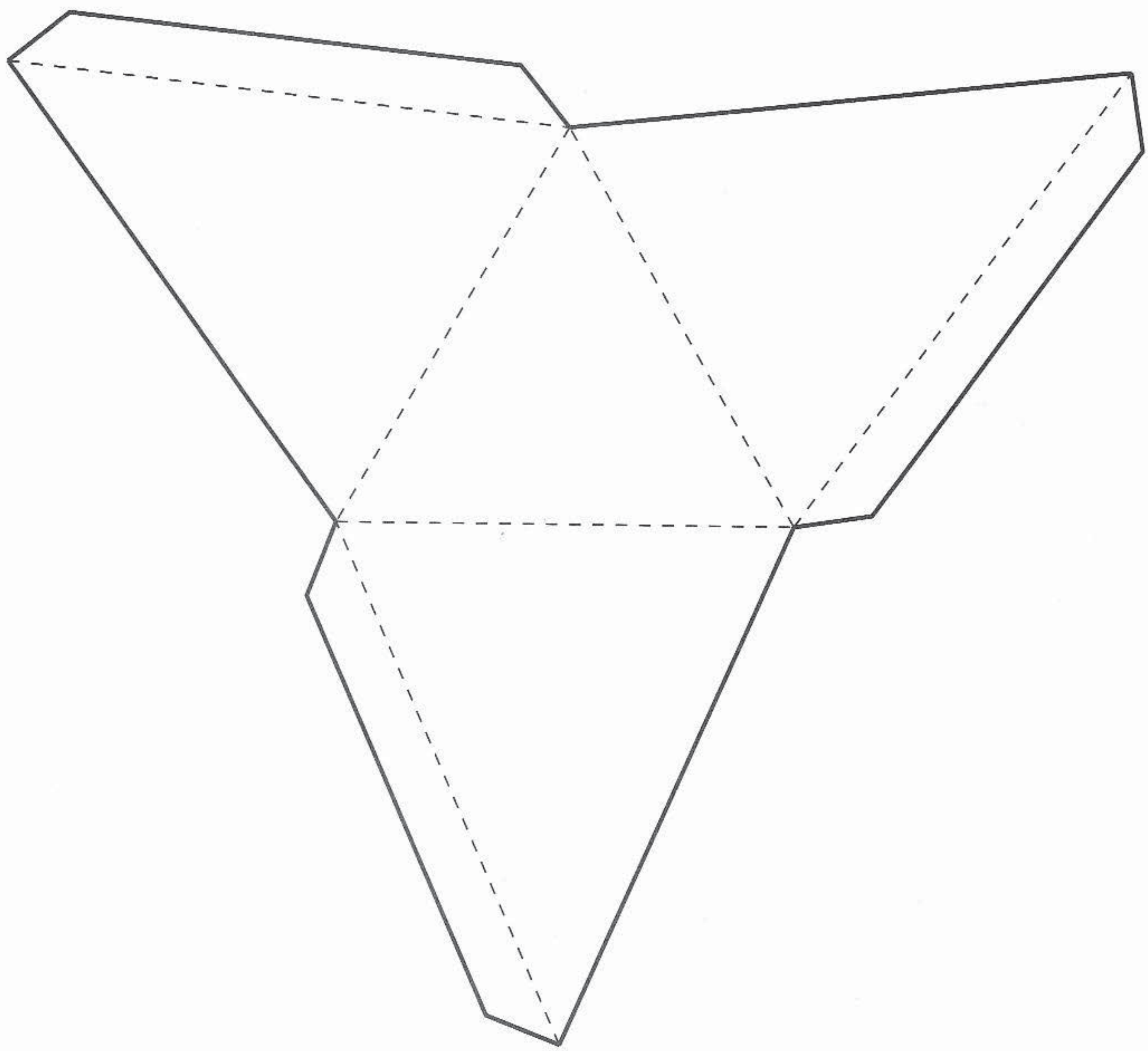
UNIT 8 Blackline Master 8

Net of Square-based Pyramid



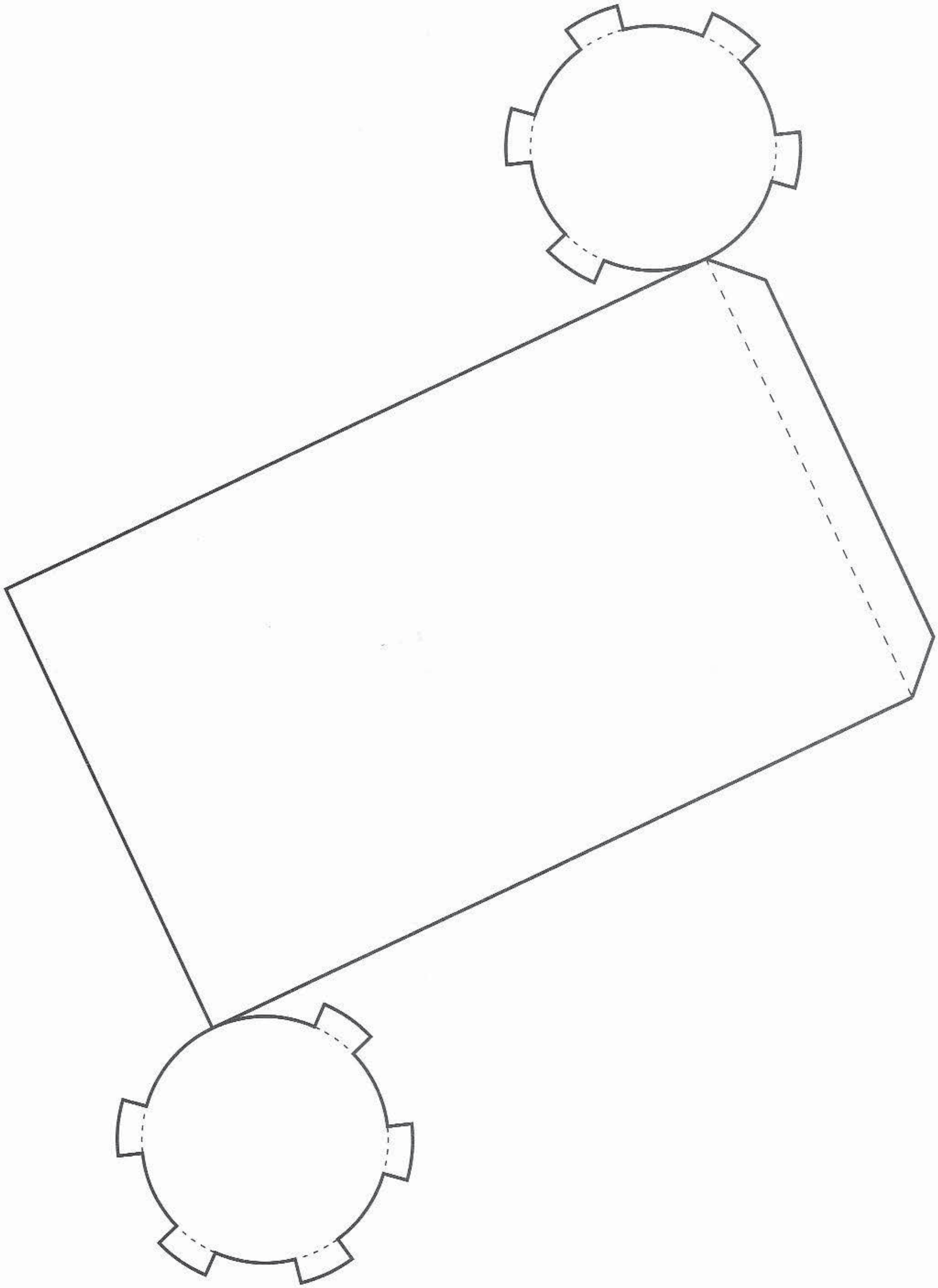
UNIT 8 Blackline Master 9

Net of Tetrahedron



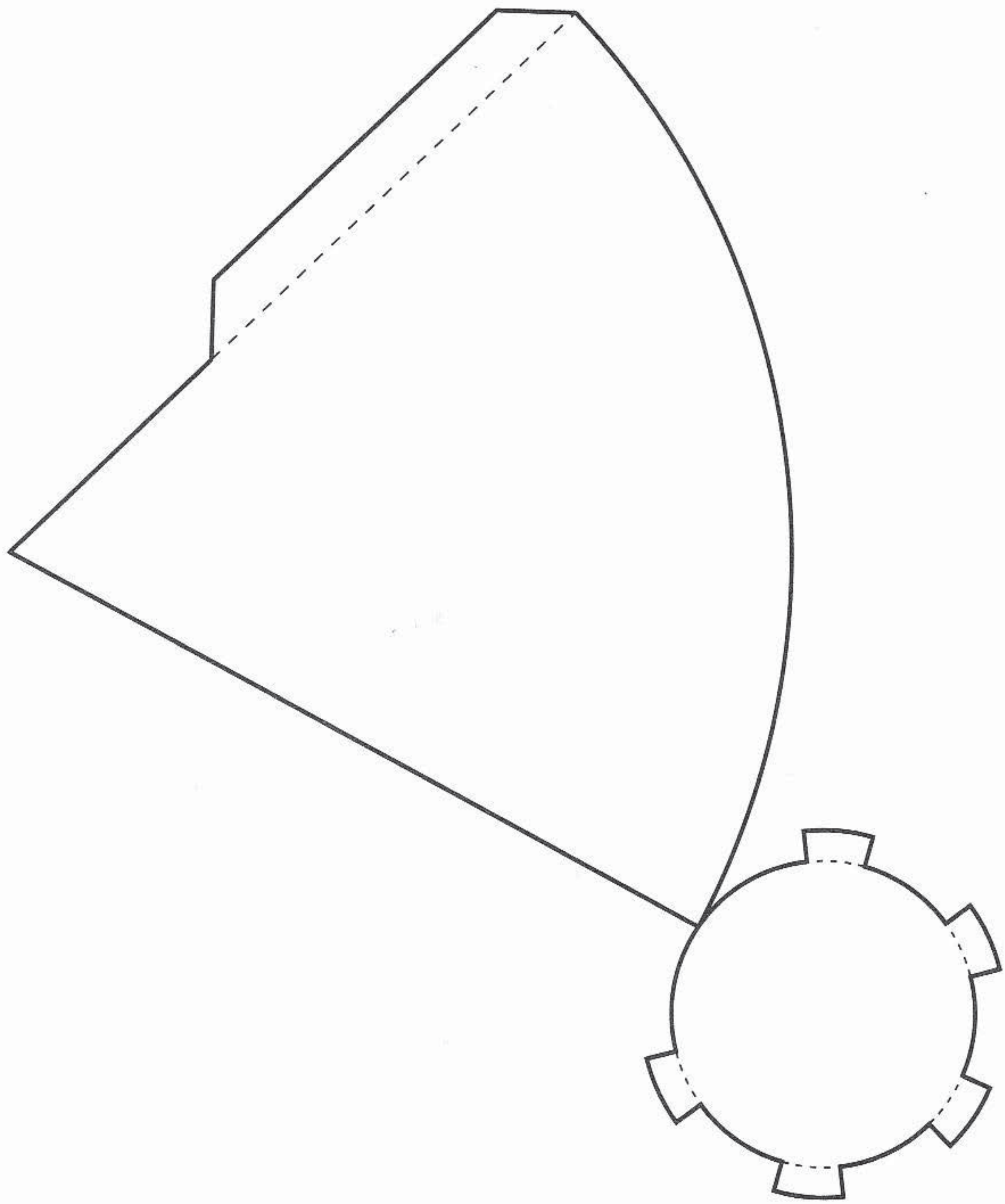
UNIT 8 Blackline Master 10

Net of Cylinder



UNIT 8 Blackline Master 11

Net of Cone



UNIT 8 Blackline Master 12

Regular Polygons and Other Shapes