Understanding Mathematics

Textbook for Class X

ཤེས་རིག
Department of School Education
Ministry of Education and Skills Development
Royal Government of Bhutan

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MINISTER

ROYAL GOVERNMENT OF BHUTAN
MINISTRY OF EDUCATION
THIMPHU : BHUTAN

FOREWORD

Provision of quality education for our children is a cornerstone policy of the Royal Government of Bhutan. Quality education in mathematics includes attention to many aspects of educating our children. One is providing opportunities and believing in our children's ability to understand and contribute to the advancement of science and technology within our culture, history and tradition. To accomplish this, we need to cater to children's mental, emotional and psychological phases of development, enabling, encouraging and supporting them in exploring, discovering and realizing their own potential. We also must promote and further our values of compassion, hard work, honesty, helpfulness, perseverance, responsibility, *thadamtsi* (for instance being grateful to what I would like to call '*Pham Kha Nga',* consisting of parents, teachers, His Majesty the King, the country and the Bhutanese people, for all the goodness received from them and the wish to reciprocate these in equal measure) and *ley-ju-drey* — the understanding and appreciation of the natural law of cause and effect. At the same time, we wish to develop positive attitudes, skills, competencies, and values to support our children as they mature and engage in the professions they will ultimately pursue in life, either by choice or necessity.

While education recognizes that certain values for our children as individuals and as citizens of the country and of the world at large, do not change, requirements in the work place advance as a result of scientific, technological, and even political advancement in the world. These include expectations for more advanced interpersonal skills and skills in communications, reasoning, problem solving, and decision-making. Therefore, the type of education we provide to our children must reflect the current trends and requirements, and be relevant and appropriate. Its quality and standard should stem out of collective wisdom, experience, research, and thoughtful deliberations.

Mathematics, without dispute, is a beautiful and profound subject, but it also has immense utility to offer in our lives. The school mathematics curriculum is being changed to reflect research from around the world that shows how to help students better understand the beauty of mathematics as well as its utility.

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The development of this textbook series for our schools, *Understanding Mathematics,* is based on and organized as per the new School Mathematics Curriculum Framework that the Ministry of Education has developed recently, taking into consideration the changing needs of our country and international trends. We are also incorporating within the textbooks appropriate teaching methodologies including assessment practices which are reflective of international best practices. The *Teacher's Guides* provided with the textbooks are a resource for teachers to support them, and will definitely go a long way in assisting our teachers in improving their efficacy, especially during the initial years of teaching the new curriculum, which demands a shift in the approach to teaching and learning of Mathematics. However, the teachers are strongly encouraged to go beyond the initial ideas presented in the Guides to access other relevant resources and, more importantly to try out their own innovations, creativity and resourcefulness based on their experiences, reflections, insights and professional discussions.

The Ministry of Education is committed to providing quality education to our children, which is relevant and adaptive to the changing times and needs as per the policy of the Royal Government of Bhutan and the wish of our beloved King.

I would like to commend and congratulate all those involved in the School Mathematics Reform Project and in the development of these textbooks.

I would like to wish our teachers and students a very enjoyable and worthwhile experience in teaching, learning and understanding mathematics with the support of these books. As the ones actually using these books over a sustained period of time in a systematic manner, we would like to strongly encourage you to scrutinize the contents of these books and send feedback and comments to the Curriculum and Professional Support Division (CAPSD) for improvement with the future editions. On the part of the students, you can and should be enthusiastic, critical, venturesome, and communicative of your views on the contents discussed in the books with your teachers and friends rather than being passive recipients of knowledge.

Trashi Delek!

Thinley Gyamtsho

MINISTER

Ministry of Education

January of 2007

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INTRODUCTION

How Mathematics Has Changed

Mathematics is a subject with a long history. Although newer mathematical ideas are always being discovered, much of what you will be learning is mathematics that has been known for hundreds of years, if not longer.

Mathematics is a study of quantity, space, structure, patterns and change. This study at the school level is divided into 5 strands of content, namely, numbers and operations, algebra, geometry, measurement, and data and probability.

Nowadays, greater emphasis is given to conceptual understanding rather than on memorizing and applying rote procedures. There are many reasons for this.

• In the long run, it is very unlikely that you will remember the mathematics you learn unless it is meaningful. It is much harder to memorize something that does not make sense than something that relates to what you already know.

• Some approaches to mathematics have not been successful; there are many adults who are not comfortable with mathematics even though they were successful in school. This indicates that a change in approach is necessary.

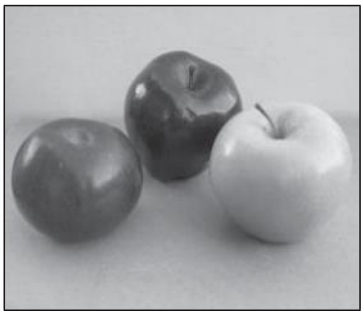
In your new textbook, the mathematics is made meaningful in many ways:

• Mathematics should be taught using contexts that are meaningful to you. They can be mathematical contexts or real world contexts. Your new textbook uses both Bhutanese and international contexts.

For example, in Unit 1 (Matrices and Networks) a task with an international context involves multiplying a matrix by a scalar to determine the price in ngultrums of several items priced in Thai baht. In Unit 6 (Data, Statistics, and Probability) you will estimate the correlation coefficient of a graph that shows the Olympic discus throw distances from 1908 to 1992 and examine data about the mass of three types of apples grown in Canada.

Tasks with Bhutanese contexts involve calculating the slope of the wheelchair ramp at the hospital in Paro (Unit 7 Trigonometry) and considering the precision of a common balance used at a local market (Unit 4 Measurement). Meaningful contexts will help you see and appreciate the value of mathematics.





Working with Thai baht, Olympic statistics, and data about apples grown in Canada



Dividing a triangular paddy field into half, calculating the slope of the wheelchair, and using a common balance

• You will be asked to explain why something is true, not simply to state that it is true. For example, you will be asked not only to demonstrate that the number of lines of symmetry of a regular shape is always equal to the order of rotational symmetry, but also to explain, using deductive reasoning, why this is so.

• When you discuss mathematical ideas, you will be expected to use the processes of problem solving, communication, reasoning, making connections (connecting mathematics to the everyday world and connecting mathematical topics to each other), and representation (representing mathematical ideas in different ways, such as graphs and tables). For example, in Unit 2 (Commercial Math and Number), you will connect radicals (both rational and irrational numbers) to the hypotenuse of right triangles, use reasoning to see how different representations of radical expressions are equivalent, and communicate your thinking while solving problems involving radicals.

• The reason you learn mathematics is to help you solve problems. In the real world, you are not told when to apply particular mathematical skills. You just need to know. You will be given opportunities to figure out when and how to apply the concepts and skills you are learning in order to solve problems.

USING YOUR TEXTBOOK

Each unit has

• a *Getting Started* section

• two or three chapters, which divide the content of the unit into sections

• regular lessons and at least one *Explore* lesson

• a *Game* (usually)

• at least one *Connections* feature

• a *Unit Revision*

Getting Started

There are two parts to each *Getting Started* section: *Use What You Know* and *Skills You Will Need*. Both will remind you of critical knowledge and terminology you should have already learned that will be required in the unit.

• *Use What You Know* is an activity that you complete with a partner or in a small group.

• *Skills You Will Need* is a review of the skills you will use in the unit.

Regular Lessons

• Lessons are numbered #.#.# — the first number tells the unit, the second number the chapter, and the third number the lesson within the chapter.

For example, Lesson 4.2.1 is Unit 4, Chapter 2, Lesson 1 (the first lesson in Chapter 2 of Unit 4).

• Each regular lesson is divided into five parts:

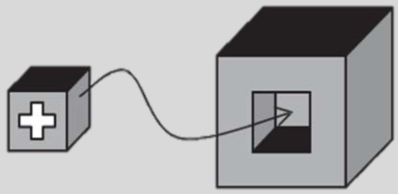
- A *Try This* task
- The exposition (the main ideas of the
- A question that revisits the *Try This*
- *Examples*
- *Practising and Applying*

Try This

• The *Try This* task is in a shaded box, like the example below from **lesson 8.1.2** on **page 275**.

A baby is playing with a small cube block that fits into a square hole in a bigger cube block.

A. How many ways can the baby fit the small cube block into the hole?



• The *Try This* is a brief task that you might do with a partner or in a small group. It is related to the new learning, but you can complete it without the concepts and skills that are the focus of the lesson. The new mathematics you are able to learn in the exposition will make more sense to you if you do some related mathematics before the teacher presents the lesson.

The Exposition

• The exposition appears in a box immediately following the *Try This*.

• The exposition presents the main concepts and skills of the lesson.

• Key mathematical terms are introduced and described. When a key term appears for the first time in a unit, it is highlighted in **bold type** to indicate that it is found in the glossary (at the back of the book).

• You are not expected to copy the exposition into your notebook either directly from the book or from your teacher's lecture.

Revisiting the Try This

• The revisiting the *Try This* question(s) follows the exposition and appears in a shaded area, like this example from **lesson 8.1.2** on **page 275**, which follows from the above *Try This* about the small cube block.

B. **i)** How does the question in **part A** relate to rotational symmetry?
ii) What is the order of turn symmetry of the small cube? Explain.

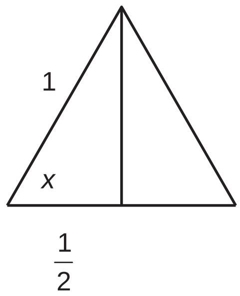
• The question shows how your new learning relates to what you already learned from the *Try This* task.

Examples

• The *Examples* provide additional instruction by modelling how to approach the questions you will meet in *Practising and Applying*. Each example is a bit different from the others so that you have many models from which to work.

• Sometimes you work through the examples independently, sometimes in pairs or in small groups, and sometimes with your teacher.

• What is special about the examples is that they show not only the formal mathematical work in the left hand *Solution* column, but also what a student might be thinking in the right hand *Thinking* column. This is intended to help you learn to think mathematically. Many of the examples present two or even three different solutions. The example below, from **lesson 7.1.4** on **page 247**, shows two possible ways to approach the task, *Solution 1* and *Solution 2*.

Example 2 Using the Reciprocal Ratio	
In a right triangle, sec x = 2. What is the value of x?	
Solution 1 If sec x = 2, then cos x = $\frac{1}{2}$.  The angle must be 60°.	**Thinking**  • I knew that sec $x = \frac{1}{\cos x}$, so I used the secant ratio to find cosine. • I knew that cosine was $\frac{1}{2}$ for one of the angles in a 30°-60°-90° triangle. I drew that triangle as half of an equilateral triangle to help me figure out whether x was 30° or 60°. • Cosine is based on the adjacent side, so the angle with cosine of $\frac{1}{2}$ must be the 60° angle.
Solution 2 If sec x = 2, then cos x = $\frac{1}{2}$. $\cos^{-1} \frac{1}{2} = 60°$	**Thinking**  • My calculator doesn't have a button for $\sec^{-1}$ that gives the angle if you enter the secant, but it does have a button for $\cos^{-1}$. That's why I used the relationship between secant and cosine to find the value of the cosine ratio.

Practising and Applying

• Depending on your teacher's choice, you might work on the *Practising and Applying* questions independently, with a partner, or in a group. You can use the exposition and examples as references.

• The questions start out like those in the *Examples* and progress to questions requiring more problem solving and more explanations. The last question brings you back to one of the main points of the lesson.

Explore Lessons

• An *Explore* lesson provides an opportunity for you to investigate some mathematics. You may work with a partner or in a small group.

• Your teacher does not lecture in an *Explore* lesson. Instead, you work through a problem by following the questions that direct your investigation.

Connections Feature

• The *Connections* feature takes many forms. Sometimes it is a relevant and interesting historical note. Sometimes it relates the mathematical content of the unit to the content of a different unit. Other times it relates the mathematical content to a real world application. For example, in Unit 4, the **CONNECTIONS** on **page 125** connects the surface area to volume ratio (efficiency of shapes) to the reason why animals have certain shapes and sizes.

• There is always one or more *Connections* feature in a unit.

• You might work in pairs or small groups to complete the task or answer the question(s).

Game

• There is usually at least one *Game* in each unit.

• The *Game* is a way to practise skills and concepts introduced in the unit with a partner or in small group.

• The required materials and rules are listed in the book. Usually there is a sample shown to help you understand the rules.

Unit Revision

• The *Unit Revision* is an opportunity to review the lessons in the unit.

• There is always a mixture of skill, concept, and problem solving questions.

• The order of the questions in the *Unit Revision* usually follows the order of the lessons in the unit.

• You can work with a partner or on your own, as your teacher suggests.

Glossary

At the end of the book, there is a glossary of new mathematical trminology and definitions. The glossary also contains other important mathematical terms from previous classes. There is also a set of instructional terms commonly used in the units (for example, justify, explain, predict, …). These are intended to help you understand what is expected of you.

Answers

• Answers to most of the numbered questions are provided in the back of the textbook. Answers that are lengthy explanations are not included; your teacher has these answers.

• Questions with letters, such as A or B, do not have answers in the back of the book. Your teacher has the answers to these questions.

• There is often more than one possible answer to a question. This is indicated in the answers by the phrase *Sample Response*. When you see an answer starting with the words *Sample Response*, your answer may still be correct even if it does not match the answer given.

ASSESSING YOUR MATHEMATICAL PERFORMANCE

Forms of Assessment

Your teacher will observe and report on your mathematical performance. Sometimes your teacher will collect information about what you understand in order to change the way you are taught. Other times your teacher will use information about your performance to give you a mark.

Assessment Criteria

- Your teacher should inform you about what mathematical content will be assessed and how it will be assessed. For example, you should know if the intent of the assessment is to focus on skills and application or on problem solving.
- Your mark and all assessments should reflect the curriculum for Class IX. The proportions of the mark assigned for each unit should reflect both the time spent on the unit and the importance of the unit.
- All assessment should have a balance of skills, applications, concepts, and problem solving. The balance will vary depending on the unit and purpose of the assessment.
- Your teacher should inform you whether a test is being marked numerically, using a letter grade, or whether a rubric is being used. A rubric is a chart that describes criteria for your work, usually in four levels of performance. If a rubric is used, your teacher should let you see it before you start on the task.

Determining a Mark or Grade

In determining your mark, your teacher might use a combination of tests, assignments, projects, performance tasks, and homework.

THE CLASSROOM ENVIRONMENT

In almost every lesson, you will be engaged in some work either on your own, in pairs, or in small groups (either in the *Try This* or during an *Explore* lesson). Being engaged in your learning helps you learn better.

While you are working on your own, in pairs or in groups, communication plays a significant role in every lesson. Through communication you can clarify your thinking and show your teacher and classmates what you understand.

You should always share your responses, even if they are different from those offered by other students. It is only in this way that you will really be engaged in the mathematical thinking instead of being a spectator.

MATHEMATICAL TOOLS

Manipulatives

• All students, including those who are already good at mathematics, can benefit from using manipulative materials. For example, Unit 2 makes frequent use of algebra tiles to represent polynomials concretely. Although some students can be successful without these materials, everyone can benefit from their use. You will start to see not only how to perform algebraic manipulations, but why they are done the way they are.


Algebra tiles for polynomials

• Manipulative materials are important in Class IX in the units on polynomials, probability, geometry, and measurement.

Appropriate Calculator Use

• In Class X, like in class IX, calculator should be used as a regular tool. At this point in your mathematical education, you are no longer being asked simply to perform routine calculations. Calculations are now part of more sophisticated mathematical tasks that are the real focus of your learning.

• You may not have the same type of calculator as your classmates, so specific instructions for how to use your calculator are not provided in the textbook. Your teacher can help you learn to use your calculator correctly.

YOUR NOTEBOOK

• It is valuable for you to have a well-organized, neat notebook to look back at to review the main mathematical ideas you have learned. However, it is also important for you to feel comfortable doing rough work in that notebook rather than doing it elsewhere and then wasting valuable time copying your rough work neatly into your notebook. If you do rough work on other paper, which will certainly happen from time to time, it may not be necessary to copy it into your notebook.

• Your teacher will sometimes point out important points to record in your notebook. You should also make your own decisions about which ideas to include in your notebook.

UNIT 1 MATRICES AND NETWORKS

Getting Started

Use What You Know

A. The numbers 1 to 9 can be arranged in a magic square so that all rows, columns, and diagonals add to the same value. Complete this magic square.

4	?	8
9	5	?
?	7	?

B. The magic sum is the sum of each row, column, or diagonal. What is the magic sum of this magic square?

C. Create your own magic square using each number from 1 to 9.

D. i) Combine the magic square in **part A** with your magic square from **part C** to create a 3 × 3 square of numbers. Do this by adding pairs of numbers that are in the same position, as shown below.

$$\begin{array}{|c|c|c|}\hline A & B & C \\ \hline D & E & F \\ \hline G & H & I \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline J & K & L \\ \hline M & N & O \\ \hline P & Q & R \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline A+J & B+K & C+L \\ \hline D+M & E+N & F+O \\ \hline G+P & H+Q & I+R \\ \hline \end{array}$$

ii) Is the new 3 × 3 square of numbers magic? How do you know?

E. i) Multiply your magic square from **part C** by 0.5 to create a new 3 × 3 square of numbers. Do this by multiplying each value in your magic square by 0.5.

ii) Is the 3 × 3 square of numbers magic? How do you know?

iii) Did you expect this? Why?

Skills You Will Need

1. Find the sum.

a) –1 + 3 **b)** –4 + (–2) **c)** 3 + (–10) **d)** –18 + (–6)

2. Find the difference.

a) –1 – 3 **b)** –4 – (–2) **c)** 3 – (–10) **d)** –18 – (–6)

3. Find the product.

a) 2 × (–2) **b)** –6 × 2 **c)** –10 × (–4) **d)** –8 × (–9)

4. Calculate.

a) 0.3 + 1.8 **b)** 2.3 – 1.8 **c)** 3 ÷ 1.5 **d)** 2.8 ÷ 0.7

5. Find the product.

a) 0.3 × 0.7 **b)** 1.2 × 30 **c)** 0.84 × 12 **d)** 1.2 × 15

Chapter 1 Matrices

1.1.1 Introducing Matrices

Try This

This chart displays information about friendships.
If two people are friends, there is a 1 in the chart.
If they are not friends, there is a 0.

	Dema	Sonam	Nima	Karma
Yuden	0	0	1	1
Thinley	1	0	0	1
Pema	0	1	0	1

A. i) Who has the most friends?
ii) How do you know?

A **matrix** (plural is **matrices**) is a rectangular **array** of items used to store and display information. The information can be numbers, shapes, or other symbols. An example of a matrix is shown below.

Matrix B

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 7 & 3 & 1 & 8 \\ 5 & 9 & 1 & 4 \end{bmatrix}$$

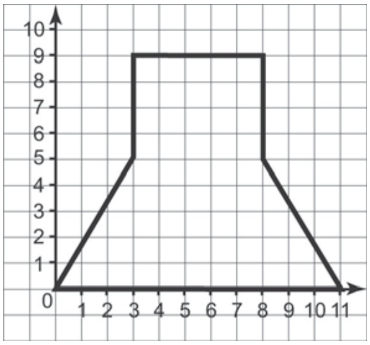
- Each item inside the matrix is an **element.** The element that is in the 3rd row, 2nd column of Matrix B is 9. You can write (3, 2) as the **address** of the element. You should write the row number first and then the column number.
- The matrix itself has no row or column labels, but sometimes column or row labels are used to make it clear what the numbers represent.
- Open square brackets are often used on the left and right of the matrix.
- The size, or **dimensions**, of a matrix tells the number of rows and then the number of columns. Matrix B is a 3-by-4 matrix, or 3 × 4 matrix, since it is 3 rows by 4 columns. You read this as “B is a three-by-four matrix.” A 3 × 4 matrix has different dimensions than a 4 × 3 matrix.
- A matrix with the same number of rows as columns is called a **square matrix**.
- A matrix with only one row is called a **row matrix**.
- A matrix with only one column is called a **column matrix**.

B. i) What part of the friendship chart above is a matrix?
ii) How do you know it is a matrix?

Examples

Example 1 Using a Matrix to Describe a Shape

What matrix might describe this shape?



Solution

$$\begin{bmatrix} 0 & 3 & 3 & 8 & 8 & 11 \\ 0 & 5 & 9 & 9 & 5 & 0 \end{bmatrix}$$

Each column describes the coordinates of each of the six vertices—the 1st row is the *x*-coordinate and the 2nd row is the *y*-coordinate.

Thinking

- I used a 2 × 6 matrix because each vertex has 2 coordinates (an *x*- and a *y*-coordinate) and there are 6 vertices.
- I could have used a 6 × 2 matrix instead, with each row describing the coordinates of one vertex.
- I started at the bottom left corner of the shape and went clockwise: (0, 0), (3, 5), (3, 9), (8, 9), (8, 5), (11, 0), but I could have started at a different point and moved in the opposite direction.

Example 2 Using Matrices to Describe Factors

Lobzang factored every multiple of 3 from 6 to 30 into prime factors. He used a matrix to show how many times each prime factor — 2, 3, 5, and 7 — appeared in each number. Create Lobzang's matrix.

Solution

$$\begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccccc} 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Each column lists the number of each prime factor (2, 3, 5, and 7) there are in the factored form of each multiple of 3 (6, 9, 12, ..., 30).

Thinking

- I used a 4 × 9 matrix because I knew there were 4 prime factors and 9 multiples.
- I began with the first column for the multiple 6. Since 6 = 2 × 3, there is 1 of prime factor 2, 1 of prime factor 3, 0 of prime factor 5, and 0 of prime factor 7.
- I completed the remaining columns the same way.

Practising and Applying

1. Identify the element in the fourth row, third column of this matrix.

$$\begin{bmatrix} A & C & F & G \\ D & S & R & E \\ B & C & A & O \\ K & M & A & C \\ D & B & F & I \end{bmatrix}$$

2. Create these matrices.

a) any 3 × 2 matrix

b) a 3 × 1 matrix with 0 in the second row

c) any size matrix with 0 at (3, 2)

3. The Government of Bhutan plans economic growth using 5-year plans. This matrix shows the first year for the first nine 5-year plans.

$$\begin{bmatrix} 1 & 1961 \\ 2 & 1966 \\ 3 & 1971 \\ 4 & 1976 \\ 5 & 1981 \\ 6 & 1987 \\ 7 & 1992 \\ 8 & 1997 \\ 9 & 2002 \end{bmatrix}$$

a) What are the dimensions of this matrix?

b) Why is a two-column matrix suitable for showing this information?

4. List four people: two people in your family and two classmates. Create a 4 × 4 matrix to show which of the four people know each other — use a 1 to show they know each other and a 0 to show they do not know each other.

5. Andu kept a record of his archery scores in his last five matches

$$\begin{bmatrix} 125 & 134 & 122 & 117 & 109 \end{bmatrix}$$

a) What are the dimensions of this row matrix?

b) How could a matrix with different dimensions show the same information?

6. Kezang, Sangay, and Choden make ghos and kiras. Here is what they have made so far this year.

$$\begin{array}{l c} & \begin{array}{cc} \text{Ghos} & \text{Kiras} \end{array} \\ \begin{array}{l} \text{Kezang} \\ \text{Sangay} \\ \text{Choden} \end{array} & \begin{bmatrix} 5 & 6 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} \end{array}$$

a) What does the sum of the elements in each row tell you?

b) What does the sum of the elements in each column tell you?

7. Each column of this matrix describes the coordinates of one of the vertices of a shape. Describe the shape.

$$\begin{bmatrix} 0 & 2 & 8 & 6 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

8. Use a matrix to show the number of times the prime factors 2, 3, 5, and 7 appear in each of the multiples of 4 from 4 to 40.

9. A multiplication table shows the results of multiplying each number from 0 to 9 by each number from 0 to 9. Why might someone call a multiplication table a square matrix?

1.1.2 Adding and Subtracting Matrices

Try This

These two matrices show the reported population of Bhutan in three age categories in 2001 and 2003.

$$\begin{array}{r} \\ \text{Under 15 years} \\ \text{15 to 64 years} \\ \text{65 years and over} \end{array} \begin{array}{c} \text{2001} \\ \begin{bmatrix} 285{,}658 \\ 419{,}308 \\ 29{,}374 \end{bmatrix} \end{array} \qquad \begin{array}{r} \\ \text{Under 15 years} \\ \text{15 to 64 years} \\ \text{65 years and over} \end{array} \begin{array}{c} \text{2003} \\ \begin{bmatrix} 294{,}258 \\ 374{,}637 \\ 30{,}055 \end{bmatrix} \end{array}$$

A. i) Use the two matrices to find the change in population in each age category from 2001 to 2003.

ii) Arrange the information about population change in a matrix.

- You can add or subtract matrices that have numerical elements, but the matrices must have the same dimensions.
- When you add or subtract matrices, you find the sum of, or difference between, the elements in the same positions in the matrices.

2 + 10 = 12

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 10 & 3 & 0.5 \\ -2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 0.5 \\ 2 & 4 & -2 \end{bmatrix}$$

2 – 10 = –8

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 3 & 0.5 \\ -2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -8 & -2 & -0.5 \\ 6 & 2 & 6 \end{bmatrix}$$

B. When you answered **part A**, did you add or subtract the matrices? Why?

Examples

Example Interpreting Sums and Differences of Matrices

Matrix A describes how many hours Deki, Dorji, and Karma spent on their English, math, and Dzongkha homework one night. Matrix B describes how many hours they spent the next night.

a) Add the matrices. What do the elements of the sum matrix tell you?

b) Subtract the matrices. What do the elements of the difference matrix tell you?

$$A = \begin{array}{c} \begin{array}{ccc} \text{English} & \text{Math} & \text{Dzongkha} \end{array} \\ \begin{bmatrix} 0.8 & 1 & 0.5 \\ 1 & 0.9 & 0.6 \\ 1 & 1 & 0.5 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \begin{array}{ccc} \text{English} & \text{Math} & \text{Dzongkha} \end{array} \\ \begin{bmatrix} 0.3 & 1.2 & 0.7 \\ 0.5 & 1.1 & 0.8 \\ 0.5 & 1.3 & 0.7 \end{bmatrix} \end{array}$$

[Continued]

Example Interpreting Sums and Differences of Matrices [Continued]

Solution

a) A + B

$$= \begin{bmatrix} 0.8+0.3 & 1+1.2 & 0.5+0.7 \\ 1+0.5 & 0.9+1.1 & 0.6+0.8 \\ 1+0.5 & 1+1.3 & 0.5+0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1 & 2.2 & 1.2 \\ 1.5 & 2.0 & 1.4 \\ 1.5 & 2.3 & 1.2 \end{bmatrix}$$

The elements describe how much time each of the three students spent altogether on their English, math, and Dzongkha homework over both nights.

b) A – B

$$= \begin{bmatrix} 0.8-0.3 & 1-1.2 & 0.5-0.7 \\ 1-0.5 & 0.9-1.1 & 0.6-0.8 \\ 1-0.5 & 1-1.3 & 0.5-0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.2 & -0.2 \\ 0.5 & -0.2 & -0.2 \\ 0.5 & -0.3 & -0.2 \end{bmatrix}$$

The elements describe how many more hours each student studied in each subject on the first night than the second.

Thinking

a) I checked to make sure the matrices were the same size so I could add them — they're both 3-by-3.

- I added the elements in the same position in each matrix.

b) Since the matrices are the same size, I knew I could subtract them.

- I subtracted A – B but I could have subtracted B – A.
- I subtracted the elements in the same position in each matrix.
- I know with A – B a negative number means the student spent more time on the second night and a positive number means the student spent more time on the first night.

Practising and Applying

1. Which matrix pairs can be added?

A. $\begin{bmatrix} 3 & 2 & 1 & 0.5 \end{bmatrix}$ $\begin{bmatrix} 4 & 2 & 0.3 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 & 2 \\ 1 & 0.5 \end{bmatrix}$ $\begin{bmatrix} 4 & 2 & 0.3 \\ 1 & 0 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 3 & 0 \\ 4 & 2 & -2 \end{bmatrix}$ $\begin{bmatrix} 4 & 2 \\ 0.3 & -2 \\ 0 & 0 \end{bmatrix}$

2. Add or subtract these matrices.

a) $\begin{bmatrix} 9 & 0.3 & 4 \\ -2 & 5 & 0.8 \end{bmatrix} + \begin{bmatrix} -4 & 0.9 & 1.9 \\ 2 & -3 & 0.3 \end{bmatrix}$

b) $\begin{bmatrix} -8 & -4 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 0.3 & -8 \\ -0.5 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 8 & -11 \\ 1.9 & 3.2 \\ -8.4 & 6 \end{bmatrix} - \begin{bmatrix} -5 & 2.3 \\ -2 & 1.8 \\ 0.4 & -2.5 \end{bmatrix}$

3. The two matrices below describe average high temperatures (Matrix A) and low temperatures (Matrix B) in degrees Celsius at three locations for three different months.

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \text{Jan.} \quad \text{May} \quad \text{Aug.} \\ \begin{bmatrix} 9.4 & 23.5 & 25.3 \\ 16.1 & 27.2 & 31.4 \\ 20.4 & 30.1 & 30.2 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Paro} \\ \text{Punakha} \\ \text{Trashigang} \end{array}$$

$$B = \begin{array}{c} \text{Jan.} \quad \text{May} \quad \text{Aug.} \\ \begin{bmatrix} -5.8 & 10.6 & 14.7 \\ 4.2 & 14.8 & 19.8 \\ 10.5 & 20.8 & 22.7 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Paro} \\ \text{Punakha} \\ \text{Trashigang} \end{array}$$

a) Calculate A – B. What do the elements in the difference matrix represent?

b) Calculate B – A. What do the elements in the difference matrix represent?

4. The matrices below describe the number of animals two farmers already had (A) and the number they have just been given by family members (B).

$$A = \begin{array}{c} \text{Cat Dog Pig Chicken} \\ \begin{bmatrix} 2 & 3 & 2 & 1 \\ 1 & 4 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Farmer 1} \\ \text{Farmer 2} \end{array}$$

$$B = \begin{array}{c} \text{Cat Dog Pig Chicken} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Farmer 1} \\ \text{Farmer 2} \end{array}$$

Create a matrix to show how many animals of each type each farmer has.

5. a) Create two different 2 × 5 matrices. Call them Matrices A and B.

b) Calculate A – B.

c) Calculate B – A.

d) What do you notice about the signs of the elements of the resulting matrices in **parts b) and c)**? Why do you think this happened?

6. The first matrix below describes the number of different Nu notes that Dorji's father and mother each had saved. The second matrix describes the Nu notes they added to their savings.

$$\begin{array}{c} \\ \text{F} \\ \text{M} \end{array} \begin{array}{c} 10 \quad 20 \quad 50 \quad 100 \\ \begin{bmatrix} 2 & 10 & 5 & 2 \\ 1 & 20 & 3 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \text{F} \\ \text{M} \end{array} \begin{array}{c} 10 \quad 20 \quad 50 \quad 100 \\ \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \end{array}$$

Create a matrix that shows the number of notes each has in savings.

7. Dorji added Matrices A and B. The sum matrix was the same as Matrix B. What did Matrix A look like?

$$B = \begin{bmatrix} -3 & 8 & 0.2 & 1.9 \\ -2.4 & -3 & 5.2 & 6.8 \end{bmatrix}$$

8. a) Create three different 3 × 2 matrices and call them Matrices A, B, and C.

b) Calculate A + B.

c) Add your answer for **part b)** to Matrix C.

d) Calculate B + C.

e) Add your answer for **part d)** to Matrix A.

f) What do you notice? Why do you think this happened?

9. Sonam added Matrices A and B. The sum matrix was Matrix C below.

$$C = \begin{bmatrix} -4 & 0 & -8 & 5 \\ 17 & -3 & 28 & -4 \end{bmatrix}$$

a) What could Matrices A and B have been? Find two possible answers.

b) Suppose A – B = C instead. What could Matrices A and B have been? Find two possible answers.

10. Describe a situation where you might add or subtract matrices.

11. Why does it make sense that matrices must have the same dimensions to be added or subtracted?

1.1.3 Multiplying a Matrix by a Scalar

Try This

The row matrix below describes the prices of a number of items in Thai currency, called baht.

$$\begin{array}{cccc} \text{Meal} & \text{Bottle of water} & \text{Candy bar} & \text{Haircut} \\ \end{array}$$
$$\begin{bmatrix} 50 & 9 & 27 & 60 \end{bmatrix}$$

A. i) Find the price of each item in ngultrums (1 baht = Nu 1.16).
ii) Arrange the information about prices in ngultrums in a matrix.

You can multiply a matrix by a single value, called a **scalar.** To do this, you multiply each element in the matrix by that value. For example, the 2 × 2 matrix below is multiplied by the scalar 0.5.

$$0.5 \times \begin{bmatrix} 22 & 16 \\ 10 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 \text{ x } 22 & 0.5 \text{ x } 16 \\ 0.5 \text{ x } 10 & 0.5 \text{ x } 8 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 5 & 4 \end{bmatrix}$$

B. When you answered **part A**, how was it like multiplying a matrix by a scalar?

Examples

Example 1 Combining Operations to Calculate With Matrices

Two matrices, A and B, are defined as shown below.

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 10 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

How many negative elements are there in 3 × A – 2 × B?

Solution

$$3A = \begin{bmatrix} 3 \text{ x } 4 & 3 \text{ x } 1 & 3 \text{ x } 0 \\ 3 \text{ x } 5 & 3 \text{ x } 10 & 3 \text{ x } 2 \end{bmatrix} = \begin{bmatrix} 12 & 3 & 0 \\ 15 & 30 & 6 \end{bmatrix}$$

$$2B = \begin{bmatrix} 2 \text{ x } 2 & 2 \text{ x } 4 & 2 \text{ x } 3 \\ 2 \text{ x } 1 & 2 \text{ x } 3 & 2 \text{ x } 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 12 & 3 & 0 \\ 15 & 30 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -5 & -6 \\ 13 & 24 & -2 \end{bmatrix}$$

There are three negative elements.

Thinking

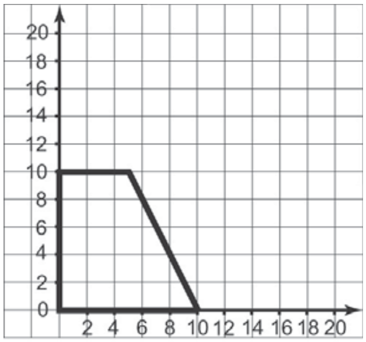
- I multiplied Matrix A by the scalar 3.
- I multiplied Matrix B by the scalar 2.
- I subtracted 3A – 2B.
- I counted the number of negative elements in the final matrix.

Example 2 Multiplying a Matrix by a Scalar to Dilatate a Shape

You can dilatate (enlarge or reduce) a shape by multiplying a matrix that describes the coordinates of its vertices by a single value.

Multiply the matrix of coordinates for this shape by 2. Describe the resulting shape.

$$\begin{bmatrix} 0 & 0 & 5 & 10 \\ 0 & 10 & 10 & 0 \end{bmatrix}$$



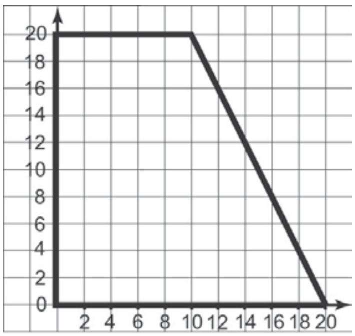
Solution

$$2 \times \begin{bmatrix} 0 & 0 & 5 & 10 \\ 0 & 10 & 10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \text{ x } 0 & 2 \text{ x } 0 & 2 \text{ x } 5 & 2 \text{ x } 10 \\ 2 \text{ x } 0 & 2 \text{ x } 10 & 2 \text{ x } 10 & 2 \text{ x } 0 \end{bmatrix}$$

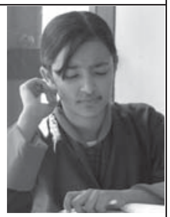
$$= \begin{bmatrix} 0 & 0 & 10 & 20 \\ 0 & 20 & 20 & 0 \end{bmatrix}$$

The resulting shape:



The new shape is similar — it's 2 times the width, 2 times the height, and 4 times the area of the original shape.

Thinking



• I multiplied each element in the shape's matrix by the scalar 2 to get the matrix of coordinates for the new shape.

• I plotted the coordinates (0, 0), (0, 20), (10, 20) and (20, 0) and connected them.

• I noticed the shape looked the same but bigger. It was a dilatation image. One of the vertices, (0, 0), didn't move so that must be the dilatation centre.

Practising and Applying

1. Multiply.

a) $3 \times \begin{bmatrix} 2 & 0.4 \\ 0.8 & 1 \end{bmatrix}$

b) $\frac{2}{3} \times \begin{bmatrix} 24 & 16 \\ 10 & 15 \\ 9 & 31 \\ 18 & 21 \end{bmatrix}$

c) $0.6 \times \begin{bmatrix} 8 & 1.2 & 4 \\ 3.5 & 0.6 & 5 \end{bmatrix}$

d) $-3 \times \begin{bmatrix} -2 & -5 \\ 0.4 & 0.9 \end{bmatrix}$

2. Given Matrices A and B below, calculate each.

a) 2A + B

b) A – 2B

c) 2A + 2B

d) B – 2A

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 9 & 1 & -2 \\ -3 & 2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & -3 & 5 \end{bmatrix}$$

3. Matrix C below describes the coordinates of the vertices of a shape.

$$C = \begin{bmatrix} 4 & 3 & 3 & 5 \\ 3 & 4 & 5 & 4 \end{bmatrix}$$

a) How do you know it is a quadrilateral?

b) Draw the shape on grid paper.

c) Create a matrix for a new shape by multiplying Matrix C by 0.5.

d) Plot the new coordinates. How has the shape changed?

4. The matrix below describes the price of 1 kg of four items in Bhutan in ngultrums. Use matrix multiplication to show the prices in United States (U.S.) dollars in a matrix. (Use the exchange rate, 1 U.S. dollar = Nu 63.5)

$$\begin{array}{cc} & \begin{array}{cccc} \text{Beef} & \text{Cheese} & \text{Rice} & \text{Flour} \end{array} \\ P = & \begin{bmatrix} 80 & 280 & 25 & 16 \end{bmatrix} \end{array}$$

1 U.S. dollar = Nu 63.5

5. The populations of some Bhutanese districts in 2005 are listed in the matrix below. Suppose each population grew by 2.1% per year for the next two years. Describe the populations of the four districts in 2006 and in 2007 using two matrices.

$$\begin{array}{c} \begin{array}{cccc} \text{Bumthang} & \text{Thimphu} & \text{Chukha} & \text{Mongar} \end{array} \\ \begin{bmatrix} 16{,}116 & 98{,}676 & 74{,}387 & 37{,}069 \end{bmatrix} \end{array}$$

6. Find the missing values.

a) $? \times \begin{bmatrix} -4 & 5 \\ 3 & 0 \\ 0.5 & -2 \end{bmatrix} = \begin{bmatrix} -20 & 25 \\ 15 & 0 \\ 2.5 & -10 \end{bmatrix}$

b) $2 \times \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} = \begin{bmatrix} -8 & 12 & 16 \\ 40 & -36 & 32 \\ 76 & 32 & -28 \end{bmatrix}$

c) $? \times \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 0.75 & 3.5 \\ -1.25 & 2 \end{bmatrix}$

7. The matrix below shows the number of cell phone minutes used by Chhimi and Gyeltshen over the past 7 days. Describe a situation where you might want to multiply this matrix by a scalar.

$$\begin{bmatrix} 30 & 10 & 20 & 0 & 0 & 30 & 12 \\ 12 & 15 & 22 & 36 & 4 & 10 & 18 \end{bmatrix}$$

8. Why can you always multiply a matrix of any size by a scalar?

1.1.4 Multiplying Matrices

Try This

A company owns two hotels — one in Paro and one in Thimphu.

• Matrix A shows the rates each hotel charged tourists and government employees during one week in April.

• Matrix B shows the number of guests paying each rate at each hotel that week.

Matrix A (Nu)

$$\begin{array}{l} \\ \text{Paro} \\ \text{Thimphu} \end{array} \begin{array}{c} \begin{array}{cc} \text{Tourist Rate} & \text{Government Rate} \end{array} \\ \begin{bmatrix} 1100 & 950 \\ 1050 & 950 \end{bmatrix} \end{array}$$

Matrix B (Guests)

$$\begin{array}{l} \text{Tourists} \\ \text{Government Employees} \end{array} \begin{bmatrix} 25 \\ 12 \end{bmatrix}$$

A. How much did each hotel earn that week?

• Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix. When comparing dimensions of matrices to see if they can be multiplied, the two inner numbers must match.

For example:

You can multiply $4 \times \underbrace{\mathbf{1} \text{ and } \mathbf{1}} \times 3$ matrices, but not $4 \times \mathbf{1}$ and $\mathbf{2} \times 3$ matrices.

• The dimensions of the product matrix are based on the number of rows in the first matrix and the number of columns in the second matrix.

For example:

Multiplying a **3** × 2 matrix by a 2 × **5** matrix results in a **3** × **5** matrix.

• To multiply matrices, you multiply each element in a row of the first matrix by the corresponding element in a column of the second matrix and then add the products.

For example:

Suppose you have Matrix A and Matrix B below. You know you can multiply them, since the number of columns in A matches the number of rows in B. You also know that the product matrix will be 3-by-3 because 3-by-2 × 2-by-3 = 3-by-3.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \qquad (3 \times \underbrace{\mathbf{2}) \quad \times \quad (\mathbf{2}} \times 3)$$

The element in the 3rd row, 2nd column of the product matrix is the result of multiplying the elements in the 3rd row of Matrix A by the elements in the 2nd column of Matrix B and adding the products (see next page).

$$\begin{array}{ccccccc} A & & B & & A \times B & & \text{Product Matrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} & \times & \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} & = & \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & 5\times8+6\times11 & ? \end{bmatrix} & = & \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & 106 & ? \end{bmatrix} \end{array}$$

Each element is found in a similar way.

For example, the element in the 2nd row, 1st column of the product matrix below comes from multiplying 3 and 4 in the 2nd row of Matrix A by 7 and 10 in the 1st column of Matrix B and adding the products.

$$\begin{array}{ccccccc} A & & B & & A \times B & & \text{Product Matrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} & \times & \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} & = & \begin{bmatrix} 1\text{x}7+2\text{x}10 & 1\text{x}8+2\text{x}11 & 1\text{x}9+2\text{x}12 \\ 3\text{x}7+4\text{x}10 & 3\text{x}8+4\text{x}11 & 3\text{x}9+4\text{x}12 \\ 5\text{x}7+6\text{x}10 & 5\text{x}8+6\text{x}11 & 5\text{x}9+6\text{x}12 \end{bmatrix} & = & \begin{bmatrix} 27 & 30 & 33 \\ 61 & 68 & 75 \\ 95 & 106 & 117 \end{bmatrix} \end{array}$$

B. Why could you have multiplied the matrices to solve **part A**?

Examples

Example 1 Multiplying Matrices to Solve a Problem

Matrix A shows four students' marks on homework, tests, and an examination. The teacher calculates a final grade by weighting homework, tests, and the exam as percentages, as described in Matrix B. What is each student's final grade?

$$A = \begin{array}{c} \text{Homework Tests Exam} \\ \begin{bmatrix} 80 & 70 & 77 \\ 90 & 78 & 82 \\ 75 & 85 & 87 \\ 85 & 71 & 78 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Zangmo} \\ \text{Wangmo} \\ \text{Phurba} \\ \text{Chitrabir} \end{array} \qquad B = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.5 \end{bmatrix} \begin{array}{l} \text{Homework} \\ \text{Tests} \\ \text{Exam} \end{array}$$

Solution

$$A \times B = \begin{bmatrix} 80 & 70 & 77 \\ 90 & 78 & 82 \\ 75 & 85 & 87 \\ 85 & 71 & 78 \end{bmatrix} \times \begin{bmatrix} 0.3 \\ 0.2 \\ 0.5 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}$$

$$= \begin{bmatrix} 80 \text{ x } 0.3 + 70 \text{ x } 0.2 + 77 \text{ x } 0.5 \\ 90 \text{ x } 0.3 + 78 \text{ x } 0.2 + 82 \text{ x } 0.5 \\ 75 \text{ x } 0.3 + 85 \text{ x } 0.2 + 87 \text{ x } 0.5 \\ 85 \text{ x } 0.3 + 71 \text{ x } 0.2 + 78 \text{ x } 0.5 \end{bmatrix}$$

Thinking

- To get a final grade for each student, I needed to multiply the homework marks by 30% (0.3), the test marks by 20% (0.2), and the exam marks by 50% (0.5), and add them together. I knew I could do that by multiplying the matrices.
- I knew I could multiply Matrices A and B, because they are 4 × **3** and **3** × 1. I also knew the product matrix would be 4 × 1 (**4** × 3 and 3 × **1**).

Solution

$$= \begin{bmatrix} 76.5 \\ 83.6 \\ 83 \\ 78.7 \end{bmatrix}$$

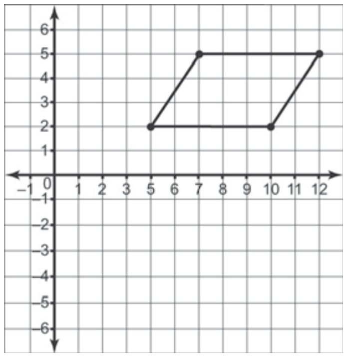
The final grades are
77 for Zangmo
84 for Wangmo
83 for Phurba
79 for Chitrabir

Thinking

- I knew that the order of the rows in the product matrix was the same as in Matrix A so I knew which mark went with each student.
- I rounded the marks to the nearest whole number.

Example 2 Multiplying Matrices to Transform a Shape

You can transform a shape by multiplying matrices. Matrix R below is a matrix that will transform a shape. Matrix C shows the coordinates of the vertices of this shape.



$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 7 & 12 & 10 \\ 2 & 5 & 5 & 2 \end{bmatrix}$$

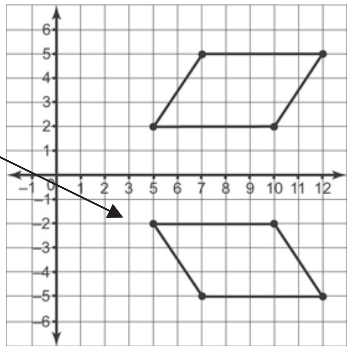
Multiply the two matrices R $\times$ C to create a matrix of vertices for a new shape. What effect did Matrix R have on the shape?

Solution

$$R \times C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 5 & 7 & 12 & 10 \\ 2 & 5 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times5+0\times2 & 1\times7+0\times5 & 1\times12+0\times5 & 1\times10+0\times2 \\ 0\times5-1\times2 & 0\times7-1\times5 & 0\times12-1\times5 & 0\times10-1\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 7 & 12 & 10 \\ -2 & -5 & -5 & -2 \end{bmatrix}$$



Thinking

- I knew I could multiply R and C because the matrices are 2 × **2** and **2** × 4.
- I plotted the coordinates of the product matrix.
- I noticed that the shape was reflected in the *x*-axis.

Practising and Applying

1. a) Which pairs of different matrices can be multiplied and in what order?

$A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix}$

b) Multiply all possible pairs of matrices from **part a).**

2. Suppose you multiplied two matrices and the product matrix was 4 by 2.

a) What size matrices might you have multiplied?

b) How do you know you are correct?

3. a) Draw a hexagon on a coordinate grid. Create matrix B for its coordinates.

b) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Multiply A × B.

c) Plot the new coordinates.

d) What happened to the shape?

4. Dorji has these notes:

Type of Nu note	Number of notes
50	10
100	20
500	6

Calculate the total value of his notes by multiplying two matrices.

5. a) What numbers are missing?

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} ? & ? \\ 0 & ? \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$$

b) What do you notice about the result in **part a)**?

6. a) Why does the order matter when you multiply these matrices?

$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 5 & 1 \end{bmatrix}$

b) Create two 2 × 2 matrices, A and B. Is A × B equal to B × A? Try a different pair. Explain.

7. In a sports tournament, teams get 2 points for a win, 1 for a tie, and 0 for a loss. Matrix A describes the number of wins, ties, and losses for five teams.

$$A = \begin{array}{c} \begin{array}{ccc} W & T & L \end{array} \\ \begin{bmatrix} 6 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 5 & 0 \\ 3 & 0 & 5 \\ 1 & 2 & 5 \end{bmatrix} \end{array}$$

a) What matrix would you multiply by to calculate the total points per team? Call this Matrix B.

b) Would you calculate A × B or B × A? Why?

8. Describe a situation for each:

a) when you might multiply a matrix by a scalar

b) when you might multiply two matrices

9. Suppose you multiplied a 3 × 2 matrix and a 2 × 4 matrix.

a) What would be the dimensions of the product matrix?

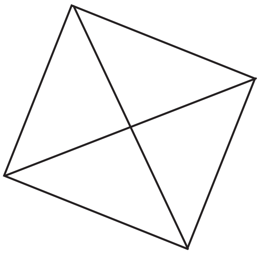
b) How would you calculate the element in the second row, third column of the product matrix?

Chapter 2 Networks

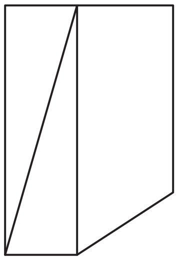
1.2.1 EXPLORE: Travelling Networks

Each set of connected roads below represents a rubbish truck route. To be efficient, the truck does not want to drive a road more than once. Each route is called a **network** and each road in the route or network is called an **edge**.

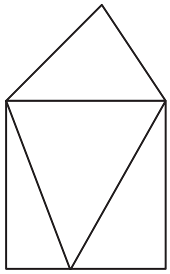
Network 1



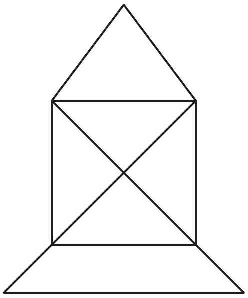
Network 2



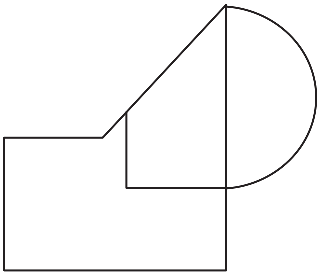
Network 3



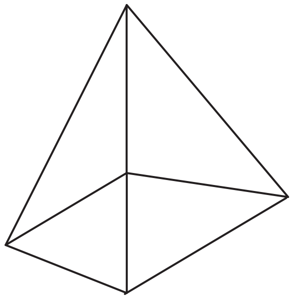
Network 4



Network 5



Network 6

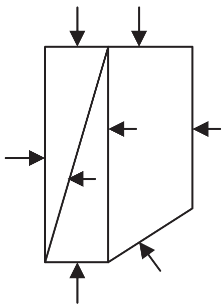


A. To determine which routes the truck can travel without driving any road more than once in either direction, follow each with your pencil. Which networks can you follow, covering each edge only once, without lifting your pencil?

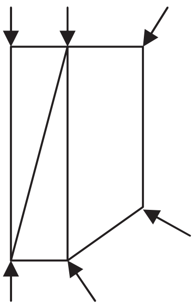
B. For each network, count the number of edges that meet at each vertex. Note which vertices have even and odd numbers of edges. What do you notice about the networks you chose in **part A**?

C. Make your own network with at least ten edges that can be travelled by covering each edge only once, without lifting your pencil.

8 edges



6 vertices



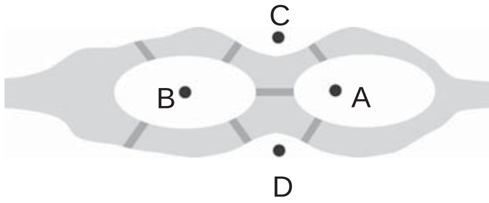
Each path is called an edge.

A **vertex** is a point where two or more edges meet.

CONNECTIONS: The Seven Bridges of Konigsberg

Leonhard Euler, pronounced *Oiler*, (1707–1783), was a famous Swiss mathematician who studied network problems. One of the problems for which he is famous is called the Seven Bridges of Konigsberg. (Konigsberg, now called Kaliningrad, is a city in Russia.)

A river ran through the city. Seven bridges were built so that people could get from one part of the city to another. Euler wondered if someone could travel across all seven bridges exactly once, travelling through each point and beginning and ending at the same point on the map to the right.



1. Do you think it is possible? Try it to find out.

2. Can you do it by starting and ending at different points? Explain.

GAME: Sprouts

Sprouts was invented by an American mathematician named John Conway.

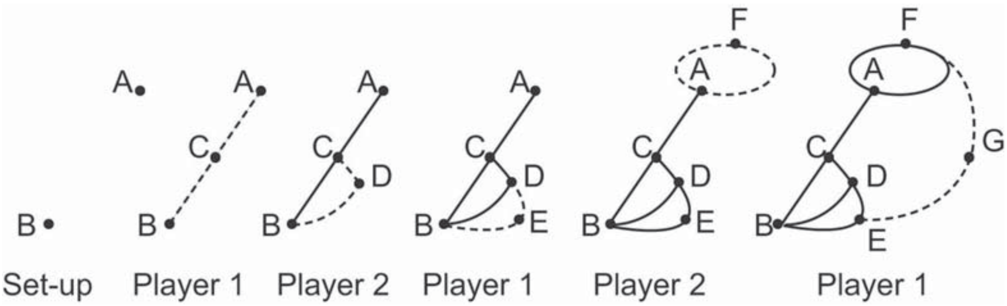
• Player 1 connects two vertices by drawing an edge (from one vertex to another) or connects a vertex to itself by a loop (an edge that starts and ends at the same vertex). He or she then creates a new vertex near the middle of that edge or loop.

• Player 2 then connects two vertices with an edge or draws a loop at a vertex and then creates a new vertex near the middle of that edge or loop.

• The two players continue in this fashion, taking turns, following these rules:

- No more than three edges can meet at a vertex.
- No edge can cross another edge.
- The winner is the last person to draw an edge or loop.

In the sample game below, Player 2 will win because he or she can draw an edge connecting F and G while creating a new vertex H. After six turns, there will be three edges at each vertex except for H, so H cannot be connected to any other vertex. And, a loop at H would result in four edges meeting at H.



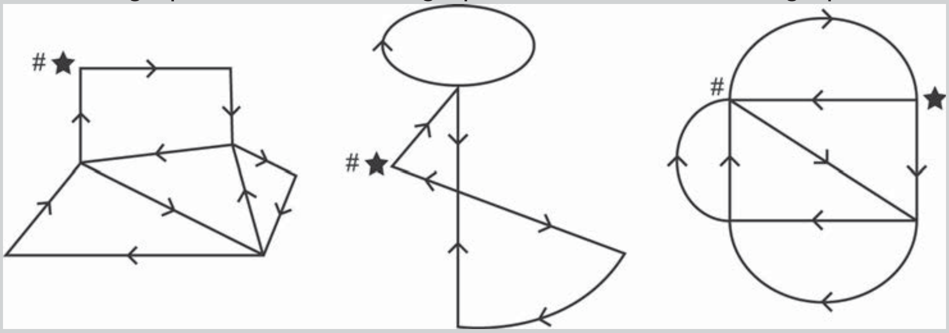
Play the game with a partner. Allow each player to play first.

1.2.2 Describing a Network with a Matrix

Try This

The networks below are called **directed graphs** (or **digraphs**) because you travel them in the direction that is shown on the edges.

Digraph 1 Digraph 2 Digraph 3



A. i) Travel each digraph. Start at the * and end at the #, following the arrows. Try to cover each edge only once, without lifting your pencil.

ii) Which networks can you travel without repeating an edge?

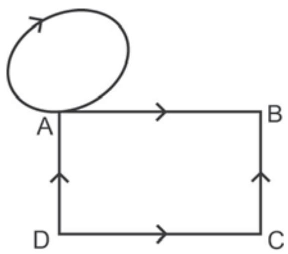
You can use a square matrix, called an **adjacency matrix,** to describe a digraph. To create an adjacency matrix

- list the vertices of the digraph across the top and down the side of the matrix
- for each row, count the number of direct edges *from* that vertex to the vertex listed for each column

For example, Matrix N below is the adjacency matrix for the digraph next to it. The first row of the matrix describes the number of edges from vertex A to each vertex: A, B, C, and D. The elements are **1 1 0 0** because there is

- **1** direct edge from A to A (a loop)
- **1** direct edge from A to B
- **0** direct edges from A to C
- **0** direct edges from A to D [Note that there is an edge between D and A, but it is from D to A, not from A to D.]

The remaining rows are filled in the same way.



Matrix N

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

← Number of edges *from* vertex A *to* each vertex: A, B, C, and D

Notice that the total of the elements in the adjacency matrix equals the total number of directed edges in the digraph, in this case, 5.

You can also create a digraph from an adjacency matrix. To create the digraph

- draw a dot for each vertex in the matrix and label it with a different letter
- for the first row of the matrix, draw each number of direct edges *from* the vertex for that row to the vertex listed at the top of each column
- repeat the previous step for each row of the matrix

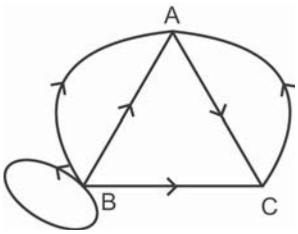
For example.

Adjacency Matrix R below is 3 by 3. That means there are 3 vertices in the digraph: A, B, and C. The second row of the matrix (vertex B), **2 1 1**, means

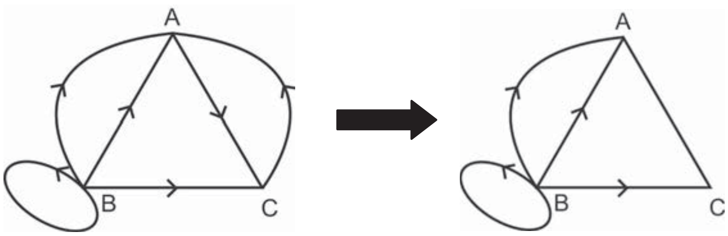
- **2** direct edges from B to A
- **1** direct edge from B to itself (a loop)
- **1** direct edge from B to C

Matrix R

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



- If there are edges in opposite directions between vertices, the edges can be replaced by one edge with no arrow. For example, the edge from A to C and the edge from C to A can be replaced by a single edge with no arrow. It is understood that the edge without an arrow between A and C is directed both from C to A and from A to C.



Simplify a digraph by replacing two opposite directed edges with one edge with no arrow

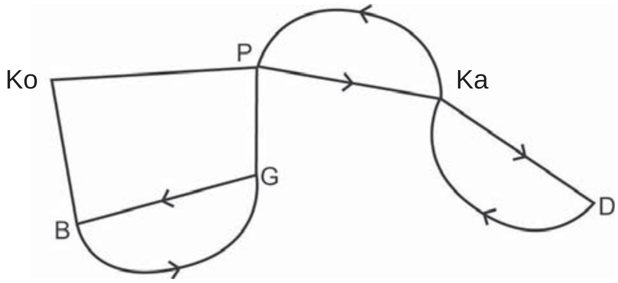
B. Create an adjacency matrix for one of the digraphs from **part A**.

Examples

Example 1 Describing a Digraph with a Matrix

a) Create a matrix to describe this digraph showing Druk Air flights.

b) Simplify the digraph further and describe what you did.



Legend

Ko = Kolkata
P = Paro
Ka = Kathmandu
B = Bangkok
G = Gaya
D = Delhi

<table>
<tr><th>Solution</th><th>Thinking</th></tr>
<tr><td>

a)

$$\begin{array}{cc} & \begin{array}{cccccc} \text{B} & \text{D} & \text{G} & \text{Ka} & \text{Ko} & \text{P} \end{array} \\ \begin{array}{c} \text{B} \\ \text{D} \\ \text{G} \\ \text{Ka} \\ \text{Ko} \\ \text{P} \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

</td><td>

a) I used a 6 × 6 matrix for the 6 cities.

• For each flight path, or edge with an arrow, I found the row where it starts and the column where it ends and then put a 1 there, e.g., for the edge from P to Ka, I put a 1 in the P row-Ka column.

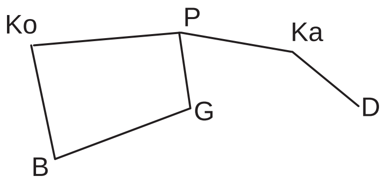
• For edges with no arrows, I put a 1 for each way because no arrow means you can travel both ways, e.g., for the edge between Ko and P, I put a 1 in the Ko row-P column and a 1 in the P row-Ko column.

• I used a 0 if there were no edges.

• I checked by counting the cities you could get to from each city and the numbers are right — 2 for Ko, Ka, B, and G, 1 for D, and 3 for P.

</td></tr>
<tr><td>

b) A simpler, but equivalent, digraph would be this one:



I replaced the two edges going in opposite directions with one edge with no arrow for the edges between P and Ka, between B and G, and between Ka and D.

</td><td>

b) I know that if you see an edge with no arrow, it's really two edges, one going one way and the other going the opposite way. So all of the edges in the simplified digraph are really two edges.

</td></tr>
</table>

Example 2 Creating a Digraph to Match an Adjacency Matrix

This matrix describes a digraph. Create a digraph for the matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

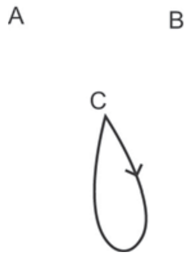
Solution

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Thinking

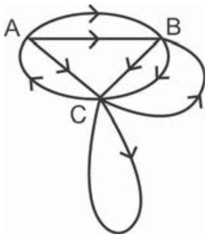
I labelled the rows and columns of the matrix A, B, and C, so it was easier to figure out.

Step 1



Step 1: I started with 3 vertices, A, B, C, because the matrix is 3 × 3. I knew C connected to itself because there is a 1 in the C row-C column, so I drew a loop at C.

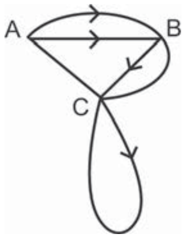
Step 2



Step 2: I drew these edges:

- 2 from A to B and 1 from A to C
- 2 from B to C
- 1 from C to A and 1 from C to B

Possible Step 3

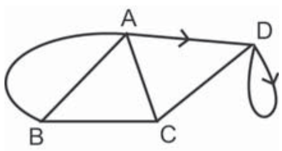


Possible Step 3: I knew that, if I wanted to simplify the digraph, I could replace any two edges going in opposite directions with one edge with no arrow.

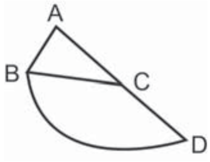
Practising and Applying

1. Describe each digraph with a matrix.

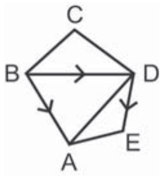
a)



b)



c)



2. Draw a digraph for each matrix.

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

3. A 4 × 4 adjacency matrix has all zeros on the diagonal. What does this tell you about loops on the digraph?

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

4. **a)** You can use a digraph to describe an ecosystem. Draw a digraph to represent this information.

Predator	Food
Insects Caterpillars Songbirds	Plants
Hawks	Toads Songbirds
Songbirds	Caterpillars
Toads Songbirds Insects	Insects

b) Create the adjacency matrix.

5. Geographical regions are interconnected with power lines so that regions can send electrical power to other regions in times of demand or excess. This matrix describes an electrical network among four regions. Draw two different digraphs to show the network.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

6. Why is there usually more than one digraph for an adjacency matrix?

7. Listed below are the number of direct flights between three cities:

- 10 from Hong Kong to Tokyo
- 11 from Tokyo to Hong Kong
- 8 from Tokyo to Bangkok
- 9 from Bangkok to Tokyo
- 5 from Hong Kong to Bangkok
- 3 from Bangkok to Hong Kong

Create a digraph for this information.

8. Look at the matrices in **question 2**. From just looking at each matrix, how can you tell each of the following?

a) the number of vertices in the digraph

b) if there are loops in the digraph

c) that two vertices are not connected

d) the total number of edges that travel away from a vertex

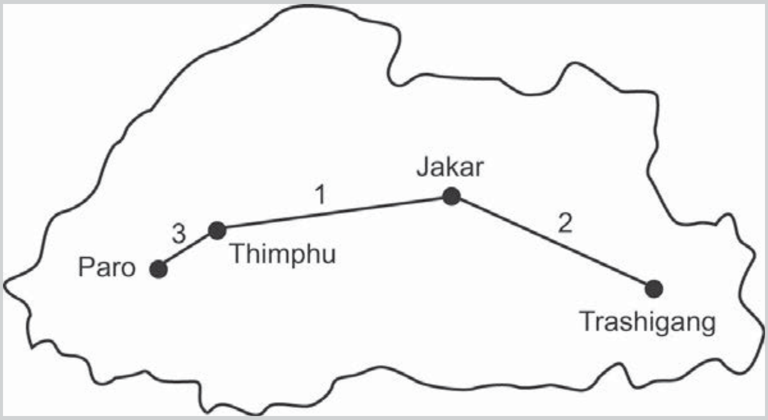
e) the total number of edges that travel toward a vertex

9. How can you tell how complicated a digraph will be from just looking at the numbers in its adjacency matrix?

1.2.3 Solving Network Problems

Try This

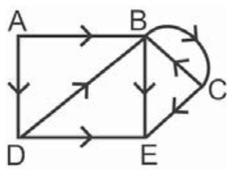
A new helicopter company wants to fly helicopters between the district centres on the map shown below. The number of daily return trips planned each way for each day is shown. For example, each day there will be 3 trips from Paro to Thimphu and 3 trips from Thimphu to Paro. To get from one place to another, sometimes you have to stop at a location between them. This is called a one-stopover trip.



A. How many one-stopover trips are there between Paro and Jakar? How do you know?

- To determine the total number of one-stopover trips between any two locations in a network, you can create a digraph and then count the trips.

For example, this is a digraph of a network:



From analysing at the digraph, you can figure out that there are 10 one-stopover trips between pairs of vertices:

- 1 from A to B (A-D-B)
- 1 from A to C (A-B-C)
- 2 from A to E (A-B-E and A-D-E)
- 1 from B to B (B-C-B)
- 1 from B to E (B-C-E)
- 1 from C to C (C-B-C)
- 1 from C to E (C-B-E)
- 1 from D to C (D-B-C)
- 1 from D to E (D-B-E)

You call it a one-stopover trip, even if you are starting and ending at the same place.

• Another way to figure out the number of one-stopover trips between vertices in a network is to create an adjacency matrix of the digraph (M^1) and then multiply it by itself, or square it. Each element in the squared matrix (M^2) is the number of one-stopover trips between vertices.

For example:

The squared matrix below (M^2) shows the number of one-stopover trips in the digraph at the bottom of **page 22**. Note that the elements of M^2 total 10, which is the same as the number of one-stopover trips found from analysing the digraph.

$$\overset{M^1}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \times \overset{M^1}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} = \overset{M^2}{\begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Each element of M^2 represents a one-stopover trip involving **2** edges, whereas each element of $M^{\mathbf{1}}$ represents a no-stopover trip involving **1** edge.

• To understand how the squared matrix (M^2) relates to the digraph, you can examine how one element is calculated.

$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The element **2** is calculated by multiplying numbers in the A row of the left-hand matrix by numbers in the E column of the right-hand matrix and adding them:

$$2 = 0 \times 0 + 1 \times 1 + 0 \times 1 + 1 \times 1 + 0 \times 0$$

Think about each part of the calculation:

0 × **0** means **0** paths from A to A and **0** paths from A to E, so there are **0** one-stopover trips from A to E through A.

1 × **1** means **1** path from A to B and then **1** from B to E, so there is **1** one-stopover trip from A to E: A-B-E.

0 × **1** means **0** paths from A to C and **1** from C to E. Since there is no way to get from A to C, there are **0** one-stopover trips from A to E through C.

1 × **1** means **1** path from A to D and then **1** from D to E, so there is **1** one-stopover trip from A to E through D: A- D- E.

0 × **0** means **0** paths from A to E and **0** paths from E to E, so there are **0** one-stopover trips from A to E through E.

• Notice that

- the row of zeros next to E in the matrix means there are 0 one-stopover trips that begin at E
- the column of zeros under A means there are 0 one-stopover trips that end at A

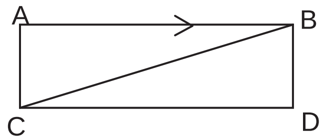
• If you were to cube the original adjacency matrix instead of squaring it, the elements in the cubed matrix (M^3) would be the number of two-stopover trips. Each element in M^3 would represent a trip involving 3 edges of the digraph.

B. Create an adjacency matrix for the helicopter network from **part A** and then square the matrix. Which element in the squared matrix is the answer to **part A**?

Examples

Example 1 Determining the Number of Trips

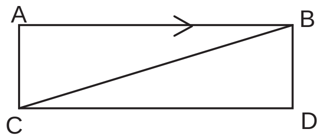
This digraph represents a network of flights.



a) How many one-stopover trips are there from A to D?

b) How many two-stopover trips are there from A to D?

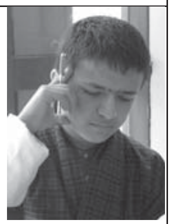
Solution 1



a) There are 2 one-stopover trips from A to D.

b) There are 2 two-stopover trips from A to D.

Thinking



a) I analysed the digraph and counted the number of one-stopover trips from A to D:

- A-B-D
- A-C-D

b) I analysed the digraph and counted the two-stopover trips from A to D:

- A-B-C-D
- A-C-B-D

Solution 2

a)

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 1 \\ C & 1 & 1 & 0 & 1 \\ D & 0 & 1 & 1 & 0 \end{array} \times \begin{array}{|cccc|} A & B & C & D \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}$$

$$= \begin{array}{c|cccc|} & A & B & C & D \\ \hline A & 1 & 1 & 1 & 2 \\ B & 1 & 2 & 1 & 1 \\ C & 0 & 2 & 3 & 1 \\ D & 1 & 1 & 1 & 2 \end{array}$$

There are 2 one-stopover trips from A to D.

Thinking

a) I created an adjacency matrix for the digraph. I knew that if I squared it, the result would be a matrix of one-stopover trips.

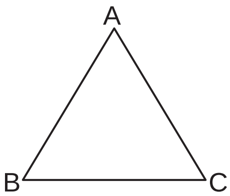
- To find the number of one stopover trips from A to D, I looked at the element in the A row-D column of the squared matrix.

Solution	Thinking
b) $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \times \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$ $$= \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 4 & 4 & 2 \\ 1 & 3 & 4 & 3 \\ 3 & 4 & 3 & 5 \\ 1 & 4 & 4 & 2 \end{bmatrix} \end{array}$$ There are 2 two-stopover trips from A to D.	 **b)** I knew that if I cubed the adjacency matrix, the result would be a matrix of two-stopover trips. • I knew that multiplying the squared matrix by the adjacency matrix is the same as cubing the adjacency matrix. I put the original matrix as the first matrix and the squared matrix as the second one. • To find the number of two-stopover trips from A to D, I looked at element in the A row-D column of the cubed matrix.

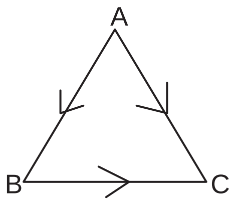
Example 2 Comparing Bi-directional and One-Directional Networks

The two networks shown below are the same except that one has edges that go in both directions and one has edges that go in one direction only.

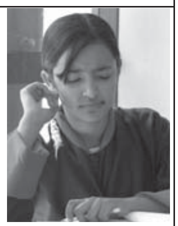
Compare the number of one-stopover trips from one vertex to another in the two networks. What do you notice? Why do you think that happens?



Network 1



Network 2

Solution	Thinking
Network 1 adjacency matrix $$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$ Network 2 adjacency matrix $$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$ [Continued]	• I created the adjacency matrices for both networks first. 

Example 2 Comparing Bi-directional and One-Directional Networks [Cont'd]

Solution

One-stopover trips for Network 1

$$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\times\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}=\begin{array}{cc} & \begin{array}{ccc}A&B&C\end{array}\\ \begin{array}{c}A\\B\\C\end{array} & \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\end{array}$$

One-stopover trips for Network 2

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\times\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{array}{cc} & \begin{array}{ccc}A&B&C\end{array}\\ \begin{array}{c}A\\B\\C\end{array} & \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\end{array}$$

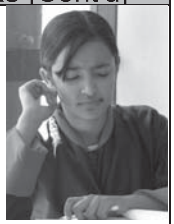
Network 1 Network 2

$$\begin{array}{cc} & \begin{array}{ccc}A&B&C\end{array}\\ \begin{array}{c}A\\B\\C\end{array} & \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{array}{ccc}A&B&C\end{array}\\ \begin{array}{c}A\\B\\C\end{array} & \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\end{array}$$

A to A	different
A to B	different
A to C	same
B to A	same
B to B	different
B to C	different
C to A	different
C to B	same
C to C	different

The only one-stopover paths that were the same described the reverses of the three one-directional paths.

Thinking

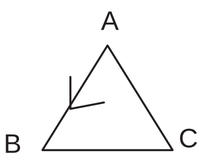


• I squared both matrices to calculate the number of one-stopover trips for each network.

• I compared the two matrices. The only elements that were the same were the one-stopover trips from A to C, from B to A, and from C to B.

Practising and Applying

1. The digraph below shows direct flights among three airports.



a) Use the digraph to count the one-stopover trips between pairs of airports. Include A to A, B to B, and C to C.

b) Square the adjacency matrix to determine the number of one-stopover trips between pairs of airports.

2. a) Create a digraph for this matrix and then use it to answer the questions below.

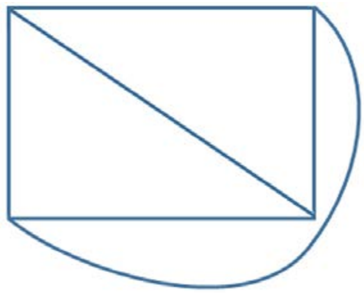
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

b) How many one-stopover trips are there between each pair of vertices?

c) How many two-stopover trips are there between each pair of vertices?

3. Check your answer to **question 2 part a)** by squaring the adjacency matrix. Check your answer to **part b)** by cubing the adjacency matrix.

4. Below is a rectangle with its diagonals shown.



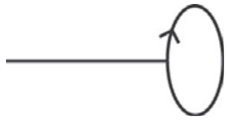
a) Create an adjacency matrix.

b) Use the matrix to calculate the number of one-stopover trips from any vertex of the rectangle to any other. What do you notice?

c) Repeat **part b)** for two-stopover trips.

5. Create a network to represent flights between three airports where some flights go in both directions and some do not. How many one-stopover paths are possible between any two of the airports? Explain.

6. a) Create an adjacency matrix for this digraph.



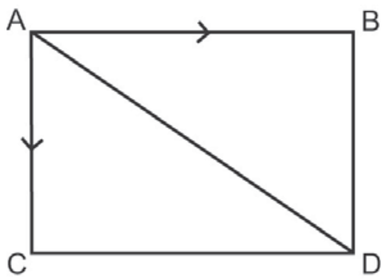
b) Calculate the matrix that shows the number of one-stopover trips between any two vertices. Why are there no zeros in the matrix?

c) Draw another digraph for which there are always one-stopover trips between any two vertices. Explain how you know your digraph is correct.

7. a) Why might someone think of the passing of the ball in a soccer game as a network?

b) Suppose you were to create an adjacency matrix to describe the passing of the ball. What would the square of that matrix tell you?

8. Suppose you created an adjacency matrix for the digraph below and then squared it. How do you know the element in the second row, fourth column of the squared matrix has to be 0?



UNIT 1 Revision

1. Create a 3 × 4 matrix with more negative than positive elements.

2. Describe three situations for which someone might use matrices.

3. a) Which pairs of matrices below can you add? How do you know?

b) Add the pairs of matrices you chose for **part a)**.

c) Which pairs of matrices can you subtract?

d) Subtract the pairs of matrices you chose for **part c)**. Why are there two possible answers?

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 1 & -5 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 2 & 3 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

4. The coordinates of the three vertices of a triangle are listed in this matrix.

$$T = \begin{bmatrix} 0 & 10 & 6 \\ 4 & 8 & -2 \end{bmatrix}$$

a) Plot the points on a grid.

b) Multiply the matrix by 1.5.

c) Plot the new coordinates.

d) What happened to the shape?

5. Suppose you were to multiply a 3 × 4 matrix by the scalar 3. What will the dimensions of the resulting matrix be?

6. Matrices A and B describe the numbers of boys and girls in three schools in March and again in November.

$$A = \begin{array}{c} \text{March} \\ \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 340 & 320 \\ 215 & 245 \\ 128 & 144 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \text{November} \\ \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 330 & 322 \\ 217 & 241 \\ 135 & 140 \end{bmatrix} \end{array}$$

a) Calculate A – B.

b) What do the elements of the resulting matrix describe?

c) Can A + B be calculated? Explain.

d) Why are the elements in the result of A + B meaningless?

7. **a)** Multiply these two matrices two ways: A × B and B × A.

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix}$$

b) What do you notice about the dimensions of the product matrices?

8. What numbers are missing in the matrices below?

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & ? & 1 & -1 \end{bmatrix} \times \begin{bmatrix} ? & 3 \\ 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & ? \\ 3 & 1 \end{bmatrix}$$

9. **a)** If Matrix A is a 3-by-4 matrix and Matrix B is a 4-by-3 matrix, why can you multiply both A × B and B × A?

b) Suppose you multiplied two matrices and the result was a 3 × 2 matrix. What could have been the dimensions of the two matrices?

10. Two of these matrices below were multiplied resulting in a product matrix.

a) Which matrix is the product matrix?

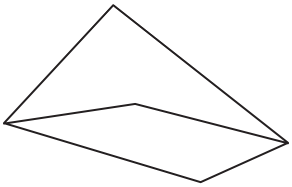
b) Which two matrices were multiplied? In what order were they multiplied?

Matrix A: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 2 & 3 \end{bmatrix}$

Matrix B: $\begin{bmatrix} 0 \\ -5 \end{bmatrix}$

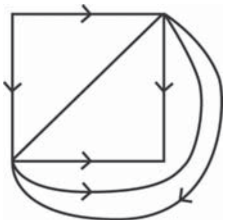
Matrix C: $\begin{bmatrix} 1 \\ -3 \\ 5 \\ -6 \end{bmatrix}$

11. The digraph below represents a network of houses. A house is located at each vertex. Where would you have to start and end to walk all the paths between the houses, visiting each house only once and travelling each path only once? Why?

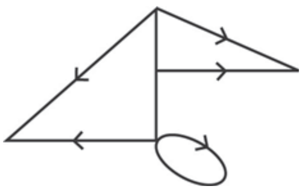


12. Create an adjacency matrix for each digraph.

a)



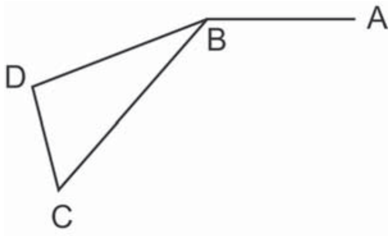
b)



13. Create a digraph for this adjacency matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

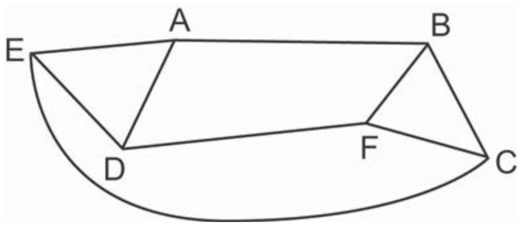
14. This graph represents a network of trekking trails.



a) Create the adjacency matrix for the network.

b) How many trips are there from A to C with one stopover? two stopovers?

15. Digraphs can be used to represent tournaments. Each vertex in the graph below represents a team and each edge represents a game played so far between two teams. Each team must play one game against each other team. How do you know the tournament is not over yet?



UNIT 2 COMMERCIAL MATH AND NUMBER

Getting Started

Use What You Know

A. Copy and complete the grid pattern. Round to four decimal places.

$1.01^2 =$	$1.02^2 =$	$1.03^2 =$
$1.01^3 =$	$1.02^3 =$	$1.03^3 =$
$1.01^4 =$	$1.02^4 =$	$1.03^4 =$
$1.01^5 =$	$1.02^5 =$	$1.03^5 =$
$1.01^6 =$	$1.02^6 =$	$1.03^6 =$

B. Do the differences across each row increase or decrease as the base increases? Explain why your answer makes sense.

C. Do the differences down each column increase or decrease as the exponent increases? Explain why your answer makes sense.

Skills You Will Need

1. The legs of a right-angle triangle are 12 cm and 5 cm.

a) What is the length of the hypotenuse?

b) A similar triangle has a perimeter of 120 cm. What is the length of its hypotenuse? How do you know?

2. How do you know that $\sqrt{40}$ is only twice as much as $\sqrt{10}$?

3. Determine the amount of interest earned over 5 years when Nu 6000 is invested at a simple interest rate of 7% per annum. Show your work.

4. a) 40 represents 25% of a number. What is the number?

b) 25 represents 40% of a number. What is the number?

c) Which is greater: 25% of 40 or 40% of 25? Explain.

5. A whole number raised to the exponent 6 must be both a perfect square and a perfect cube. Why?

2.1.1 Purchasing Decisions

Try This

Store A marked the price of an item at Nu 2000 and then offered a discount of 10%. Store B marked the identical item at a different price and offered a discount of 20%. In the end, the selling prices were the same.

A. What was the marked price at Store B?

• A **discount** is the amount that is taken off the **marked price** (the initial price assigned to the item), usually to encourage a purchase. A **percent discount** is the percentage of the marked price that the discount represents.

• The **selling price** is found by applying the percent discount to the marked price. It can be calculated in different ways.

For example, an item with a marked price of Nu 800 is discounted by 25%. The selling price can be calculated in these two ways:

Subtract the discount from the marked price

800 – 25% × 800

= 800 – 0.25 × 800 = 800 - 200

= Nu 600

Subtract the percent discount from 100% and calculate that percentage of the marked price

(100 – 25)% × 800 = 75% × 800

= 0.75 × 800

= Nu 600

Both of these methods work because a discount of 25% means the selling price is 100% – 25% = 75% of the marked price.

• If the selling price and the marked price are known, the percent discount can be calculated in different ways.

For example, the selling price of an item is Nu 2250 and the marked price is Nu 3000. The percent discount can be calculated in these two ways:

Find the percent of the marked price that the selling price represents

$$\frac{\text{selling price}}{\text{marked price}} \times 100$$

$$= \frac{2250}{3000} \times 100$$

= 0.75 × 100

= 75%

Subtract that percent from 100%

100% – 75% = 25%

Find the discount

marked price – selling price

= 3000 – 2250

= Nu 750

Find the percentage of the marked price the discount represents

$$\frac{750}{3000} \times 100$$

= 0.25 × 100

= 25%

• The **markup** is the amount by which the marked price exceeds the price the seller paid to obtain the goods, the **cost price**. Sellers mark up items in order to make a profit. The **percent markup** is the percentage of cost price the markup represents.

For example, the marked price of an item in a store is Nu 3700. The storeowner paid a cost price of Nu 2500 for the item. The markup and percent markup can be calculated as shown below:

Subtract the cost price from the marked price

3700 – 2500 = Nu 1200

Find the percentage of the cost price that the markup represents

$\frac{1200}{2500} \times 100 = 0.48 \times 100 = 48\%$

• If you know the cost price and percent markup, you can calculate the selling price. For example, a storeowner paid a cost price of Nu 1750 for an item and marks it up 75%. The selling price can be calculated as shown below:

Cost price × (1 + percent markup as a decimal) = 1750 × 1.75 = Nu 3062.5

• **Commission** is an amount a salesperson earns based on his or her sales. Commission is intended to encourage the salesperson to bring in more business. A salesperson's entire income can be based on commission or commission can be a supplement to his or her income. The **percent commission**, or **rate of commission**, is the percentage of sales that the salesperson earns.

For example, a car salesperson, who receives a rate of commission of 10% on each sale, sells a car for Nu 400,000. The commission can be calculated like this: 400,000 × 10% = 400,000 × 0.10 = Nu 40,000

B. The percent discount applied to the same item in Store B is double that of Store A. However, both stores sell the item for the same selling price. What is the ratio of Store B's marked price to Store A's marked price? Explain.

Examples

Example 1 Calculating Percent Discount

A restaurant meal usually costs Nu 80. A special rate of Nu 60 is offered for lunch on Thursdays only. Calculate the percent discount.

Solution 1	Thinking
$\frac{\text{selling price}}{\text{marked price}} \times 100$ $= \frac{60}{80} \times 100 = 75\%$ 100% – 75% = 25% Percent discount was 25%.	• I calculated the ratio of the selling price to the marked price and multiplied by 100 to figure out what percentage the selling price was of the marked price. • I subtracted that percent from 100%. 
Solution 2	**Thinking**
marked price – selling price = 80 – 60 = Nu 20 $\frac{20}{80} \times 100 = 25\%$ Percent discount was 25%.	• I calculated the discount by subtracting the selling price from the marked price • I found what percent the selling price was of the marked price. 

Example 2 Calculating Percent Markup

A compound bow is sold for Nu 45,000.

a) Determine the percent markup if the shop owner paid a cost price of Nu 15,000.

b) The shop owner paid Nu 2500 for delivery and other expenses in addition to the cost price of Nu 15,000. Re-calculate the percent markup.

Solution

a) *Markup*

45,000 – 15,000 = Nu 30,000

Percent markup

$\frac{30{,}000}{15{,}000} \times 100 = 200\%$

The percent markup is 200%.

b) *Cost price and related expenses*

15,000 + 2500 = Nu 17,500

Markup

45,000 – 17,500 = Nu 27,500

Percent markup

$\frac{27{,}500}{17{,}500} \times 100 = 157.14\%$

The percent markup is 157.14%.

Thinking

a) I calculated the markup by subtracting the cost price from the selling price.

• I then used a ratio to compare the markup to the cost price and multiplied by 100 to calculate a percentage.

b) I calculated how much the shop owner paid for the item, including cost price, delivery, and other expenses.

• I calculated the markup amount by subtracting what the shop owner paid for the item from its selling price.

• I then used a ratio to compare the markup amount to what the shop owner paid and multiplied by 100 to calculate a percentage.

Example 3 Calculating Commission Amount and Rate of Commission

A salesperson had total sales of Nu 60,000.

a) The salesperson is awarded a commission amount of Nu 3000. Determine the rate of commission.

b) Suppose the salesperson is to be paid a rate of commission of 7%. Determine the amount of commission the salesperson would earn.

Solution

a) $\frac{3000}{60{,}000} \times 100 = 5\%$

The rate of commission is 5%.

b) 7% of 60,000 $= 0.07 \times 60{,}000$

$= \text{Nu } 4200$

The commission amount would be Nu 4200.

Thinking

a) I calculated the ratio of the commission amount to total sales and multiplied by 100 to figure out what percentage the commission was of sales.

b) All I had to do was calculate 7%, or 0.07, of 60,000.

Practising and Applying

1. A vendor sells 1 kg of oranges for Nu 25. If you buy 4 kg, you pay Nu 80.

a) How much per kilogram do you pay if you buy 4 kg?

b) What is the percent discount per kilogram when you buy 4 kg compared to 1 kg?

2. The marked price of an item was Nu 140 before it was discounted to Nu 112. Jigme mentally calculated the percent discount. He figured that, because the price was reduced by Nu 28, which is $\frac{1}{4}$ of Nu 112, the percent discount had to be 25%. Is Jigme's thinking correct? Explain.

3. Which of the following options is better for the buyer? Explain.

Option A:
20% markup on an item with a cost price of Nu 480

Option B:
15% discount on the same item that has a marked price of Nu 700

4. A shop owner is trying to get rid of items to make room for new items so she is offering huge discounts. She has discounted an item by 50% to a selling price which is the same as the cost price of the item. What percent markup did the shop owner originally place on this item? Justify your answer with an example.

5. Dorji is a car salesperson. He is paid Nu 1000 each week plus an additional 3% commission on sales.

a) Determine Dorji's total income for a week in which sales were Nu 70,000.

b) Dorji's goal is to earn Nu 5000 each week. What is the minimum amount of weekly sales required to earn this level of income?

6. Explain why a percent markup of more than 100% is reasonable to consider but a percent discount of more than 100% is not reasonable.

7. Give an example of a situation where commission paid as a percent would be less than commission paid as an amount of Nu 500 per sale.

8. **a)** Explain why you cannot determine whether a markup of Nu 100 or a markup of 7% is greater.

b) Give an example of a situation where a markup of Nu 100 would be greater than a 7% markup.

b) Give an example of a situation where a markup of Nu 100 would be less than a 7% markup.

9. Why do you think discounts, markups, and commissions are usually described as percents?

2.1.2 Compound Interest

Try This

A product costs Nu 30 to manufacture. The cost is marked up 20% when the manufacturer sells it to a local business person. The business person marks it up another 20% before selling the product to the shopkeeper. Finally, the shopkeeper marks up the price another 20% before selling the product to the customer.

A. Compared to the original manufacturing cost of the product, determine the percent markup actually charged to each person:

i) the business person **ii)** the shopkeeper **iii)** the customer

• Recall that **simple interest** is a percentage of the **principal**, which is the amount of money that is borrowed or invested. Simple interest only applies to the principal.

For example, if you invested Nu 10,000 at 4% simple interest once a year, also written **p.a.** (**per annum**), your investment will be worth

Nu 10,000 + 4% of Nu 10,000 = Nu 10,400 at the end of one year

Nu 10,400 + 4% of Nu 10,000 = Nu 10,800 at the end of two years

Nu 10,800 + 4% of Nu 10,000 = Nu 11,200 at the end of three years, and so on

• **Compound interest** is different from simple interest in that you also earn or pay interest on interest already earned. Compound interest is described using a per annum percentage rate and a **frequency of compounding**.

For example, the following tables compare two Nu 10,000 investments over a five-year period, one at 4% simple interest p.a. and the other at 4% compound interest p.a. compounded **annually**.

Simple Interest on Nu 10,000

Year	Interest earned (4%)	Total in bank at end of year
1	400.00	10,400.00
2	400.00	10,800.00
3	400.00	11,200.00
4	400.00	11,600.00
5	400.00	12,000.00

Compound Interest on Nu 10,000

Year	Interest earned (4%)	Total in bank at end of year
1	400.00	10,400.00
2	416.00	10,816.00
3	432.64	11.248.64
4	449.95	11,698.59
5	467.94	12,166.53

Notice that the two types of interest yield the same amount at the end of Year 1 but after that the total amount earned grows faster with compound interest than with simple interest. Compound interest yields Nu 16 more than simple interest at the end of Year 2, Nu 48.64 more at the end of Year 3, Nu 98.59 more at the end of Year 4, and Nu 166.53 more at the end of Year 5.

• Compound interest is usually compounded annually, but sometimes it is more frequent: compounded **semi-annually** (every six months), **quarterly** (every three months), **monthly** (every month), or **daily** (every day). The more frequently the interest compounds, the more you earn or pay, since you start earning or paying interest on interest before the year is over.

• When an interest rate is stated, it is a per annum rate. However, if the **interest period** is more frequent than annual, the actual interest rate for the period is the appropriate fraction of the per annum rate.

For example:

5% p.a. compounded semi-annually is $\frac{0.05}{2}$ or 2.5% per half year (semi-annual).

5% p.a. compounded quarterly is $\frac{0.05}{4}$ or 1.25% per three months (quarterly).

The formula for compound interest calculations is $A = P(1 + \frac{r}{n})^{nt}$, where

A = amount earned or paid
P = initial amount invested or borrowed, known as the principal
r = interest rate for a year
n = number of interest periods in one year
t = time expressed in years

For example, the value of a Nu 10,000 investment at 4% p.a. compounded annually for 5 years could be represented as

$$A = P(1 + \frac{r}{n})^{nt} \rightarrow A = 10{,}000\ (1 + \frac{0.04}{1})^{1 \times 5} = 12{,}166.53$$

The table below shows why this value and the formula make sense.

Year	1	2	3	4	5
Amount to start	10,000.00	10,400.00	10,816.00	11,248.64	11,698.59
Interest	400.00	416.00	432.64	449.95	467.94
Amount at end	10,400.00	10,816.00	11,248.64	11,698.59	12,166.53
Multiplier of 10,000	1.04^1 = 1.04	1.04^2 = 1.0816	1.04^3 = 1.1249	1.04^4 = 1.1699	1.04^5 = 1.2167

• The frequency of the compounding affects the result.

For example, if the Nu 10,000 investment had been at a rate of 4% p.a. compounded *semi-annually* (twice a year) for 5 years, the result would be more:

$$A = P(1 + \frac{r}{n})^{nt} \rightarrow A = 10{,}000\ (1 + \frac{0.04}{2})^{2 \times 5} = 12{,}189.94 \text{ (instead of 12,166.53).}$$

There is Nu 23.41 more because interest is earned on interest more frequently.

• Although compounding is favourable to investors, it also adds to debts that must be repaid when money is borrowed.

B. i) Explain how the idea of a compound interest rate of 20% per period could be useful in explaining how the markup process worked in **part A**.

ii) Calculate $(1.20)^3$. Compare the result with the percent markup that the customer paid.

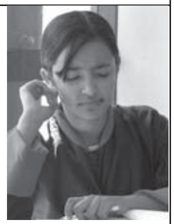
Examples

Example 1 Compound Interest on Investments

Determine the value of a Nu 5000 investment after two years if it is invested at an interest rate of 7% p.a. compounded at each frequency:

a) annually **b)** semi-annually **c)** quarterly **d)** monthly **e)** daily

What do you notice?

Solution	**Thinking**
$A = P(1 + \frac{r}{n})^{nt}$	I used the compound interest formula and substituted the correct values. 
a) $A = 5000(1.07)^{1 \times 2}$ $= \text{Nu } 5724.50$	**a)** The exponent was 1 × 2 because the interest is calculated 1 time each year for 2 years.
b) $A = 5000(1 + \frac{0.07}{2})^{2 \times 2}$ $= \text{Nu } 5737.62$	**b)** The exponent was 2 × 2 because the interest is calculated 2 times a year for 2 years. I divided the interest rate of 0.07 by 2, since that's the rate for each half-year.
c) $A = 5000(1 + \frac{0.07}{4})^{4 \times 2}$ $= \text{Nu } 5744.41$	**c)** The exponent 4 × 2 means there are 8 interest periods, 4 each year for 2 years. The interest rate is divided by 4 because that's the rate for each quarter of a year.
d) $A = 5000(1 + \frac{0.07}{12})^{12 \times 2}$ $= \text{Nu } 5749.03$	**d)** The exponent 12 × 2 means there are 24 interest periods, 12 each year for 2 years. The interest rate is divided by 12 because that's the rate for each month.
e) $A = 5000(1 + \frac{0.07}{365})^{365 \times 2}$ $= \text{Nu } 5751.29$	**e)** The exponent 365 × 2 means there are 730 interest periods, 365 each year for 2 years. The interest rate is divided by 365 because that's the rate for each day.
The interest earned increased as the compounding frequency increased.	• The interest earned was Nu 724.50 (annually), Nu 737.62 (semi-annually), Nu 744.41 (quarterly), Nu, 749.03 (monthly), and Nu 751. 29 (daily).

Example 2 Compound Interest on Loans

Ugyen borrows Nu 3000 from a lender who charges an interest rate of 18% p.a. compounded monthly. Ugyen agrees to pay Nu 1000 at the end of each month until the loan has been repaid.

a) Will Ugyen be able to pay back the loan in three months?

b) How much will Ugyen owe after making his first payment?

c) Determine how many payments Ugyen will make and the amount of his final payment.

d) Find the amount of interest to be paid by Ugyen over the term of the loan.

<table>
<tr><th>Solution</th><th>Thinking</th></tr>
<tr><td>

a) No, he will not be able to pay it back in three months because he will only have paid back Nu 3000, which will not cover the interest on the loan.

</td><td>

a) I knew that Ugyen had to pay back the principal of the loan *plus* interest.

</td></tr>
<tr><td>

b) *Interest in the first month*

$3000 \times \frac{0.18}{12} = \text{Nu } 45$

Amount owing after the first month

3000 + 45 = Nu 3045

Amount still owing after first payment

3045 – 1000 = Nu 2045

</td><td>

b) To calculate the amount of interest for one month, I divided the p.a. rate of 18% by 12.

</td></tr>
<tr><td>

c) *Interest in the second month*

$2045 \times \frac{0.18}{12} = \text{Nu } 30.68$

Amount owing after the second month

2045 + 30.68 = Nu 2075.68

Amount still owing after second payment

2075.68 – 1000 = Nu 1075.68

</td><td>

c) To calculate how much he will owe after the second month, I added the Nu 2045 he owed after the first month to the interest charged on that amount for the second month.

</td></tr>
<tr><td>

Interest in the third month

$1075.68 \times \frac{0.18}{12} = \text{Nu } 16.14$

Amount owing after the third month

1075.68 + 16.14 = Nu 1091.82

Amount still owing after third payment

1091.82 – 1000 = Nu 91.82

</td><td>

• To calculate how much he will owe after the third month, I added the Nu 1075.68 he owed after the second month to the interest charged on that amount for the third month.

</td></tr>
<tr><td>

Interest in the fourth month

$91.82 \times \frac{0.18}{12} = \text{Nu } 1.38$

Amount owing after the fourth

91.82 + 1.38 = Nu 93.20

There are four payments, three at Nu 1000 and a final payment of Nu 93.20.

</td><td>

• Since he would only owe Nu 91.82 after the third month, I knew the next and fourth payment will be his last.

</td></tr>
<tr><td>

d) *Total amount of interest paid*

Total payments – principal
= (1000 + 1000 + 1000 + 93.20) – 3000
= Nu 93.20

</td><td>

d) Alternatively, I could have added up the interest he will pay each month:

45 + 30.68 + 16.14 + 1.38 = Nu 93.20

</td></tr>
</table>

Example 3 Calculating an Interest Rate

Karma borrowed Nu 20,000. He repaid the loan at the end of 4 years with a single payment of Nu 35,680. What interest rate was charged, if the compounding was semi-annual?

Solution

Create an equation

$$A = P(1 + \frac{r}{n})^{nt}$$

$A = 35{,}680$ $P = 20{,}000$

$n = 2$ $t = 4$ $r = ?$

$$35{,}680 = 20{,}000(1 + \frac{r}{2})^{2 \times 4}$$

Solve the equation for r

$$35{,}680 = 20{,}000(1 + \frac{r}{2})^{8}$$

$$35{,}680^{\frac{1}{8}} = \left[20{,}000 \times (1 + \frac{r}{2})^{8}\right]^{\frac{1}{8}}$$

$$35{,}680^{\frac{1}{8}} = 20{,}000^{\frac{1}{8}} \times (1 + \frac{r}{2})$$

$$3.7073 = 3.4485(1 + \frac{r}{2})$$

$$1.0750 = 1 + \frac{r}{2}$$

$$1.0750 - 1 = 1 - 1 + \frac{r}{2}$$

$$0.0750 = \frac{r}{2}$$

$$0.0750 \times 2 = \frac{r}{2} \times 2$$

$$0.15 = r$$

The interest rate was 15% p.a. compounded semi-annually.

Thinking

- I substituted the values I knew into the compound interest formula.

- I wanted to get rid of the exponent 8 from the expression $(1 + \frac{r}{2})^8$ in order to isolate the variable *r*, so I raised both sides of the equation to the $\frac{1}{8}$ power.

- I knew $35{,}680^{\frac{1}{8}} = \sqrt[8]{35{,}680}$ and $20{,}000^{\frac{1}{8}} = \sqrt[8]{20{,}000}$.

- I finished solving for *r* to find the rate of interest.

Example 4 Relating Interest Rates

Pema’s bank is paying an interest rate of 14% p.a. compounded monthly. What would the equivalent annual interest rate be if it were paid over each time period?

a) i) 1 year compounded annually **ii)** 5 years compounded annually

b) What simple interest rate earned over 5 years would be equivalent?

Solution

a) i) Calculate A for Nu 1 invested for 1 year at 14% p.a. compounded monthly:

$$A = P(1 + \frac{r}{n})^{nt} = 1(1 + \frac{0.14}{12})^{12 \times 1} = \text{Nu } 1.1493$$

Nu 0.1493 interest is earned in 1 year and Nu 0.1493 is 14.93% of Nu 1.

An interest rate of 14% p.a. compounded monthly is equivalent to an interest rate of 14.93% p.a. compounded annually for 1 year.

ii) An interest rate of 14% p.a. compounded monthly is equivalent to an interest rate of 14.93% p.a. compounded annually for each of the 5 years.

b) Calculate A for Nu 1 invested for 5 years at 14% p.a. compounded monthly:

$$A = P(1 + \frac{r}{n})^{nt} = 1(1 + \frac{0.14}{12})^{12 \times 5} = \text{Nu } 2.01$$

Nu 1.01 simple interest earned over 5 years means Nu 0.20 (1.01 ÷ 5 = 0.202) is earned in 1 year.

Nu 0.20 is 20.2% of Nu 1.

14% p.a. compounded monthly for 5 years is equivalent to 20.2% simple interest for 5 years.

Thinking

a) i) I used a principal of Nu 1 to make the calculation simpler.

- I substituted $r = 0.14$, $n = 12$, and $t = 1$ into the formula.
- I thought of A as Nu 1 + r, with r representing the interest rate.

ii) It doesn't matter if it's 1 year or 5 years, the annual interest rate is the same because the frequency of the compounding is the same each year (annually) and interest rates are annual (that's what p.a. means).

b) I used a principal of Nu 1 again and substituted $r = 0.14$, $n = 12$, and $t = 5$ into the formula.

- I thought of A as Nu 1 + 5r, where r represented the simple interest rate p.a.

Practising and Applying

1. Pema invested Nu 3200 in an account for four years. The value of the investment at the end of four years was Nu 4000. Determine the rate of simple interest that was earned.

2. An amount of Nu 600 is invested in a savings account earning 4% interest p.a. compounded quarterly.

a) How much money will be in the account at the end of the first year?

b) How much will be in the account at the end of the second year?

3. Repeat **question 2** using a simple interest rate of 4% p.a. What do you observe?

4. Sonam borrowed Nu 9000 from a bank. Calculate the amount of interest she will pay in one year at each interest rate:

a) 12% p.a. compounded monthly

b) 12% p.a. compounded semi-annually

c) 12% p.a. compounded annually

5. A loan for Nu 10,000 is to be repaid in monthly payments of Nu 300. What is the balance on the loan after each period of time, if the interest rate is 14% p.a. compounded monthly?

a) one month

b) two months

6. Phuntso borrowed Nu 30,000 at 16% p.a. compounded annually. She agreed to the following payment plan:

• Nu 10,000 at the end of the first year

• Nu 15,000 at the end of the second year

• A final payment of the remaining amount at the end of the third year

a) Determine the amount of the final payment.

b) Find the amount of interest Phuntso paid altogether.

7. Deki borrowed Nu 12,000 at a rate of interest compounded quarterly. The balance was Nu 10,950 after making her first quarterly payment of Nu 1500. What was the interest rate?

8. A bank advertises a lending rate of 12% p.a. compounded monthly. What is the equivalent rate compounded annually?

9. Nu 25,000 is invested at each interest rate:

I: 9% p.a. compounded quarterly

II: 9% p.a. compounded annually

III: 9% p.a. simple interest

a) What is the value of each investment after 1 year? 10 years? 20 years?

b) Graph each investment on the same grid, with time (years) along the horizontal axis and amount (Nu) along the vertical axis. Use a different colour for each graph.

c) What do the shapes of the graphs tell you about the effect of the different interest rates?

10. Explain how compound interest behaves like exponential growth whereas simple interest reflects linear (constant) growth.

CONNECTIONS: The Rule of 72

The Rule of 72 suggests that an investment will approximately double in 72 ÷ *r* years at a rate of *r* % compounded annually.

1. Check the validity of the Rule of 72 using any investment amount and two different rates for *r*.

2. Would you expect a rate of 12% p.a. compounded monthly to more than double an investment in six years? Explain.

3. How many months does it actually take for an investment earning 12% compounded monthly to more than double in value?

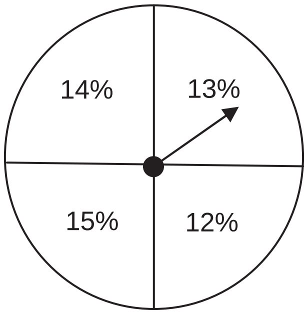
GAME: Target 200

Play this game with two spinners and two dice.

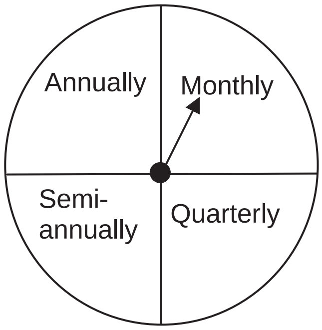
Two to four people can play.

- On your turn, spin a per annum interest rate using Spinner A and how often the interest is compounded using Spinner B.
- Roll a pair of dice. Multiply the two-digit number you roll by 100 to determine the principal you will invest. For example, a roll of 2 and 3 could be Nu 2300 or Nu 3200.
- Your score is the sum of the digits of the total amount your investment will be worth in one year, rounded to the nearest ngultrum.
- The first player with 200 or more points wins.

Spinner A



Spinner B



For example:

Spin for interest rate: 13% monthly

Roll for principal: 3 and 5, so Nu 5300 or Nu 3500

Calculate investment value: $5300(1+\frac{0.13}{12})^{12} = 6032$

Score: 6 + 0 + 3 + 2 = 11 points

Calculate investment value: $3500(1+\frac{0.13}{12})^{12} = 3983$

Score: 3 + 9 + 8 + 3 = 23 points

Choose Nu 3500 and score 23 points.

2.1.3 Dividends and Stocks

Try This

Sonam wants to start a small business. To get started, she needs Nu 100,000. She asks ten members of her family to invest in the business to provide her some of this money. She promises to pay back the money with interest in a year.

A. If ten family members each invest Nu 10,000 and Sonam promises to pay 12% simple interest, how much will she owe each person at the end of the year?

• One way to earn money is to buy **shares**, also called **stocks**, in corporations. This money is used by the corporation to do its business. As a shareholder, you earn money in the form of **dividends** from the company. Dividends are usually given annually as a percentage of the value of the shares, called the **dividend rate**, which is often higher than interest rates offered for money invested in a bank account. Shareholders might earn even more money by selling their shares in the future for a price that is higher than the price they paid. They can also earn more if they reinvest their dividend earnings each year.

• Companies can choose to offer them for sale to the public after getting approval from the government. There are usually two types of shares:

- Equity shares, where profit is shared equally among the shareholders
- Preferential shares, where the percentage applied to calculate the dividend is fixed by the company

• The printed price on the share is called the **face value**. The selling price of the share might be higher, lower, or the same as the face value. The price at which shares are sold is called the **market price**.

- If market price is greater than face value, you say the shares were sold at a premium.
- If market price is less than face value, you say the shares were sold at a discount.
- If market price and face value are the same, you say the shares were sold at par.

• To decide whether an investment is a good one, you can calculate a **yield percentage**, which is the ratio of the dividends earned to the money invested.

This table shows part of a stock report for six companies (November, 2006).

Listed company	Issue price	Best sell price	Last sale price
BBCL	100	200	100
BBPL	100		100
BCCL	1,000	8,500	6,000
BDL	100	140	100
BFAL	100		510
BNB	100	750	700

Notice that the shares listed here were sold either at par or at a premium.

B. i) How is Sonam's arrangement in **part A** like selling shares in a company?
ii) Is it more like equity shares or like preferential shares? Explain.

Examples

Example 1 Calculating How Many Shares Can be Purchased	
Dawa invests Nu 125,000 in RICB shares with a face value of Nu 100 but are being sold at a premium of 25%. How many shares can Dawa buy?	
Solution 1 1.25×100 $= \text{Nu } 125$ $125{,}000 \div 125$ $= 1000$ Dawa can buy 1000 shares.	**Thinking**  • Dawa is buying shares at a premium of 25%, so the market price is 25% more than the face value. I multiplied Nu 100 by 1.25 (125%) to calculate the market price of each share. • To figure out how many shares she can buy, I divided the amount she is investing by the market price.

Example 2 Calculating Dividends Earned	
BNB declares a 25% dividend rate on its stock. Rinchen owns 200 shares, each with a face value of Nu 100. **a)** What dividend amount will Rinchen receive? **b)** Rinchen bought the shares originally at a premium of 30%. What is the yield percentage? **c)** Would Rinchen have been better off investing his money in a savings account paying 8% simple interest? Explain.	
Solution **a)** Dividend amount = $FV \times r \times n$ - FV is the face value of each share - r is the dividend rate - n is the number of shares Dividend amount = $100 \times 0.25 \times 200$ = Nu 5000 **b)** Dividend amount = Nu 5000 Original Investment = $200 \times (100 \times 1.3)$ = Nu 26,000 $\frac{5000}{26000} = \frac{5}{26} = 19.23\%$ The yield percentage is 19.23%. [Continued]	**Thinking**  **a)** I calculated the dividend amount by finding 25% or 0.25 of the face value of each share ($FV \times r$) and then multiplying by the number of shares (n). **b)** To calculate the yield percentage, I compared the dividend amount to the original investment amount using a ratio. • To calculate the original investment amount, I multiplied the number of shares (200) by the premium price of each share (Nu 100 × 130%) — the premium price is what he paid for the shares originally.

Example 2 Calculating Dividends Earned [Continued]	
Solution	**Thinking**
c) No, a yield percentage of 19.23% is much better than an 8% interest rate, so the stock was the better choice.	**c)** Investing in stocks may seem attractive because of the potential for high yields but stocks are riskier than savings accounts. If Rinchen had to sell his shares at a huge discount, the savings account might have been the better investment. 

Practising and Applying

1. What is the market price of each stock?

Stock	Face value (Nu)	Number of shares	Premium or discount
A	100	200	10% premium
B	500	100	20% discount
C	1000	50	15% discount

2. Kinley has Nu 85,000 to invest in a stock. For each stock in **question 1** how many shares can he purchase?

3. The companies issuing each of the stocks in **question 1** each declare a 22% dividend rate.

a) Calculate the dividend amount for 150 shares of each stock.

b) Calculate the yield percentage for each stock.

c) Which was the best investment? Explain.

4. Yangdon bought 50 shares of a stock that had a face value of Nu 100 but were selling at a discount of 15%. A 25% dividend rate was paid at the end of one year. She then sold the stock at a 10% premium.

a) How much profit did she make?

b) How much was her profit as a percentage of her investment?

5. Penjor Wangdi earned a dividend amount of Nu 990 from 50 shares of stock with a face value of Nu 100.

a) What was the dividend rate?

b) How much more would he have earned if the rate had been 5% higher?

c) If he sells his shares at a premium of 10%, how much money will he receive from the sale?

6. You want to buy Nu 20,000 shares, with a face value of Nu 100, at par. To be able to do this, you borrow 60% of the price at an interest rate of 12% p.a. compounded annually. The annual dividend rate is 18%. At the end of the first year, how much more will you earn in dividends than pay in interest on your loan?

7. Suppose you had Nu 25,000 to invest in shares that are being sold at par. How much more would you earn if the shares paid a dividend rate of 30% than if they paid a dividend rate of 5%?

8. What might lead a company to pay higher dividend rates? Explain.

9. a) Why might someone invest in stocks instead of earning interest in a savings account?

b) Why might someone prefer a savings account over investing in stocks?

2.1.4 Using Commercial Math

Try This

Kailash has a choice of investing Nu 1000 with Bank A at 3.93% p.a. compounded monthly or with Bank B at 4% p.a. compounded annually. He predicts the investment with Bank A will be better of the more frequent compounding.

A. Without calculating, do you agree with Kailash's prediction? Explain.

B. Compare how much each option will yield after each period of time.

i) 1 year **ii)** 10 years **iii)** 20 years

Before making a decision about investing, it is important to compare the financial alternatives. When information about investments is given in different forms, the comparison will require some calculations.

For example, suppose you have Nu 1500 to invest:

- If you have the choice of 3.75% p.a. compounded semi-annually or 3.75% p.a. compounded quarterly, the comparison is easy — the more frequent compounding (quarterly) is better.

- If you have the choice of 3.75% p.a. compounded semi-annually or 3.5% p.a. compounded semi-annually, the comparison is also easy — the higher interest rate (3.75%) is better.

- However, if you have the choice of the following three investments, the comparison is not so easy and some calculations are required:

Investment A	*Investment B*	*Investment C*
4% p.a. simple interest	3.75% p.a. compounded semi-annually	3.5% p.a. compounded quarterly

Value of Nu 1500 After 1 year

Investment A	*Investment B*	*Investment C*
Nu 1500 × (1.04) = Nu 1560	$A = P(1 + \frac{r}{n})^{nt}$ $= 1500(1 + \frac{0.0375}{2})^{2}$ = Nu 1556.77	$A = P(1 + \frac{r}{n})^{nt}$ $= 1500(1 + \frac{0.035}{4})^{4}$ = Nu 1553.19

Investment A is best after 1 year.

Value of Nu 1500 After 10 years

Investment A	*Investment B*	*Investment C*
Nu 1500 + 10 × Nu 60 = Nu 2100	$A = P(1 + \frac{r}{n})^{nt}$ $= 1500(1 + \frac{0.0375}{2})^{20}$ = Nu 2174.92	$A = P(1 + \frac{r}{n})^{nt}$ $= 1500(1 + \frac{0.035}{4})^{40}$ = Nu 2125.36

Investment B is best after 10 years.

• Another way to compare investments is to calculate **equivalent interest rates**.

For example, if you are given the following two investment options, you could find the equivalent interest rate for one of the options under the same compounding conditions as the other and then compare them.

Investment 1: Nu 1000 at 14% p.a. compounded monthly for 1 year	*Investment 2*: Nu 1000 at 14.5% p.a. compounded annually for 1 year

Nu 1000 at 14% p.a. compounded monthly for 1 year

= Nu 1000 at 14.93% p.a. compounded annually for 1 year

Now you can compare the two investment options because they have same compounding frequency:

Refer to Lesson 2.1.2, Example 4, page 41, *Relating Interest Rates* for how to find equivalent rates

Investment 1: Nu 1000 at 14.93% p.a. compounded annually for 1 year	*Investment 2*: Nu 1000 at 14.5% p.a. compounded annually for 1 year

Investment 1 is the better option.

C. i) Which of Kailash's investment options in **part A** is the better choice?

ii) What three factors do you have to consider when comparing investment options? Explain.

Examples

Example 1 Paying Off a Loan

Yeshey wishes to purchase a car priced at Nu 300,000 (3 Lakhs).

• He must pay a 25% **down payment** but can borrow the other 75% from the bank.

• The bank charges an interest rate of 15% p.a. compounded monthly on the loan.

• His monthly payments will be Nu 5000 for as long as necessary.

a) Determine the **balance** of the loan at the end of the first month.

b) How much less interest will he pay in the second month than in the first month?

c) He is hoping to pay off the loan in 4 years. Do you think that is realistic? Explain.

Solution

a) *Amount of the loan*

75% of 300,000 = 0.75 × 300,000 = 225,000

Interest charged in first month

$225{,}000 \times \frac{0.15}{12} = \text{Nu } 2812.50$

Balance after first month

225,000 + 2812.50 = Nu 227,812.50

Balance after first payment

227,812.50 – 5000 = Nu 222,812.50

Thinking

a) I found 75% of the cost of the car to calculate how much he will have to borrow.

• I calculated the interest on the loan for one month. Since the interest is compounded monthly, the monthly interest charge is calculated by dividing 15% (0.15) by 12.

Solution	Thinking
b) *Interest charged in second month* $222{,}812.50 \times \frac{0.15}{12} = \text{Nu } 2785.16$ *Comparison of interest in two months* 2812.50 – 2785.16 = Nu 27.34 He will pay Nu 27.34 less interest in the second month.	**b)** I multiplied the balance of the loan after one month by the monthly interest rate to calculate the interest for the second month. • I compared the interest charged in the two months by subtracting.
c) Nu 225,000 ÷ Nu 5000/month = 45 months 45 months is 3 years and 9 months. That leaves him 3 months of payments of Nu 5000 per month to pay off all the interest on the loan. He could pay it off if the total interest were Nu 15,000 or less. He owed interest of Nu 2812.50 and Nu 2785.16 in the first two months, so the interest in the first year alone is probably more than 12 × Nu 2000 = Nu 24,000. Yeshey cannot pay off the loan in four years.	**c)** I calculated how many years it would take him to pay off just the principal, without any interest, based on monthly payments of Nu 5000. • I estimated the interest for the first year.

Example 2 Comparing Payment Options

Kinley is purchasing a new computer for Nu 50,000 and has been offered two payment options:

Option A: Pay Nu 2250 at the end of each month for 30 months. No down payment is required.

Option B: Pay 25% as a down payment and then make payments of Nu 10,000 every six months until the loan has been repaid. The interest charged on any outstanding balance after each payment is 15% p.a. compounded semi-annually.

Which option would you recommend to Kinley? Why?

Solution	Thinking
Interest amount for Option A (30 × Nu 2250) – Nu 50,000 = Nu 17,500 [Continued]	 • I compared the interest charges for the two options. • With Option A, the interest charged is hidden — it is the difference between the amount Kinley pays for the computer (making 30 payments of Nu 2250) and the actual cost (Nu 50,000).

Example 2 Comparing Payment Options [Continued]

Solution

Loan amount for Option B

0.75 × Nu 50,000 = Nu 37,500

Amount still owing after 6 months

$37{,}500 \times (1 + \frac{0.15}{2}) - 10{,}000$

= Nu 30,312.50

Amount still owing after 12 months

$30{,}312.50 \times (1 + \frac{0.15}{2}) - 10{,}000$

= Nu 22,585.94

Amount still owing after 18 months

$22{,}585.94 \times (1 + \frac{0.15}{2}) - 10{,}000$

= Nu 14,279.89

Amount still owing after 24 months

$14{,}279.89 \times (1 + \frac{0.15}{2}) - 10{,}000$

= Nu 5350.87

Last payment

$5350.87 \times (1 + \frac{0.15}{2})$

= Nu 5752.19

Total interest paid using Option B

4 × 10,000 + 5752.19 – 37,500

= Nu 8252.19

Total interest paid using Option A

Nu 17,500

I recommend Kinley take Option B, if she can afford the down payment, because she would pay less than half as much interest.

Thinking

• Calculating the interest amount for Option B was more difficult because the interest amount changed each time a payment was made. I had to look at each six-month period to see the balance and the interest.

• I first calculated how much the loan would be. Since the down payment was 25% of Nu 50,000, the loan would be 75% of Nu 50,000.

• For each six-month period, I multiplied the amount still owing by $1 + \frac{0.15}{2}$ to calculate the amount owing and then subtracted the Nu 10,000 payment.

• After 24 months, there was a balance of Nu 5350.87, so I calculated the interest charged on that amount to figure out what the last payment would be.

• I calculated the total interest by subtracting the amount of the loan (Nu 37,500) from the total amount Kinley paid (four payments of Nu 10,000 each plus a final payment of Nu 5752.19).

• It makes sense that Option B would charge less interest because she wouldn't pay interest on 25% of the cost (the down payment).

Practising and Applying

1. Which is the best option to pay off a loan of Nu 20,000? Which is the worst? Show your work.

Option 1: Pay Nu 23,000 at the end of one year

Option 2: Pay off the loan at the end of one year at an interest rate of 14.6% p.a. compounded semi-annually

Option 3: Pay off the loan at the end of one year at an interest rate of 14% p.a. compounded monthly

2. You intend to invest Nu 60,000 in a bank for 10 years to have enough to pay for your child's education. What interest rate compounded monthly do you have to earn if you want the investment to grow to Nu 90,000 in 10 years?

3. An investment is placed in an account that guarantees an increase of 25% in three years.

a) What interest rate compounded annually must the account earn?

b) Will the investment increase from its original value by 50% in six years (assuming the same rate used in **part a)** applies)? Explain.

4. A company offers a "no interest" option on their computer sales.
A computer can be paid for in a single payment at the end of one year with no interest. However, an administration fee of Nu 4000 is required at the end of the year, in addition to the full payment of the price of the computer.

a) If the computer is sold for Nu 50,000, what interest rate would be equivalent to this arrangement?

b) If the computer is sold for Nu 25,000, what interest rate would be equivalent to this arrangement?

5. The winner of a lottery can choose to receive Nu 10,000 monthly for the rest of his or her life or receive a one-time amount of Nu 1,000,000.

a) Which option would you choose? Why?

b) Why might someone choose the other option?

6. A university wishes to establish a scholarship fund. The fund must earn enough interest so that scholarships worth a total value of Nu 2,000,000 can be awarded annually from the interest earned. The fund is guaranteed to earn at least 8% annually. What principal must be invested to be able to earn enough interest?

7. A car lot owner pays his salespeople a commission of 10% on total sales. The owner wants to make a profit of 1 Lakh on each car. The cost price of a car is Nu 90,000. What must the selling price of the car be for the owner to make a profit and pay commission?

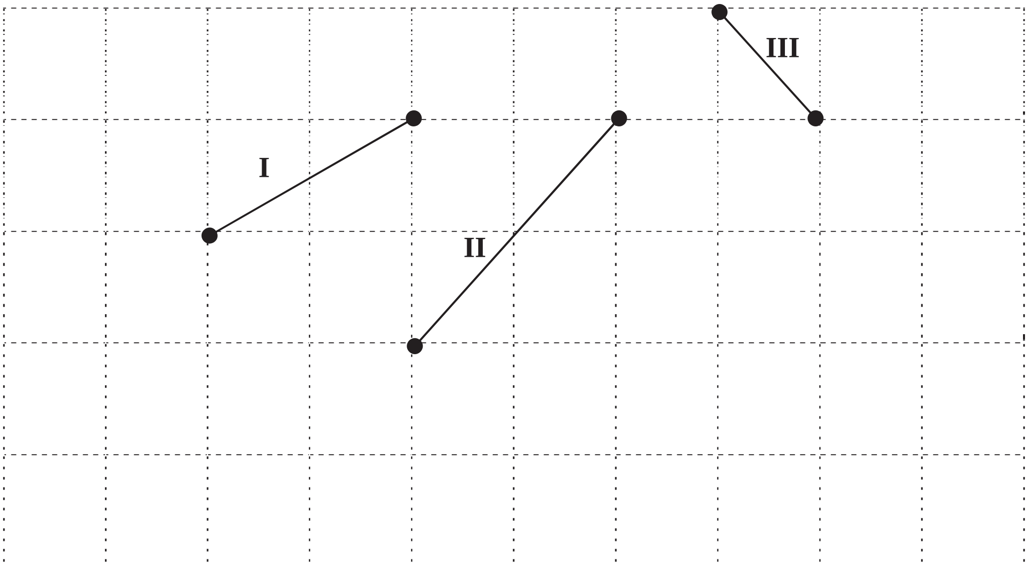
8. a) Sangay repaid a loan of Nu 10,500 by paying Nu 12,000 at the end of a year. What interest rate compounded annually was Sangay charged?

b) Suppose Sangay was charged the same interest rate but the compounding was semi-annual and he made a payment of Nu 6,000 after six months. How much would Sangay have to pay at the end of a year to pay off the loan?

9. A loan can be repaid in a year (principal and interest) by paying Nu 1,000 at the end of each month or by paying Nu 12,000 at the end of the year. Without calculating, decide which option has the lower interest rate. Explain your thinking.

2.2.1 EXPLORE: Representing Square Roots

You can use the **Pythagorean theorem** to help you represent and understand **square roots**.



A. The three line segments above join intersection points on the grid. Which segment represents $\sqrt{2}$? $\sqrt{5}$? $\sqrt{8}$? How do you know?

B. Use the diagram above to help you explain why $\sqrt{8}$ could also be described as $2 \times \sqrt{2}$ or $2\sqrt{2}$.

C. On a piece of grid paper, mark off a 10 square-by-10 square grid. Find the lengths of all other possible line segments that join two grid intersection points. Express each length as an integer, as a square root, or as an integer multiple of a square root, for example, $2\sqrt{2}$.

D. i) Which lengths in **part C** are you certain represent **rational numbers**? Explain.

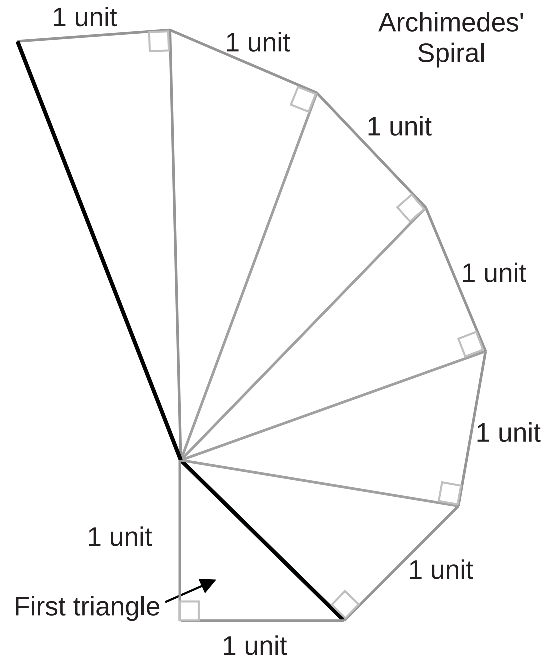
ii) Which lengths do you think are not rational numbers? Explain.

2.2.2 Simplifying Radicals

Try This

The spiral to the right, called Archimedes' spiral, was created by starting with a right triangle with **legs** of length 1 unit and adding on right triangles, each with a base of 1 unit, using the **hypotenuse** of the previous triangle as the other leg.

A. Measure the lengths of the hypotenuses of the first triangle and of the seventh triangle. How do the lengths compare?



- You have already been introduced to **irrational numbers** like $\sqrt{2}$.
They are numbers that cannot be written as repeating or terminating decimals.

- When you calculate $\sqrt{2}$ on a calculator, you get the approximate value of the number, 1.414213562... . When it is written in **radical** form, $\sqrt{2}$, the value is exact.

- The square root of 2 can also be written as a power: $\sqrt{2} = 2^{\frac{1}{2}}$
We know this because of the exponent laws.

If $\sqrt{2} \times \sqrt{2} = 2$ and $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2$, then $\sqrt{2} = 2^{\frac{1}{2}}$.

In general, $\sqrt{n} = n^{\frac{1}{2}}$.

- The relationship between the root of a number and a fractional exponent can be extended to other roots besides square roots.
For example.

- The **cube root** of 8, which is written as $\sqrt[3]{8}$, is 2 since 2 × 2 × 2 or $2^3 = 8$.

$\sqrt[3]{8}$ can also be written as $8^{\frac{1}{3}}$.

- The fourth root of 16, which is written as $\sqrt[4]{16}$, is 2 since 2 × 2 × 2 × 2 or $2^4 = 16$.

$\sqrt[4]{16}$ can also be written as $16^{\frac{1}{4}}$.

In general, $\sqrt[q]{n} = n^{\frac{1}{q}}$.

• Sometimes it is possible to write a radical in an equivalent or simplified form.

For example, $\sqrt{300}$ can be written as $10\sqrt{3}$ because $\sqrt{300} = \sqrt{100 \times 3}$

$= \sqrt{100} \times \sqrt{3} = 10\sqrt{3}$

We know this because of the exponent laws.

$\sqrt[n]{ab} = (ab)^{\frac{1}{n}}$ and $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$.

In general, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

• An expression such as $\sqrt{300}$ is called an **entire radical**. An entire radical can be rational like $\sqrt{100}$, but most are irrational like $\sqrt{3}$ (1.732050808...).

• An expression like $10\sqrt{3}$ is called a **mixed radical** because consists of an integer multiplied by a radical that is irrational, $10 \times \sqrt{3} = 10\sqrt{3}$.

• When you **simplify a radical**, you keep the number in an exact form but create a mixed radical with the least possible value under the root sign, $\sqrt{300} = 10\sqrt{3}$, or you get rid of the root sign altogether, $\sqrt{100} = 10$.

• One way to simplify a radical is to look for **perfect powers** as factors under the root sign and then take their roots outside the root sign.

For example: $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

$\sqrt{150} = \sqrt{25 \times 6} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6}$

$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2}$

• The principles that apply to numerical **radical expressions** also apply to algebraic radical expressions.

For example: $\sqrt{b^8} = \sqrt{b^4 \times b^4} = b^4$ $\quad$ $\sqrt[4]{b^8} = \sqrt[4]{b^2 \times b^2 \times b^2 \times b^2} = b^2$

$\sqrt[3]{b^8} = \sqrt[3]{b^2 \times b^2 \times b^2 \times b^2} = \sqrt[3]{b^2 \times b^2 \times b^2} \times \sqrt[3]{b^2} = b^2 \times \sqrt[3]{b^2}$

B. i) Use the Pythagorean theorem to complete the chart below. Express each length as an entire radical and, if possible, as a simplified mixed radical.

Triangle	1	2	3	4	5	6
Hypotenuse length						

ii) What do you notice about the hypotenuse lengths of the first and seventh triangles? How does that explain what you observed in **part A**?

C. If the spiral were extended, which triangle would have a hypotenuse that is twice the length of the seventh triangle's hypotenuse? How do you know?

Examples

Example Simplifying Radicals

Express each radical in a simplified form.

a) $\sqrt{20}$ **b)** $\sqrt{200}$ **c)** $\sqrt{180}$ **d)** $\sqrt[3]{80}$

Solution	Thinking
a) $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$ **b)** $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$ **c)** $\sqrt{180} = \sqrt{9 \times 20} = \sqrt{9} \times \sqrt{20} = 3\sqrt{20}$ $3\sqrt{20} = 3\sqrt{4 \times 5} = 3 \times \sqrt{4} \times \sqrt{5} = 3 \times 2 \times \sqrt{5}$ $= 6\sqrt{5}$ **d)** $\sqrt[3]{80} = \sqrt[3]{8 \times 10} = \sqrt[3]{8} \times \sqrt[3]{10} = 2\sqrt[3]{10}$	 For each radical in **parts a) to c)**, I looked for factors that were perfect squares: 4, 100, and 9. For **part d)**, I looked for a factor that was a perfect cube: 8. In **part c)**, I realized that 3 $\sqrt{20}$ could be simplified further because 20 = 4 × 5.

Practising and Applying

1. Write each radical in simplified form.

a) $\sqrt{48}$ **b)** $\sqrt{1000}$

c) $\sqrt[3]{32}$ **d)** $\sqrt{110}$

2. Order the following expressions from greatest to least by expressing them as entire radicals.

$3\sqrt{13}$ $4\sqrt{7}$ $9\sqrt{2}$ 11 $6\sqrt{3}$

3. a) Simplify $\sqrt{18}$ and $\sqrt{45}$.
b) Use your answers from **part a)** to help you simplify $\sqrt{18} \times \sqrt{2}$ and $\dfrac{\sqrt{45}}{\sqrt{5}}$.

4. Without using a calculator, decide which is greater: $13\sqrt{5}$ or $\sqrt{920}$. Explain your thinking.

5. Give an example of a value of n for each, if possible. If a value is not possible, explain why.

a) $\sqrt{n}$ is a whole number but $\sqrt[3]{n}$ is not

b) $\sqrt{n}$ in its simplified form is an entire radical and $\sqrt[3]{n}$ is a whole number

c) $\sqrt{n}$ is a mixed radical and $\sqrt[3]{n}$ is a whole number

6. Is it possible to find a value of n

a) for which $\sqrt[4]{n}$ is a whole number but $\sqrt{n}$ is not a whole number? Explain.

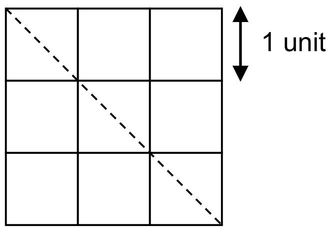
b) for which $\sqrt[4]{n}$ is not a whole number but $\sqrt{n}$ is a whole number? Explain.

7. $M = 3^6 \times 5^a \times 11^3$

a) What values for a, if any, would make M a perfect square? Explain your thinking.

b) What values for a, if any, would make M a perfect cube? Explain your thinking.

8. How does this diagram show that $\sqrt{18} = 3\sqrt{2}$?



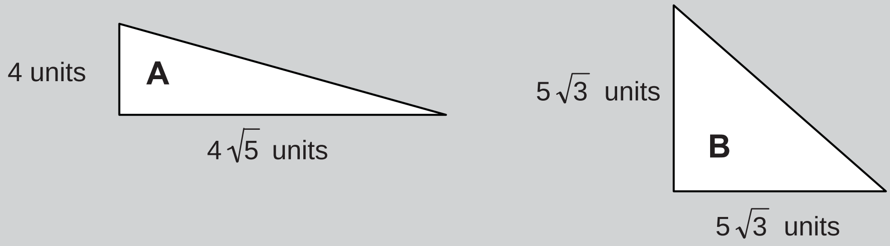
9. Explain how

a) to simplify a radical of the form $\sqrt{n}$.

b) to create an entire radical from a mixed radical of the form $m\sqrt{n}$.

2.2.3 Operations with Radicals

Try This



A. Which triangle has the longer hypotenuse? About how much longer is it?

- Adding and subtracting radicals requires like terms in the same way that adding and subtracting fractions requires common denominators, and adding and subtracting algebraic expressions requires like terms.

For example, $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$ because 3 of something plus 2 more of the same thing is 5 altogether. However, the two terms in $3\sqrt{2} + 2\sqrt{3}$ cannot be added together because they do not have **like terms**.

- Sometimes radicals can be added or subtracted but only after simplifying first to create like terms.

For example: $\sqrt{2} + \sqrt{8} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

$\sqrt{175} - \sqrt{28} = 5\sqrt{7} - 2\sqrt{7} = 3\sqrt{7}$

As with fractions and algebraic expressions, the like term (in this case the radical) remains unchanged. Only the coefficients of the radical are added or subtracted.

- These strategies also work with more complex radical expressions.

For example:

$\sqrt{5} - 5\sqrt{5} - 2\sqrt{5} + 13\sqrt{5} = 7\sqrt{5}$

(since 1 – 5 – 2 + 13 = 7)

$$\sqrt{10} - \sqrt{20} + 3\sqrt{5} - \sqrt{14} + 5\sqrt{40}$$
$$= \sqrt{10} - 2\sqrt{5} + 3\sqrt{5} - \sqrt{14} + 10\sqrt{10}$$
$$= 11\sqrt{10} + \sqrt{5} - \sqrt{14}$$

- The principles for adding and subtracting numerical radical expressions also apply to those with variables.

For example: $\sqrt{x^9} + \sqrt{9x^5} = \sqrt{x^4 \times x^4 \times x^1} + \sqrt{9 \times x^2 \times x^2 \times x^1}$

$= x^4\sqrt{x} + 3x^2\sqrt{x}$

- You can factor expressions with radicals just like with numerical expressions.

For example: $x^4\sqrt{x} + 3x^2\sqrt{x} = x^2\sqrt{x}\,(x^2 + 3)$

• Division and multiplication of radicals can be done following these models:

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} \text{ and } \sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

For example: $\sqrt{12} \times \sqrt{3} = \sqrt{36} = 6 \quad \sqrt{90} \div \sqrt{40} = \sqrt{\frac{90}{40}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

However, both calculations could have been approached differently, by simplifying one or both of the terms:

$\sqrt{12} \times \sqrt{3} = 2\sqrt{3} \times \sqrt{3} = 2 \times \sqrt{3} \times \sqrt{3} = 2 \times 3 = 6$

$\sqrt{90} \div \sqrt{40} = 3\sqrt{10} \div 2\sqrt{10} = \frac{3 \times \sqrt{10}}{2 \times \sqrt{10}} = \frac{3}{2}$

• Any of the four operations can be combined in a radical expression.
For example:

$$(2\sqrt{3} + \sqrt{5})(\sqrt{3} - 4\sqrt{5}) = (2\sqrt{3})(\sqrt{3}) + (2\sqrt{3})(-4\sqrt{5}) + (\sqrt{5})(\sqrt{3}) + (\sqrt{5})(-4\sqrt{5})$$
$$= 6 - 8\sqrt{15} + \sqrt{15} - 20 = -14 - 7\sqrt{15}$$

$$(\sqrt{17} + \sqrt{5})(\sqrt{17} - \sqrt{5}) = (\sqrt{17})(\sqrt{17}) + (\sqrt{17})(-\sqrt{5}) + (\sqrt{5})(\sqrt{17}) + (\sqrt{5})(-\sqrt{5})$$
$$= 17 + 0 + (-5) = 12$$

• Sometimes you have to apply the rules for the order of operations.
For example, in this calculation you multiply before adding:

$\sqrt{2} + \sqrt{3} \times \sqrt{27} = \sqrt{2} + \sqrt{81} = \sqrt{2} + 9$

• When you add, subtract, multiply, and divide radicals in a radical expression, you are simplifying the expression.

B. Calculate exactly how much longer is the longer hypotenuse in **part A** by using simplification of radicals to calculate. Express each hypotenuse as a mixed radical.

Examples

Example 1 Adding and Subtracting Radicals

Simplify each.

a) $\sqrt{12} + \sqrt{75}$ **b)** $\sqrt{45} + \sqrt{75} - \sqrt{49}$

c) $\sqrt{68} - \sqrt{17} - \sqrt{8} + \sqrt{98}$ **d)** $-\sqrt{27x^6} + \sqrt{4x} - \sqrt{12x^2} + \sqrt{x^5}$

Solution

a) $\sqrt{12} + \sqrt{75} = \sqrt{2 \times 2 \times 3} + \sqrt{5 \times 5 \times 3}$
$= 2\sqrt{3} + 5\sqrt{3}$
$= 7\sqrt{3}$ [Continued]

Thinking

a) After simplifying each term, I had like terms that I could add.

Example 1 Adding and Subtracting Radicals [Continued]

Solution

b) $\sqrt{45} + \sqrt{75} - \sqrt{49}$

$= \sqrt{9 \times 5} + \sqrt{25 \times 3} - 7$

$= 3\sqrt{5} + 5\sqrt{3} - 7$

c) $\sqrt{68} - \sqrt{17} - \sqrt{8} + \sqrt{98}$

$= \sqrt{4 \times 17} - \sqrt{17} - \sqrt{4 \times 2} + \sqrt{49 \times 2}$

$= 2\sqrt{17} - \sqrt{17} - 2\sqrt{2} + 7\sqrt{2}$

$= \sqrt{17} + 5\sqrt{2}$

d) $-\sqrt{27x^6} + \sqrt{4x} - \sqrt{12x^2} + \sqrt{x^5}$

$= -\sqrt{9 \times 3 \times x^3 \times x^3} + 2\sqrt{x} - \sqrt{4 \times 3 \times x^2} + \sqrt{x^4 \times x^1}$

$= -3x^3\sqrt{3} + 2\sqrt{x} - 2x\sqrt{3} + x^2\sqrt{x}$

$= (2 + x^2)\sqrt{x} - (3x^3 + 2x)\sqrt{3}$

Thinking

b) I was able to simplify the terms but I couldn't add them because they were not like terms.

c) I simplified the expression to two radicals involving roots of 17 and 2.

d) After simplifying each term, I was able to factor out $\sqrt{x}$ from two terms and $\sqrt{3}$ from the other two terms.

Example 2 Multiplying and Dividing Radicals

Simplify each.

a) $\sqrt{12} \times \sqrt{75}$ **b)** $\sqrt{72} \div \sqrt{18}$ **c)** $\sqrt{24} \div \sqrt{60}$

d) $\dfrac{\sqrt{30} \times \sqrt{5}}{\sqrt{6}}$ **e)** $\sqrt{6x^3} \times \sqrt{2x^5}$ **f)** $9 + 9\sqrt{x^3} \div 2\sqrt{x^5}$

Solution

a) $\sqrt{12} \times \sqrt{75} = \sqrt{4 \times 3} \times \sqrt{25 \times 3}$

$= 2\sqrt{3} \times 5\sqrt{3}$

$= 2 \times 5 \times 3 = 30$

b) $\sqrt{72} \div \sqrt{18} = \sqrt{72 \div 18} = \sqrt{4} = 2$

c) $\sqrt{24} \div \sqrt{60} = \sqrt{\dfrac{24}{60}} = \sqrt{\dfrac{6}{15}} = \sqrt{\dfrac{2}{5}}$

d) $\dfrac{\sqrt{30} \times \sqrt{5}}{\sqrt{6}} = \dfrac{\sqrt{6 \times 5} \times \sqrt{5}}{\sqrt{6}} = \dfrac{\sqrt{6} \times \sqrt{5} \times \sqrt{5}}{\sqrt{6}}$

$= \dfrac{\sqrt{6} \times 5}{\sqrt{6}} = 5$

Thinking

a) I could have multiplied 12 × 75 first and then taken the square root of 900 instead.

b) I could have simplified each radical and then divided, but this way was easier.

c) I tried simplifying each radical first and then dividing but it didn't work ($\sqrt{24} \div \sqrt{60} = 2\sqrt{6} \div 2\sqrt{15}$).

d) Since the expression involved 30, $\sqrt{6}$, and $\sqrt{5}$, I figured it would help to write 30 = 6 × 5.

Solution

e) $\sqrt{6x^3} \times \sqrt{2x^5} = \sqrt{6x^3 \times 2x^5}$

$= \sqrt{6 \times 2 \times x^3 \times x^5}$

$= \sqrt{12x^8}$

$= \sqrt{4 \times 3 \times x^4 \times x^4}$

$= 2x^4\sqrt{3}$

f) $9 + 9\sqrt{x^3} \div 2\sqrt{x^5} = 9 + \dfrac{9\sqrt{x^3}}{2\sqrt{x^5}}$

$= 9 + \dfrac{9x\sqrt{x}}{2x^2\sqrt{x}}$

$= 9 + \dfrac{9}{2x}$

$= 9\,(1 + \dfrac{1}{2x})$

Thinking

e) I multiplied the expressions to get them all under the same root sign and then I moved anything I could out from under the root sign.

f) Because of the order of operations, I was able to create a fraction out of the second and third terms.

• I factored 9 from each term.

Example 3 Expanding Expressions with Radicals

Simplify each.

a) $(\sqrt{3} + 2\sqrt{5})(\sqrt{5} + \sqrt{3})$ **b)** $(4 - \sqrt{7})(4 + \sqrt{7})$

c) $(3\sqrt{2} - \sqrt{7})(2\sqrt{5} - \sqrt{6})$ **d)** $(\sqrt{x} - \sqrt{2y})(\sqrt{x} + \sqrt{y})$

Solution

a) $(\sqrt{3} + 2\sqrt{5})(\sqrt{5} + \sqrt{3})$

$= \sqrt{3} \times \sqrt{5} + \sqrt{3} \times \sqrt{3} + 2\sqrt{5} \times \sqrt{5} + 2\sqrt{5} \times \sqrt{3}$

$= \sqrt{15} + 3 + 10 + 2\sqrt{15}$

$= 13 + 3\sqrt{15}$

b) $(4 - \sqrt{7})(4 + \sqrt{7}) = 16 + 4\sqrt{7} - 4\sqrt{7} - \sqrt{7} \times \sqrt{7}$

$= 16 - 7$

$= 9$

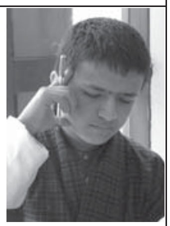
c) $(3\sqrt{2} - \sqrt{7})(2\sqrt{5} - \sqrt{6})$

$= 6\sqrt{10} - 3\sqrt{12} - 2\sqrt{35} + \sqrt{42}$

$= 6\sqrt{10} - 6\sqrt{3} - 2\sqrt{35} + \sqrt{42}$

[Continued]

Thinking



a) I expanded the expression to four partial products and combined like terms.

b) I was able to simplify to a whole number.

c) I expanded the expression to four partial products. I was able to simplify one term but I couldn't combine any terms.

Example 3 Expanding Expressions with Radicals [Continued]

Solution	Thinking
d) $(\sqrt{x} - \sqrt{2y})(\sqrt{x} + \sqrt{y})$ $= \sqrt{x^2} + \sqrt{xy} - \sqrt{2xy} - \sqrt{2y^2}$ $= x + \sqrt{xy} - \sqrt{2xy} - y\sqrt{2}$	**d)** I expanded the expression to four partial products and then simplified two of the terms. 

Practising and Applying

1. Simplify.

a) $\sqrt{48} + \sqrt{12}$

b) $\sqrt{x^3} - \sqrt{49x}$

c) $7\sqrt{k} + \sqrt{9k}$

d) $\sqrt{44} - \sqrt{121} + \sqrt{55}$

2. Simplify.

a) $\sqrt{12} \times \sqrt{3}$

b) $\sqrt{4x^5} \times \sqrt{5x^3}$

c) $\sqrt{6} \times \sqrt{7} \times \sqrt{2}$

3. Simplify.

a) $\sqrt{48} \div \sqrt{12}$

b) $\sqrt{90} \div \sqrt{20}$

c) $\sqrt{27x^7} \div \sqrt{3x^3}$

d) $\frac{\sqrt{28} \times \sqrt{10}}{\sqrt{21}}$

4. Simplify.

a) $(3\sqrt{5} - \sqrt{11})(4 + 2\sqrt{11})$

b) $(\sqrt{13} - \sqrt{5x})(\sqrt{13} + \sqrt{5x})$

5. Give an example of a value of n for which each result would be an integer.

a) $\sqrt{7} \times n$

b) $4\sqrt{5} \div n$

c) $\sqrt{14} + \sqrt{56} - n$

d) $(\sqrt{6} + \sqrt{8})(\sqrt{n} - \sqrt{8})$

6. Find each missing value.

a) $\sqrt{30} \times \sqrt{m} = 5\sqrt{6}$

b) $\frac{\sqrt{30} \times \sqrt{p}}{\sqrt{p^2}} = \sqrt{5}$

c) $(\sqrt{24} - \sqrt{k})(\sqrt{24} + \sqrt{k}) = 5$

d) $\sqrt{32} - \sqrt{s} = -4\sqrt{2}$

7. a) In each expression below, the numbers under the root sign all add to the same number, 10. Determine which expression is greatest.

$\sqrt{7} + \sqrt{3}$ $\quad \sqrt{6} + \sqrt{4}$ $\quad \sqrt{8} + \sqrt{2}$

b) What do you notice about the numbers under the root signs in the expression that was greatest in **part a)** compared to the others?

c) Use what you noticed in **part b)** to predict which expression below is greatest. Justify your prediction.

$\sqrt{15} + \sqrt{4}$ $\quad \sqrt{11} + \sqrt{8}$ $\quad \sqrt{17} + \sqrt{2}$

d) Check your prediction.

8. a) Simplify $\frac{3}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}$.

b) Explain why your answer in **part a)** must be equal to $\frac{3}{\sqrt{11}}$.

8. c) Suppose you wanted to make the denominator of $\frac{6}{\sqrt{13}}$ an integer. By what number could you multiply $\frac{6}{\sqrt{13}}$ without changing its value?

9. If you were advising someone on how to simplify an expression involving all four operations and multiples of $\sqrt{5}$, $\sqrt{20}$, $\sqrt{x}$, and $\sqrt{x^3}$, what would you tell them to look for?

GAME: Five Radicals

Materials

- A deck of cards numbered $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{12}$, $\sqrt{15}$, $\sqrt{18}$, $\sqrt{20}$, and $\sqrt{24}$ with an equal quantity of each number in a deck.
- For four players, four of each number, or a deck of 44 cards, is recommended.

Rules

- Deal five cards to each player.
- The object of the game is to create at least one integer value from 0 to 50 using some or all of the cards. The same card cannot be used more than once in a round.
- A game consists of five rounds.
- In each round, each player receives 10 points for each card used and additional points are awarded equal to the product of the integers created. Return all cards to the deck and shuffle before dealing the next round.
- The person with the highest score after five rounds wins.

Example

Jigme is dealt the following cards in the first round:

$\sqrt{3}$	$\sqrt{6}$	$\sqrt{15}$	$\sqrt{20}$	$\sqrt{6}$

He creates two integer values using all five of his cards:
$\sqrt{6} \times \sqrt{6} = 6$ and $\sqrt{20} \times \sqrt{3} \times \sqrt{15} = 30$.

Since he used 5 cards and created the integers 6 and 30, he scores 330 points altogether for the round:

$5 \times 10 + 6 \times 30 = 150 + 180 = 330$ points

Unit 2 Revision

1. a) The marked price of an item is Nu 200. A discount of 20% is to be applied. Determine the selling price.

b) An item that cost the seller Nu 160 is to be sold at Nu 200. Determine the percent markup.

c) A shopkeeper offers a 25% discount on an item marked at Nu 300. An identical item at another shop is marked at Nu 280 but is on sale at a discount of 15%. Which item can be purchased for less money?

2. a) You invest Nu 1500 in a bank savings account that earns 5% p.a. compounded annually. Determine the value of the investment after one year.

b) What will be the value of the investment after one year if 5% p.a. simple interest was earned instead?

c) Suppose you invest the money for two years instead. Would you be wise to accept a change in the rate from 5% p.a. compounded annually to 5% p.a. simple interest? Explain.

3. What interest rate compounded monthly is equivalent to 11.5% p.a. compounded annually?

4. You invest each amount of money as indicated below. For each, calculate

• the equivalent simple interest rate for the time period indicated

• the equivalent compound interest rate, compounded annually

Round to two decimal places.

a) Nu 10,000 for 5 years at 4% p.a. compounded monthly

b) Nu 10,000 for 10 years at 3.75% p.a. compounded quarterly

4. c) Nu 10,000 for 10 years at 4.5% p.a. compounded daily

d) Nu 20,000 for 10 years at 3.9% p.a. compounded semi-annually

5. Dechen Choden buys stock at a discount of 28%. Each share has a face value of Nu 100.

a) How many shares can she buy with Nu 25,000?

b) If the company pays a dividend of 17% at the end of the first year, what dividend amount will she earn?

c) What will be her yield percentage?

6. Gom Raj bought 200 shares that had a face value of Nu 100 for Nu 85 each. She sold the shares for Nu 220 each.

a) What was her percent profit?

b) If she also earned a dividend of 17% before she sold the shares, what dividend amount did she earn?

7. A car sells for Nu 180,000. The cost price of the car was Nu 110,000.

a) Determine the percent markup, or dealer's profit.

b) Suppose you are the salesperson and can be paid any of the following three ways:

Option 1: a commission amount of Nu 5,000

Option 2: an 8% commission rate based on the dealer's profit

Option 3: a 3.5% commission rate based on the selling price

Which would you choose and why?

8. Tika borrowed Nu 22,000 at an interest rate compounded quarterly. The balance was Nu 19,770 after making her first payment of Nu 3000. What was the interest rate?

9. You purchase a car for 0.25 million by making a 25% down payment and monthly payments of Nu 4000 for as long as necessary. The interest rate charged is 11.5% p.a. compounded monthly. If you could pay off the loan at the end of the fourth month, how much would your fourth payment be?

10. You agree to pay Nu 5000 monthly to pay off a car loan with an interest rate of 12% p.a. compounded monthly. You purchased the car for 0.28 million and your down payment was 25%. What fraction of your first payment is interest?

11. You are offered a loan with a choice of two rates:

• 14.75% p.a. compounded monthly

• 15% p.a. compounded semi-annually

Which rate would you choose? Why?

12. Determine which expressions below represent integer values. Justify your answer.

A $\sqrt{80}+\sqrt{5}-\sqrt{125}$

B $\dfrac{\sqrt{32}+\sqrt{50}-\sqrt{8}}{\sqrt{2}}$

C $\sqrt{49}\times\sqrt{64}\times\sqrt{108}$

13. Write each radical in simplified form.

a) $\sqrt{27}$

b) $\sqrt{300}$

c) $\sqrt[3]{250}$

d) $\sqrt{30}$

14. Simplify.

a) $\sqrt{28}+\sqrt{18}-\sqrt{63}+\sqrt{72}$

b) $(5\sqrt{3}+\sqrt{6})(2\sqrt{2}-\sqrt{3})$

c) $\dfrac{\sqrt{20}\times\sqrt{7}}{\sqrt{35}}$

d) $\sqrt{80}\div\sqrt{45}$

e) $\sqrt[3]{27x^5}\times\sqrt[3]{x}$

15. Explain why both $\sqrt[3]{2007^6}$ and $\sqrt{2007^6}$ are integers, but $\sqrt[3]{6^{2007}}$ is an integer and $\sqrt{6^{2007}}$ is not.

16. Find the missing values.

a) $\sqrt{96}\div\sqrt{k}=2\sqrt{6}$

b) $\dfrac{-\sqrt{30p}\times\sqrt{p}}{\sqrt{5}}=-\sqrt{54}$

c) $(\sqrt{n}+\sqrt{17})(\sqrt{n}-\sqrt{17})=11$

d) $\sqrt{36x^m}\times\sqrt{48}=24x^7\sqrt{3x}$

UNIT 3 LINEAR FUNCTIONS AND RELATIONS

Getting Started

Use What You Know

A. Draw the next three figures in this pattern.

Figure 1 Figure 2 Figure 3 Figure 4

B. Complete the table of values for the pattern.

Figure number	Number of circles
1	3
2	7
3	11
4	15
5	
6	
7	

C. Is the data discrete or continuous? How do you know?

D. Without drawing a graph, predict the type of relationship between the number of circles and the figure number. Explain your prediction.

E. Plot the number of circles against the figure number in a scatter plot. What relationship does the graph suggest?

F. Write an equation in slope and y-intercept form to represent this relation.

G. How does your equation from **part F** relate to the graph you drew in **part E**?

Skills You Will Need

1. Examine the three patterns.

• Pattern A grows by adding two squares to the previous figure.

• Pattern B grows by forming double squares of squares, each time one square wider than in the previous figure.

Figure	Pattern A	Pattern B	Pattern C
1	■ ■ ■	■ ■	■
2	■ ■ ■ ■ ■	■ ■ ■ ■ ■ ■ ■ ■	■ ■ ■
3	■ ■ ■ ■ ■ ■ ■	■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■	■ ■ ■ ■ ■ ■ ■ ■ ■

• Pattern C grows by tripling the number of squares in the previous figure.

a) For each pattern, draw a scatter plot that relates the number of squares to the figure number.

b) Determine whether each pattern is linear, quadratic, or exponential. Justify each choice.

2. What are the values of *a, b*, and *c*, if the relationship between *x* and *y* is linear?

x	10	12	14	16	18	20
y	80	*a*	72	*b*	*c*	60

3. What are the slope and *y*-intercept of the graph of each equation?

a) $y = 2x + 3$

b) $-\frac{3}{4}x - 2 = y$

c) $3y = 6 - 9x$

d) $1.5x - 3.3 = 2y$

4. Sketch a graph for each.

a) a line with a negative slope and a positive *y*-intercept

b) a line with a positive slope and a positive *y*-intercept

5. Sketch the graph for each equation.

a) $y = -\frac{2}{3}x + 1$

b) $2x + 3y = 6$

6. Sketch the graph for each. Write the equation for each line in standard form.

a) *x*-intercept of 3 and *y*-intercept of 4

b) *x*-intercept of –2 and *y*-intercept of 6

c) slope of 3 and *y*-intercept of 1.5

7. Solve each equation.

a) $5x - 4 = -11$

b) $2a + 3 = 5a - 9$

8. Solve each inequality.

a) $5a + 7 < 3a$

b) $-4x + 2 > -2x - 2$

9. All values of *s* greater than 4 satisfy the inequality $10 - 2s < \square$.
What is the value of $\square$?

10. Write an algebraic expression to describe each situation.

a) total cost of a cell phone plan that charges a base fee of Nu 200 and a per minute rate of Nu 2

b) cost of a set of 300 stamps, if 5 stamps cost *s* ngultrums

c) cost of food for *n* weeks, if each week costs Nu 1000

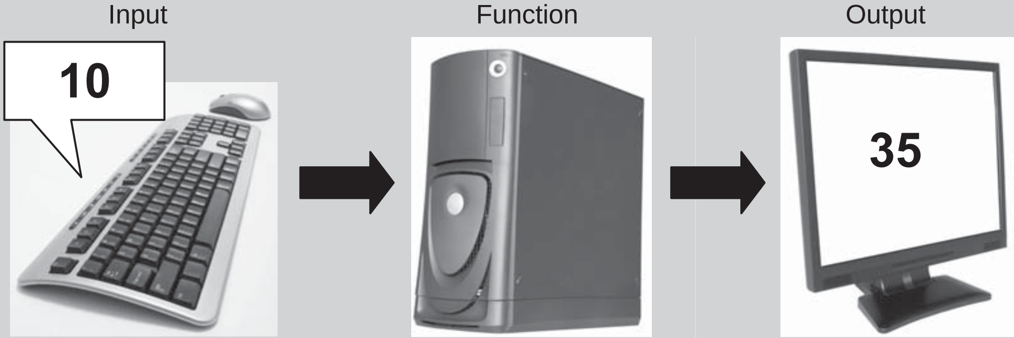
d) amount of salt in a *n* L solution, if the salt is 4% of the solution

Chapter 1 Linear Functions and Relations

3.1.1 Linear Functions

Try This

A computer program has been designed to perform a function, or calculation. When you input a number, it displays the result of the calculation, called the output, on the monitor. The program displays 35 for an input of 10.



Enter a number using the keyboard.

The computer program performs the calculation.

The computer displays the result.

A. Write three equations that might describe the calculation rule used by the program. Use x to represent the input value and y to represent the output value. One example is $y = 4x - 5$.

• A **relation** describes how two variables are connected. For example:

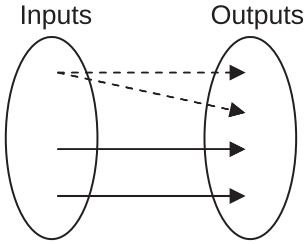
- The relation represented by $y < x$ indicates y is less than x.
- The relation represented by $y = 2x + 3$ indicates y is 3 more than double x.

For $y < x$, there are many output values of y for a particular input value of x. For example, if $x = 6$, then y could be 5, 4, 3, 2.5, and so on. But for $y = 2x + 3$, there is only one output value of y for a particular input value of x. For example, if $x = 6$, then y must be 15.

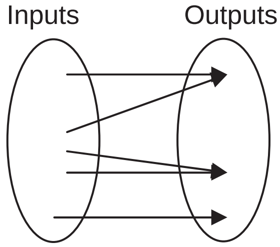
• A **function** is a special kind of relation where there is only one value of the output variable for a particular value of the input variable. That means $y = 2x + 3$ is a function but $y < x$ is not.

• You can visualize a relation and a function using the pictures below.

- In a relation, you can have more than one arrow from an input value, as shown by the dashed arrows.
- In a function there is never more than one arrow from an input value.



A relation



A function

• Note that, in a function, there can be more than one input value for the same output value.

For example, consider the function $y = x^2 - 3$. For any input value x, there is only one output value y, but there are sometimes two input values x for the same output value y. For an output of 1, the input could be 2 or –2.

• Mathematicians like things to be well-defined — with a function, you can predict exactly what is going to happen, but with a relation the outcome is uncertain.

• It is useful to think of a function as a set of instructions that a machine such as a computer performs. The machine accepts an input value (x) and produces a corresponding output value (y) and there is only one possible output value for each input. The input and output values are usually, but not always, numbers.

For example:

- If the instruction, or function, was *subtract input number from 100*, then an input of 3 would result in an output of 97.

- If the function was *figure out the month number for an input month name (assuming January = 1)*, then an input of March would result in an output of 3.

• The output of a function is the **dependent variable** and the input is the **independent variable** because the output value *depends on* the input value.

• A **function rule** shows how the value of the dependent variable, usually y, can be calculated from the value of the independent variable, usually x. A function rule is written as an equation using y, or in **function notation**, $f(x)$, which is read as "f of x". We usually use y for graphing, and $f(x)$ can always be used.

For example, the function $y = 6x - 7$ can also be written as $f(x) = 6x - 7$.

• For y to be a function of x,

- each value of y normally depends on a value of x, and
- there can only be one possible value of y that corresponds to each value of x.

• Functions are named according to the type of algebraic expression they use.

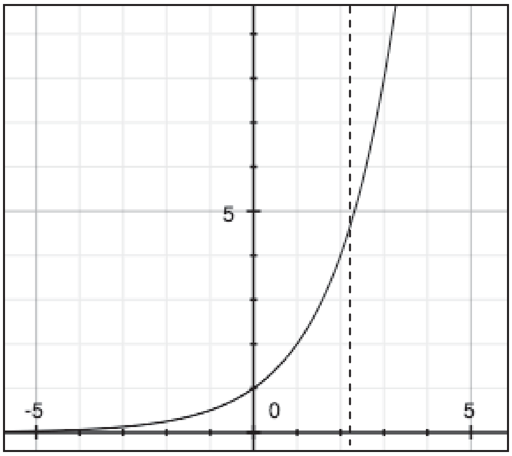
Form of expression	Type of function	Example using function notation
$ax + b$	**linear**	$f(x) = 2x + 1$
$ax^2 + bx + c$	**quadratic**	$f(x) = x^2 + 1$
$ab^x + c$	**exponential**	$f(x) = -3^x + 4$

• A function can be represented in different ways:

- as a set of ordered pairs
- a table of values
- a function rule using words
- an algebraic equation to represent the function rule
- a graph

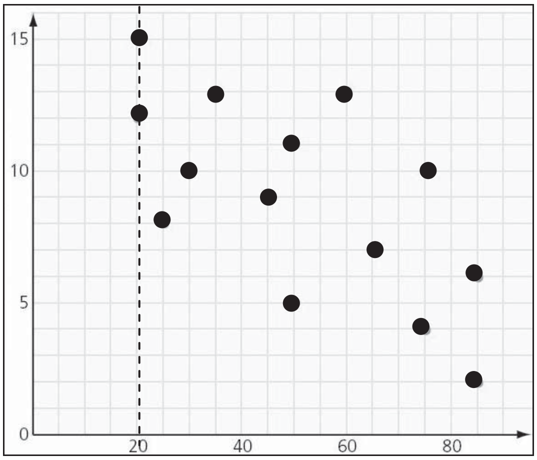
• You can tell from looking at a graph of a relation if it is a function. Imagine a vertical line drawn anywhere on the graph. If the line touches the graph more than once, the graph is not the graph of a function.

Sample graph of a function



If you can draw a vertical line anywhere on a graph and only one value is on that line, it is a function.

Sample graph of a non-function



If you can draw at least one vertical line on a graph and find more than one value on that line, it is not a function.

B. i) Suppose the computer program in **part A** uses a linear function to compute the output values. It displays 11 when you input 2 and it displays 35 when you input 10. Write the function rule for this program.

ii) What value would you input to result in an output value of 20?

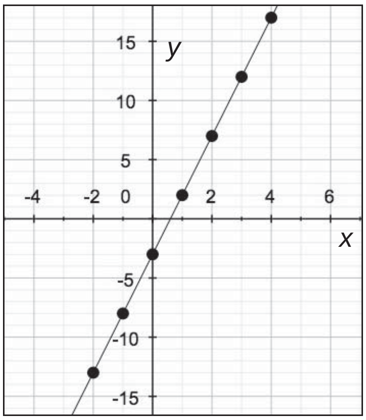
Examples

Example 1 Graphing a Linear Function

Graph the function $f(x) = 5x - 3$.

Solution

x	$f(x) = 5x - 3$
–2	–13
–1	–8
0	–3
1	2
2	7
3	12



$f(x) = 5x - 3$

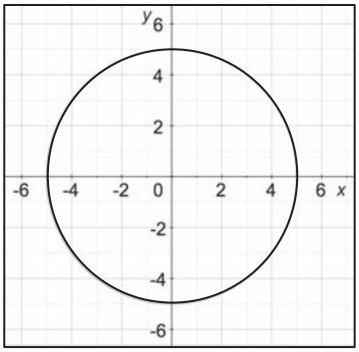
Thinking

• I made a table of values that showed the values of the outputs for several input values for x.

• I plotted the ordered pairs and joined them with a solid line (I assumed that the variables x and y are continuous since there is nothing to suggest they're not).

Example 2 Determining Whether a Graph Represents a Function

Determine whether this graph represents a function. Justify your decision.



Solution

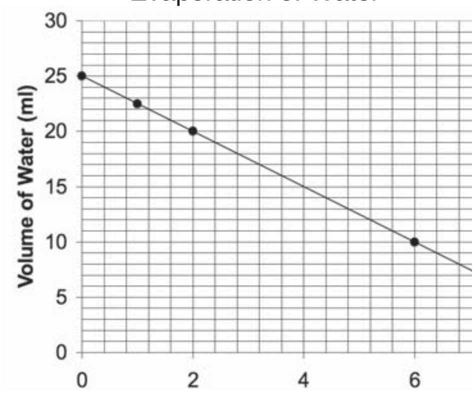
The graph does not represent a function because there are points on the graph that have the same *x*-coordinate, but different *y*-coordinates, for example, (0, 5) and (0, –5).

Thinking

- You only have to find one example where there's more than one *y*-value for an *x*-value to know it's not a function.
- Because the graph is a circle, it wouldn't pass the vertical line test.

Example 3 Using Function Notation to Describe a Real-World Situation

Water is poured into a container and allowed to evaporate. This graph shows the change in the volume of water over several hours. Use function notation to write the function rule.



Solution

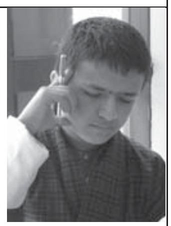
y-intercept: 25

Slope: Using the points (0, 25) and (2, 20), the slope is $\frac{20-25}{2-0} = \frac{-5}{2}$.

The equation of the line is $y = \frac{-5}{2}x + 25$.

The function rule is $f(x) = \frac{-5}{2}x + 25$.

Thinking

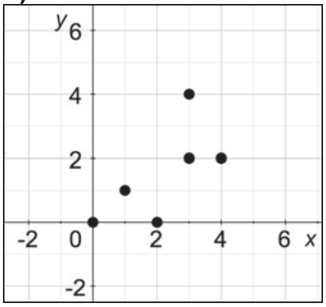


- I used the graph to determine the slope and *y*-intercept.
- I wrote the equation in slope and *y*-intercept form.
- I used the equation to write the function rule in function notation.

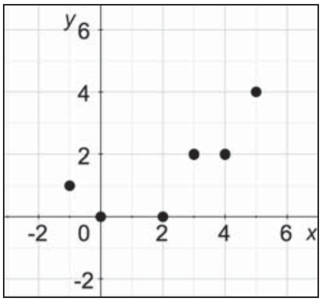
Practising and Applying

1. Which of the following graphs represent functions? Justify your decision.

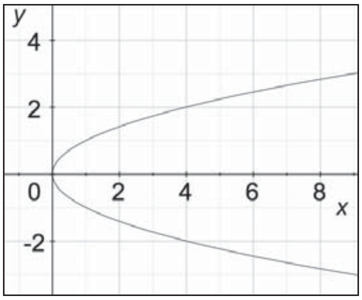
a)



b)



c)



2. a) Suppose you made a table of values. The input values are the names of the students in your class and the output values are their ages. Would the table represent a function? Justify your decision.

Student's name	Student's age

b) Suppose you made a table that showed number of siblings as the input and the names of students in the class with that many siblings as the output. Would the table represent a function? Justify your decision.

Number of siblings	Name(s) of student(s) with each number of siblings

3. These are the first two figures in a pattern.

Figure 1 Figure 2

a) The pattern rule is a linear function. Draw the next three figures.

b) The input is the figure number, n, and the output is the number of stars. Write the function in function notation.

4. a) Create a table of values to represent the function $f(x) = 4x - 3$ for at least four values of x.

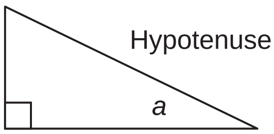
4. b) Explain why the table suggests that this is a function.

c) How can you be sure it is a function?

5. a) Create a table of values and draw the graph of $f(x) = x^2 - 1$.

b) How do you know the graph represents a function?

6. Sonam drew several different right-angle triangles, each with a base of 10 cm, but with different angles at the base. For each different angle, she measured the hypotenuse. Is this a function? Explain.



Base is 10 cm

Angle a (°)	Hypotenuse (cm)
10	10.2
20	10.6
30	11.6
40	13.1
50	15.6
60	20.0
70	29.2

7. a) How can you tell if a table of values represents a function?

b) How can you tell if a graph represents a function?

3.1.2 Applications of Linear Functions

Try This

Dechen has a collection of white and black balls. She chose 15 of them and put them into a bag.

A. i) Draw a graph to show the possible combinations of black and white balls that might be in the bag.

ii) Did you use a dashed or solid line? Why?

There are 15 balls in the bag.

• An equation of a linear relation in **slope and *y*-intercept form** can be rewritten easily in function notation.

For example, $y = \frac{-3}{2}x + 3$ is $f(x) = \frac{-3}{2}x + 3$ in function notation.

• When a linear relation with two variables is written in **standard form**, you can isolate one variable in terms of the other. The resulting equation expresses the isolated variable as a function of the other variable.

For example, you could solve the relation $5a + 2b = 10$ in these two ways:

You could isolate, or solve for a.

$$\begin{aligned} 5a + 2b &= 10 \\ 5a + 2b - 2b &= 10 - 2b \\ 5a &= 10 - 2b \\ 5a \div 5 &= (10 - 2b) \div 5 \\ a &= -\frac{2}{5}b + 2 \end{aligned}$$

In this instance, a is the dependent variable and is a function of the independent variable b.

$$f(b) = -\frac{2}{5}b + 2$$

You could isolate, or solve for b.

$$\begin{aligned} 5a + 2b &= 10 \\ 5a + 2b - 5a &= 10 - 5a \\ 2b &= 10 - 5a \\ 2b \div 2 &= (10 - 5a) \div 2 \\ b &= -\frac{5}{2}a + 5 \end{aligned}$$

In this instance, b is the dependent variable and is a function of the independent variable a.

$$f(a) = -\frac{5}{2}a + 5$$

• In a two-variable relation, it is important to know which variable is the independent variable and which one is the dependent variable.

- Sometimes the context of the situation will tell you which is which.

For example, if you are measuring the height of a ball at different points in time after it has been thrown, it makes sense for time to be the independent variable.

- Sometimes the context allows for either variable to the independent variable.

- If an equation is provided without a context, you can often choose which variable to use as the independent variable.

• Sometimes, in a two-variable relation, one variable is a function of the other, but the reverse is not true. That means that even if there is no context, there is no choice about which is the independent variable.

For example, for $y = x^2$, there is only one value of y for each value of x. So $y = x^2$ is a function with x as the independent variable. However, for any non-zero value of y, there are two values of x (the positive and negative square roots), so x cannot be written as a function of y, and y cannot be the independent variable.

B. i) Write an equation that describes the relationship between the number of white balls and the number of black balls in **part A**.

ii) Why is it possible to use either the number of white balls or black balls as the independent variable?

iii) Write the function rule that tells how to calculate the number of white balls if you know the number of black balls.

Examples

Example 1 Transforming Standard Form to Slope and *Y*-intercept Form

a) Transform the linear equation $3x - 4y = 12$ to slope and y-intercept form.

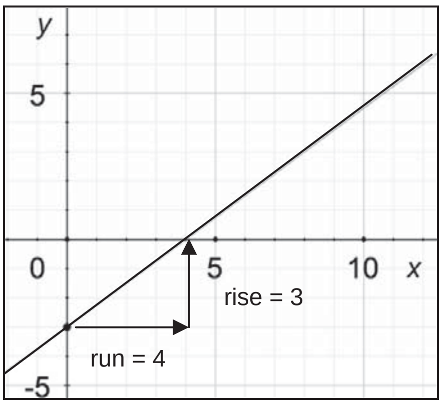
b) Sketch the graph of the resulting linear function.

Solution

a) *Solve for y*

$$\begin{aligned} 3x - 4y &= 12 \\ -4y &= 12 - 3x \\ -4y \div (-4) &= (12 - 3x) \div (-4) \\ y &= -3 + \frac{3}{4}x \\ y &= \frac{3}{4}x - 3 \end{aligned}$$

b)



Thinking

a) To isolate y, I subtracted $3x$ from both sides and then divided both sides by -4.

• I rearranged the equation to look like $y = mx + b$.

b) I was able to determine the slope and y-intercept from the equation because it was in $y = mx + b$ form:

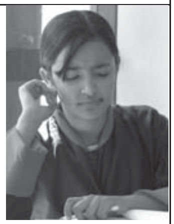
- The slope (m) is $\frac{3}{4}$.
- The y-intercept (b) is -3.

• I used the slope and y-intercept to sketch the graph:

- I plotted the y-intercept.
- I used the slope ($\frac{3}{4}$) to get a second point by going right 4 units (run) and up 3 units (rise).
- I joined the points.

Example 2 Using a Linear Function to Solve a Financial Problem

Dechen invests some of her money so that it will earn 3% simple interest per year. She invests another amount in a slightly riskier investment to earn 4% simple interest per year. Determine one combination of investments that would earn Dechen a total of Nu 1500 interest in one year.

Solution	**Thinking**
Determine the variables a is the amount invested at 3%, or 0.03 b is the amount invested at 4%, or 0.04 Total interest earnings: Nu 1500	 • I needed two variables because the amounts invested at each interest rate could be different.
Write an equation My equation represents the total interest earnings from both investments: $0.03a + 0.04b = 1500$	• I wrote an equation to model the situation: - I multiplied a by 0.03 to represent the interest at 3%. - I multiplied b by 0.04 to represent the interest at 4%.
Write the equation as a function of a $0.03a + 0.04b = 1500$ $a = \frac{1500 - 0.04b}{0.03}$ $f(b) = \frac{1500 - 0.04b}{0.03}$	• I rearranged the equation to isolate a so I could express it as a function of b (I could have solved for b instead and written it as a function of a.).
Use the equation to find values of a and b If $b = 300$, then $f(300) = a$ $f(300) = \frac{1500 - 0.04 \times 300}{0.03}$ $= \frac{1500 - 12}{0.03}$ $= \frac{1488}{0.03}$ $= 49{,}600$	• I picked a value of b that would make the calculation easy and used it to determine the value of a.
Check the values 4% on Nu 300 is $0.04 \times 300 = 12$ 3% on Nu 49,600 is $0.03 \times 49{,}600 = 1488$ Total interest would be $12 + 1488 = 1500$ Dechen could invest Nu 300 at 3% and Nu 49,600 at 4% to earn Nu 1500.	• I checked the values by substituting them into the original equation.

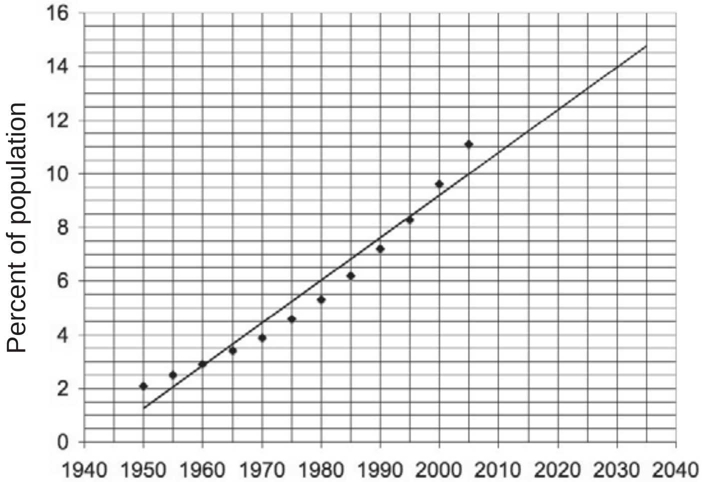
Example 3 Determining a Function to Represent a Line of Best Fit

This graph shows how the percentage of Bhutan's population living in urban areas has increased since 1950.

a) What is the equation of the line of best fit?

b) Describe the relationship between the year and the percentage of the population that is urban as a function.

Percent of Population in Bhutan that is Urban



Solution

a) *Find the slope*

(1980, 6) and (2005, 10) are on the line, so

the slope is $\frac{10 - 6}{2005 - 1980} = \frac{4}{25}$.

Write an equation

If x represents the year and y represents the percent of the population, the equation is

$$y = \frac{4}{25}x + b$$

Use the equation to find b

Substitute (2005, 10) into the equation. Solve for b:

$$\begin{aligned} y &= \frac{4}{25}x + b \\ 10 &= \frac{4}{25}(2005) + b \\ 10 &= 320.8 + b \\ -310.8 &= b \end{aligned}$$

The equation is $y = \frac{4}{25}x - 310.8$. [Continued]

Thinking

a) I determined the slope using two points on the line that were easy to read.

• I used the slope to write an equation in $y = mx + b$ form, with b still unknown.

• I substituted a pair of values that I knew were on the line, $x = 2005$ and $y = 10$, into the equation to solve for b.

Example 3 Determining a Function to Represent a Line of Best Fit [Cont'd]

Solution

b) *Write the equation in function form*

$y = \frac{4}{25}x - 310.8 \rightarrow f(x) = \frac{4}{25}x - 310.8$

Check the function for (1980, 6)

$$\begin{aligned} f(x) &= \frac{4}{25}x - 310.8 \\ f(1980) &= \frac{4}{25}(1980) - 310.8 \\ &= 316.8 - 310.8 \\ &= 6 \end{aligned}$$

$f(x) = \frac{4}{25}x - 310.8$ describes the relationship between the year and the percentage of the population that is urban.

Thinking

b) I knew that (1980, 6) was on the line, so I checked to make sure my function got an output of 6 for an input of 1980.

Practising and Applying

1. $2t + 5m = 4$

a) Write m as a function of t.

b) Write t as a function of m.

2. You withdraw Nu 2000 in Nu 20 and Nu 50 notes from the bank.

a) Write an equation to model this situation.

b) Write a function that tells the number of Nu 20 notes if you know the number of Nu 50 notes.

3. On a 100 chart, you can make L shapes by shading five numbers: four numbers vertically and two horizontally, as shown below.

1	2	3	4	5	6
11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46

b) What is the least value in the L if the sum of the values in the L is 211?

c) Write a function to describe the least value in the L in terms of the sum, s, of the L.

4. Yuden invested some money in a bank account earning 4.2% interest. The bank then improved its rates, so she invested in an account earning 4.5% interest.

a) Write an equation to describe the total interest.

b) Write a function that will calculate the amount invested at 4.5% if you know the amount at 4.2%.

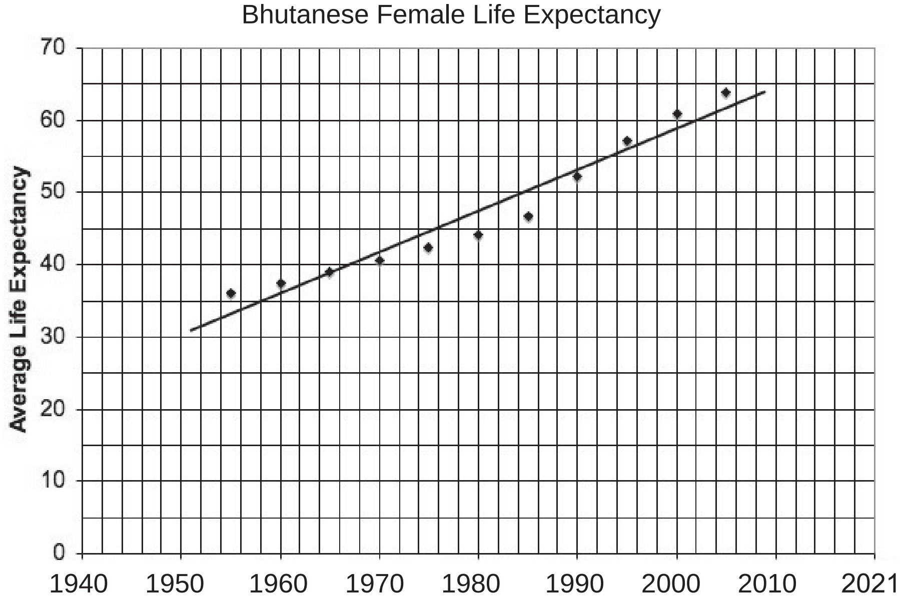
5. Lhakpa drives an average of 30 km/h for r hours and an average of 20 km/h for the remaining s hours. He travels a total of 51 km.

a) Write an equation to model this situation.

b) Write a function that tells how many hours he travels at 30 km/h if you know how many hours he travels at 20 km/h.

c) How long does he travel at 30km/h, if he travels 1.2 h at 20km/h?

6. The United Nations reports the average life expectancy for its member nations. This graph shows the data for females in Bhutan.



a) Identify two ordered pairs on the line of best fit.

b) Use the ordered pairs from **part a)** to write a function for the line of best fit.

c) Use the function in **part b)** to predict the lifespan of a Bhutanese female in 2010.

d) Use the function in **part b)** to predict the lifespan of a Bhutanese female in 1970. How close to the actual plotted value was the predicted value?

7. a) The relationship between Fahrenheit and Celsius temperatures is linear. If 100°C = 212°F and 0°C = 32°F, write the Fahrenheit temperature as a function of the Celsius temperature.

b) Write the Celsius temperature as a function of the Fahrenheit temperature.

8. Deki and Thinley are practicing for a 100 m race. Deki gives Thinley a 12 m head start. Deki runs at a speed of 8 m/s and Thinley runs at 7 m/s.

a) Write each sprinter's distance as a function of time.

8. b) What will each sprinter's distance be after 9 s?

c) Who will win the race? How do you know?

d) Why is it more appropriate to write distance as a function of time than time as a function of distance?

9. If *y* is a linear function of *x*, is *x* always a linear function of *y*? How do you know?

3.1.3 Graphs of Linear Inequalities

Try This

A small sport equipment company manufactures table tennis paddles and badminton racquets. They produce

- no more than 80 table tennis paddles per day
- no more than 50 badminton racquets per day
- no more than 110 table tennis paddles and badminton racquets in total per day

The profit on each item is

- Nu 20 for each table tennis paddle
- Nu 30 for each badminton racquet

A. i) Determine five possible combinations of table tennis paddles and badminton racquets that could be manufactured.

ii) Calculate the profit the company would make with each combination in **part i)**.

- A **linear inequality** is created when the equals sign (=) in a linear equation is replaced with an inequality (<, >, ≤, or ≥) symbol. For example, the linear equation $y = 3x + 2$ can become the linear inequality $y < 3x + 2$.
- Linear inequalities are never functions. When you examine the ordered pairs that satisfy the inequality, there are always many ordered pairs where the x-coordinate is the same, but there are different y-coordinates. Consider these examples:

For the inequality $y < 3x + 2$

x	y	$3x + 2$	$y < 3x + 2$
0	–1	2	Yes
0	–2	2	Yes
0	–3	2	Yes

The ordered pairs (0, –1), (0, –2), and (0, –3) all satisfy the inequality, so it cannot be a function.

For the inequality $3x + 2y \geq 6$

x	y	$3x + 2y$	$3x + 2y \geq 6$
0	4	8	Yes
0	5	10	Yes
0	6	12	Yes

The ordered pairs (0, 4), (0, 5), and (0, 6) all satisfy the inequality, so it cannot be a function.

- You can graph a linear inequality by following these steps:

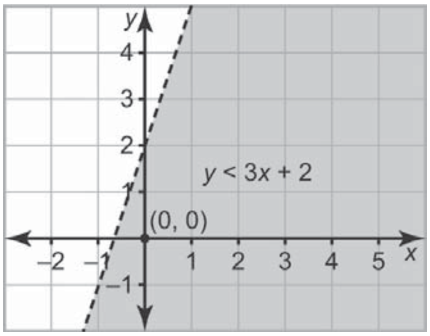
- Graph the related linear equation — this line will form the boundary line of the region that contains the points that satisfy the inequality.

- Draw a solid boundary line if the inequality uses ≤ or ≥, or a dashed boundary line if the inequality uses < or >.

- Shade the region on the side of the boundary line that contains the ordered pairs that satisfy the inequality. You can determine this by testing any point that is not on the boundary line. If that point satisfies the inequality, shade that side. If it does not, shade the other side.

- When you graph an inequality, you are representing its solutions.

Here is an example, the graph of $y < 3x + 2$.



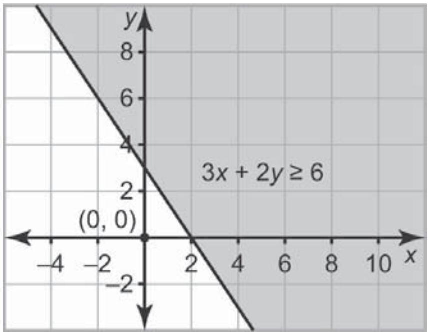
- Graph the line $y = 3x + 2$ using a dashed line because y **<** $3x + 2$.
- Substitute $x = 0$ and $y = 0$ to see if (0, 0) is in the region: $y < 3x + 2$

$0 < 3(0) + 2$

$0 < 2$ *True*

- Shade the region on the side of the boundary that contains (0, 0).

Here is another example, the graph of $3x + 2y \geq 6$.



- Graph the line $3x + 2y = 6$ using a solid line because $3x + 2y$ **≥** 6.
- Substitute $x = 0$ and $y = 0$ to see if (0, 0) is in the region: $3x + 2y \geq 6$

$3(0) + 2(0) \geq 6$

$0 \geq 6$ *Not true*

- Shade the region on the side of the boundary that does not contain (0, 0).

B. i) How do the inequalities $p \leq 80$ and $r \leq 50$ and $p + r \leq 110$ describe the situation in **part A**?

ii) How do you know there are more combinations that can be manufactured than the five you listed in **part A i)**?

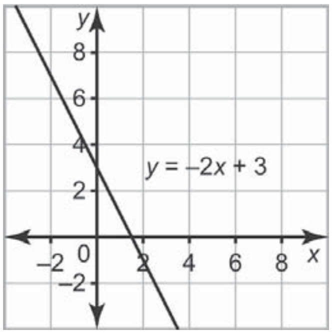
Examples

Example 1 Graphing an Inequality in Slope and *Y*-intercept Form

Sketch the graph of $y \geq -2x + 3$.

Solution

Graph the related equation $y = -2x + 3$



[Continued]

Thinking

- I drew the graph of $y = -2x + 3$ to create the boundary line.
- I drew a solid line because ≥ means "greater than or equal to." The "equal to" part means the values on the line are included.

Example 1 Graphing an Inequality in Slope and *y*-intercept Form [Continued]

Solution [Cont'd]

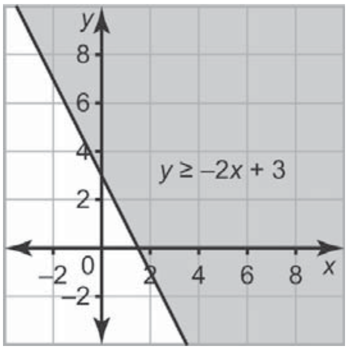
Test a point to decide where to shade

Substitute (0, 0) into $y \geq -2x + 3$:

$$0 \geq -2(0) + 3$$

$$0 \geq 3 \quad \textit{Not true}$$

Shade the inequality region



Thinking

- I tested (0, 0) to see which side to shade. (I could have used another point as long as it wasn't on the boundary line.)
- (0, 0) does not make the inequality true so the inequality region is the side of the boundary line that does not contain (0, 0).

Example 2 Graphing an Inequality in Standard Form

Sketch the graph of $10x - 3y < 60$.

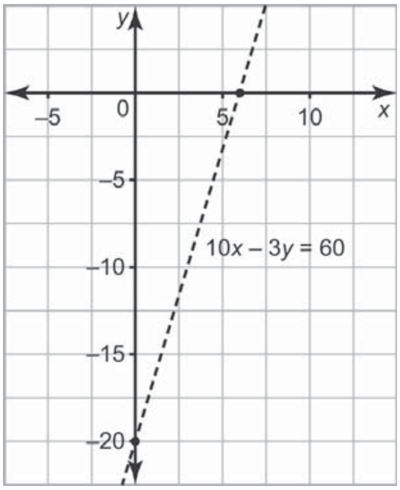
Solution

Graph the related equation $10x - 3y = 60$

To determine the *x*-intercept, substitute $y = 0$ and solve for x: $10x = 60$ so $x = 6$

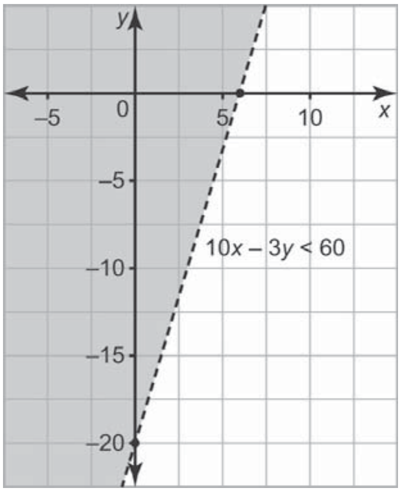
To determine the *y*-intercept, substitute $x = 0$ and solve for y: $-3y = 60$ so $y = -20$

x-intercept is 6 and *y*-intercept is –20



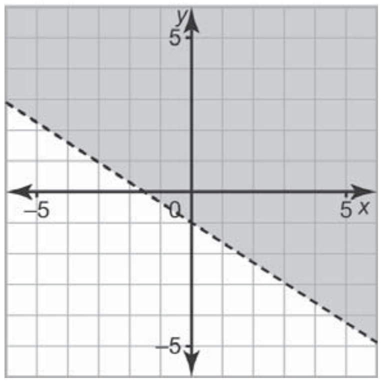
Thinking

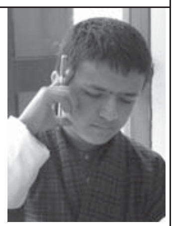
- I graphed the equation $10x - 3y = 60$ to create the boundary line — I used a dashed line since < means the values along the line are not included.

Test a point to decide where to shade Substitute (0, 0) into $10x - 3y < 60$: $10(0) - 3(0) < 60$ $0 < 60$ *True* *Shade the inequality region* 	• (0, 0) makes the inequality true so the inequality region is on the same side of the boundary line as (0, 0).

Example 3 Express an Inequality Algebraically From Its Graph

Write the linear inequality that describes this graph.



Solution	Thinking
Write the equation for the boundary line The y-intercept of the boundary line is –1. The slope of the boundary line is $\frac{1 - (-1)}{-3 - 0} = -\frac{2}{3}$. The equation of the boundary line is $y = -\frac{2}{3}x - 1$. [Continued]	 • I read the y-intercept from the graph. • I also used to graph to find two points, (0, -1) and (-3, 1), to determine the slope.

Example 3 Express an Inequality Algebraically From Its Graph [Continued]

Solution [Cont'd]

Determine the inequality sign

The inequality is either < or >.

(0, 0) is in the shaded region, so substitute (0, 0) into the linear relation, $y = -\frac{2}{3}x - 1$.

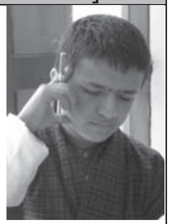
Left side: $y = 0$ Right side: $\frac{2}{3}(0) - 1 = -1$

Since $0 > -1$, the inequality sign is >.

Write the inequality

The inequality is $y > -\frac{2}{3}x - 1$.

Thinking



- I knew it was < or > because the line was dashed.
- I tested a point, (0, 0), to see if the inequality was < or >.

Practising and Applying

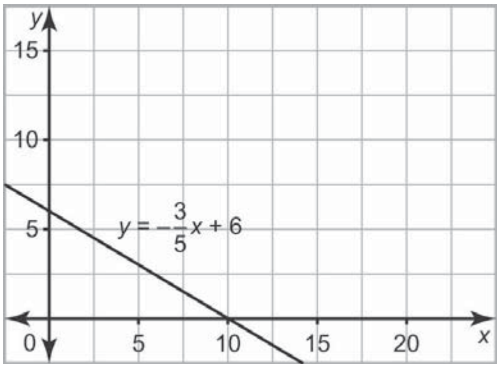
1. Is (0, 0) in the shaded region of each graph?

a) $y + 4 < 5x - 3$

b) $2y - 3x \leq 7$

c) $3x + 5y > 9$

2. The graph of $y = -\frac{3}{5}x + 6$ is shown. Is the region that describes $y < -\frac{3}{5}x + 6$ above or below the boundary line? How do you know?



3. Without graphing, describe how the graphs of each pair of inequalities are alike and different.

a) $y > 2x + 3$ and $y < 2x + 3$

b) $2y \leq x - 5$ and $2y < x - 5$

4. Sketch the graph of each inequality.

a) $3x + 4y \geq 12$

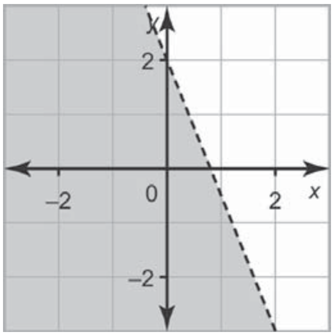
b) $5x - 2y < 10$

c) $3x + 4y \geq 24$

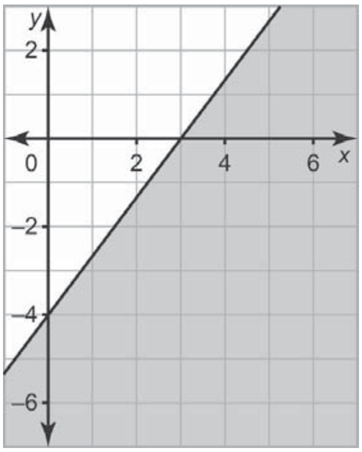
d) $5x - 2y < 20$

5. Write the inequality for each graph.

a)



b)



5. c) How would your answer to **part a)** change if the line were not dashed? Write the new inequality.

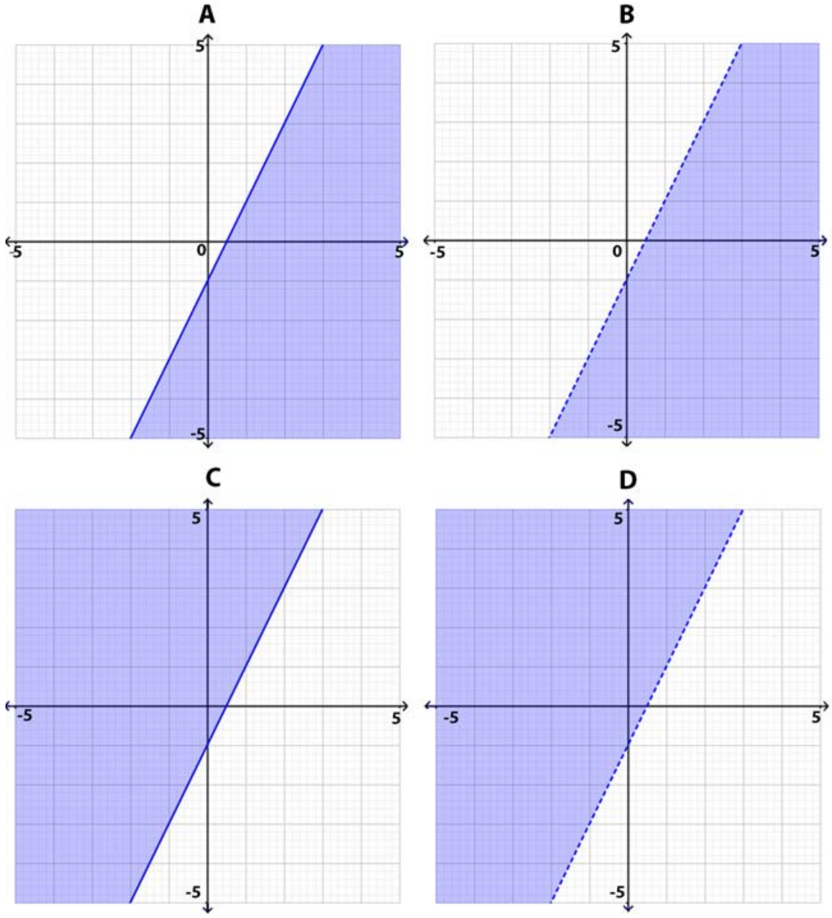
d) How would your answer to **part b)** change if the shaded region were above the line? Write the new inequality.

6. a) Rajesh works part-time at two. Last month he made at least Nu 300. Write an inequality to represent his total income from both jobs last month.

b) Graph the inequality in **part a)**.

7. For each inequality, identify the correct graph below.

a) $y \leq 2x - 1$ **b)** $y > 2x - 1$



c) How did you know which graph matched **part a)?**

d) How did you know which graph matched **part b)?**

8. Why is an inequality never a function?

9. Explain the steps you would follow to graph $2x - 4y < 10$.

3.1.4 EXPLORE: Transforming Graphs of Linear Functions

- When you evaluate a function for a value of the independent variable, you substitute that value of x into the function.

For example, to evaluate $f(x) = 3(x) + 2$ for $x = 2 \rightarrow f(2) = 3(2) + 2 = 8$

- You can also substitute expressions into functions.

Here are some examples using the function $f(x) = 3x + 2$:

$$f(x) = 3x + 2 \rightarrow f(b - 1) = 3(b - 1) + 2 = 3b - 3 + 2 = 3b - 1$$

$$f(x) = 3x + 2 \rightarrow f(x + 3) = 3(x + 3) + 2 = 3x + 9 + 2 = 3x + 11$$

- Replacing x with an algebraic expression changes the function rule and creates a new function.

Here are some examples using $f(x) = 3x + 2$. The letter g is used to represent the new function.

$$g(x) = f(x) + 2 = (3x + 2) + 2 = 3x + 4$$

$$g(x) = f(x) - 2 = (3x + 2) - 2 = 3x$$

$$g(x) = -f(x) = -(3x + 2) = -3x - 2$$

$$g(x) = -2f(x) = -2(3x + 2) = -6x - 4$$

The new function is related to the original function.

A. Draw the graph of $f(x) = 3x + 2$ and the graph of each of the following new functions on the same grid.

i) $f(x - 1)$ **ii)** $f(x + 1)$ **iii)** $f(x + 3)$ **iv)** $f(-x)$

B. Draw the graph of $f(x) = 3x + 2$ and the graph of each of the following new functions on the same grid.

i) $g(x) = f(x) + 2$ **ii)** $g(x) = f(x) - 2$

C. Draw the graph of $f(x) = 3x + 2$ and the graph of each of the following new functions on the same grid.

i) $g(x) = -f(x)$ **ii)** $g(x) = 2f(x)$ **iii)** $g(x) = 0.5f(x)$

D. Each graph in **parts A, B, and C** is the image of the graph of the original function $f(x) = 3x + 2$ after a transformation. Describe the transformation for each new function.

GAME: True or False

You will need scissors and light cardboard or paper to make cards for this game.

• Make two sets of variable cards:

21 *y*-variable cards, each marked with *y* on the back and the integers –10 to 10 on the front

11 *x*-variable cards, each marked with *x* on the back and the integers –5 to 5 on the front

• Make 16 linear equation cards, each marked with an L on the back:

Write these equations on the front of eight cards:

• $y = 2x + 1$ • $y = -2x + 1$
• $y = 2x - 1$ • $y = -2x - 1$
• four more equations of your choice

Write these inequalities on the front of the remaining eight cards:

• $y < 2x + 1$ • $y > -2x + 1$
• $y < 2x - 1$ • $y > -2x - 1$
• four more inequalities of your choice

Play in groups of three or more.

• Players take turns acting as dealer. The dealer does not play.

• The dealer shuffles the deck of *y*-variable cards and deals one card to each player. Then the dealer does the same for the *x*-variable cards.

• The dealer shuffles the L cards and turns over the top card for all players to see.

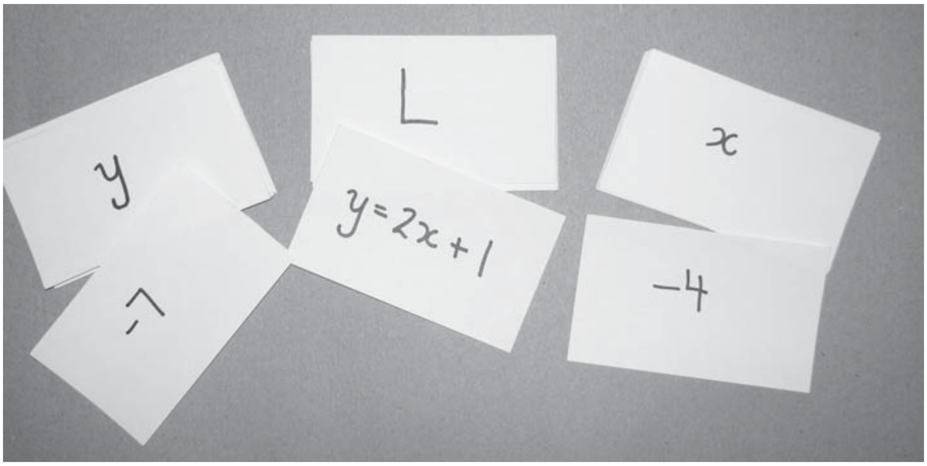
• Each player must decide if the values he or she has on his or her *y*- and *x*-variable cards make the equation or inequality true. He or she must say "true" or "false" by the time the dealer counts to 10 (or less, if all players agree).

• Each player who is correct receives a point.

• Return the cards to their decks and continue the game with a different player acting as dealer until each player has had a turn as dealer.

• The player or players with the most points at the end of the game wins.

y-variable cards Linear equation cards *x*-variable cards



$y = 2x + 1 \longrightarrow \mathbf{-7} = 2(\mathbf{-4}) + 1$ True!

3.2.1 Solving Algebraically — The Comparison Strategy

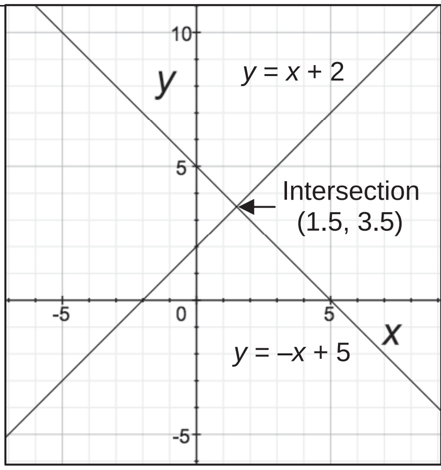
Try This

Passang plans to earn money by selling biscuits. He estimates that it will cost him Nu 4800 to buy equipment and Nu 5 to make each biscuit. He plans to sell each biscuit for Nu 20.

In business, the *break-even point* is the point at which your costs for running the business equal the amount of money you make in sales. In his first month of business, Passang's goal is to at least break even.

A. How many biscuits must Passang sell to break even? How do you know?

A **linear system** is a set of linear equations that represents multiple relationships. The **solution of a system of linear equations** involving the variables x and y is the x- and y-values that satisfy all the equations in the set.



The solution of a system of linear equations is the intersection point of the graphs of the equations.

- The solution of a system of linear equations can be determined graphically. The solution is the intersection of the graphs of the equations in the system.

For example, the linear system $y = -x + 5$ and $y = x + 2$ is shown to the right. The intersection point (1.5, 3.5) is its solution. This makes sense since (1.5, 3.5) is the only ordered pair that is on both graphs and it is the only pair of numbers that makes both equations true.

- It is often impossible to be exact about the intersection point of two graphs, so the coordinates might be just an estimate. If you need to be more exact, you can solve the system of equations algebraically.

- There are several methods for solving a system of equations algebraically. One method is the comparison strategy, which can be used when both equations in the system have the same variable term isolated.

For example:

$$y = \underline{-x + 5} \text{ and } y = \underline{x + 2}$$

$$-x + 5 = x + 2$$

$$x = 1.5$$

In both equations, the variable y is isolated on one side, which means the other sides of the equations are equal. That means you can create a one-variable equation by comparing the other sides of the equation. You can then solve this equation to determine the value of x.

$y = x + 2$
$y = 1.5 + 2$
$y = 3.5$
Solution: $x = 1.5$, $y = 3.5$

You can substitute $x = 1.5$ into one of the equations to find the value of y.

$y = -x + 5 \rightarrow 3.5 = -1.5 + 5$
$y = x + 2 \rightarrow 3.5 = 1.5 + 2$

To check the solution, you can substitute $x = 1.5$ and $y = 3.5$ into both equations to see if those values make both equations true.

• A system of equations can be used to represent and solve a problem situation. Here are two examples:

- An Internet service provider offers two payment plans and you want to compare them. You could write an equation to represent each plan and then solve the system of equations. The solution is the point at which both plans cost the same.

- You are working at two jobs, each paying a different hourly rate, and you have a goal of earning a certain amount of money in a certain number of hours. You could write an equation to represent the earnings for each job and then solve the system of equations. The solution would be the number of hours you would need to work at each job to achieve your goal.

B. i) Write a system of equations to determine Passang's break-even point in **part A**.

ii) Solve the system to determine the break-even point and check your answer from **part A**.

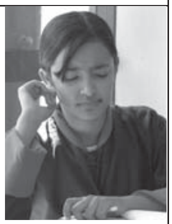
Example 1 Solving a System to Solve a Problem

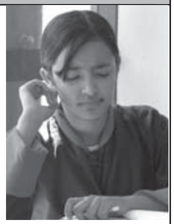
Meto wants to work at Mountain Trekkers and Tours as a sales agent. He has been offered a choice of two income plans:

Plan A: 3% commission on total sales

Plan B: an annual base salary of Nu 12,000 and 1% commission on total sales

For what amount of sales will both plans pay the same income?

Solution	Thinking
Create equations I represents income s represents total sales Plan A: $I = 0.03s$ Plan B: $I = 12,000 + 0.01s$ [Continued]	• I wrote an equation for each income plan. 

Example 1 Solving a System to Solve a Problem [Continued]	
Solution [Cont'd] *Compare expressions and solve for s* If $I = \underline{0.03s}$ and $I = \underline{12{,}000 + 0.01s}$, then $0.03s = 12{,}000 + 0.01s$ $0.02s = 12{,}000$ $s = 600{,}000$ *Substitute for s to solve for I* If Plan A is $I = 0.03s$ and $s = 600{,}000$, then $I = 0.03(600{,}000) = 18{,}000$ *Check solution* Substitute $s = 600{,}000$ into $I = 12{,}000 + 0.01s$ to see if I is 18,000: $12{,}000 + 0.01s$ $= 12{,}000 + 0.01(600{,}000)$ $= 18{,}000$ When sales are Nu 600,000, the two plans pay the same income (Nu 18,000).	**Thinking**  • Since I was already isolated in both equations, I compared the expressions involving s and then solved for s. • I substituted the value for s into the equation for Plan A because it involved the least computation. I could have used the other equation instead. • I checked the values with Plan B's equation to see if it made the equation true. I already knew it worked for Plan A's equation.

Example 2 Finding the Intersection Point of a System of Equations	
Determine the coordinates of the intersection point of the graphs of $6x - 8y = 3$ and $2x + 4y = 6$. Graph the system to check your answer.	
Solution *Isolate the variable term in the first equation* $6x - 8y = 3$ $6x = 3 + 8y$ $6x - 3 = 8y$ *Isolate the same variable term in the second equation* $2x + 4y = 6$ [Multiply by 2] $4x + 8y = 12$ $8y = 12 - 4x$	**Thinking**  • To use the comparison strategy, I needed the same variable term isolated for both equations. • I isolated $8y$ for the first equation. • I multiplied the second equation by 2 to get the same variable term as the first equation, $8y$, and then I isolated $8y$.

Compare expressions and solve for x

If $\underline{6x - 3} = 8y$ and $8y = \underline{12 - 4x}$, then

$$6x - 3 = 12 - 4x$$
$$10x = 15$$
$$x = \frac{15}{10}$$
$$x = \frac{3}{2}$$

• I wrote an equation that compared both expressions involving x and solved for x.

Substitute for x to solve for y

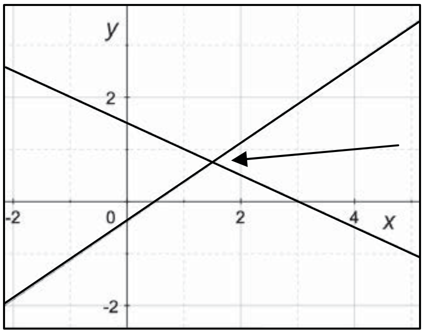
If $2x + 4y = 6$ and $x = \frac{3}{2}$, then $2(\frac{3}{2}) + 4y = 6$

$$3 + 4y = 6$$
$$y = \frac{3}{4}$$

• I substituted the value for x into the second equation and solved for y.

The intersection point is $(\frac{3}{2}, \frac{3}{4})$.

Check solution by graphing



$(\frac{3}{2}, \frac{3}{4})$

Practising and Applying

1. A phone service provider offers two different monthly plans.

Plan	Monthly fee	Price per minute
A	Nu 1200	Nu 1.60
B	Nu 1400	Nu 1.40

a) What does the equation $1200 + 1.6m = 1400 + 1.4m$ represent? How do you know?

b) For what number of minutes do the plans cost the same? How do you know?

c) If you expect to use the phone for 600 min per month, which plan would you choose? What about 1200 min?

2. Determine the point of intersection for each system of linear equations.

a) $y = 2x - 7$ and $y = 5x - 4$

b) $y = 3x - 1$ and $y = 5x + 3$

c) $y = -x + 2$ and $y = 4x - 4$

3. The sum of two numbers is 48 and the difference is 34. What are the two numbers?

4. Half of one number is one third of another. The sum of the two numbers is 95. What are the two numbers?

5. A team bought 20 basketballs for a total of Nu 8800. Practice balls cost Nu 400 and official balls cost Nu 600. Use a system of equations to determine how many of each type of ball they bought.

6. A vehicle has a mass of 1295 kg and uses petrol. Another vehicle has a mass of 1290 kg and uses diesel fuel.

• 1 L of petrol has a mass of 737 g.

• 1 L of diesel has a mass of 820 g.

How many litres of fuel will result in the two vehicles having the same mass? Round to the nearest tenth of a litre.

7. Chandra sold some stamps to a group of tourists. They bought some stamps for Nu 4 and some for Nu 9. Altogether they spent Nu 440 and bought 60 stamps. How many of each type of stamp did they buy?

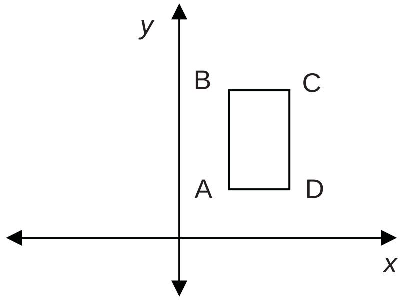
8. Determine the point of intersection for each system of linear equations.

a) $y = 2x + 3$ and $3x - y = 2$

b) $x - 2y = -2$ and $x + y = 4$

c) $3x - 4y = -15$ and $2x + 3y = 7$

9. A rectangle has these vertices:
A(2, 2), B(2, 6), C(4, 6), and D(4, 2)



a) Determine the equation of each diagonal, BD and AC.

b) Determine the coordinates of the point of intersection of the diagonals.

10. A plane travelling in one direction has a tailwind and takes 3 h to travel 960 km. When it makes the return trip, it has a headwind and takes 4 h to travel the same distance. Assuming the tailwind speed was the same as the headwind speed, what was the speed of the wind? What was the speed of the plane? [Hint: Think of the faster speed as the plane speed plus the tailwind speed.]

11. Solve each linear system.

a) $5a + 3.5b = 17$ and $2a + 0.5b = 5$

b) $x - \frac{3}{4}y = -4$ and $x + \frac{1}{4}y = 0$

c) $x - \frac{1}{3}y = 2$ and $\frac{2}{3}x - \frac{1}{4}y = 1$

12. Why is it important to isolate the same variable term of each equation in order to use the comparison strategy?

3.2.2 Solving Algebraically — The Substitution Strategy

Try This

Maya's store sells masks in two different sizes, one for Nu 1200 and a smaller one for Nu 1000. Last year she sold 100 masks and had total sales of Nu 108,200.

A. i) If she had sold 50 of each mask, what would be her total sales?

ii) How could you use your answer to **part i)** to estimate the number of each type of mask Maya actually sold for sales of Nu 108,200?

iii) Estimate the number of masks sold at each price. Show your work.

• In the previous lesson you learned about the comparison strategy for solving a system of equations. This lesson introduces another strategy, called the substitution strategy. To use this strategy, you rearrange one of the equations in the system to isolate one of the variables. Then, you substitute the expression for that variable into the other equation.

For example, consider the system of equations $y = x - 1$ and $5x + 2y = 12$:

$y = x - 1$ and $5x + 2y = 12$

In the first equation, y is already isolated so it does not have to be rearranged.

$$\begin{aligned} 5x + 2y &= 12 \\ 5x + 2(x - 1) &= 12 \\ 5x + 2x - 2 &= 12 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Substitute the expression for y into the second equation and then solve for x.

$$\begin{aligned} y &= x - 1 \\ y &= (2) - 1 \\ y &= 1 \end{aligned}$$

Next, substitute the value for x into either of the equations and solve for y.

Solution: $x = 2$ and $y = 1$

$$\begin{aligned} 5x + 2y = 12 \rightarrow 5(2) + 2(1) &= 12 \\ 10 + 2 &= 12 \end{aligned}$$

To check the solution, substitute $x = 2$ and $y = 1$ into the other equation to see if those values make the equation true.

B. Create and solve a system of linear equations to determine how many masks Maya sold at each price.

Examples

Example 1 Solving a Problem by Solving a System of Equations

A 45 minute fitness program is designed to burn 400 calories and involves riding a bicycle and jogging. Riding a bicycle burns 8 cal/min. Jogging burns 10 cal/min. How much time should be spent on each activity to meet the program goals?

Solution	**Thinking** 
Create equations If b is number of minutes biking and j is number of minutes jogging, • total time spent exercising is $b + j = 45$ • total calories burned is $8b + 10j = 400$	• I wrote a system of linear equations to model the situation — one equation to model the amount of time spent exercising and the other to model the number of calories burned.
Isolate the variable b $b + j = 45$ $b = 45 - j$	• I rearranged the first equation to isolate *b*. (I could have used the other equation instead but the first equation was simpler.)
Substitute the expression for b into the other equation $8b + 10j = 400$ $8(45 - j) + 10j = 400$ $360 - 8j + 10j = 400$ $360 + 2j = 400$ $2j = 40$ $j = 20$	• I substituted the expression for *b* into the other equation and solved for *j*.
Substitute the value for j into the first equation $b + j = 45$ $b + (20) = 45$ $b = 25$ $b = 25$ and $j = 20$	• I substituted the value for *j* into the first equation to determine the value for *b*.
Check the solution $8b + 10j = 400 \rightarrow 8(25) + 10(20)$ $= 200 + 200$ $= 400$ The program should have 20 min of jogging and 25 min of biking.	• I checked my solution to make sure the total number of calories burned was 400 calories, since I already knew that 25 min + 20 min = 45 min.

Example 2 Solving a System of Linear Equations

Determine the solution of this system of equations: $3x + 2y = 12$ and $x - 2y = -2$

Solution

Isolate the variable term 2y

$$3x + 2y = 12$$
$$2y = 12 - 3x$$

Substitute the expression for 2y into the other equation

$$x - 2y = -2$$
$$x - (12 - 3x) = -2$$
$$x - 12 + 3x = -2$$
$$4x = 10$$
$$x = 2.5$$

Substitute the value for x into the second equation

$$x - 2y = -2$$
$$2.5 - 2y = -2$$
$$-2y = -4.5$$
$$y = 2.25$$

The solution is $x = 2.5$ and $y = 2.25$.

Thinking

- I noticed both equations had the same variable term, $2y$, so I isolated $2y$ in the first equation.
- I substituted the expression for $2y$ into the second equation and solved the resulting equation for x.
- I substituted the value for x into the second equation to find the value for y.

Example 3 Solving a Linear System with Fractional Coefficients

Determine the point of intersection of the lines with the following equations:

$$\frac{1}{2}x + \frac{3}{5}y - 2 = 0 \text{ and } 6x + 4y = 8$$

Solution

Create an equivalent equation with a 6x-term

$$12\left(\frac{1}{2}x + \frac{3}{5}y - 2\right) = 12(0) \rightarrow 6x + \frac{36}{5}y - 24 = 0$$

Isolate the variable term 6x in the other equation

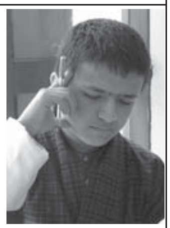
$$6x + 4y = 8$$
$$6x = 8 - 4y$$

Substitute the expression for 6x into the first equation

$$8 - 4y + \frac{36}{5}y - 24 = 0$$
$$\frac{16}{5}y - 16 = 0$$
$$16y = 80$$
$$y = 5$$

[Continued]

Thinking



- I multiplied the first equation by 12 so that it would have a $6x$-term, just like in the second equation.
- Then I isolated the $6x$-term in the second equation.
- I substituted the expression $8 - 4y$ for $6x$ into the first equation and solved for y.

Example 3 Solving a Linear System with Fractional Coefficients [Continued]

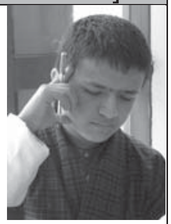
Solution [Cont'd]

Substitute the value for y into the second equation

$$\begin{aligned} 6x + 4y &= 8 \\ 6x + 4(5) &= 8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The point of intersection is (–2, 5).

Thinking



• I substituted the value for *y* into the second equation and solved for *x*. (I could have used the first equation instead but it would have been more complicated).

Practising and Applying

1. Use the substitution strategy to solve each system of linear equations.

a) $y = 4x - 1$ and $2x + 3y = 11$

b) $y = 3x + 5$ and $x - 3y = 1$

c) $y = 1 - x$ and $4x + 2y = 3$

d) $y = x + \frac{1}{12}$ and $3x + 4y - 5 = 0$

2. a) Lhamo is 5 years older than Devika. Write an equation that models this relationship.

b) The sum of their ages is 29 years. Write an equation to model this relationship. Use the same variables as you used in **part a)**.

c) Solve the system of equations to determine each person's age.

3. Find the point of intersection of the graphs for each system.

a) $x - y = 1$ and $3x - y = -1$

b) $2x + 5y = 0$ and $x + y = 3$

c) $4x - 2y = 3$ and $-3x + y = -2$

d) $2x + 2y = 3$ and $6x - 6y = -1$

4. Therchu is thinking of two numbers.

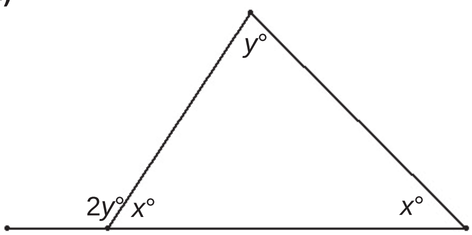
• The sum of the numbers is 212.

• The difference between them is 88.

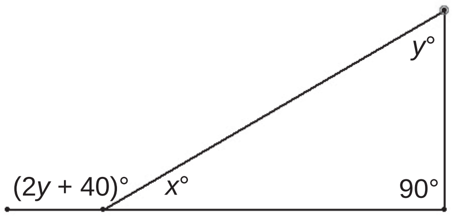
Solve a system of equations to determine the value of each number.

5. Determine the unknown values in each diagram. Show your work. [Hint: interior angles of a triangle add to 180°.]

a)



b)



6. Solve each system.

a) $\frac{1}{2}x + \frac{2}{3}y = 6$ and $\frac{3}{4}x - \frac{1}{3}y = 1$

b) $-\frac{1}{6}x + \frac{3}{4}y = -8$ and $y + \frac{2}{3}x = 0$

c) $4x - y = 6.9$ and $2x + 3y = 14.3$

d) $x + y = 0.6$ and $2y + 3x + 0.5 = 0$

7. The comparison strategy and the substitution strategy both involve isolating terms. In what ways do the strategies differ?

3.2.3 Solving Algebraically — The Elimination Strategy

Try This

Dorji and Deki both have stamp collections that contain Bhutanese stamps and stamps from other countries.

- Dorji has 200 stamps altogether. Deki has 450 stamps altogether.
- Deki has twice as many Bhutanese stamps as Dorji and three times as many stamps from other countries as Dorji.

A. i) Write two equations to describe the stamp collections in terms of Bhutanese stamps and stamps from other countries.

ii) How many Bhutanese stamps does each person have?

- Another strategy for solving systems of linear equations algebraically is the elimination strategy. This strategy is used when the equations in the linear system are in a form that allows you to eliminate one variable by adding or subtracting the equations.

For example, consider the system of equations $3x + 4y = 13$ and $3x + 2y = 11$. Both equations have a $3x$-term that could be eliminated if one equation were subtracted from the other.

You can find the value of y by eliminating the $3x$-terms from the system by subtracting the equations and then solving for y:

$$\begin{array}{rr} & 3x + 4y = 13 \\ - & \underline{3x + 2y = 11} \\ & 2y = 2 \\ & y = 1 \end{array}$$

Then you can substitute the value for y into one of the equations to determine the value for x:

$$\begin{aligned} 3x + 2y &= 11 \\ 3x + 2(1) &= 11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

The solution of $3x + 4y = 13$ and $3x + 2y = 11$ is $x = 3$ and $y = 1$.

The solution can be checked by substituting these values into the other equation, $3x + 4y = 13$, to see if they make the equation true:

$$3x + 4y = 13 \;\rightarrow\; 3(3) + 4(1) = 13 \;\rightarrow\; 9 + 4 = 13$$

• Sometimes equations can be added instead of subtracted to eliminate a variable when variable terms have opposite values.

For example, consider the system of equations $3x + 4y = 14$ and $3x - 2y = 11$. You can eliminate the y-terms by creating an equivalent equation for the second equation that has the same y-term but with an opposite value as the first equation, and then adding the equations.

Create an equivalent equation, add the equations, and then solve for x:

$$2(3x - 2y) = 2(11) \rightarrow 6x - 4y = 22$$

$$\begin{array}{rl} & 3x + 4y = 14 \\ + & \underline{6x - 4y = 22} \\ & 9x = 36 \\ & x = 4 \end{array}$$

Substitute the value for x into one of the equations to determine the value for y:

$$\begin{aligned} 3x + 4y &= 14 \\ 3(4) + 4y &= 14 \\ y &= 0.5 \end{aligned}$$

The solution of $3x + 4y = 14$ and $3x - 2y = 11$ is $x = 4$ and $y = 0.5$.

B. Describe how to use the elimination strategy to solve **part A ii)**.

Examples

Example 1 Solving a System of Linear Equations

Determine the solution of this system of linear equations.

$$\frac{1}{2}x + \frac{2}{3}y = 2$$

$$\frac{3}{4}x - \frac{1}{3}y = 11$$

Solution

Create an equivalent equation

$$\frac{3}{4}x - \frac{1}{3}y = 11 \rightarrow \frac{3}{2}x - \frac{2}{3}y = 22$$

Add equations and solve for x

$$\begin{array}{rl} & \frac{1}{2}x + \frac{2}{3}y = 2 \\ + & \underline{\frac{3}{2}x - \frac{2}{3}y = 22} \\ & 2x \quad = 24 \\ & x = 12 \end{array}$$

Substitute for x to solve for y

$$\frac{3}{4}x - \frac{1}{3}y = 11 \rightarrow \frac{3}{4}(12) - \frac{1}{3}y = 11$$

$$-\frac{1}{3}y = 2$$

$$y = -6$$

The solution of the system of equations is $x = 12$ and $y = -6$.

Thinking

• I doubled the second equation so it would have a $-\frac{2}{3}y$-term, like the other equation.

• When I added the equations, the y-terms cancelled out because $\frac{2}{3}y + (-\frac{2}{3}y) = 0$.

Example 2 Solving a Mixture Problem Using a System of Equations

A lab technician needs 3 L of 8% saline (salt) solution. The stock room has only 5% and 9% solution available. What volume of each must she mix?

Solution	Thinking
a is the volume of 5% solution needed b the volume of 9% solution needed The equation that models the total volume of the final mixture is: $a + b = 3000$ The equation that models the amount of salt in the final mixture is: $0.05a + 0.09b = 0.08(3000)$ $0.05a + 0.09b = 240$	• I represented the volumes of the two original solutions with variables.  • I created two equations: - one to describe what I knew about the volumes and - one to describe what I knew about the amount of salt.
$100(0.05a + 0.09b) = 100(240)$ $5a + 9b = 24{,}000$	• I multiplied the terms on both sides of the salt equation by 100 to get rid of the decimals.
$a + b = 3000 \rightarrow 5a + 5b = 15{,}000$	• I multiplied the volume equation by 5 so that its coefficient of a was the same as the coefficient of a in the new salt equation. I did this so I could subtract the equations and eliminate the a-terms.
$\begin{array}{rl} 5a + 9b &= 24{,}000 \\ -\quad 5a + 5b &= 15{,}000 \\ \hline 4b &= 9000 \\ b &= 2250 \end{array}$	• I subtracted the new volume equation from the new salt equation to eliminate the a-terms and solve for b.
$a + (2250) = 3000$ $a = 750$	• I substituted the value for b into the original volume equation.
The technician should mix 750 mL of the 5% solution and 2250 mL of the 9% solution.	

Practising and Applying

1. Use the elimination strategy to solve each system of linear equations.

a) $2x + 4y = 6$ and $2x + 3y = 4$

b) $5x - 3y = 2$ and $7x + 3y = 10$

c) $4x + 3y = -3$ and $2x + y = 1$

d) $x - 2y = 6$ and $2x + y = -8$

2. Yangchen invested a total of Nu 2500 in two investments. One investment earned 4% interest and the other earned 5% interest. The total interest earned was Nu 115.

a) Write an equation to model the total amount invested. [Cont'd]

2. [Cont'd] **b)** Write an equation to model the total interest earned.

c) Solve the system of equations to determine the amount invested at each interest rate.

3. A factory uses steel and aluminium to manufacture small trucks and passenger cars. The chart below shows how much of each material is needed for each vehicle.

Vehicle	Truck	Car
Steel	500 kg	375 kg
Aluminium	250 kg	150 kg

Last year the factory used 125,000 kg of steel and 55,000 kg of aluminium.

a) Write a system of linear equations to model this situation.

b) Solve the system to determine the number of cars and trucks produced last year.


4. Karma wrote a 20-question multiple-choice test. He answered each question. It was scored as follows:

• gain 4 points for each correct answer

• lose 1 point for each incorrect answer

Karma received 60 points on the test. Determine the number of questions he answered correctly.

5. Use the elimination strategy to solve each system of equations.

a) $0.5a + 0.2b = 80$
$0.7a - 0.3b = 25$

b) $1.5a + 1.5b = 1.5$
$4.5a + 3.0b = 1.5$

c) $0.06a - 0.02b = 12$
$0.03b - 0.04a = 42$

6. Champa makes two grades of recycled paper using scrap paper and cloth. One batch of each grade requires a different combination of paper and cloth as shown below.

Grade	Scrap Cloth	Scrap Paper
Deluxe	4.4 kg	19.8 kg
Fine	1.1 kg	16.5 kg

Champa has 11 kg of scrap cloth and 72.6 kg of scrap paper and she wants to use up all her supplies. How many batches of each grade can she make?

7. Use the elimination strategy to determine the point of intersection for each pair of lines.

a) $\frac{1}{2}x + \frac{1}{3}y = 9$ and $\frac{3}{5}x - \frac{3}{4}y = -3$

b) $\frac{1}{2}y + \frac{3}{4}x = -8$ and $\frac{3}{4}y - \frac{1}{2}x = 14$

c) $\frac{3}{4}x - \frac{2}{3}y = 3$ and $\frac{1}{2}x - \frac{1}{2}y = 3$

8. A dye company mixes red and blue dyes to make two types of purple dye.

• One batch of light purple dye uses $\frac{1}{4}$ package of blue and $\frac{1}{6}$ package of red.

• One batch of deep purple dye uses $\frac{1}{6}$ package of blue and $\frac{1}{12}$ package of red.

How many batches of each shade of purple dye can the company make with 900 packages of blue dye and 500 packages of red dye?

9. Give an example of a pair of equations where you might use each strategy to solve them. Explain your thinking.

a) elimination

b) substitution

c) comparison

CONNECTIONS: Matrix Solution of a Linear System

Matrices can also be used to solve systems of equations.

1. Show that the matrix equation below represents the given system of equations.

System of equations

$8x + 5y = 11$

$3x + 2y = 4$

Matrix equation

$$\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

Notice that the first matrix on the left side of the matrix equation is made up of the coefficients of the variables and the matrix on the right side of the equation is made up of the constants.

2. a) Multiply each side of the matrix equation by the matrix $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ as the first factor, as shown here: $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 11 \\ 4 \end{bmatrix}$

Keep multiplying until there is one matrix on each side of the equation.

b) What does the resulting equation tell you about the solution of the system of equations?

3. a) Create a matrix equation to represent this system of equations:

$5x + 9y = 7$ and $x + 2y = 2$

b) Multiply each side of the matrix equation by $\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix}$ as the first factor.

Keep multiplying until there is one matrix on each side of the equation.

c) What is the solution of the original system of equations?

4. a) Create a matrix equation to represent this system of equations:

$4x + 7y = 3$ and $5x + 9y = 4$

b) Predict what matrix you could multiply by to solve the system of equations.

c) Check your prediction.

3.2.4 EXPLORE: Counting Solutions for Different Systems

In all the situations you have seen so far in this chapter, you have worked with systems of two linear equations, each with two variables. There has always been one solution for each system.

For example, there is only one solution for $2x + 3y = 2$ and $5x – 6y = 32$, $x = 4$ and $y = –2$, because these are the only values that make both equations true.

You might wonder if this is always the case. You also might wonder how many solutions there could be if there were different numbers of equations or different numbers of variables.

A. Show that there is more than one solution to each equation.

i) $2x + 3y = 2$ **ii)** $5x – 6y = 32$ **iii)** $3x – 6y + z = 8$

B. Show that there is more than one solution to each system of equations.

i) $2x + 3y = 2$ and $4x + 6y = 4$ **ii)** $5x – 6y = 32$ and $18y = 15x – 96$

C. Show that there is only one solution to each system of equations.

i) $2x + 6y = 26$ and $3y + 8x = – 1$ **ii)** $6x + 4y = 14$ and $8y – x = 2$

D. Show that there is only one solution to each system of equations.

i)
$2x + 6y = 26$
$3y + 8x = – 1$
$10x + 9y = 25$

ii)
$6x + 4y = 14$
$8y – x = 2$
$11x + 16y = 30$

iii)
$6x + 4y = 14$
$8y – x = 2$
$17x – 32y = 18$

E. Explain why there is no solution to each system of equations.

i) $2x + 6y = 26$ and $2x + 6y = 24$ **ii)** $6x + 4y = 14$ and $3x + 2y = 5$

F. Explain why there is no solution to this system of three equations.

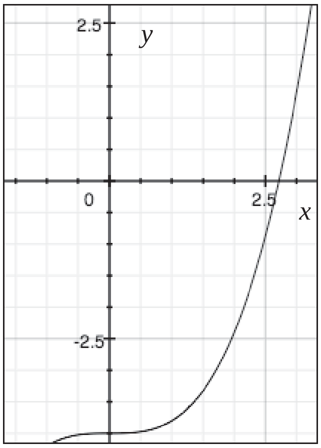
$2x + 6y = 26$ $3y + 8x = – 1$ $x + y = 4$

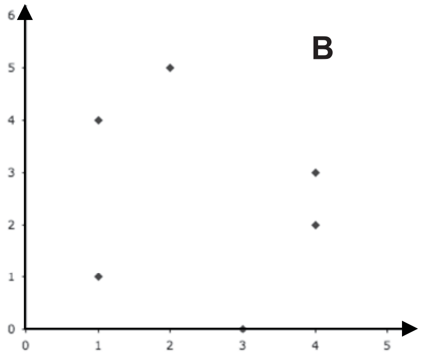
G. Summarize what you have discovered in this exploration.

UNIT 3 Revision

1. Which graph shows a function? How do you know?

A





2. a) Pema says $y = 3$ is not a function. Is Pema right? Justify your answer.

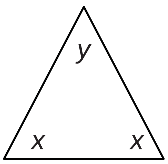
b) Kinley says $x = 3$ is not a function. Is Kinley right? Justify your answer.

3. a) Create a table of values for $f(x) = 10 - 3x$.

b) How do you know it is a function?

4. a) Write x as a function of y.

b) Write y as a function of x.



c) Why can either variable be the independent variable?

d) What is the value of $f(20)$, if y is a function of x?

5. a) Rewrite $10c - 3d = 30$ to describe c as a function of d.

b) Sketch the graph of this function.

6. Rewrite $3x - 2y = 8$ in slope and y-intercept form and graph the line.

7. Write a linear function that describes the mean of five consecutive integers, if x is the second smallest of the numbers.

8. Data values were collected for six students.

Student	Thumb length	Longest finger length
1	8.9	11.4
2	7.6	9.8
3	8.2	10.5
4	8.3	10.7
5	7.3	9.4
6	7.9	9.9

a) Draw a scatter plot a line of best fit.

b) What is the equation of the line?

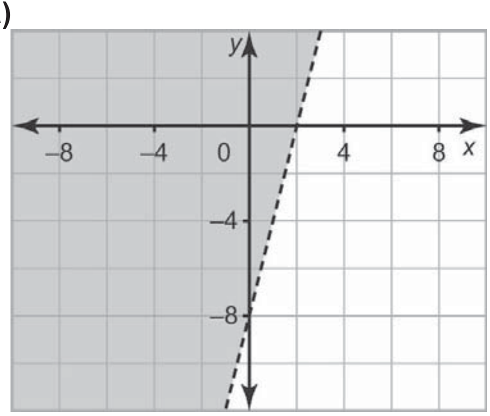
9. Graph each relation.

a) $y \le -3x - 5$ **b)** $3x + 2 \ge y$

c) $2 - 4x < y$

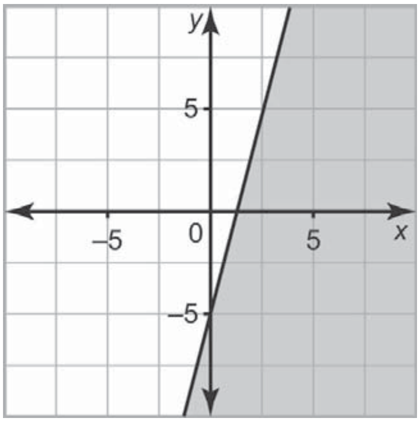
10. Write an inequality for each graph.

a)



[Cont'd]

10. [Cont'd]]
b)



11. Predict how each pair of graphs will be alike and different. Explain your prediction.

a) $3y < 2x + 4$ and $3y \leq 2x + 4$

b) $4y + 2x < 8$ and $4y + 2x < 10$

12. How will the graph of each function compare with the graph of $f(x) = 5x - 3$?

a) $f(x + 4)$

b) $f(x - 4)$

c) $4f(x)$

d) $-4f(x)$

13. Determine the point of intersection of each pair of lines.

a) $y = 5x - 3$ and $y = 4x + 2$

b) $2y + 5x = 10$ and $y = 3x - 0.5$

c) $3y = 5x - 2$ and $y + 4x = 22$

d) $y - 5x = 7$ and $y + 4x = 25$

14. Anjali wrote a 30-item multiple choice exam and answered every question.

- She got 8 points for each correct item.
- She lost 2 points for each incorrect item.
- Her total score was 150 points.

How many items did she answer correctly?

15. The perimeter of one rectangle is 120 cm. Another rectangle with twice the length and one third the width has a perimeter of 170 cm. What are the dimensions of the two rectangles?

16. Use the substitution or comparison strategy to solve each system.

a) $2x + 3y = 17$ and $3y = 15 - x$

b) $3x + 5y = 14$ and $6x - y = -16$

c) $x + 6y = 35$ and $5x - 10 = 3y$

17. Use the elimination strategy to solve each system of equations.

a) $4y - 3x = 11$ and $2y + 3x = 31$

b) $5y + 2x = 24$ and $3y + 4x = 20$

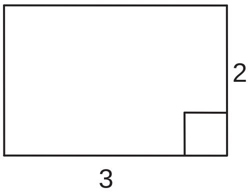
c) $x + 0.2y = 3.1$ and $0.7x - 0.8y = 1.7$

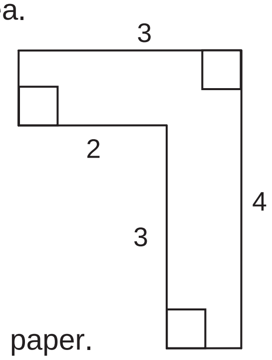
18. When might you choose to use a graph to solve a linear system of equations instead of solving it algebraically?

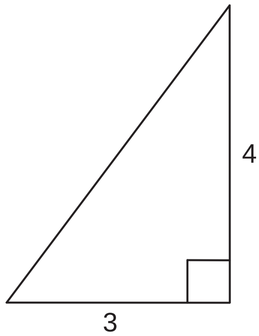
Getting Started

Use What You Know

A. These three shapes can be called **equitable shapes** because they all have the same area.







i) Draw the shapes on grid paper.

ii) Determine the area of each shape in square units to show that they all have the same area.

iii) Predict which shape has the greatest perimeter and which shape has the shortest perimeter. Explain your predictions.

iv) Check your predictions by finding the perimeter of each shape.

B. i) Create three other shapes on grid paper that have the same area as the shapes in **part A**. Label them with their dimensions.

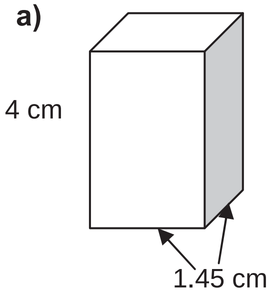
ii) Show that the area of each shape in **part i)** is the same as the area of the shapes in **part A**.

iii) Which of your new shapes has the greatest perimeter? What is its perimeter? How could you have predicted it would be greatest?

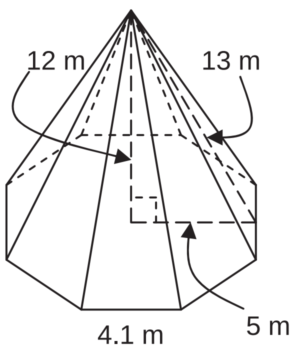
Skills You Will Need

1. Determine the volume and total surface area of each shape. Record your answers to one decimal place.

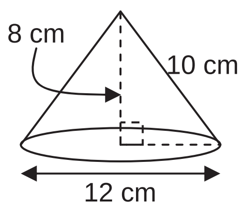
a)



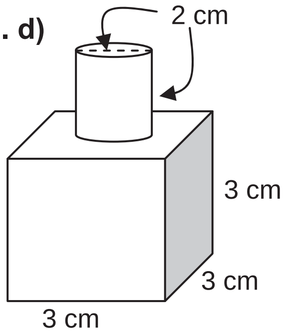
b)



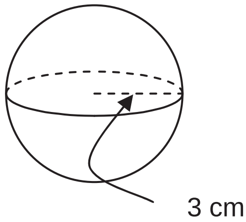
c)



1. d)



e)



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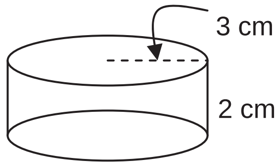


2. Fill in the blanks to make equivalency statements.

a) 32 mm = _____ cm

b) 1.7 kg = __ g

c) 270 mL = ___ L

d) 2 m^2 = ___ cm^2

e) 7 mL = ___ cm^3

f) 4.3 L = ___ cm^3

g) 8.2 cm^3 = ___ mm^3

3. Construct each triangle.

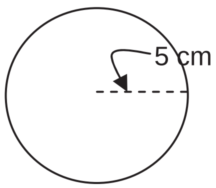
a) ΔABC with AB = 12 cm, BC = 9 cm, and AC = 6 cm

b) ΔDEF with DE = 7.8 cm, EF = 23 mm, and $\angle$E = 110º

c) ΔJKL with JK = 84 mm, $\angle$J = 53º, and $\angle$L = 97º

4. Calculate the circumference and area of each circle. Express your answer as an exact value using π and as a decimal rounded to 1 decimal place.

a)



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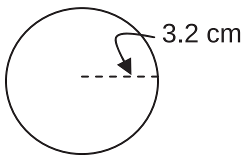
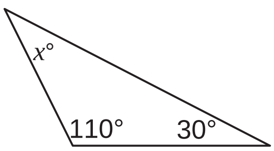


5. What is the value of the missing angle in each triangle?

a)



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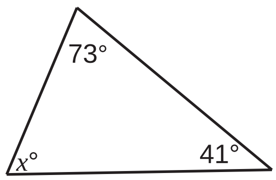
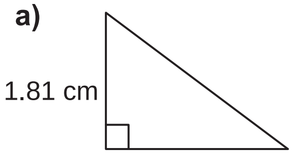


6. Calculate the length of the hypotenuse of each. Round to two decimal places.

a)



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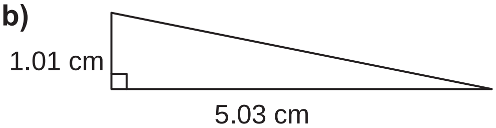

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Chapter 1 Precision

4.1.1 Precision and Accuracy

Try This

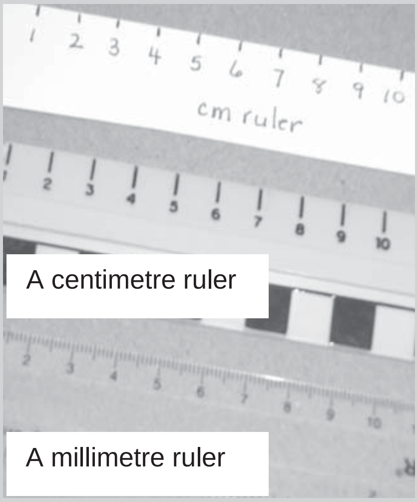
Dorji and Dodo were asked to each measure the dimensions of a textbook. Dorji used a ruler with centimetre markings and Dodo used a ruler with millimetre markings. They then compared their results. Work with a partner and follow the steps below to see what they discovered.

A. Find two rulers, one marked only in centimetres and one with millimetre markings. (If you do not have a centimetre ruler, make one with a strip of paper as shown to the right.)

i) Measure the dimensions (length and width) of your textbook using the centimetre ruler. Record the measurements to the nearest centimetre.

ii) Measure the dimensions of your book using the millimetre ruler. Record the measurements to the nearest millimetre. Then rewrite them in the form [ ].[ ] cm.

iii) How do the results in **i)** and **ii)** compare?

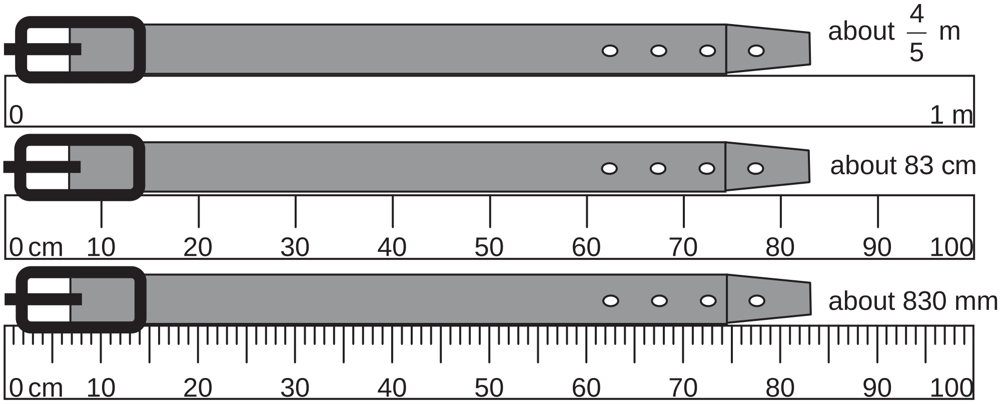


Measuring always involves estimation because, for every unit of measure, there is always a smaller, more precise unit. For example, if three people measure the same desk; one might describe the measurement as 0.4 m, another as 41 cm, and another as 412 mm or 41.2 cm, depending on the tool and unit used. All three are reasonable estimates of the desk's length, even though they differ.

Precision The **precision** of a measurement relates to the scale of the instrument used. The smaller the unit, the more precise the measurement can be.

For example, this belt, when measured with a 1 m stick, is about $\frac{4}{5}$ m long;

when measured with a centimetre ruler, it is about 83 cm; and
when measured with a millimetre ruler, it is about 830 mm, or 83.0 cm.



• Precision also applies to other types of measurement.

For example, a thermometer can have a scale marked in tenths of a degree (3.2ºC) or a less precise scale marked in whole degrees (3ºC).

• When measuring, it is important to think about both the precision required and the availability of tools that will allow for that precision. Some measurements do not need to be very precise. For example, when deciding on what to wear to school, you do not need to know that it is 18.3ºC outside; it is enough to know that it is almost 20ºC. Other times, precision is very important, like when comparing measurements that are close to one another. For example, if two objects are each about 12 cm long, you might discover when you re-measure them with more precision, that one is 11.9 cm (or 119 mm) and the other is 12.0 cm (or 120 mm).

Signficant figures One way to assess the precision of a measurement is to count the number of **significant figures** (sometimes called significant digits) in the number. In the belt example on **page 105**, you can tell 83.0 cm is more precise than 83 cm because it has more significant figures. The abbreviation SF can be used to refer to significant figures.

• To determine the number of significant figures, count all non-zero digits (1 to 9) and all zero digits, except those that are there only to indicate place value.
For example, <u>8</u>0, <u>8</u>00, and <u>8</u>000 each have only one significant figure, the 8. The zeros are not significant because they are there to indicate the place value of the digit 8, that is, whether the 8 is 8 tens, 8 hundreds, or 8 thousands. This is true for numbers in standard form like 80 and numbers in **scientific notation** like 8×10^1.

• The reason a measurement such as <u>8</u>0 cm is has only <u>one significant figure</u> is because it is not possible to tell if the person reporting the measurement was using 80 cm as a rounded value. The actual measurement may have been somewhere between 75 cm and 85 cm. Because this is not clear, scientists choose to not make any assumptions about the precision of the measurement and 80 cm is said to have only one significant figure.

• The reason a measurement such as <u>82</u> cm has <u>two significant figures</u> is because it is obvious the person did not use a rounded value. The actual measurement might be somewhere between 81.5 cm and 82.5 cm, which is a more precise range than 75 cm to 85 cm.

• The reason a measurement such as <u>802</u> m has <u>three significant figures</u> and not two is because it is obvious that the person did not use a rounded value.
A measurement like 802 m indicates that it is closer to 800 m than to 810 m.
The 0 in 8<u>0</u>2 plays a role in indicating this, so it has significance and is counted.

Here are some rules for counting significant figures with whole numbers

- Count all non-zero digits, for example,
<u>5371</u> has <u>four significant figures</u>.

- Count the zeros between non-zero digits, for example,
<u>5001</u> has <u>four significant figures</u>.

- Do not count the zeros to the right of the last non-zero digit, for example,
<u>53</u>00 has <u>two significant figures</u>.

• When decimals are used, zeros after the decimal place are seen as significant. For example, 2.0 cm has two significant figures since the zero indicates the measurement is closer to 2.0 cm than to 1.9 cm or to 2.1 cm, both of which have two significant figures. A measurement of 2 cm has one significant digit because it indicates that it is closer to 2 cm than to 1 cm or to 3 cm, which is a less precise range of measurements than 1.9 cm to 2.1 cm.

Here are some rules for counting significant figures with decimal numbers

- Count all non-zero digits. For example,
both 2.4 and 2.4×10^2 have two significant figures.

- Count all zeros between significant figures. For example,
20.04 has four significant figures. So does 2.004×10^1.

- Count all the zeros to the right of the decimal other than the leading zeros (those that come before the first non-zero digit to the right of the decimal point), if the whole number part of the decimal is 0. For example,
0.0400 has three significant figures. So does 4.00×10^{-2}.

- Count all the zeros to the right of the decimal if the whole number part of the decimal is 1 or greater. For example,
2.0400 has five significant figures, as does 2.0400×10^n, no matter what *n is.*

• Even when the same measurement is reported using different units, the level of precision can still be compared.

For example, one person reports a measurement as 8.2 cm and another reports it as 82 mm. Both are equally precise because they each have two significant figures, even though the units are different.

• Mathematicians often do not worry about the precision of a measurement. They simply recognize that the value they have calculated is not exact and that there will be a margin of error. However, the more precise a measurement is, the smaller that margin of error will be.

• Sometimes people apply calculation rules to ensure that their calculated measurement is as close as possible to the actual measurement. There are different rules for this, but the two that are most often used are described below:

- Round your final calculated measurement to the number of significant figures in the least precise measurement you used in your calculations.

OR

- Round your final calculated measurement to the least number of decimal places in the measurements you used in your calculations.

For example, the area of a 2.4 m by 3 m rectangle calculates to 7.2 m^2:

- round 7.2 m^2 to 1 SF (since 3 m has 1 SF) to get 7 m^2

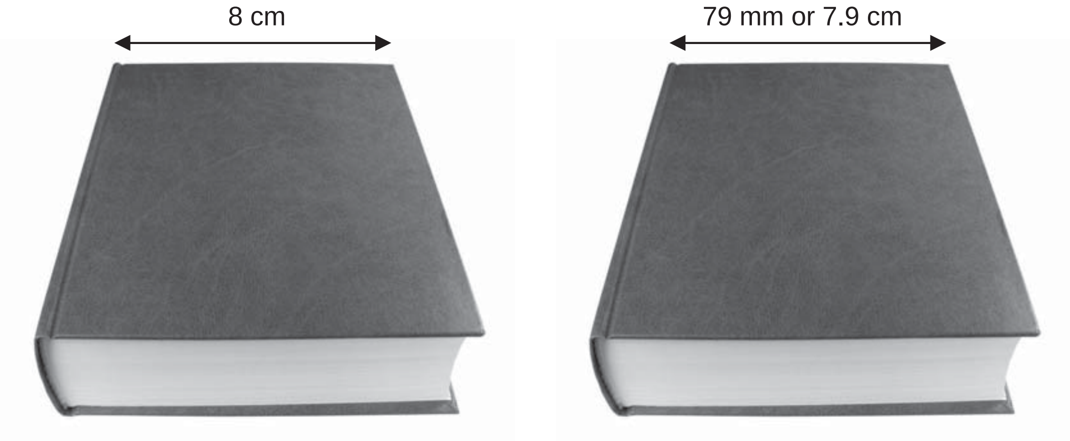
OR

- round 7.2 m^2 to a whole number (since 3 m is a whole number) to get 7 m^2

The two rules do not always result in the same value. In this textbook, the second rule will be used.

Accuracy A measurement is **accurate** when the measurement is taken correctly.

For example, if you say a book is 8 cm wide and a classmate says it is 79 mm or 7.9 cm wide, your measurements can both accurate, as long as you both measured from one side of the book along a straight distance to the other side and you both read your rulers correctly. The difference is that your classmate measured with more precision than you did, probably because he or she had a more precise tool.



Both measurements are accurate but one is more precise than the other.

• Inaccurate measurements can be caused by carelessness in using the tool or in reading or interpreting the tool.

• Inaccuracy also occurs in situations where it is difficult to measure accurately.

For example, it is difficult to measure the length of your forearm since it is hard to decide where to start and end the measurement. If two people each measured their forearms, it is unlikely that they would both do it in exactly the same way.

• Precision and accuracy often get confused, perhaps because of how the words are used in everyday language. It is important to keep in mind that a measurement is either accurate or not, while there are different levels of precision.

B. Based on your results from **part A**, what do you think Dorji and Dodo discovered about the precision of their measurements?

Examples

Example 1 Counting Significant figures

How many significant figures are there in each number?

89 7.5×10^3 2.03×10^2 120.0 0.005 0.90 4.03

Solution	Thinking
	Counting significant figures is mostly about understanding the role or significance of the zero digits in a number. 
89 has two significant figures.	• 89 has two non-zero digits so we count both of them.
7.5×10^3, which is 7500, has two significant figures.	• The zeros in 7500 don't count because they are there only to show the place value of the digits 7 and 5. The power in a number in scientific notation has no effect on how many significant figures it has.
2.03×10^2, which is 203, has three significant figures.	• The zero in 2.03 counts because it is between non-zero digits. The power 10^2 in 2.03×10^2 doesn't count.
120.0 has four significant figures.	• The decimal at the end of 120.0 is there to show that the measurement was taken to the nearest tenth, so it should be counted. • The 0 before the decimal counts since it is between significant figures. • The 2 is significant because it's non-zero. • The last 0 is significant because it's after the decimal point.
0.005 has one significant figure.	• The digit 5 in 0.005 means 5 thousandths. The only significant digit is 5, as the zeros simply tell the place value of 5 (thousandths).
0.90 has two significant figures.	• The zero to the right of the 9 in 0.90 is included to show the level of precision, not the place value of the digit 9, so it counts.
4.03 has three significant figures.	• The zero in 4.03 counts because it is between non-zero digits.

Example 2 Reading measurements

Dechen weighs herself on a scale and sees this result.

a) What possible measurements might result from the situation?

b) What are some possible causes of inaccuracy?

Solution

a) She might say that she weighs about 44 kg, 45 kg, or 40 kg.

b) The scale may not have been set properly (some scales have to be set to zero first).

She might not be putting her full weight on the scale.

She may not be in the correct position when reading the scale to distinguish whether the line is closer to 44 kg or to 45 kg.

Thinking

a) Depending on how precise she wants to be, she might round to the nearest 1 kg (44 kg or 45 kg) or to the nearest 10 kg (40 kg).

b) Inaccuracies are often the result of not using or reading the measuring tool correctly.

Example 3 Considering accuracy and precision

A group of boys wondered who was the fastest in their group. They timed each other running from the school to the flagpole and back. They could not all run at the same time so they had to rely on reported times to decide who won the race.

Dorji 41 s Penjor 36 s Devi 41 s Pema 37.0 s Dago 35.8 s

They ranked the times from fastest to slowest: 35.8 s, 36 s, 37.0 s, 41 s, 41 s

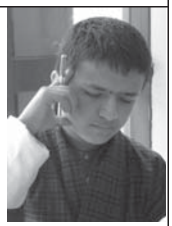
a) Based on what you know about precision, can you be certain that Dago was the fastest? that Devi and Dorji were equally slow?

b) What are some possible causes of inaccuracy in this situation?

Solution

a) You cannot be certain that Dago was fastest because it is possible that Penjor was faster, since his time of 36 s might have been rounded up from 35.5 s, 35.6 s, or 35.7 s, which is faster than Dago at 35.8 s.

Thinking



a) There are different numbers of significant figures in the measurements — some are to the nearest tenth of a second (3 SFs) and some only to the nearest second (2 SFs).

• Penjor's time was to the nearest second, so his time could have been anywhere between 35.5 s and 36.5 s.

You cannot be certain that Devi and Dorji were equally slow since 41 s could be anywhere from 40.5 s to 41.4 s. That means Dorji could have had a time as fast as 40.5 s and Devi could have had a time as slow as 41.4 s, or vice versa.	Both 40.5 s and 41.4 s round to 41 s, when rounded to the nearest whole second.
b) One thing that would cause inaccuracy is if the stopwatch was not started and stopped correctly.	**b)** Inaccuracy can be caused by human and mechanical errors.

Practising and Applying

1. Count the significant figures in each.

a) 5.380×10^3 mL **b)** 8081 km

c) 98,070 cm^3 **d)** 1.24×10^1 cm

e) 0.6 L **f)** 0.04 km

g) –20º **h)** 0.0930 kg

i) 0.4030 m^2 **j)** 2×10^{-3} km

2. Round each number as indicated.

a) 0.837 to one significant figure

b) 4712 to two significant figures

c) 3.19 to two significant figures

3. a) Write two different lengths in millimetres that could be written as 12 cm when recorded to the nearest centimetre.

b) Write two different capacities in millilitres that could be written as 7.3 L when recorded to the nearest tenth of a litre.

4. a) Write a number that has three zeros and three significant figures.

b) Is it possible to write a five-digit number for **part a)**? Explain.

c) Write a number that has seven digits but only one is a significant figure.

d) What is the greatest and least number you could write for **part c)**?

5. What measurement might result from measuring the length of this calculator? Comment on the precision and accuracy in the measurement.

6. Novin says the distance from Thimphu to Paro is 65 km. Dodo says the distance from Thimphu to Wangdi Phodrang is 70 km.

a) Can you be certain that Paro is closer than Wangdi Phodrang to Thimphu? Explain.

b) Comment on the precision and accuracy of these measurements.

7. a) Describe a situation in which two capacity measurements are difficult to compare because they have different levels of precision.

b) Explain why it is difficult to compare them.

CONNECTIONS: Precision Instruments

People have been measuring things for thousands of years. Part of the development of human culture includes the development of increasingly precise instruments for measurement.

• Sundials have been used for many years to tell time. Later, mechanical devices, such as analogue watches, were invented to measure time with greater precision.


Sundial


Analogue watch

• Instruments to describe location have become more precise. Sextants, devices used in the past to estimate one's latitude based on the angle required to sight the moon, were considered precise instruments at the time, but new global positioning system (GPS) receivers are much more precise.


Sextant


GPS receiver

• A pan balance, used in many stores for measuring the mass of items, is a more precise measuring instrument than the common balance used at the market.


Pan balance


Common balance

1. Look around your classroom, home, and community for measuring instruments. For each, describe what it measures and comment on its level of precision.

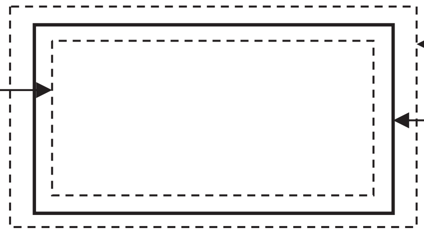
4.1.2 EXPLORE: Measurement Error

Even when you measure an object as precisely as you can, there is some margin of error since it is always possible to measure more precisely. When you calculate with any measurement, the margin of error increases.

For example:

Measuring to the nearest centimetre

The rectangle below is 32 cm long × 14 cm wide, measured to the nearest centimetre. Because the measurements are to the nearest centimetre, the actual measurements could be anywhere from 31.5 cm × 13.5 cm to 32.5 cm × 14.5 cm (as shown by the dashed rectangles).



This possible range of measurements increases when you calculate the area of the rectangle, as shown below:

Calculated area of rectangle, using 32 cm × 14 cm, is 448 cm^2

Least possible area of rectangle, using 31.5 cm × 13.5 cm, is 425.25 cm^2

Greatest possible area of rectangle, using 32.5 cm × 14.5 cm, is 471.25 cm^2

This means the actual area could be anywhere from 425.25 cm^2 to 471.25 cm^2.

Expressing the margin of error as a percentage of the calculated value helps show how significant the margin is. You can calculate the possible error as a percent using the greatest or least possible area measurement:

471.25 cm^2 – 448 cm^2 ≈ 23 cm^2, which is about 5% of 448 cm^2 (23 ÷ 448 ≈ 0.05)
448 cm^2 – 425.25 cm^2 ≈ 23 cm^2, which is also about 5% of 448 cm^2
This means the actual area could be greater or less than 448 cm^2 by about 23 cm^2 or 5%.

Measuring to the nearest ten centimetres

If the rectangle had been measured to the nearest ten centimetres, the margin of error would have been even greater:

Calculated area of rectangle, using 30 cm × 10 cm, is 300 cm^2

Least possible area of rectangle, using 25 cm × 5 cm, is 125 cm^2

Greatest possible area of rectangle, using 35 cm × 15 cm, is 525 cm^2

This time the actual area could be greater or less than 300 cm^2 by about 225 cm^2 or 75%.

• Factors other than precision play a role in affecting the margin of error, for example, the size of the measurements and the type of calculations.

• If you are not told to what precision a measurement is taken, you can use what you know about significant figures and precision to make assumptions.

For example:

- If a measurement is reported to be 30 cm, you can assume, because it has one significant figure, that it was rounded to the nearest ten centimetres.
- If a measurement is reported to be 31 cm, you can assume, because it has two significant figures, that it was rounded to the nearest centimetre.

A. Investigate the margin of error in each measurement situation below by finding the calculated value, the least possible value, and the greatest possible value. Use these values to calculate the possible percent error.

	Shape	Measured dimensions	Measurement to calculate
i	rectangle	40 cm by 20 cm	perimeter
ii	rectangle	40 cm by 20 cm	area
iii	rectangle	140 cm by 120 cm	perimeter
iv	rectangle	140 cm by 120 cm	area
v	rectangle	42 cm by 21 cm	perimeter
vi	rectangle	42 cm by 21 cm	area
vii	rectangular prism	40 cm by 20 cm by 10 cm	total surface area
viii	rectangular prism	40 cm by 20 cm by 10 cm	volume
ix	rectangular prism	140 cm by 120 cm by 110 cm	total surface area
x	rectangular prism	140 cm by 120 cm by 110 cm	volume
xi	rectangular prism	21 cm by 31 cm by 11 cm	total surface area
xii	rectangular prism	21 cm by 31 cm by 11 cm	volume

B. Compare your results from each of these pairs from **part A** to decide whether the size of the measurements affects the margin of error. Describe your findings.

i and **iii**   **ii** and **iv**   **vii** and **ix**   **viii** and **x**

C. Compare your results from each of these pairs from **part A** to decide whether the level of precision affects the margin of error. Describe your findings.

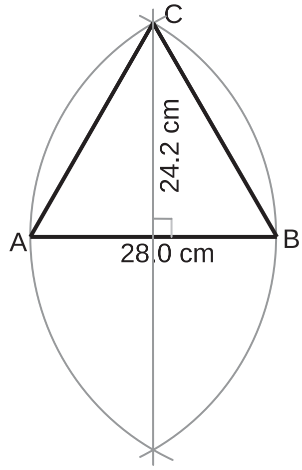
i and **v**   **ii** and **vi**   **vii** and **xi**   **viii** and **xii**

D. What would you consider to be a reasonable percentage of measurement error? Explain.

4.2.1 EXPLORE: Regular Polygons with a Constant Perimeter

These instructions show how to draw an equilateral triangle with a perimeter of 84.0 cm and then find its area:

- Calculate the side length: 84.0 ÷ 3 = 28.0 cm
- Draw one 28.0 cm side (base) of the triangle, AB.
- Use a compass to mark vertex C, which is 28.0 cm from vertex A and 28.0 cm from vertex B.
- Draw side lengths AC and BC (each 28.0 cm).
- Construct a line through C that is perpendicular to AB and use it to measure the height of the triangle.
- Calculate the area of the triangle:

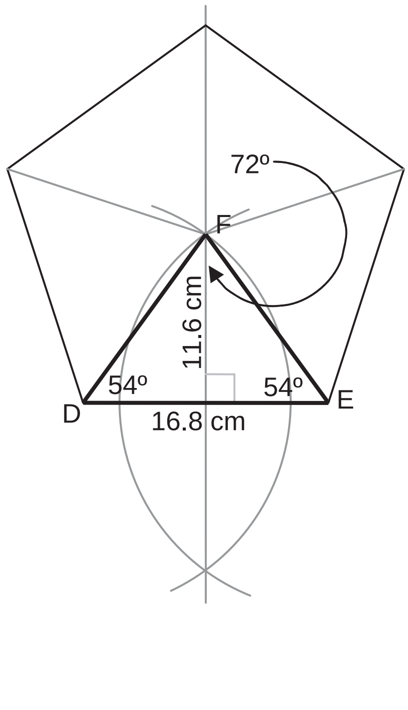


$$A = \frac{1}{2}bh = \frac{1}{2}(28.0)(24.2)$$
$$= 338.8 \text{ cm}^2$$
$$\approx 339 \text{ cm}^2 \text{ (rounded to 3 SFs)}$$

These instructions show how to find the area of a regular pentagon with a perimeter of 84.0 cm:

- Calculate the side length: 84.0 ÷ 5 = 16.8 cm
- A regular pentagon is made up of five congruent isosceles triangles, so consider one triangle:

- The angle of the triangle that is at the centre of the pentagon is 360º ÷ 5 = 72º and each other angle is (180º – 72º) ÷ 2 = 54º.
- Draw one 16.8 cm side of the triangle, DE, and 54º angles on each endpoint.
- Use the 54º angles to draw side lengths DF and EF.
- Construct a line through F perpendicular to DE. Use it to measure the height of the triangle.
- Calculate the area of the triangle:



$$A = \frac{1}{2}bh = \frac{1}{2}(16.8)(11.6) = 97.44 \text{ cm}^2$$

- The pentagon is made up of five of these triangles so the area of the pentagon is

$$5 \times 97.44 = 487.2 \text{ cm}^2$$
$$\approx 487 \text{ cm}^2 \text{ (rounded to 3 SFs)}$$

A. Use the instructions on **page 115** to find the areas of other regular polygons, each with a perimeter of 84.0 cm. Record your results in a chart.

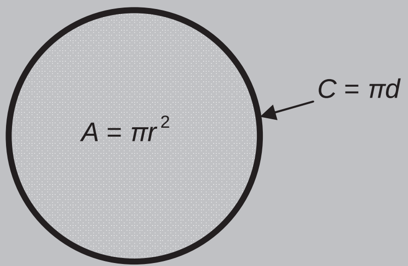
Polygon	Triangle 3 sides	Square 4 sides	Pentagon 5 sides	Hexagon 6 sides	...	Decagon 10 sides
Area	339 cm^2		487 cm^2			

B. Graph the information from the chart with the number of sides as the independent variable and the area as the dependent variable.

C. What do you notice about the area, as the number of sides increases?

D. i) Use your observation in **part C** to predict the area of a circle with a perimeter, or circumference, of 84.0 cm. Explain your prediction.

ii) Calculate the area of a circle with a perimeter of 84.0 cm. Was your prediction in **part D i)** reasonable?



E. How is a circle like a regular polygon?

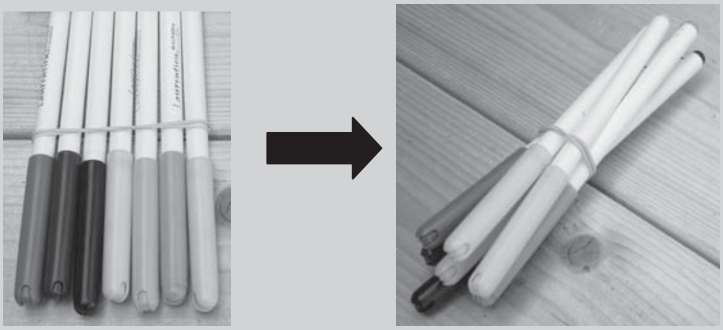
F. Do you think there is another shape with a perimeter of 84.0 cm that has an area greater than the area of the circle? Explain.

G. What shape has the greatest area for a given perimeter?

4.2.2 2-D Efficiency

Try This

Seldon had to carry seven markers home. She laid them out in a row and put an elastic around them. The elastic contracted and the pens snapped together.



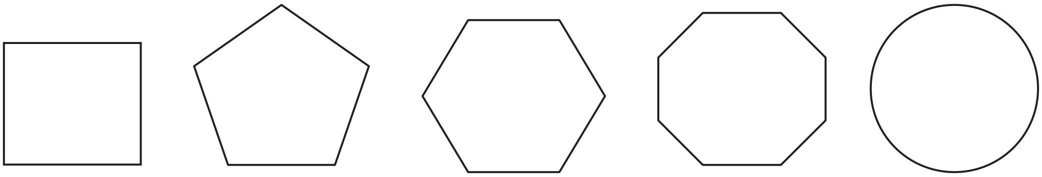
A. The diameter of one marker is 6 mm. Estimate the length of the elastic when the pens were laid out in a row.

B. After the pens were snapped together, the elastic formed a roughly-shaped circle with a diameter of about 20 mm. Estimate the length of the elastic now.

• When you form a boundary or perimeter around an area, it is often beneficial to enclose as much area as possible.

For example, if you have a certain amount of fencing material for a garden, you might want to set up the fence in such a way as to maximize the area of the garden. This is an efficient use of the fencing material because it costs less money and takes less time to build it, and you end up with the largest garden possible.

• As a shape with a constant perimeter becomes increasingly circular, the enclosed area increases.



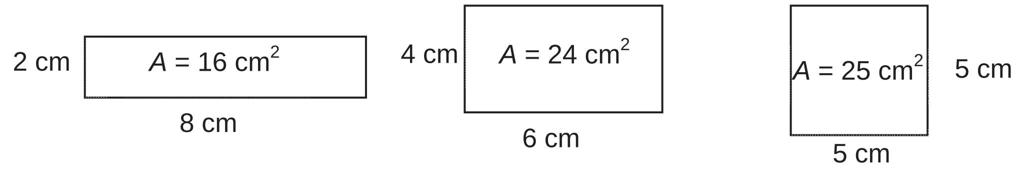
If the perimeter remains constant, the area increases as the number of sides increases.

• For **two-dimensional shapes**, or **2-D shapes**, a circle is the most efficient shape for enclosing the maximum area for a given perimeter. This also means that if you enclose a certain area, the shortest possible perimeter would be circular.

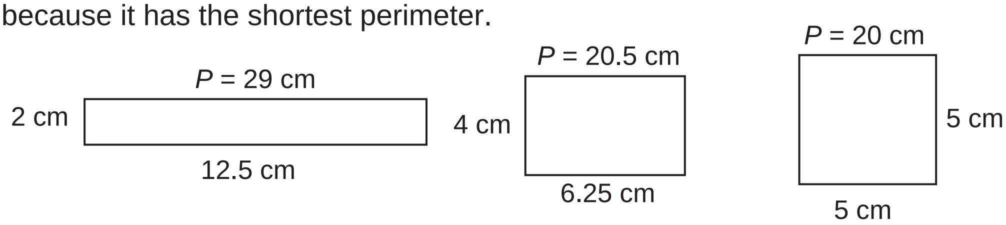
• You can use this knowledge about efficiency to help you compare the efficiency of various shapes.

• For certain classifications of polygons, the shape that is most like a circle is the most efficient shape. For example, a square is the most efficient rectangle.

- For rectangles with a perimeter of 20 cm, the square is the most efficient because it has the greatest area.



- For rectangles with an area of 25 cm^2, the square is the most efficient because it has the shortest perimeter.

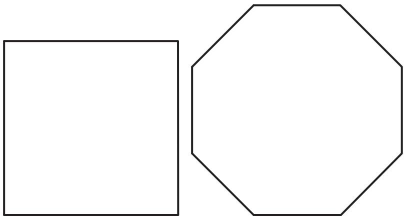


C. How does what you observed with the elastic in **parts A and B** relate to the notion of a circle being the most efficient shape?

Examples

Example 1 Comparing the Areas of Two Same-Perimeter Polygons

Both of these regular polygons have the same perimeter. Predict which polygon encloses more area, explain your prediction, and then check your prediction.



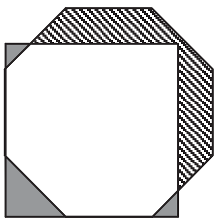
Solution 1

Prediction

I predict the octagon will have the greater area because all the angles are obtuse, making it more circle-like than the square, which has all 90º angles.

Check

The octagon has the greater area.



Thinking

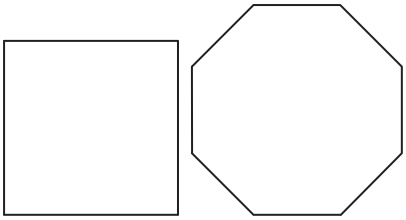
To check, I superimposed one on the other:

- The white area is what they have in common, so I only had to compare the striped and grey areas.
- The striped area of the octagon has a greater area than the grey area of the square, so the octagon has the greater area.

Solution 2

Prediction

I think the octagon has the greater area because it just looks larger and is more circular.



Check

The area of the square is 4.8 cm^2.

The area of the octagon is 5.7 cm^2.

The octagon has the greater area.

Thinking

To check, I calculated each area:

- For the square, I measured one side length and squared it:

$$2.2^2 = 4.84, \text{ or } 4.8 \text{ cm}^2$$

- For the octagon, I thought of it as eight congruent triangles, found the area of one triangle, and then multiplied by 8:

- To find the area of one triangle, I found its base by dividing the perimeter by 8 (8.8 ÷ 8 = 1.1 cm), and then measured its height, 1.3 cm. I then used the formula for the area of a triangle:

$$A = \frac{1}{2}bh = \frac{1}{2} \times 1.1 \times 1.3 = 0.715 \text{ cm}^2$$

- To find the octagon's area, I multiplied the triangle's area by 8:

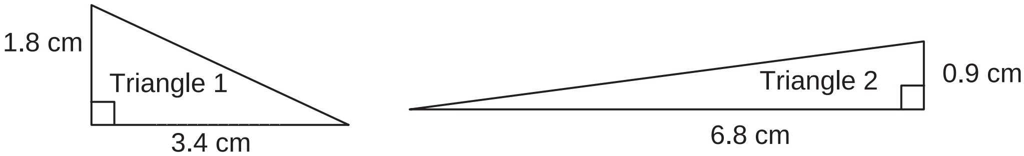
$$8 \times 0.715 = 5.72, \text{ or } 5.7 \text{ cm}^2$$

Example 2 Comparing the Areas of Two Same-Perimeter Triangles

These triangles have the same area but different perimeters.

a) Predict which triangle has the shorter perimeter. Explain your prediction.

b) Check your prediction. Show your work.



Solution

a) I think Triangle 1 has a shorter perimeter because it is more circle-like.

b) $P_1 = 1.8 + 3.4 + \sqrt{1.8^2 + 3.4^2}$

$\approx 1.8 + 3.4 + 3.8$

$= 9.0$ cm

$P_2 = 6.8 + 0.9 + \sqrt{6.8^2 + 0.9^2}$

$\approx 6.8 + 0.9 + 6.9$

$= 14.6$ cm

Triangle 1 has a shorter perimeter.

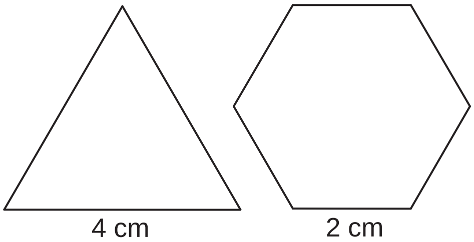
Thinking

a) For a given perimeter, the more circular a shape, the greater the area. So it makes sense that, for a given area, the more circular the shape, the shorter the perimeter.

b) I used the Pythagorean theorem to find each hypotenuse so that I could calculate the perimeters.

Practising and Applying

1. Predict which regular polygon has the greater area. Check your prediction.

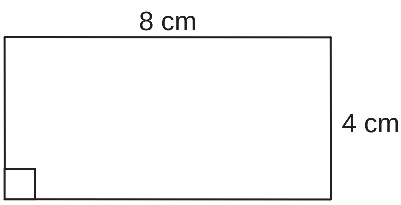


2. Sketch two different hexagons, each with a perimeter of 36 cm. Which has the greater area? How do you know?

3. a) Give the dimensions of two other non-square rectangles, each with the same area as the one shown below but with a shorter perimeter.

b) Explain your strategy for finding rectangles to meet the condition in **part a)**.

4. a) Give the dimensions of two other quadrilaterals, each with the same perimeter as the one shown below but with a smaller area. Only one of the quadrilaterals can be a rectangle and it cannot be a square.



b) Explain your strategy for finding rectangles to meet the condition in **part a)**.

5. Therchu is buying material to fence a new garden. He buys two strands of barbed wire and a post for every 3 m of fence. Each post costs Nu 300 and barbed wire costs Nu 20 per metre. What is the shape of the largest area Therchu can enclose if he spends Nu 3000 on fence material? Show your work.

6. a) Drakpa is building a rectangular table with an area of 21,000 cm^2. He wants to put wood trim around the four edges. What is the shortest length of trim he could use?

b) How much less trim would he need if the table were round and the trim was flexible?

7. A right triangle has a perimeter of 100 cm.

a) What must be the length of each the three sides for the triangle to have the maximum area possible?

b) How do you know?

8. Can two regular polygons with different numbers of sides have the same area and perimeter? Explain.

9. Describe one situation when you would want to do each.

a) maximize the area for a given perimeter

b) minimize the perimeter for a given area

4.2.3 3-D Efficiency

Try This

A farmer plans to use some insulation panels to construct the walls of a fruit storage room. He wants the room to store as much fruit as possible.

He considers two shapes for the room:

Option 1 a cube with edges 4.0 m long

Option 2 a rectangular prism that is 1.8 m by 2.4 m by 10.4 m

A. i) Find the total surface area of the inside walls of each room option.

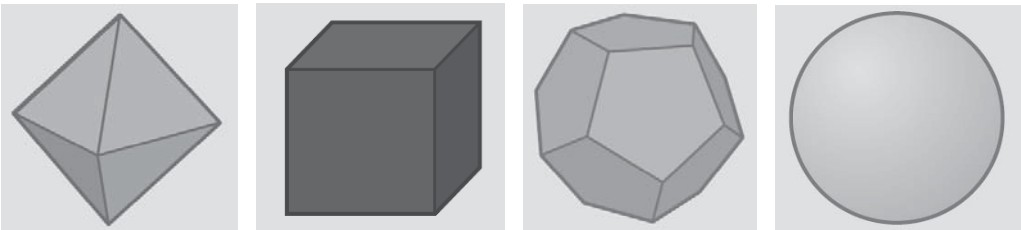
ii) Find the greatest volume of fruit that can be stored in each room.

B. Which option should he choose? Explain.

• When enclosing space, it is often beneficial to have as much capacity or volume as possible with the least total surface area.

For example, if you are manufacturing cans for canning vegetables and you want to keep your costs down, you would want to make a can using as little metal as possible to hold a given volume of vegetables.

• With 2-D shapes, if the perimeter remains constant but becomes increasingly circular, the area increases. It is similar with **three-dimensional shapes**, or **3-D shapes** — if the total surface area remains constant, the capacity or volume increases as the shape becomes increasingly spherical.



If the total surface area remains constant, the volume/capacity increases as the shape becomes more spherical.

• You can use this knowledge to help you compare the efficiency of shapes:

- For shapes with the same total surface area, the most spherical shape will have the greatest capacity or volume and is therefore most efficient.
- For shapes with the same capacity or volume, the most spherical shape will have the least total surface area and is therefore most efficient.

• When designing containers, efficiency of shape is only one consideration. Though spherical containers use the least material, they are difficult to make and transport, they are not practical because they roll in every direction, and they are not efficient to pack. For this reason, cylinders and cubes make better containers.

- A cube is the most efficient rectangular prism because it is most like a sphere.
- A cylinder with a height equal to its diameter is the most efficient cylinder because it is most like a sphere.

C. Explain how you could have answered **part B** without calculating.

Examples

Example 1 Exploring the Efficiency of Cylinders

Determine the dimensions of the most efficient cylindrical tin can that will hold 480 mL.

Solution

Convert capacity to volume

480 mL → 480 cm^3

Try a diameter of 20 cm

$r = 20 \div 2 = 10$ cm

$$V = \pi r^2 h$$
$$480 = \pi(10)^2 h$$
$$480 = 314h$$
$$1.53 \text{ cm} = h$$

$$SA = 2\pi r^2 + 2\pi rh$$
$$= 2\pi(10)^2 + 2\pi(10)(1.53)$$
$$= 724 \text{ cm}^2$$

d (cm)	h (cm)	SA (cm^2)
20.0	1.53	724
10.0	6.11	349
8.0	9.55	341
7.0	12.47	351
7.5	10.86	344
8.5	8.46	339
8.4	8.66	339
8.6	8.26	339

The can should have a diameter of 8.5 cm and a height of 8.5 cm.

Thinking

- I expected the result would be the cylinder that looked most like a sphere — a cylinder with a height equal to its diameter.
- I used trial and error and organized my results using a chart.
- My calculations each time were basically the same, so I only showed my work for the first trial using a diameter of 20 cm (I used 3.14 for π).
- The total surface area (SA) decreased until I reached a diameter of 7, so I knew the diameter was greater than 7 cm.
- I tried different values for the diameter. It looked like 8.5 cm was about right because I tried a diameter a little larger and a little smaller and the total surface area was the same (after rounding).
- The can's diameter and height are about the same, so my conjecture was correct.

Example 2 Exploring the Efficiency of Square-Based Prisms

A rectangular prism box has a total surface area of 1200 cm^2.
What is the maximum capacity the box can hold?

Solution

1200 cm^2 ÷ 6 = 200 cm^2

$e = \sqrt{200}$ cm

$V = e^3 = (\sqrt{200})^3$
$= 2828.427128$
≈ 2800 cm^3

2800 cm^3 → 2800 mL

The box will have a capacity of 2800 mL.

Thinking

• I knew the box had to be a cube because it is the most efficient rectangular prism.

• To find its capacity,

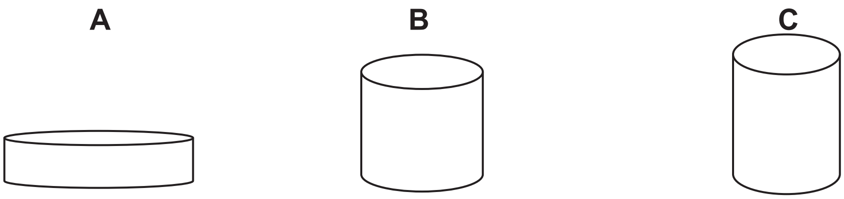
- I divided the surface area by 6 to find the area of each face, since a cube has 6 congruent faces.
- I calculated the square root of the area to find the edge length (e), since each face is a square.
- I cubed the edge length to find the volume (V), since $V = e^3$.
- I converted cm^3 to mL, since 1 cm^3 = 1 mL.

Example 3 Calculating Total Surface Area to Volume Ratios

The surface area to volume ratio can be used to determine the most efficient shape for a given volume or surface area.

a) Find the ratio for each cylinder below. Which is least efficient? Explain.

b) Why might this information be useful?



A	B	C
V = 196 cm^3	V = 196 cm^3	V = 196 cm^3
SA = 220 cm^2	SA = 190 cm^2	SA = 196 cm^2

Solution

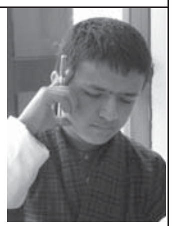
a)

Cylinder	*SA*	*V*	*SA:V*
A	220	196	1.12
B	190	196	0.97
C	196	196	1.00

Cylinder A is least efficient because it has the greatest total surface area to volume ratio.

b) A company that manufactures cylindrical containers may want to know which dimensions result in a container that holds the most but uses the least material.

Thinking



• A smaller ratio means that less SA is required for the same V.

• A greater ratio means that more SA is required for the same V.

• This ratio only works for comparing efficiency of shapes when the SA is the same or the volume is the same.

Practising and Applying

1. Four cylindrical barrels each have a capacity of 100 L but have different diameters and heights. Determine the height of the barrel for each diameter.

a) 40.0 cm

b) 45.0 cm

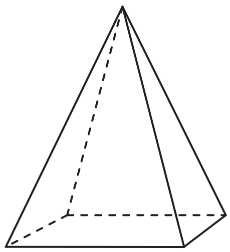
c) 50.0 cm

d) 55.0 cm

2. Use your results from **question 1** to predict which barrel has the least total surface area and then explain your prediction. Check your prediction by calculating the total surface areas.

3. Which barrel from **question 1** has the least total surface area to volume ratio? How do you know?

4. Determine the dimensions of the most efficient square-based pyramid with a capacity of 320 mL.



5. Four square-based prisms each have a total surface area of 96 cm^2. What is the height of a prism with each square base?

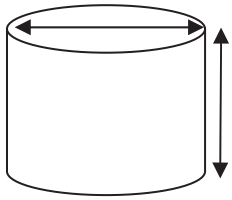
a) 3.0 cm × 3.0 cm

b) 4.0 cm × 4.0 cm

c) 5.0 cm × 5.0 cm

d) 6.0 cm × 6.0 cm

7. a) Determine the total surface area of a cylinder with a diameter of 24 cm and a height of 24 cm.



b) Determine the radius of a sphere with the same total surface area.

c) How much greater (as a percent) is the volume of the sphere?

8. a) Determine the edge length of a cube with the same total surface area as the cylinder and the sphere in **question 7**.

b) Predict how the volume of the cube relates to the volume of the cylinder and the sphere.

c) Calculate the volume to check your prediction.

9. Sonam is using fabric to cover a rectangular prism box. The box has a volume of 0.7 m^3 and the fabric costs Nu 50 for 1 m^2. What is the least it could cost to cover the box with fabric?

10. a) Determine the volume and total surface area of a cylinder with a diameter of 16 cm and a height of 16 cm.

b) Determine the total surface area of a sphere with the same volume.

c) Which shape has the lesser total surface area to volume ratio?

d) Why might you have predicted this?

11. Explain how the total surface area to volume ratio can be used to compare the efficiency of shapes.

CONNECTIONS: Animal Shapes and Sizes

Animals have many different shapes and sizes for many reasons. For example, the shape and size of an animal significantly affects the kinds of conditions in which it can survive. If an animal has a large volume and a small total surface area, it can maintain its body temperature relatively well because there is less total surface area for body heat to escape through.

You have explored how more spherical shapes are more efficient. This helps explain why in colder regions we find more rounded or spherical animals (such as the yak) and in hotter regions we find animals that are thinner (such as the impala, an African animal, which is shown here).


Rounded shapes keep animals like yaks warm; thin shapes keep animals like impalas cool

Of course, there are other features of animals that help with temperature. Hair helps insulate. Wrinkles or extra skin can add surface area to round animals living in hot temperatures, for example, elephants.

Size is another feature that influences the ratio between total surface area and volume. In two animals that have the same shape, the total surface area to volume ratio is greater in the smaller animal, thus making the smaller animal more vulnerable to losing its body heat.

Baby elephants have a greater *SA* to *V* ratio

Recall the surface area and volume formulas for a sphere, $SA = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

1. Experiment with the effect of size on the ratio of total surface area to volume by comparing different spheres. Find the total surface area and volume for the spheres described in the chart and then calculate the ratio.

Radius (cm)	20.0 cm	10.0 cm	5.0 cm	2.0 cm	1.0 cm	0.5 cm
Total surface area (cm^2)						
Volume (cm^3)						
Total surface area ÷ volume (*SA*:*V* ratio)						

2. Complete a similar chart for cubes using edge length instead of radius.

UNIT 4 Revision

1. How many significant figures are in each number?

a) 4.7×10^2 **b)** 0.05000 **c)** 4.03

2. Write a number for each.

a) less than 100 with 3 SFs

b) greater than 1 million with 2 SFs

c) less than 0.5 with 4 SFs

3. Dawa says his mass is 70 kg. Nima claims to weigh 67 kg. Can Nima conclude that he is lighter than Dawa? Explain. Explain using the concepts of precision and accuracy in your explanation.

4. Meto measured the side of a triangle to be 73 cm using a metre stick that measures to the nearest centimetre.

a) If she had used a metre stick that measured to the nearest 10 cm, what might she have found the length to be?

b) If she had used a metre stick that measured to the nearest millimetre, what might she have found the length to be?

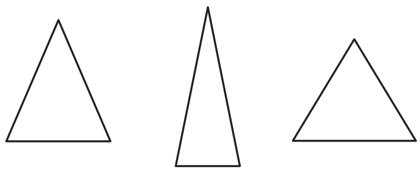
5. You are told that a rectangle has measurements of 20 cm by 34 cm.

a) Why is the area not likely to be exactly 680 cm^2?

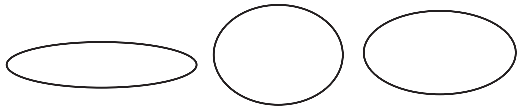
b) What is the least area the rectangle could have if the measurements are accurate?

c) What is the greatest area the rectangle could have if the measurements are accurate?

6. Without measuring, predict which of these shapes with the same perimeter would have the greatest area. Explain.



7. Without measuring, predict which of these shapes with the same area would have the shortest perimeter. Explain.



8. a) Determine the area of an equilateral triangle with perimeter 24 cm.

b) Determine the area of a square with perimeter 24 cm.

c) Explain why the area of one shape in **parts a) and b)** is greater than the other.

d) Describe another shape with a perimeter of 24 cm, but with an area greater than either the square or the triangle.

9. It costs Nu 20 for each metre of border edge for a rectangular area. What is the greatest area Chencho can enclose by spending Nu 4500?

10. Four cylinders each have a total surface area of 180 cm^2 but different dimensions. Determine the height of the cylinder with each diameter.

a) 5.0 cm **b)** 6.0 cm

c) 7.0 cm **d)** 8.0 cm

11. Use your results from **question 10** to predict which cylinder has the greatest volume. Explain your prediction. Calculate the volumes to check your prediction.

12. a) Determine the volume of a sphere with a diameter of 1.30 m.

b) Determine the dimensions of a cube with the same volume as the sphere.

c) Predict which of the above shapes has the greater total surface area. Explain your prediction.

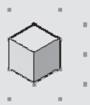
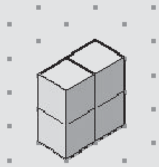
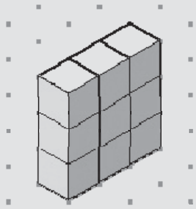
d) Calculate the total surface areas to check your prediction.

UNIT 5 NON-LINEAR FUNCTIONS AND EQUATIONS

Getting Started

Use What You Know

Imagine joining cubes to create square walls of various sizes and then painting the outside surface of each wall, including the bottom.

1-cube high wall	2-cube high wall	3-cube high wall
 6 painted cube faces	 16 painted cube faces	 30 painted cube faces

A. Complete a table of values to show the area of the painted surface for each wall up to a 5-cube high wall.

Height of wall (number of cubes)	Area of painted surface (number of painted cube faces)

B. i) Calculate the first and second differences for the values in the table.

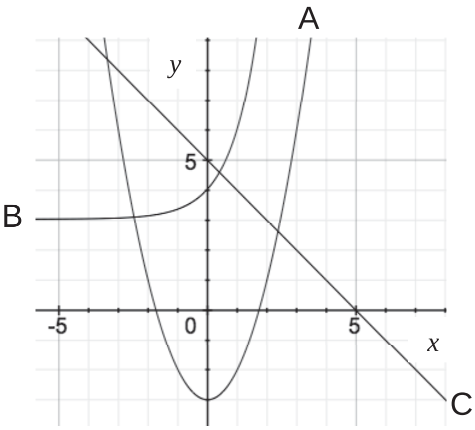
ii) How can you tell from the table that there is a quadratic relationship between the area of the painted surface and the height of the wall?

C. i) Write an equation that describes the relationship in **part B ii)**.

ii) How can you tell from the equation that it is a quadratic relationship?

Skills You Will Need

1. Which of the graphs to the right, A, B, or C, represents a quadratic relation?
How do you know?



2. Use $f(x) = 3x + 2$ to determine each.

a) $f(-3)$

b) $-3f(x)$

c) $f(x) + 2$

d) $f(x + 2)$

3. Solve each equation.

a) $2x - 8 = 0$

b) $2x + 3 = 7$

c) $3 - 4x = 5 + 3x$

4. Determine the image of the point (2, –3) after each transformation.

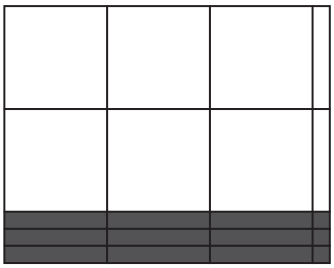
a) $(x, y) \rightarrow (x, -y)$

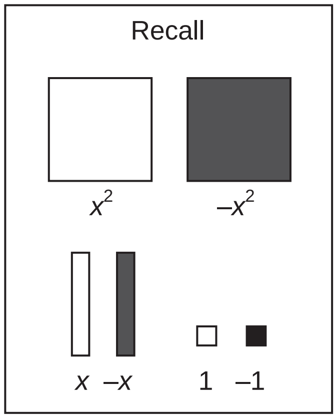
b) $(x, y) \rightarrow (x + 2, y + 1)$

c) $(x, y) \rightarrow (2x, y)$

d) $(x, y) \rightarrow (\frac{1}{2}x, y)$

5. a) Write the polynomial multiplication represented by the algebra tile diagram below.





b) Write the polynomial product in simplified form.

6. Multiply.

a) $(x + 3)(x + 4)$

b) $(2x + 3)(3x + 2)$

c) $(4 - x)(5 - x)$

d) $(5 - 2x)(3x + 1)$

7. Determine each quotient.

a) $(6y + 2y^2) \div 2y$

b) $(x^2 + 7x + 10) \div (x + 2)$

Chapter 1 Graphing Functions

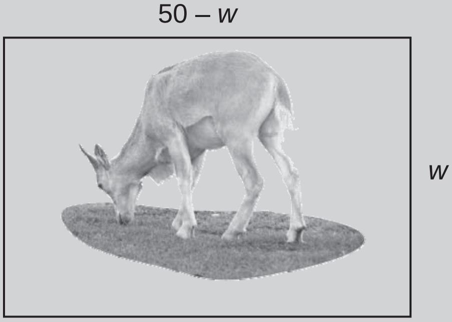
5.1.1 Forms of Quadratic Functions

Try This

Choki has 100 m of bamboo fencing to enclose a rectangular area for her goats.

A. If w represents the width of the pen, why does $50 - w$ represent the length?

B. Write an equation to represent the area of Choki's goat pen in terms of its width.



• A **quadratic function** is any function that can be expressed using a degree 2 single-variable polynomial. The graph of a quadratic function is a **parabola**.

• Quadratic functions may be written in different forms:

Standard form	$f(x) = ax^2 + bx + c$	In standard form, the degree 2 polynomial can have one, two, or three terms. For example: $2x^2 + 4x + 1$ $x^2 + x - 2$ $0.5x^2 + 3x$ $3x^2$
Factored form	$f(x) = a(x - p)(x - q)$	Factored form is only possible if the degree 2 polynomial can be written as the product of two linear factors. Sometimes they are multiplied by a constant. For example: $(x - 1)(x + 2)$ $x(x - 3)$ $\frac{(x+2)(x-1)}{5}$
Vertex form	$f(x) = a(x - h)^2 + k$	In vertex form, the degree 2 polynomial is written as the square of a linear term with an x-coefficient of 1. It can be multiplied by a constant and/or added to another constant. For example: $(x - 2)^2 + 3$ $(x - 1)^2 - 1$ $3(x - 2)^2$ $(x - 3)^2$

If a = 0, the function is not quadratic; if b or c = 0 the function can still be quadratic.

• **Equivalent functions** produce the same output value for every input value. You can prove that two functions, $f(x)$ and $g(x)$, are equivalent by

- showing their graphs are identical

- showing that the algebraic expression for $f(x)$ is equivalent to the algebraic expression for $g(x)$

You can prove that two functions, $f(x)$ and $g(x)$, are *not* equivalent by finding at least one value of x for which $f(x) \neq g(x)$.

• To move from vertex or factored form to standard form, you can **expand**.

For example:

$3(x - 2)^2 - 7$ can be expanded to $3(x^2 - 4x + 4) - 7$ and then $3x^2 - 12x + 5$.

$4(x + 2)(x - 6)$ can be expanded to $4x^2 - 16x - 48$.

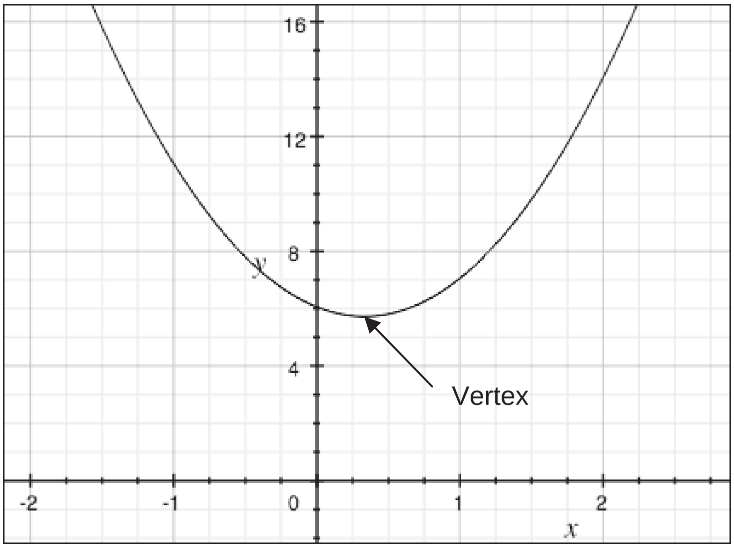
Later in the unit you will see how to move from standard form to factored form.

• A quadratic function can be graphed by creating a table of values, plotting the ordered pairs, and then sketching the curve formed by plotted points. Knowing that the function is quadratic and that the curve will be a parabola allows you to sketch relatively few values and still have a sense of what the graph looks like.

Here are a table of values and graph for $y = 3x^2 - 2x + 6$:

x	*f*(x)
–1	11
0	6
1	7
2	14

The **vertex** of this parabola is the function's minimum value. For a parabola that opens downward, the vertex is the maximum value.



The vertex is the point where the parabola changes direction.

• It is possible for two quadratic functions that are not equivalent to have the same output value for one or two input values. However, once they have the same output value for at least three inputs, they must be equivalent. For example:

For $x = 0$ and $x = 1$, $f(x) = 3x^2 + x + 2$ and $g(x) = 4x^2 + 2$ have the same values:

$f(x) = 3x^2 + x + 2 \rightarrow f(0) = 3(0)^2 + 0 + 2 = 2$

$f(1) = 3(1)^2 + 1 + 2 = 6$

$g(x) = 4x^2 + 2 \rightarrow g(0) = 4(0)^2 + 2 = 2$

$g(1) = 4(1)^2 + 2 = 6$

But for $x = 2$, the values are different; $f(2) = 16$ and $g(2) = 18$.

Therefore, $f(x)$ and $g(x)$ are not equivalent.

C. How do you know from the equation in **part B** that the function that relates the area of the goat pen and its width is quadratic?

D. i) Use the equation from **part B** to complete a table of values that shows the area for goat pens of width 5 m, 10 m, 15 m, 20 m, and 25 m.

ii) Describe what you think the graph of the function will look like.

Examples

Example 1 Determining Equivalent Functions

a) How do you know the equations below are functions?

b) Which functions are equivalent?

$f(x) = 2x^2 + 4x - 6$ $\quad$ $g(x) = 2(x - 1)(x + 3)$ $\quad$ $h(x) = 2(x + 1)^2 - 3$

a) Solution

- The expression in $f(x) = 2x^2 + 4x - 6$ is in $ax^2 + bx + c$ form, or standard form for a function.
- The expression in $g(x) = 2(x - 1)(x + 3)$ is equal to $2x^2 + 4x - 6$ but it is in factored form.
- The expression in $h(x) = 2(x + 1)^2 - 3$ is equal to $2x^2 + 4x - 6$ but it is in vertex form.

Thinking

- I knew if I expanded the expressions in the second and third functions, they'd be in standard form too.

b) Solution 1

Evaluate each for $x = 1$

$$f(x) = 2x^2 + 4x - 6 \rightarrow f(1) = 2(1)^2 + 4(1) - 6 = 2 + 4 - 6 = 0$$

$$g(x) = 2(x - 1)(x + 3) \rightarrow g(1) = 2(1 - 1)(1 + 3) = 2(0)(4) = 0$$

$$h(x) = 2(x + 1)^2 - 3 \rightarrow h(1) = 2(1 + 1)^2 - 3 = 2(2)^2 - 3 = 8 - 3 = 5$$

Evaluate $f(x)$ and $g(x)$ for $x = 0$ and $x = 2$

x	f(x)	g(x)
1	0	0
0	–6	–6
2	10	10

$f(x)$ and $g(x)$ are equivalent.

Thinking

- I found the value of each for $x = 1$ to do a quick check — if the values were different, the functions couldn't be equivalent.
- Since $h(1) \neq f(1)$ or $g(1)$, I knew that $h(x) \neq f(x)$ or $g(x)$. Since $f(1) = g(1)$, I knew I needed to check further.
- I needed to check only three values to be sure $f(x)$ and $g(x)$ were equivalent, so I tried two more, $x = 0$ and $x = 2$.

b) Solution 2

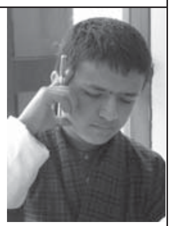
$f(x) = 2x^2 + 4x - 6$

$$g(x) = 2(x - 1)(x + 3) = 2(x^2 - x + 3x - 3) = 2(x^2 + 2x - 3) = 2x^2 + 4x - 6$$

$$h(x) = 2(x + 1)^2 - 3 = 2(x + 1)(x + 1) - 3 = 2(x^2 + 2x + 1) - 3 = 2x^2 + 4x + 2 - 3 = 2x^2 + 4x - 1$$

$g(x)$ and $f(x)$ are equivalent.

Thinking

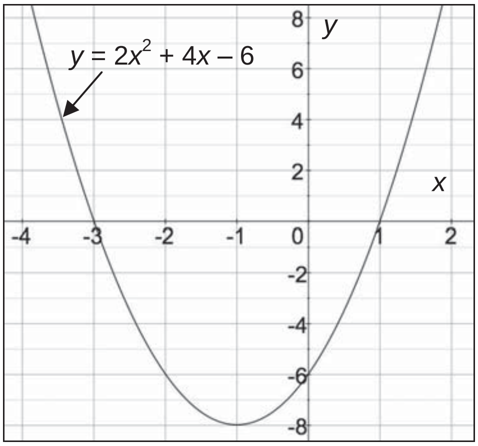


I knew if I changed the expressions for $g(x)$ and $h(x)$ to standard form, I could compare them with the expression in $f(x)$ to see if they were the same — $g(x)$ and $f(x)$ were the same.

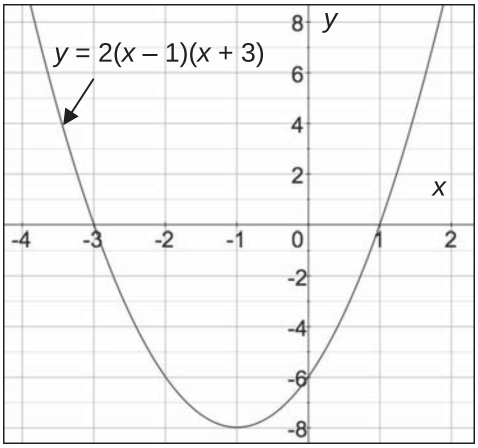
Example 1 Determining Equivalent Functions [Continued]

b) Solution 3

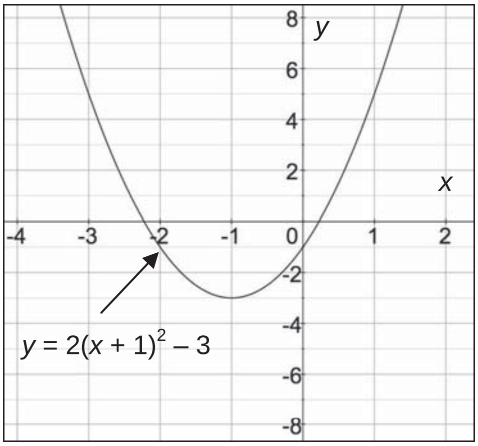
The graph of $f(x) = 2x^2 + 4x - 6$



The graph of $g(x) = 2(x - 1)(x + 3)$



The graph of $h(x) = 2(x + 1)^2 - 3$



$f(x)$ and $g(x)$ are equivalent functions.

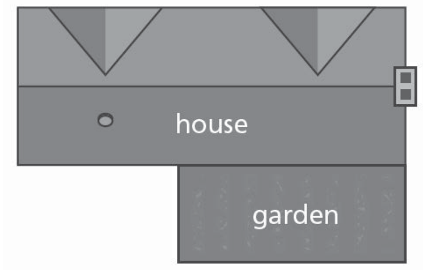
Thinking

- I knew that if the graphs of the functions are the same, they are equivalent, so I graphed the three functions.
- When you graph a function, you usually use y instead of $f(x)$ for the equation and vertical axis.
- The graphs of $f(x)$ and $g(x)$ were the same because they shared three points — they had the same y-intercept and the same two x-intercepts, so I knew $f(x)$ and $g(x)$ were equivalent.
- The graph of $h(x)$ had the same shape as the graphs of $f(x)$ and $g(x)$ but it was shifted up 5 units — the vertex moved from –8 to –3. That meant that $h(x) \neq f(x)$ or $g(x)$.

Example 2 Using a Quadratic Function and Its Graph to Solve a Problem

Karma has 25 m of fencing to enclose a garden. Since the garden is next to his house, he needs to fence only three of its sides.

About how wide should his garden be, if he wants the fence to enclose the greatest area possible? Estimate the area.

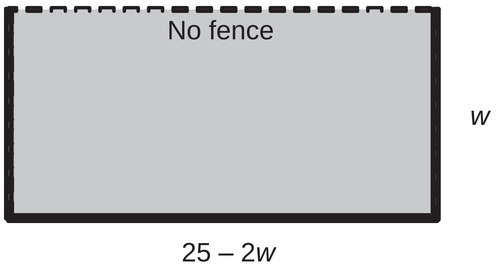


Solution

Create a function

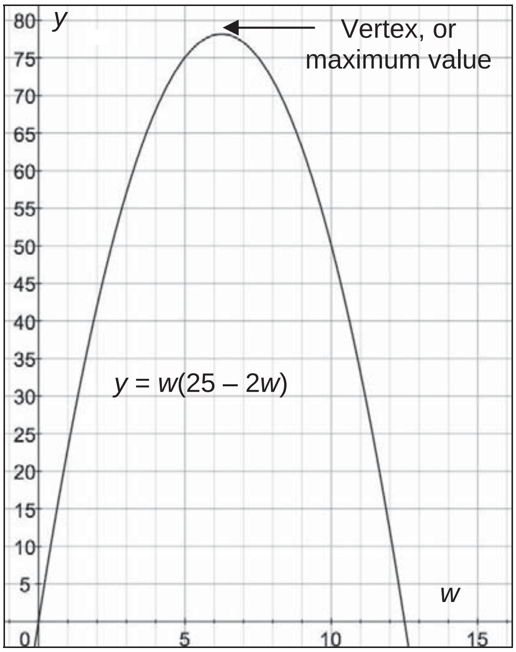
If he has 25 m of fencing for three sides: 2 widths and 1 length, then $2w + l = 25$.

If $2w + l = 25$, then $l = 25 - 2w$.



Since $A = w \times l$, then $A = w(25 - 2w)$, or $a(w) = w(25 - 2w)$, in function notation.

Graph the function



A width of about 6.3 m results in the greatest area possible, which is about 80 m^2.

Thinking

- I drew a diagram to represent and better understand the problem situation.
- I realized if I created a function that related the area to the width and then graphed it, the parabola's vertex would represent the maximum area.
- I used the algebraic expressions for the length and width to write the area as a function of the width.
- I created a table of values for several values of *w* and then plotted the ordered pairs to create the parabola.
- The vertex was halfway between 5 m and 7.5 m on the *w*-axis, or about 6.3 m. It was a bit greater than 78 m^2 on the *y*-axis, or about 80 m^2.

Practising and Applying

1. Which functions are quadratic?

A $f(x) = 2x^2 - x$

B $f(x) = 3(x - 5)^2$

C $f(x) = -3x + 2$

D $f(x) = -2(x + 1) + 3$

E $f(x) = 3(x - 2)(x + 2)$

F $f(x) = x^3 + 2x^2 + 3x + 1$

2. a) Create a table of values for both functions.

$f(x) = (x + 3)^2$ $\qquad$ $g(x) = x^2 + 6x + 9$

b) Explain how the table shows that the functions are equivalent.

3. a) Evaluate $g(x)$ and $h(x)$ to show that they are not equivalent.

$f(x) = 3(x + 3)(x - 1)$

$g(x) = 3x^2 + 6x - 12$

$h(x) = 3(x + 1)^2 - 12$

b) Express $f(x)$ and $h(x)$ in standard form to show that they are equivalent.

4. a) Sketch the graph of the function $f(x) = 3x^2 + 6x - 2$.

b) Estimate the coordinates of the vertex. Is it the minimum or maximum value of the function?

c) Estimate the x-intercepts and the y-intercept.

5. Determine if $f(x)$, $g(x)$, and $h(x)$ are equivalent by graphing.

$f(x) = -(x + 1)(x - 3)$

$g(x) = -x^2 + 2x + 3$

$h(x) = -(x - 1)^2 + 4$

7. a) Sketch a graph to represent the relationship between the width and area of Choki's goat pen from the **Try This**.

b) How could you use the graph to determine the dimensions of the pen that would enclose the greatest area?

c) What are the dimensions that enclose the greatest area? Explain.

8. Deki sells flour. If she sells 1 kg for Nu 20, she expects to sell about 50 kg per day. For every increase in price of Nu 1, she expects her daily sales to decrease by about 1 kg.

a) Write an algebraic expression to

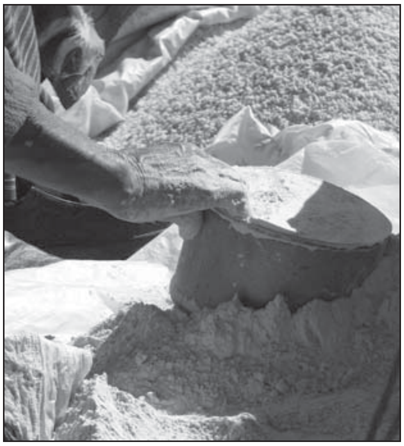
i) represent the price of each kilogram, if Deki increases the price by Nu x.

ii) represent the number of kilograms Deki expects to sell for each price increase of Nu x.

b) Use your answers from **part a)** to write a function representing Deki's total sales for each price increase of Nu x.

c) Sketch the graph of the function from **part b)**.

d) Use your graph to estimate the price that will result in the greatest daily sales and the total daily sales at this price.



9. Two linear functions are equivalent if they have the same output value for *two* input values. Use an example to show why two quadratic functions need to have the same output value for *three* input values to be equivalent.

5.1.2 Graphs of Quadratic Functions in Factored Form

Try This

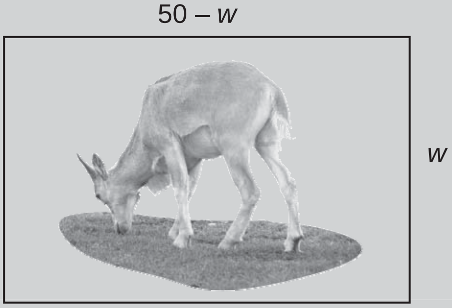
The diagram below shows Choki's goat pen from **lesson 5.1.1**. For **question 7** on **page 134**, you sketched a graph of the function $a(w) = -w^2 + 50w$ to determine the maximum area Choki could enclose with 100 m of fencing. The graph also provides information about the pen's area and dimensions for other conditions.

A. Which ordered pair on the graph represents each?

i) a pen with a width of 0 m

ii) a pen with an area of 0 m^2

iii) a pen with the maximum possible area



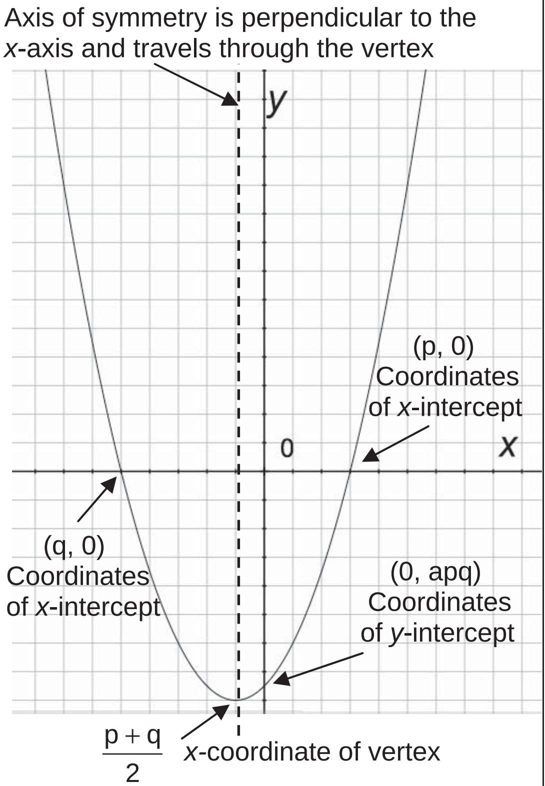
• When a quadratic function is in factored form, the factors can be used to determine the *x*-intercepts.

- The **zeros** of the function $f(x)$ are the *x*-intercepts, that is, the two possible values of *x* when $f(x) = 0$.

- For a function in factored form, that is $f(x) = a(x - p)(x - q)$, $0 = a(x - p)(x - q)$. Since $a \neq 0$ (or it would not be a quadratic function), either $x - p = 0$ or $x - q = 0$. That means *x* is either p or q and the coordinates of the *x*-intercepts are (p, 0) and (q, 0).

• You can use the *x*-intercepts or zeros to determine the coordinates of the parabola's vertex.

- The vertex of a parabola is on the parabola's **axis of symmetry**, which is located mid-way between the two *x*-intercepts. That means the *x*-coordinate of the vertex is mid-way between the zeros, p and q, at $\frac{p+q}{2}$.



- Once you know the value of the *x*-coordinate of the vertex, you can substitute that value into the function to determine its *y*-coordinate.

• You can determine the coordinates of the *y*-intercept by substituting $x = 0$ into the function: $f(x) = a(x - p)(x - q)$, $f(0) = a(0 - p)(0 - q) = a(-p)(-q) = apq$.

B. The function $a(w) = -w^2 + 50w$ is $a(w) = w(50 - w)$ in factored form.
Use the factored form of the function to determine the coordinates of each.
i) the x-intercepts **ii)** the vertex **iii)** the y-intercept

Examples

Example 1 Sketching the Graph of a Quadratic Function in Factored Form

Sketch the graph of the function $f(x) = \frac{1}{2}(x - 4)(x + 2)$.

Solution

Determine the zeros, or x-intercepts

$f(x) = \frac{1}{2}(x - 4)(x + 2)$

$0 = \frac{1}{2}(x - 4)(x + 2)$

$x - 4 = 0$ or $x + 2 = 0$, so $x = 4$ or -2
The coordinates of the x-intercepts are $(4, 0)$ and $(-2, 0)$.

Determine the coordinates of the vertex

x-coordinate: $\frac{-2 + 4}{2} = 1$

y-coordinate: $f(1) = \frac{1}{2}(1 - 4)(1 + 2) = \frac{-9}{2}$

The coordinates of the vertex are $(1, \frac{-9}{2})$.

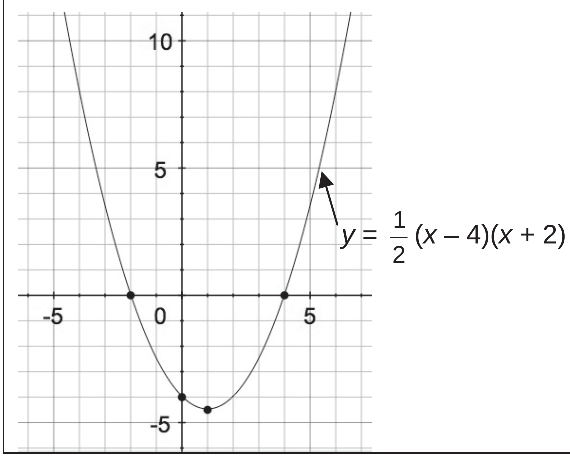
Determine the coordinates of the y-intercept

x-coordinate: 0

y-coordinate: $f(0) = \frac{1}{2}(0 - 4)(0 + 2) = -4$

The coordinates of the y-intercept are $(0, -4)$.

Plot the four points and sketch the parabola

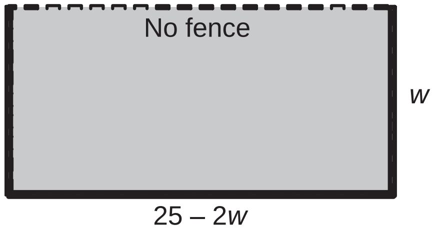


Thinking

- I knew if I determined the two x-intercepts or zeros, I could use them to figure out the coordinates of the vertex. That would give me three points to plot and connect to make a parabola.
- The x-coordinate of the vertex is midway between the x-coordinates of the x-intercepts, so I just calculated the mean of the two zeros.
- I realized I could also figure out a fourth point to help with the graph — the y-intercept.

Example 2 Using the Coordinates of the Vertex to Solve a Problem

In **lesson 5.1.1** on **page 133**, Karma wanted to enclose a garden on three sides using 25 m of fencing. To estimate the maximum possible area and width at that area, the graph of the function $a(w) = w(25 - 2w)$ was sketched. What are the exact maximum area and width?



Solution

Determine the zeros

$a(w) = w(25 - 2w) \rightarrow 0 = w(25 - 2w)$

$w = 0$ or $25 - 2w = 0$, so $w = 0$ or $12\frac{1}{2}$

Determine the coordinates of the vertex

If $w = 0$ or $12\frac{1}{2}$, the w-coordinate $= \dfrac{12\frac{1}{2} + 0}{2}$

$= 6\frac{1}{4}$

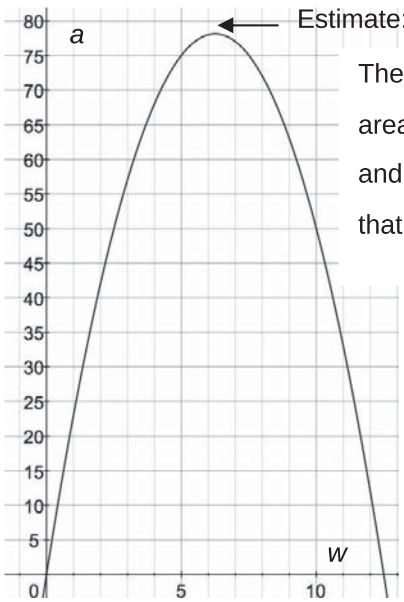
If the w-coordinate is $6\frac{1}{4}$, the y-coordinate is

$a(6\frac{1}{4}) = 6\frac{1}{4} \times (25 - 2 \times 6\frac{1}{4}) = 6\frac{1}{4} \times 12\frac{1}{2}$

$= 78\frac{1}{8}$

The coordinates are ($6\frac{1}{4}$ m, $78\frac{1}{8}$ m^2).

Compare answer with the estimate



The maximum area is $78\frac{1}{8}$ m^2 and the width at that area is $6\frac{1}{4}$ m.

Thinking

- The graph of the function was a parabola that opened downward, so I knew its vertex was a maximum value.
- I knew the coordinates of the vertex represented the maximum area and the width at that area.
- I was able to figure out the exact coordinates, by finding the zeros and using them to calculate the coordinates.
- I checked my answer by comparing it to my previous estimate of 6.3 m and 80 m^2 from the graph. My answer of $6\frac{1}{4}$ m and $78\frac{1}{8}$ m^2 seemed reasonable.

Practising and Applying

1. a) State the zeros and y-intercept for the graph of each quadratic function.

i) $f(x) = (x + 2)(x - 3)$

ii) $f(x) = (x - 5)(x + 5)$

iii) $f(x) = (2x + 3)(4x + 2)$

iv) $f(x) = 3(x - 2)(x + 2)$

v) $f(x) = -2(x + 1)(x + 2)$

b) Write the coordinates of the vertex for each function in **part a)**.

c) Sketch each parabola in **part a)** using the information from **parts a) and b)**.

2. In order to graph the function $f(x)$, Mindu factored $f(x)$ as shown below:

$f(x) = 2x^2 + 10x - 12 = 2(x - 1)(x + 6)$

a) How do you know that $2x^2 + 10x - 12 = 2(x - 1)(x + 6)$?

b) Why would Mindu do this before graphing the function?

c) Sketch the graph of the function.

3. A store sells about 30 ghos each week for Nu 800 each. The owner expects to lose one sale each week for every increase in price of Nu 40.

a) Write an expression to represent the

i) new price of a gho after n Nu 40 price increases.

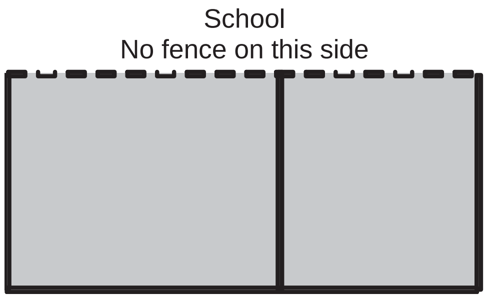
ii) expected number of ghos that will be sold weekly after n price increases.

b) Write a function to represent the expected weekly sales as a function of the number of price increases of Nu 40.

c) Use the function to determine the price that will maximize total sales.

d) Sketch the graph of the function to see if your answer is reasonable.

4. Sonam wants to fence a playground next to the school and to build a dividing fence to make separate play areas for younger and older students. The budget allows for 210 m of fencing.



a) Write a function that represents the total area of the playground as a function of its width.

b) Determine the width that will result in the maximum total area that can be enclosed using 210 m of fencing. Sketch the graph of the function to see if your answer is reasonable.

5. A store sells an average of 100 kg of carrots weekly at Nu 50/kg. For every increase of Nu 5/kg, the manager expects his sales to decrease by 2 kg. Write a function that could be used to determine the price per kilogram that would result in the greatest total weekly sales.

6. Why is the x-coordinate of the vertex of a parabola the mean of the zeros of the function?

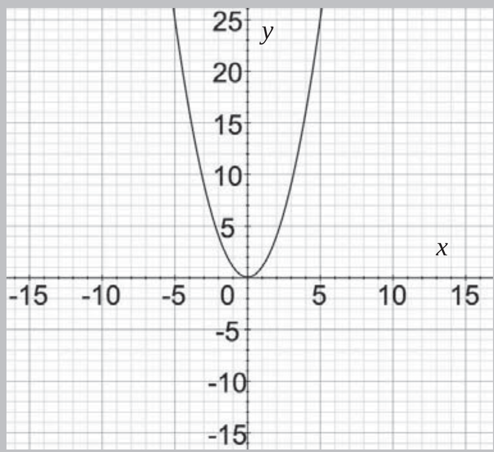
5.1.3 EXPLORE: Transforming Quadratic Function Graphs

The graph of every quadratic function is a parabola. Particular changes to the algebraic expression that defines the function result in corresponding changes to the size, shape, position on the grid, and direction of opening of the parabola.

A. i) Draw *x*- and *y*-axes on grid paper:

ii) Prepare a table of values for the function $f(x) = x^2$. Use *x*-values of –3, –2, –1, 0, 1, 2, and 3.
(Allow room for three more columns to be added in **part B**.)

iii) Sketch the graph of $f(x) = x^2$ (which should look like the graph shown here).



B. i) Add columns to your table from **part A** to display values for $g(x)$, $h(x)$, and $i(x)$ for the same values of *x* you used for $f(x)$. Sketch each graph on your grid.

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = -2x^2$,	$i(x) = \frac{1}{2}x^2$
–3				
–2				
–1				
0				

ii) How do the values in the table explain how the graphs of $g(x)$, $h(x)$, and $i(x)$ compare to the graph of $f(x)$?

iii) Why might someone call the functions $g(x)$ and $i(x)$ **dilatations** of $f(x)$?
Why might someone call $h(x)$ a dilatation and a **reflection** of $f(x)$?

C. i) Complete a table of values for the functions shown below. (Use the values in your table from **part A** for $f(x)$ to complete the second column of the table.) Then sketch the graphs of all five functions on the same grid.

x	$f(x) = x^2$	$f(x-1)$	$f(x-2)$	$f(x+1)$	$f(x+2)$
–3					
–2					
–1					
0					

ii) What transformation would have to be applied to the graph of $f(x)$ to result in each of the other graphs?

iii) How could you have used the values in the table to predict the size and direction of the transformation?

iv) Suppose you replaced x with $x - 1$, $x - 2$, $x + 1$, and $x + 2$ for the functions $g(x)$, $h(x)$, and $i(x)$ and graphed the new functions. For example:

$g(x) = 2x^2 \rightarrow g(x-1) = 2(x-1)^2$ $\qquad$ $g(x) = 2x^2 \rightarrow g(x-2) = 2(x-2)^2$

Without graphing, describe in general how the new graphs would compare to the graphs of the original functions.

D. i) Complete a table of values for the functions shown below and then sketch the graphs of all five functions on the same grid.

x	$f(x) = x^2$	$f(x) - 1$	$f(x) - 2$	$f(x) + 1$	$f(x) + 2$
–3					
–2					
–1					
0					

ii) Repeat **ii) and iii)** from **part C** for the functions you graphed in **part D i)**.

iii) Suppose you changed each function $g(x)$, $h(x)$, and $i(x)$ in the same way as $f(x)$ was changed in **part D i)** and graphed the new functions. For example:

$g(x) = x^2 \rightarrow g(x) = x^2 - 1$ $\qquad$ $g(x) = x^2 \rightarrow g(x) = x^2 - 2$

Without graphing, describe in general how the new graphs would compare to the graphs of the original functions.

E. Sketch the graphs of the functions below using what you learned in **parts B, C, and D**. Which functions would you call negative dilatations?

$f(x) = x^2$ $\quad$ $g(x) = \frac{1}{2}x^2$ $\quad$ $h(x) = -\frac{1}{2}x^2$ $\quad$ $i(x) = -\frac{1}{2}(x-4)^2$ $\quad$ $j(x) = -\frac{1}{2}(x-4)^2 + 5$

F. What transformations are involved in transforming the graph of $f(x) = x^2$ to the graph of $g(x) = -\frac{1}{2}(x-4)^2 + 5$? How do you know?

5.1.4 Relating Graphs of Quadratic Functions

Try This

The height of a football kicked into the air is represented by the quadratic function $h(t) = -5(t - 2)^2 + 20$, where t represents the time in seconds the ball is in the air and h represents the height of the ball in metres.

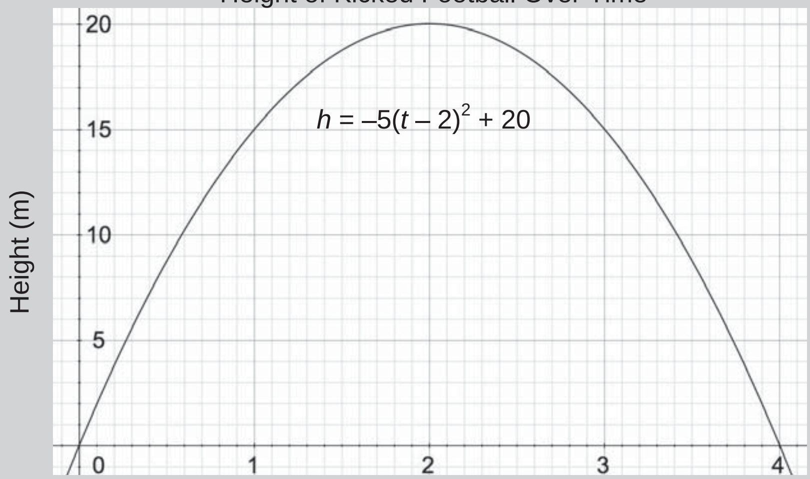
A. Use the graph of $h(t) = -5(t - 2)^2 + 20$ below to estimate each.

i) The amount of time the ball was in the air

ii) The maximum height reached by the ball

iii) The time at which the ball reached its maximum height

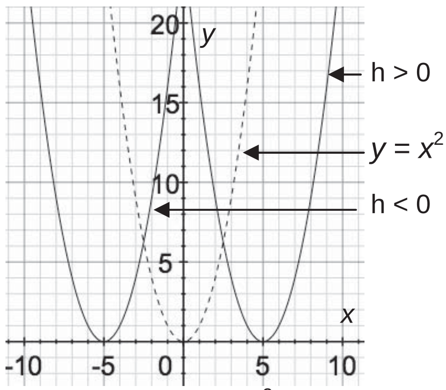
Height of Kicked Football Over Time



- The graph of a quadratic function is a parabola. The parabola's size, shape, position on the grid, and direction of opening are the result of applying one or more transformations to the graph of the function $f(x) = x^2$.
- A transformation can be represented using **mapping notation**, which describes what happens to the coordinates of each point on the graph as a result of the transformation. It gives you a "map" of where each point moves.

Horizontal translations

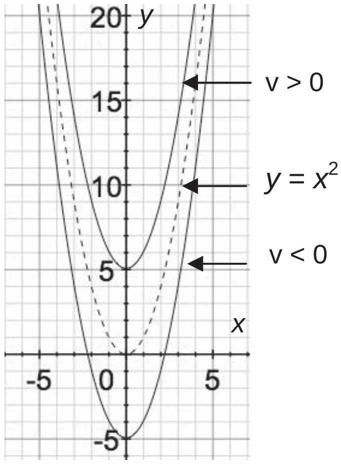
Function	Transformation in mapping notation
$f(x) = (x - \text{h})^2$	$(x, y) \rightarrow (x + \text{h}, y)$
• When h < 0, for example, $y = (x - (-5))^2$ or $y = (x + 5)^2$, the parabola is translated left h units. • When h > 0, for example, $y = (x - (+5))^2$ or $y = (x - 5)^2$, the parabola is translated right h units.	



$f(x) = (x - \text{h})^2$

Vertical translations

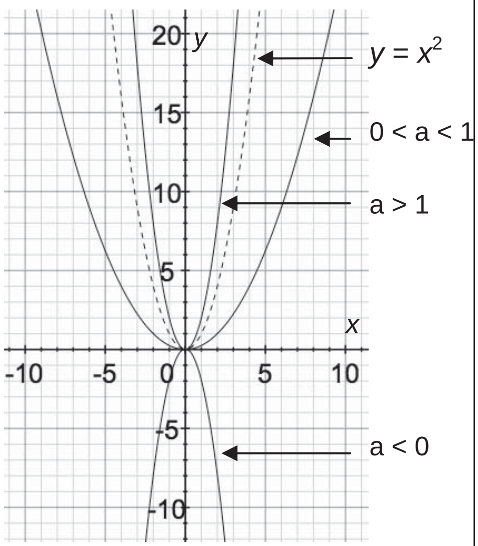
Function	Transformation in mapping notation
$f(x) = x^2 + v$	$(x, y) \rightarrow (x, y + v)$
• When v > 0, for example, $y = x^2 + 5$, the parabola is translated up *v* units. • When v < 0, for example, $y = x^2 - 5$, the parabola is translated down *v* units	



$f(x) = x^2 + v$

Vertical stretches and compressions (dilatations)

Function	Transformation in mapping notation
$f(x) = ax^2$	$(x, y) \rightarrow (x, ay)$
When a > 0 • When 0 < a < 1, for example, $y = \frac{1}{2}x^2$, the parabola is compressed vertically. • When a > 1, for example, $y = 2x^2$, the parabola is stretched vertically. These are called *vertical* compressions and *vertical* stretches because only the *y*-coordinate of each point on the parabola is affected. *When* a < 0 • When a = –1, for example, $y = -x^2$, the parabola is reflected in the *x*-axis. • When –1 < a < 0, for example, $y = -\frac{1}{2}x^2$, the parabola is reflected in the *x*-axis and compressed vertically. • When a < –1, for example, $y = -2x^2$, the parabola is reflected in the *x*-axis and stretched vertically. These are called negative dilatations because the negative value of *a* changes the *y*-coordinate of every point on the parabola to its opposite value, resulting in a reflection in the *x*-axis.	



$f(x) = ax^2$

• When you apply a composite transformation to the graph of $y = x^2$, you have some choice about the order in which you apply the separate transformations.

One possible sequence of transformations that will change the graph of $f(x) = x^2$ to the graph of $f(x) = a(x - h)^2 + v$ is described below:

A. A vertical stretch or compression (dilatation), with a reflection in the x-axis if $a < 0$:

$(x, y) \rightarrow (x, ay)$ $\qquad y = \mathbf{a}x^2$

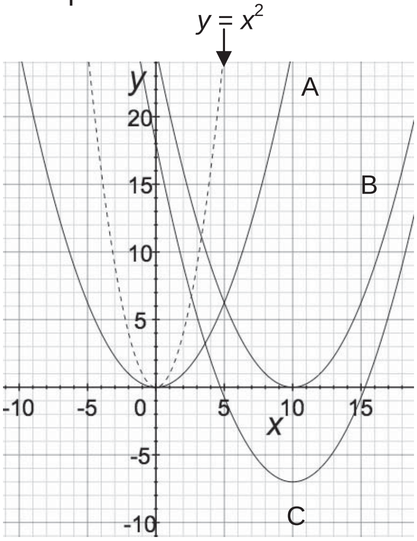
B. A horizontal translation:

$(x, y) \rightarrow (x + h, ay)$ $\qquad y = a(x \mathbf{- h})^2$

C. A vertical translation:

$(x, y) \rightarrow (x + h, ay + v)$ $\qquad y = a(x - h)^2 \mathbf{+ v}$

Other possible orders are A, C, B and B, A, C.



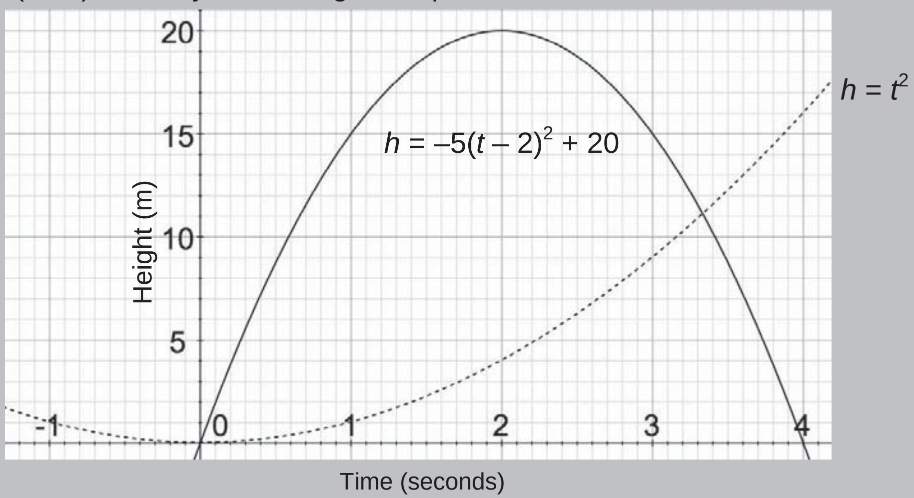
Note that C, A, B or B, C, A will not work since both result in a different parabola. This is because C, the vertical translation, is supposed to define the final location of each point on the parabola. The y-intercept should end up at (0, c), but if A, the dilatation, follows C, the y-intercept will end up at (0, ac) instead.

The horizontal translation can be done before or after the vertical dilatation (**B**, **A**, C or **A**, **B**, C) and before or after the vertical translation (A, **B**, **C** or A, **C**, **B**). This is because the horizontal translation (B) affects only the x-coordinates and the two vertical transformations (A and C) affect only the y-coordinates.

B. Examine the equation of the function $h(t) = -5(t - 2)^2 + 20$.

i) Which transformation(s) would change the graph of $h = t^2$ to the graph of $h = -5(t - 2)^2 + 20$? How do you know?

ii) How can you predict the coordinates of the vertex of the graph of $h = -5(t - 2)^2 + 20$ by examining the equation?



Examples

Example 1 Transforming the Graph of $y = x^2$ to Sketch Another Graph

Sketch the graph of the function $f(x) = -3(x + 4)^2 + 3$.

Solution

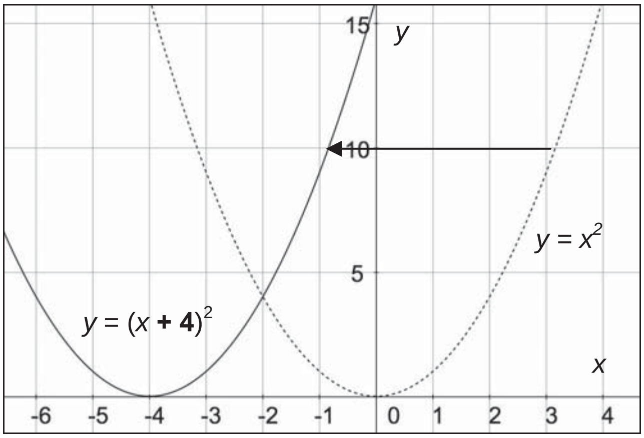
Use the equation to determine the transformations

A	$y = -3(x + 4)^2 + 3$ or $-3(x - (\mathbf{-4}))^2 + 3$	Horizontal translation left 4 units: $(x, y) \rightarrow (x\ \mathbf{-\ 4}, y)$
B	$y = \mathbf{-3}(x + 4)^2 + 3$	Reflection in the x-axis and a vertical stretch by 3: $(x, y) \rightarrow (x, \mathbf{-3}y)$
C	$y = -3(x + 4)^2\ \mathbf{+\ 3}$	Vertical translation up 3 units: $(x, y) \rightarrow (x, y\ \mathbf{+\ 3})$

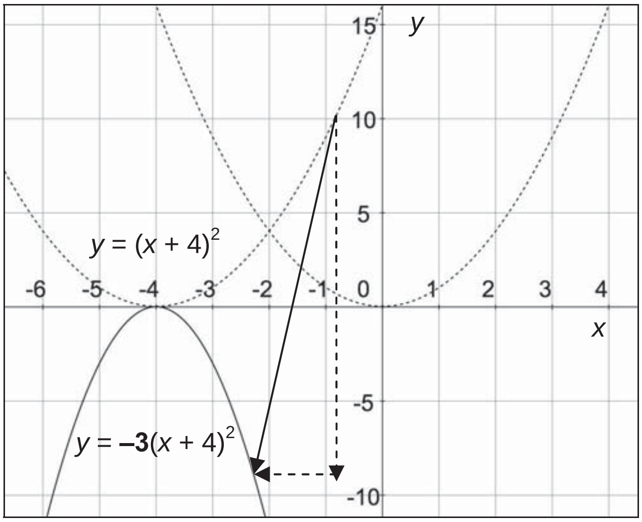
Composite transformation: $(x, y) \rightarrow (x - 4, -3y + 3)$

Apply each transformation to $y = x^2$

A. Horizontal translation left 4 units: $(x, y) \rightarrow (x - 4, y)$



B. Reflection in the x-axis and a vertical stretch by 3:
$(x, y) \rightarrow (x, -3y)$



Thinking

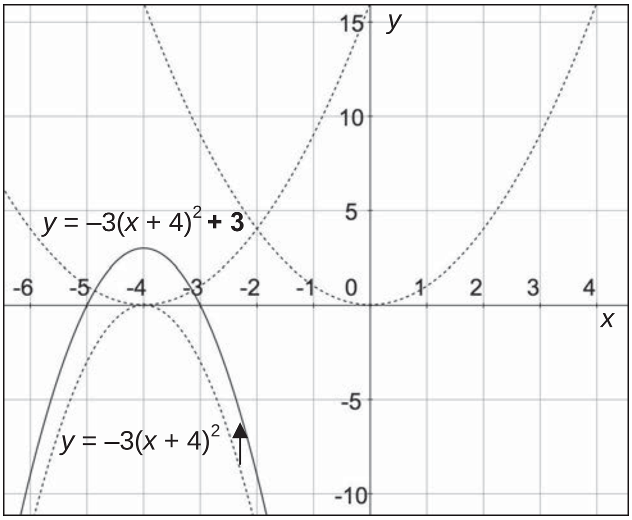
I began with the graph of $y = x^2$ and applied the transformations as shown by the equation one at a time.

• I could follow the order B, C, A or B, A, C, as long as C followed B.

A. I sketched $y = x^2$ by plotting several points and then sketching a parabola through the points. I then translated each of those points using the mapping to plot points for $y = (x + 4)^2$. I then sketched the parabola.

B. I dilatated the plotted points on the graph of $y = (x + 4)^2$, using the mapping, to plot points for the graph of $y = -3(x + 4)^2$. I then sketched the parabola.

C. Vertical translation up 3 units: $(x, y) \rightarrow (x, y + 3)$



Check points on final parabola

$(x, y) \rightarrow (x - 4, -3y + 3)$

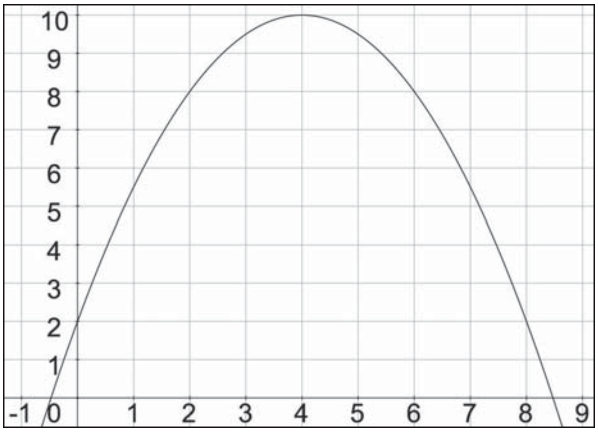
$(0, 0) \rightarrow (-4, 3)$ $(1, 1) \rightarrow (-3, 0)$ $(2, 4) \rightarrow (-2, -9)$

C. I translated the plotted points on the graph of $y = -3(x + 4)^2$, using the mapping, to plot points for the graph of $y = -3(x + 4)^2 + 3$. I then sketched the parabola.

• I checked three of the plotted points on the final parabola, using the mapping for the composite transformation.

Example 2 Determining the Equation of a Function from Its Graph

Determine the equation of this parabola.



Solution 1

Determine the translations

The vertex of $y = x^2$ is (0, 0) and the vertex of the parabola is (4, 10), so the coordinates of the vertex moved 4 units right and 10 units up:

$$(x, y) \rightarrow (x + 4, y + 10)$$

The equation so far is $y = a(x - 4)^2 + 10$.

[Continued]

Thinking

• I compared the coordinates of the vertices of $y = x^2$ and the parabola to determine the vertical and horizontal translations.

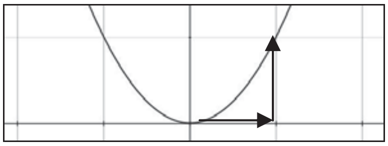
• I used that information to begin writing an equation.

Example 2 Determining the Equation of a Function from Its Graph [Cont'd]

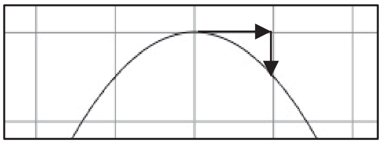
Solution 1 [Continued]

Determine the dilatation factor, a

If you move 1 unit right (or left) from the vertex of $y = x^2$, the corresponding point is 1 unit up.



If you move 1 unit right (or left) from the parabola's vertex, the corresponding point is $\frac{1}{2}$ unit down.



That means the parabola is compressed by $\frac{1}{2}$ and reflected in the x-axis.

The equation is $y = -\frac{1}{2}(x - 4)^2 + 10$.

Thinking

- I compared the points that were one unit to the right of the vertex on both $y = x^2$ and on the parabola to determine the dilatation factor.
- Because the parabola was upside down compared to the graph of $y = x^2$, I knew a reflection in the x-axis was involved.

Solution 2

Determine the translations

Since $(0, 0) \rightarrow (4, 10)$, then $(x, y) \rightarrow (x + 4, y + 10)$.

The equation so far is $y = a(x - 4)^2 + 10$.

Determine the dilatation factor, a

The point (0, 2) is on the parabola.

Substitute $x = 0$ and $y = 2$ into $y = a(x - 4)^2 + 10$ and then solve for a:

$$2 = a(0 - 4)^2 + 10$$
$$2 = 16a + 10$$
$$-8 = 16a$$
$$-\frac{1}{2} = a$$

The equation is $y = -\frac{1}{2}(x - 4)^2 + 10$.

Check

$$2 = -\frac{1}{2}(8 - 4)^2 + 10$$
$$2 = -\frac{1}{2}(16) + 10$$
$$2 = -8 + 10$$

Thinking

- I compared the coordinates of the vertex of $y = x^2$ to the coordinates of the vertex of the parabola to determine the translations and write an initial equation.
- I substituted the values from a point that I knew was on the parabola into the equation to figure out the value of a.
- I knew the equation worked for (0, 2) and (4, 10), so I tried a third point, (8, 2), in the equation to check.

Practising and Applying

1. a) What are the coordinates of the vertex for the graph of each function?

i) $f(x) = x^2 + 4$

ii) $f(x) = (x - 8)^2$

iii) $f(x) = (x - 3)^2 - 2$

iv) $f(x) = -2(x + 1)^2 - 1$

v) $f(x) = \frac{1}{5}(x - 1)^2 + 1$

vi) $f(x) = -\frac{1}{5}(x - 2)^2 + 2$

b) For each function in **part a)**, indicate

- the direction the parabola opens
- the dilatation factor compared to the graph of $y = x^2$, if applicable

c) Sketch the graph of each function in **part a).**

2. Determine the equation of the parabola that would result from applying each transformation or composite transformation to the graph of $y = x^2$.

a) $(x, y) \rightarrow (x + 4, y)$

b) $(x, y) \rightarrow (x, -\frac{1}{2}y)$

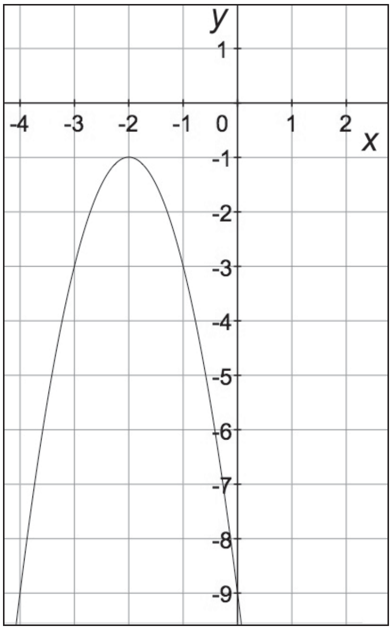
c) $(x, y) \rightarrow (x - 4, y - 3)$

d) $(x, y) \rightarrow (x + 4, \frac{1}{4}y)$

e) $(x, y) \rightarrow (x + 4, -5y)$

f) $(x, y) \rightarrow (x + 4, -3y + 6)$

3. Determine the equation of the parabola.



4. a) Why can a reflection in the x-axis also be called a negative dilatation?

b) Why is a dilatation by a factor of 2 called a vertical stretch?

5. $f(x) = a(x - h)^2 + k$ is called the vertex form of a quadratic function. Why is that an appropriate name? Use an example to support your explanation.

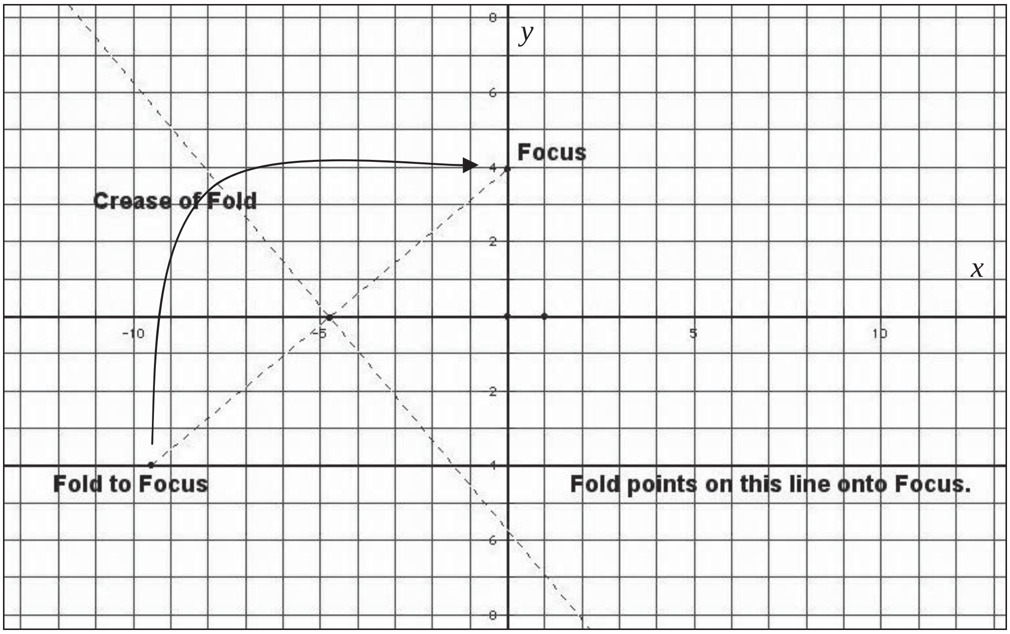
CONNECTIONS: Parabolas and Paper Folding

You can make multiple folds in a piece of paper to create a parabola.

A. Draw a coordinate grid.

B. Choose a point on the *y*-axis. Label it “Focus”. Draw a horizontal line on the other side of the *x*-axis from the Focus.

C. Mark a point on the line you drew in **step B** and then fold the paper so that the point folds onto the Focus point.



D. Mark another point on the line and fold it onto the Focus point. Repeat this several times. You should notice that the fold lines begin to form a parabola.

1. How does the position of the vertex of the parabola relate to the position of the Focus point you chose in **step A** and the line you drew in **step B**?

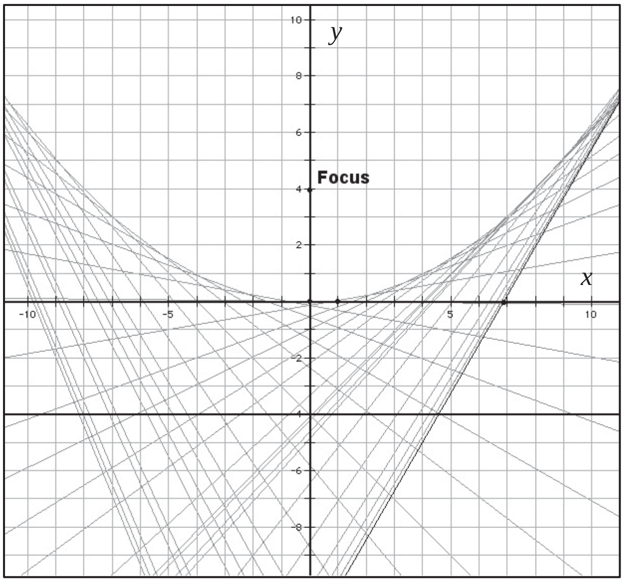
2. Investigate other positions for the Focus and the line. Try

• locating them farther apart

• locating the Focus and line so they are different distances from the *x*-axis

• moving the Focus away from the *y*-axis

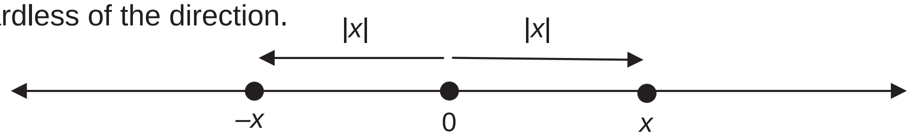
What happens to the parabola each time?



5.1.5 EXPLORE: The Absolute Value Function

The **absolute value** of a number x, $|x|$, is the value of x without regard to its sign.

You can think of $|x|$ as the distance that x or $-x$ lies from 0 on a number line, regardless of the direction.

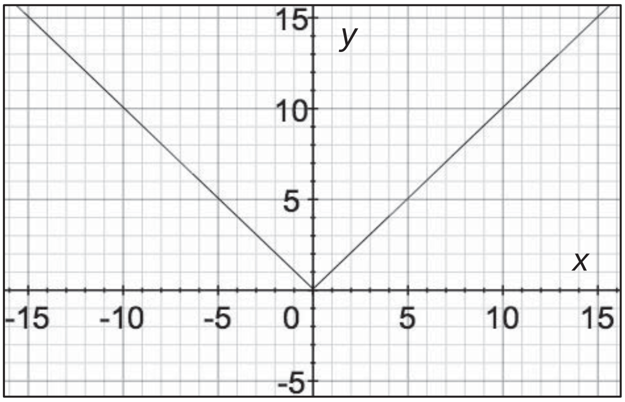


This table shows some sample values.

x	0	1.5	–1.5	5	–5
$\lvert x\rvert$	$\lvert 0\rvert = 0$	$\lvert 1.5\rvert = 1.5$	$\lvert -1.5\rvert = 1.5$	$\lvert 5\rvert = 5$	$\lvert -5\rvert = 5$

An **absolute value function** involves absolute values, for example, $f(x) = |x|$.

The graph of $f(x) = |x|$ has the shape of two lines meeting to form a symmetrical "V" with the y-axis as the line of reflection.



The graph of $y = |x|$

A. Why is the graph of $f(x) = |x|$ symmetrical about the y-axis?

B. Create a table of values for each function and then graph the function.

a) $f(x) = f(x - 1)$ **b)** $f(x) = f(x + 1)$ **c)** $f(x) = f(x) - 1$

d) $f(x) = f(x) + 1$ **e)** $f(x) = -f(x)$ **f)** $f(x) = 2f(x)$

C. Use mapping notation to describe a single transformation that would change the graph of $f(x) = |x|$ to each graph in **part B**. For example:

$(x, y) \rightarrow (x + 5, y)$ would change $f(x) = |x|$ to $g(x) = f(x - 5)$.

D. Use what you know about transforming graphs of a quadratic function to predict the series of transformations that must be applied to change the graph of $f(x) = |x|$ to the graph of $f(x) = -\frac{1}{2}|x - 4| + 5$.

Chapter 2 Solving Non-Linear Equations

5.2.1 Factoring Quadratic Expressions

Try This

Kinley multiplied two binomials and got a trinomial product, as shown below. Each shape represents an integer value.

$$(\blacktriangle x + \bullet)(\blacksquare x + \blacklozenge) = 2x^2 + x - 6$$

A. i) What is the value of each expression below? How do you know?

$\blacktriangle \times \blacksquare$ $\qquad$ $\bullet \times \blacklozenge$ $\qquad$ $\blacktriangle \times \blacklozenge + \bullet \times \blacksquare$

ii) Use the values in **part i)** to determine the value of each shape.

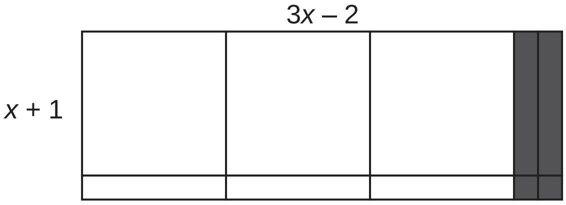
$\blacktriangle$ $\qquad$ $\blacklozenge$ $\qquad$ $\bullet$ $\qquad$ $\blacksquare$

• You can use algebra tiles to determine the product of two degree 1 binomials.

- The factors are represented by the dimensions of the rectangle.
- The product is represented by the area of the rectangle.

For example:

The product of $(3x - 2)(x + 1)$ can be modelled as shown:



This model shows that

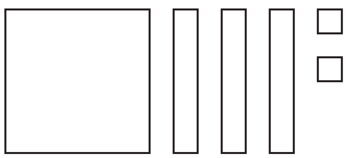
$$(3x - 2)(x + 1)$$
$$= 3x^2 - 2x + 3x - 2$$
$$= 3x^2 + x - 2$$

Notice that the area model has four parts to it, each representing one of the four partial products in $3x^2 - 2x + 3x - 2$ before it was simplified to $3x^2 + x - 2$.

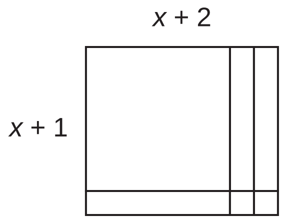
	$3x$	-2
x	$3x^2$	$-2x$
1	$3x$	-2

• You can also use algebra tiles to **factor**, or **factorise** by creating a rectangle with a given area (the degree 2 polynomial) to find its dimensions (the factors).

For example, to factor $x^2 + 3x + 2$, arrange tiles worth $x^2 + 3x + 2$ to form a rectangle. The dimensions of the rectangle are the factors of the polynomial.



$x^2 + 3x + 2$



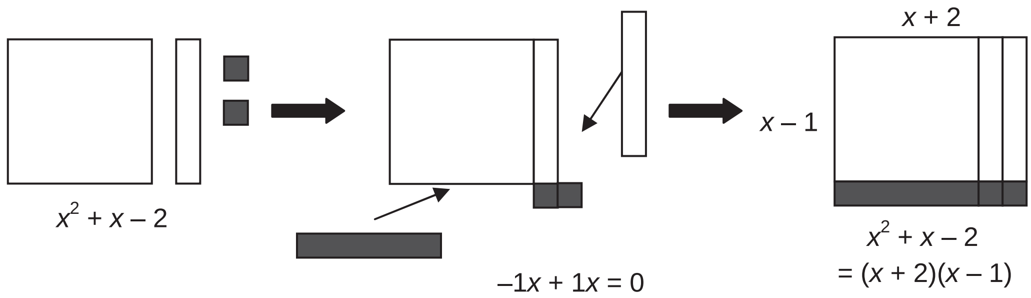
The factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$.

If negatives are involved, you sometimes have to add tiles to make a rectangle. To do this without changing the value of the polynomial, you add positive and negative tiles of equal value.

For example:

The polynomial $x^2 + x - 2$ can be factored as shown below:

If you add a vertical $+x$-tile and a horizontal $-x$-tile, you can make a rectangle without changing the value of the polynomial.



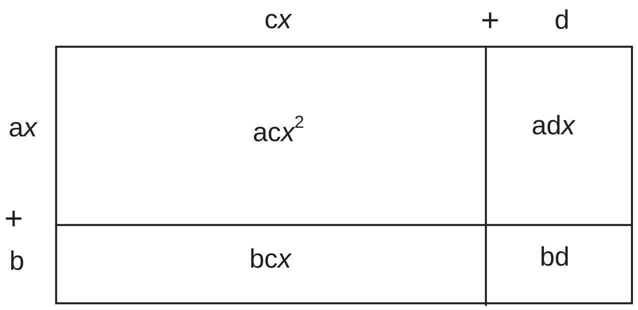
• Another way to factor a degree 2 polynomial is to use an algebraic model.

To develop the model,

- Represent the unknown factors as ax + b and cx + d (a, b, c, and d are integers).
- Multiply (ax + b)(cx + d) to create an algebraic expression for the product.

$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd$$

The sketch of the area model below shows how the factors (dimensions) and product (area) are related.



The total area is $acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd$.

Here is how you can use the model to factor $x^2 + 2x - 3$:

Step 1 Compare $x^2 + 2x - 3$ to $acx^2 + (ad + bc)x + bd$ to find the value of ac and bd:

If $\mathbf{1}x^2 + 2x \mathbf{- 3} = \mathbf{ac}x^2 + (ad + bc)x \mathbf{+ bd}$,

then ac = 1 and bd = –3.

[Continued]

Step 2 Find possible values for a and c, and for b and d:

If ac = 1, possible pairs of values are

a	c
–1	–1
1	1

If bd = –3, possible pairs of values are

b	d
1	–3
–1	3
–3	1
3	–1

Step 3 Compare $x^2 + 2x - 3$ to $acx^2 + (ad + bc)x + bd$ to find the value of ad + bc:

If $x^2 + \mathbf{2}x \mathbf{-} 3 = acx^2 + (\mathbf{ad + bc})x + bd$, then ad + bc = 2.

Step 4 Substitute possible values for a, c, b, and d into ad + bc = 2 to find a combination that works:

a = 1, c = 1, b = –1, d = 3 works [It works because ad + bc = (1)(3) + (–1)(1) = 2, ac = (1)(1) = 1, and bd = (–1)(3) = –3.]

Step 5 Replace a, b, c, and d in $(ax + b)(cx + d)$ to create factors:

If a = 1, b = –1, c = 1, and d = 3, then $(ax + b)(cx + d) = (x - 1)(x + 3)$.

$x^2 + 2x - 3$ factors as $(x - 1)(x + 3)$.

• There is never more than one pair of factors (two binomials, or a monomial and a binomial) for a quadratic expression. It is also possible for there to be no factors.

B. i) Explain why you can factor $2x^2 + x - 6$ to check your answer to **part A ii)**.
ii) Factor $2x^2 + x - 6$ to check your answer.

Examples

Example 1 Factoring a Trinomial

Factor $6x^2 - 13x - 5$.

Solution 1

$(ax + b)(cx + d)$
$= \mathbf{ac}x^2 + (ad + bc)x + \mathbf{bd}$
$= \mathbf{6}x^2 - 13x \mathbf{- 5}$

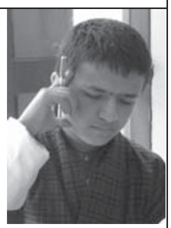
ac = 6

a	c
1	6
6	1
3	2
2	3
–1	–6
–6	–1
–3	–2
–2	–3

bd = –5

b	d
1	–5
–1	5
–5	1
5	–1
1	–5
–1	5
–5	1
5	–1

Thinking



• I compared the algebraic model of the product to the trinomial to figure out the values of ac and bd. (I knew that bd was –5 because $6x^2 - 13x - 5 = 6x^2 - 13x + (-5)$.)

• I listed all the possible pairs of values for ac and for bd.

• To figure out the actual values of a, c, b, and d, I knew that the pairs had to stay together. For example:

- if a were 1, c would have to be 6.
- if b were 1, d would have to be –5.

<table>
<tr>
<td>
$acx^2 + (\mathbf{ad + bc})x + bd$
$= 6x^2 \mathbf{-13}x - 5$
$ad + bc = -13$

a = 3, b = 1, c = 2, d = –5 works because
• ac = (3)(2) = 6
• bd = (1)(–5) = –5
• ad + bc = 3(–5) + 2(1) = –13

If a = 3, b = 1, c = 2, d = –5,
$(ax + b)(cx + d) = (3x + 1)(2x - 5)$

$6x^2 - 13x - 5 = (3x + 1)(2x - 5)$
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• I used the model again to figure out the value of ad + bc. (I knew ad + bc was –13 because $6x^2 - 13x - 5 = 6x^2 + (-13x) - 5$.)
• Two combinations worked:
a = 3, b = 1, c = 2, d = –5
a = –3, b = –1, c = –2, d = 5
Since only one pair of factors is possible,
I predicted they'd result in equivalent factors:
a = 3, b = 1, c = 2, d = –5 → $(3x + 1)(2x - 5)$
a = –3, b = –1, c = –2, d = 5 → $(-3x - 1)(-2x + 5)$
$(3x + 1)(2x - 5) = (-3x - 1)(-2x + 5)$ because, each results in the same product, $6x^2 - 13x - 5$. It's like 2 × 3 = -2 × -3.
This means you don't need to consider negative pairs of values when you list the possible values for a and c in ac.
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Solution 2
Find ac × bd, and ad + bd
$(ax + b)(cx + d) = \mathbf{ac}x^2 + (\mathbf{ad + bc})x + \mathbf{bd}$
$= \mathbf{6}x^2 \mathbf{- 13}x \mathbf{- 5}$
ac × bd = 6 × (–5) = –30 and ad + bc = –13

Find two numbers that multiply to ac × bd and add to ad + bc
5| 30
3| 6   Prime factors are 5, 3, and 2.
2   2 + 3(–5) = –13
2 × (–15) = –30   2 + (–15) = –13

Use the numbers to rewrite the middle term
$6x^2 \mathbf{- 13x} - 5 \rightarrow 6x^2 \mathbf{+ 2x - 15x} - 5$

Factor out the common factor in each pair
$6x^2 + 2x - 15x - 5 = (6x^2 + 2x) - (15x + 5)$
$= 2x(3x + 1) - 5(3x + 1)$

Factor out the common factor
$2x(3x + 1) - 5(3x + 1) = (2x - 5)(3x + 1)$

$6x^2 - 13x - 5 = (3x + 1)(2x - 5)$
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Thinking

• I found the prime factors of 30 (I ignored the negative sign for –30 at first) and then combined them in different ways until I had two numbers that added to –13. (I made one factor negative each time because they're factors of –30.)

• I broke the middle term into two terms so I'd have four terms.

• $6x^2 + 2x$ had a common factor of $2x$ and $15x + 5$ had a common factor of 5.

• $2x(3x + 1) - 5(3x + 1)$ had a common factor of $3x + 1$.
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Example 2 Factoring Using a Common Factor

Factor $6x^2 + 15x$.

Solution

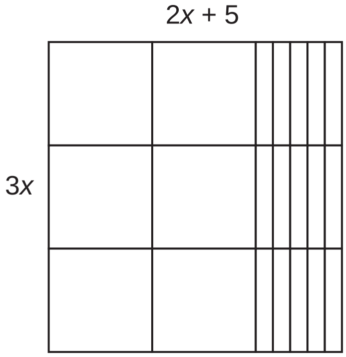
Factor using a common factor

$(6x^2 + 15x)\ 3x = 3x(2x) + 3x(5)$

Write the factors

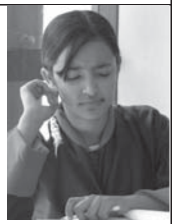
$6x^2 + 15x = 3x(2x + 5)$

Check using algebra tiles



$6x^2 + 15x = 3x(2x + 5)$

Thinking



• I noticed both terms had a common factor, so I factored $3x$ out of each term.

• I used an algebra tile model to check. I gathered 6 x^2-tiles and 15 x-tiles to represent $6x^2 + 15x$ and formed them into a rectangle with dimensions $3x$ by $2x + 5$.

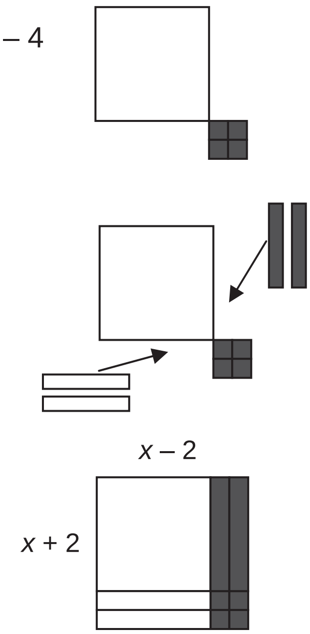
Example 3 Factoring a Difference of Squares

Determine the factors of each difference of squares.

a) $x^2 - 4$ **b)** $9x^2 - 25$

a) Solution

$x^2 - 4$



$x^2 - 4 = (x - 2)(x + 2)$

Thinking

• $x^2 - 4$ is called a difference of squares because x^2 and 4 are both perfect squares.

• I used algebra tiles to create an area model:

- I gathered tiles to represent $x^2 - 4$.
- Since the x^2-term and the constant were squares, I began by making two squares in those parts of the area model.
- I added tiles with no value ($-2x + 2x = 0$) to complete the rectangle.
- The dimensions of the rectangle were the factors.

b) Solution

$9x^2 - 25$

$9x^2$	$-?x$
$?x$	-25

	cx	$+ \; d$
ax	$acx^2 = 9x^2$	$-?x = adx$
$+ \; b$	$?x = bcx$	$bd = -25$

ac = 9

a	c
1	9
9	1
3	3

bd = –25

b	d
1	–25
–1	25
–25	1
25	–1
5	–5
–5	5

Since $-?x = adx$ and $?x = bcx$, then $-ad = bc$.

a = 3, c = 3, b = 5, d = –5 works because

- $ac = 3(3) = 9$
- $bd = 5(-5) = -25$
- $-ad = bc \rightarrow -(-5)(3) = (5)(3)$

If a = 3, b = 5, c = 3, d = –5, then
$(ax + b)(cx + d) = (3x + 5)(3x - 5)$

$9x^2 - 25 = (3x + 5)(3x - 5)$

Thinking

- I sketched an area model to represent $9x^2 - 25$:

- I knew the x^2-term and the constant so I filled them in.

- Since there was no x-term, the remaining two parts of the model had to add to 0, so I marked them as opposites, $?x$ and $-?x$.

- I used an algebraic expression to represent each dimension.

- I used the model to figure out the values of ac and bd and then listed possible pairs. (I didn't list the negative combinations for ac since you just end up with equivalent factors (see **example 1**).)

- I could tell from the model that –ad = bc.

- I tried combinations of a, b, c, and d until I found values that met all three conditions.

- a = 3, c = 3, b = –5, and d = 5 also worked but I ended up with the same answer:

$(3x - 5)(3x + 5)$
$= (3x + 5)(3x - 5)$

Example 4 Factoring a Perfect Square

Factor $x^2 - 20x + 100$.

Solution

$(x + a)(x + b) = x^2 + (a + b)x + ab$

$x^2 \mathbf{- 20}x + \mathbf{100} = x^2 + (\mathbf{a + b})x + \mathbf{ab}$
So $a + b = -20$ and $ab = 100$.

$a = -10$ and $b = -10$ because
- $a + b = -10 + (-10) = -20$
- $ab = (-10)(-10) = 100$

Since $a = -10$ and $b = -10$, then

$$\begin{aligned}(x + a)(x + b) &= (x - 10)(x - 10)\\ &= (x - 10)^2\end{aligned}$$

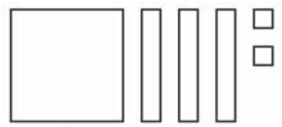
$x^2 - 20x + 100 = (x - 10)^2$

Thinking

- I used a different algebraic model to factor this polynomial because the coefficient of the x^2-term was 1. (The other model would still have worked but this was simpler.)
- -10 was the only value for a and b that satisfied both conditions.
- The two factors were identical, so I wrote them as a square. I guess that's why $x^2 - 20x + 100$ is called a perfect square.

Practising and Applying

1. a) Write the polynomial represented by these algebra tiles.



b) Sketch what the tiles would look like if they were rearranged in a rectangle.

c) Use your sketch to determine the factors of the polynomial.

2. a) Use algebra tiles to represent the polynomial $x^2 + x - 2$.

b) Add equal numbers of x-tiles and $-x$-tiles to form a rectangle.

c) What are the factors of $x^2 + x - 2$?

3. Use algebra tiles to factor each.

a) $4x^2 + 6x$ **b)** $x^2 - 2x - 3$
c) $x^2 + 4x + 3$ **d)** $x^2 - 9$
e) $9x^2 - 1$ **f)** $3x^2 + 5x + 2$

4. What is the value of each shape?

a) $(x + 3)(x + \square) = x^2 + \blacklozenge x + 12$
b) $(x + \square)(x - \blacklozenge) = x^2 - 81$
c) $(\square x + 2)(\blacklozenge x - \bigcirc) = 5x^2 + 9x - 2$
d) $(2x + \square)(\blacklozenge x - 2) = 6x^2 + \bigcirc x - 6$

5. Factor each difference of squares.

a) $x^2 - 121$ **b)** $x^2 - 400$
c) $25x^2 - 1$ **d)** $36x^2 - 25$

6. Factor.

a) $12x^2 + 18x$ **b)** $15x^2 - 25x$
c) $ax^2 + ax$

7. What is the value of each shape?

a) $(x + 3)^2 = x^2 + \square x + \blacklozenge$
b) $(x + \square)^2 = x^2 + \blacklozenge x + 36$
c) $x^2 - 12x + \square = (x - \blacklozenge)^2$
d) $x^2 + \square x + 49 = (x + \blacklozenge)^2$

8. Factor.

a) $x^2 + 8x + 15$ **b)** $x^2 - 7x + 6$
c) $x^2 + x - 12$ **d)** $x^2 - 6x - 16$
e) $x^2 - 8x + 16$ **f)** $x^2 - 64$

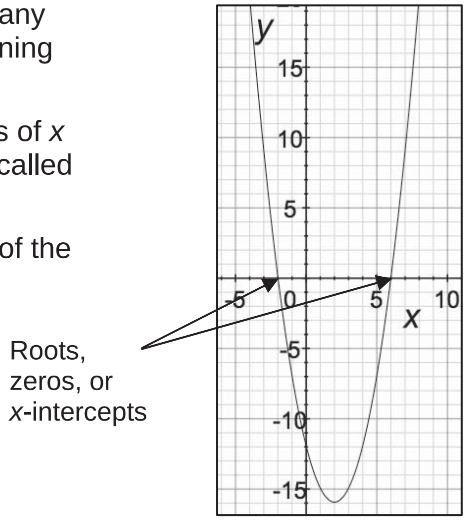
9. Factor.

a) $4x^2 - 12x + 9$ **b)** $2x^2 - x - 6$
c) $3x^2 - 11x - 4$ **d)** $10x^2 + 3x - 1$
e) $3x^2 - 11x + 6$ **f)** $49x^2 - 100$

10. How is factoring quadratic expressions similar to factoring numbers? How is it different?

5.2.2 EXPLORE: Roots of Quadratic Equations

- You can create a **quadratic equation** from any quadratic function $f(x) = ax^2 + bx + c$ by assigning a value, d, to $f(x)$. That is, $ax^2 + bx + c = d$.
- If d = 0, that is, if $ax^2 + bx + c = 0$, the values of x are the solutions of the equation, sometimes called the **zeros** or **roots** of the equation.
- The zeros or roots are also the x-intercepts of the graph of the function.



A. Graph each quadratic function.

i) $f(x) = (x - 2)(x + 2)$ **ii)** $f(x) = (x - 4)(x - 3)$

iii) $f(x) = (x + 5)(x - 1)$ **iv)** $f(x) = (x - 5)(x - 5)$

B. Describe how the roots of the equations below can be determined by examining the graphs in **part A**.

i) $(x - 2)(x + 2) = 0$ **ii)** $(x - 4)(x - 3) = 0$

iii) $(x + 5)(x - 1) = 0$ **iv)** $(x - 5)(x - 5) = 0$

C. i) Which equation in **part B** has only one root?

ii) How can you tell by examining the graph? the equation?

D. For each equation in **part B**, find all equivalent equations below.

i) $x^2 - 4 = 0$ **ii)** $x^2 - 10x + 25 = 0$ **iii)** $x^2 - 7x + 12 = 0$

iv) $x^2 + 5x - x - 5 = 0$ **v)** $x^2 - 5x - 5x + 25 = 0$ **vi)** $x^2 + 2x - 2x - 4 = 0$

vii) $x^2 - 7x + 16 = 4$ **viii)** $x^2 = 4$ **ix)** $x^2 - 10x = -25$

x) $x^2 + 4x - 5 = 0$

E. i) Show that these three equations are equivalent.

$(x - 6)(x + 1) = 0$ $x^2 - 5x - 6 = 0$ $x^2 - 5x = 6$

ii) Determine the roots.

iii) Which form in **part i)** did you use to determine the roots? Why?

F. How does the graph of $f(x) = x^2 + 9$ tell you that $x^2 + 9 = 0$ has no roots?

5.2.3 Solving Quadratic Equations by Factoring

Try This

Yangchen and Passang are playing a mystery number game. Yangchen says, "I'm thinking of a number. When you add it to its own square, the answer is 56."

A. Show that 7 is a possible value for the mystery number. Find another value.

- A quadratic equation can sometimes be written in factored form:

$$ax^2 + bx + c = 0 \rightarrow (ax + b)(cx + d) = 0$$

- The **zero product rule** states that, if a product of factors is zero, then at least one of the factors must be zero. So it makes sense that $(ax + b)(cx + d) = 0$ when $ax + b = 0$ or $cx + d = 0$.
- The solutions of the equation $(ax + b)(cx + d) = 0$ will be the solutions of the equations $ax + b = 0$ and $cx + d = 0$.

For example, to solve $x^2 – 7x + 10 = 0$:

$x^2 – 7x + 10 = 0 \rightarrow (x – 5)(x – 2) = 0$

$x – 5 = 0$ or $x – 2 = 0$, so $x = 5$ or 2

The two solutions of $x^2 – 7x + 10 = 0$ are $x = 5$ and $x = 2$.

You can check solutions by substituting them into the original equation.

- Before you can solve a quadratic equation this way, you need to change it to $ax^2 + bx + c = 0$ form.

For example, to solve $2x^2 + x = 15$:

$2x^2 + x = 15 \rightarrow 2x^2 + x \mathbf{– 15} = 15 \mathbf{– 15}$

$2x^2 + x – 15 = 0$

$(2x – 5)(x + 3) = 0$ $\quad$ $2x – 5 = 0$ or $x + 3 = 0$,

so $x = 2\frac{1}{2}$ or $–3$

The two solutions of $2x^2 + x = 15$ are $x = 2\frac{1}{2}$ and $x = –3$.

- You can model a real-world problem situation using a quadratic equation — one or both solutions of the equation will help you solve the problem. Even though mathematically both solutions are possible, in real-world applications, often only one of the solutions makes sense.
- The solutions of a quadratic equation are sometimes called the roots or zeros.
- If you were to graph the quadratic equation $ax^2 + bx + c = 0$, you would discover that the solutions of the equation are the x-intercepts of the parabola.

B. i) Write an equation to represent Yangchen's description in **part A**. How do you know it is a quadratic equation?

ii) Solve the equation. Are the solutions the same as your answers in **part A**?

iii) Why are only two solutions possible?

Examples

Example 1 Using Factoring to Solve a Quadratic Equation

Solve $6x^2 + 5x - 6 = 0$.

Solution

$6x^2 + 5x - 6 = 0$

$(3x - 2)(2x + 3) = 0$

$3x - 2 = 0$, so $x = \frac{2}{3}$

$2x + 3 = 0$, so $x = \frac{-3}{2}$

The solutions of $6x^2 + 5x - 6 = 0$ are $x = \frac{2}{3}$ and $\frac{-3}{2}$.

Thinking

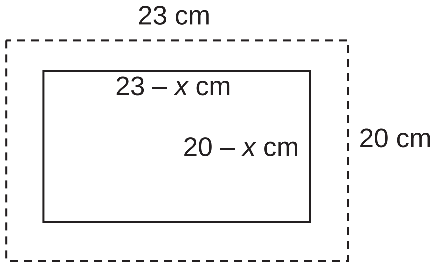
- I factored the quadratic expression in the equation.
- Since the product of the factors was zero, I knew at least one factor had a value of zero. That's why I found the values of x that made each factor equal to zero.

Example 2 Solving a Problem Using a Quadratic Equation

Buthri had a rectangular piece of paper 20 cm wide and 23 cm long. When she decreased the length and width by the same amount, the area decreased by 120 cm^2. Determine the dimensions of the new rectangle.

Solution

x represents the amount each dimension was decreased



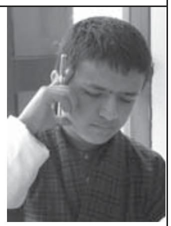
The dimensions of the new rectangle are $23 - x$ cm by $20 - x$ cm.

The area of the new rectangle is
$20 \text{ cm} \times 23 \text{ cm} - 120 \text{ cm}^2$
$= 460 \text{ cm}^2 - 120 \text{ cm}^2$
$= 340 \text{ cm}^2$

$$340 = (23 - x)(20 - x)$$
$$340 = 460 - 43x + x^2$$
$$340 - 340 = 460 - 340 - 43x + x^2$$
$$0 = 120 - 43x + x^2$$
$$0 = x^2 - 43x + 120$$

[Continued]

Thinking



- I drew a diagram to help me visualize the problem and organize all the given information.
- I knew the area of the new rectangle was 120 cm^2 less than the area of the original rectangle, which was 20 cm × 23 cm.
- I wrote an equation to represent the area of the new rectangle in terms of x.
- I expanded the factors and then rearranged the equation to get 0 on one side.

Example 2 Solving a Problem Using a Quadratic Equation [Continued]

Solution [Continued]

$0 = x^2 - 43x + 120 \rightarrow 0 = (x - 40)(x - 3)$

The possible solutions are $x = 40$ or 3.

- If $x = 3$, the new dimensions are 17 cm ($20 - 3 = 17$) and 20 cm ($23 - 3 = 20$).
- If $x = 40$, the dimensions are negative, so 40 cannot be a solution.

The new dimensions are 17 cm by 20 cm.

Thinking

- I factored the quadratic expression by finding two numbers that multiplied to 120 and added to –43. The numbers that worked were –40 and –3.

Example 3 Determining the *x*-intercepts and Vertex of a Parabola

Determine the *x*-intercepts and the coordinates of the vertex for the graph of the function $f(x) = 10x^2 - 13x - 3$.

Solution

Write a quadratic equation equal to 0

$f(x) = 10x^2 - 13x - 3 \rightarrow 10x^2 - 13x - 3 = 0$

Factor and then solve equation

$10x^2 - 13x - 3 = 0 \rightarrow (5x + 1)(2x - 3) = 0$

$5x + 1 = 0$, so $x = -\frac{1}{5}$ or $2x - 3 = 0$, so $x = \frac{3}{2}$.

The solutions, or *x*-intercepts are $-\frac{1}{5}$ and $\frac{3}{2}$.

Use x-intercepts to find x-coordinate of vertex

$(-\frac{1}{5} + \frac{3}{2}) \div 2 = \frac{13}{20}$

Use x-coordinate of vertex to find y-coordinate

$$f(\tfrac{13}{20}) = 10 \times \frac{169}{400} - 13 \times \frac{13}{20} - 3$$

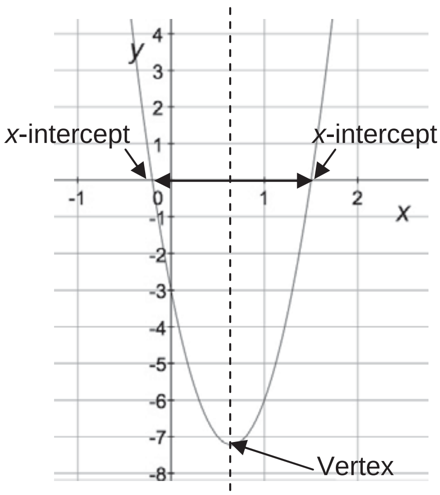
$$= -\frac{289}{40}$$

$$= -7\frac{9}{40}$$

The vertex coordinates are $(\frac{13}{20}, -7\frac{9}{40})$.

Thinking

- I knew the solutions of the equation were the *x*-intercepts of the graph of *f*(*x*). (This makes sense since the *x*- intercepts are the values of the *x*-coordinates when the *y*-coordinates are 0.)
- I found the *x*-coordinate of the vertex by calculating the mean of the *x*-intercepts (because the vertex is halfway between them). Then I used the *x*-coordinate to calculate the *y*-coordinate.



Practising and Applying

1. Solve each equation.

a) $(x - 4)(x - 2) = 0$

b) $(x + 3)(x - 9) = 0$

c) $(2x + 5)(3x - 1) = 0$

d) $(5x + 4)(2x - 9) = 0$

e) $15x(x - 10) = 0$

f) $(x + 2)(x - 2) = 0$

2. Factor and then solve each equation.

a) $x^2 + 8x + 15 = 0$

b) $x^2 - x - 110 = 0$

c) $10x^2 + 30x = 0$

d) $x^2 - 36 = 0$

e) $x^2 - 20x + 100 = 0$

f) $5x^2 + 20x + 20 = 0$

3. Factor and then solve each equation.

a) $2x^2 - 5x - 25 = 0$

b) $8x^2 + 2x - 1 = 0$

c) $2x^2 + 5x - 3 = 0$

d) $6x^2 - 6x - 12 = 0$

4. Factor and then solve each equation.

a) $x^2 - 6x = 7$ **b)** $x^2 + 10 = 7x$

c) $x^2 + 5x + 12 = 12x$

d) $x^2 + 3x = 3x + 25$

5. When the squares of two consecutive integers are added, the sum is 221.

a) Write an algebraic equation to represent the situation.

b) Solve the equation.

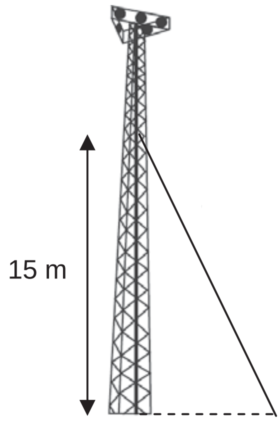
c) What is the meaning of the solutions of the equation?

6. The length of a rectangle is 6 cm longer than its width. The area of the rectangle is 91 cm^2. Determine the dimensions of the rectangle.

7. One leg of a right triangle is 2 cm longer than the other leg. The hypotenuse is 10 cm long. Determine the length of the shorter leg.

8. A rectangular play area is twice as long as it is wide. If you increase the length by 4 m and decrease the width by 3 m, the new area will be 532 m^2. Determine the dimensions of the original rectangle.

9. A support wire is attached to a radio antenna at a height of 15 m. The wire is 1 m longer than twice the distance from the antenna's base to the place where the wire is anchored to the ground. Determine the distance from the base of the antenna to the place where the wire is anchored to the ground.



10. The hypotenuse of a right triangle is 1 m longer than the longer leg. The other leg is 7 m shorter than the longer leg. Determine the lengths of the three sides of the triangle.

11. When the squares of two consecutive odd integers are added, the sum is 290. What are the integers?

12. A model rocket was launched with an initial speed of 200 m/s. The function $h(t) = -5t^2 + 200t$ calculates the height of the rocket t seconds after the launch. After how many seconds was the rocket at a height of 1500 m?

13. Look back at the questions on this page. In what sorts of situations might you encounter problems where solving a quadratic equation would help you solve the problem?

5.2.4 EXPLORE: Absolute Value Equations

You can represent the solutions of an absolute value function by graphing it, just like you can with a quadratic function. You can use your graph to find specific solutions, or values of x when $f(x)$ is a certain value. And, just like with quadratic functions, you can write an **absolute value equation** and solve it to find specific solutions.

For example, $f(x) = |x - 5| + 2$ can be written as $|x - 5| + 2 = 7$ and then solved to find possible values of x when $f(x) = 7$.

Recall that an absolute value can be thought of as a distance from 0 on a number line, regardless of direction. The absolute values $|5|$ and $|-5|$ are both equal to 5, since they are both 5 units away from 0. When you solve $|x - 5| + 2 = 7$, you are finding values of x that are $7 - 2$ units away from 5.

A. i) Graph the absolute value function $f(x) = |x - 5| + 2$.

ii) Use the graph to determine the values of x when $f(x) = 7$.

iii) Why are these values also the solutions of the equation $|x - 5| + 2 = 7$?

B. i) If $|x - 5| + 2 = 7$, what is the value of $|x - 5|$? Why?

ii) What are the values of $x - 5$? Why is there more than one value?

iii) What are the values of x? How do you know?

C. In **part B**, you used a "working backwards" approach to solving an absolute value equation. Use the same approach to solve each equation below. Verify each pair of solutions by graphing its function.

i) $|x + 7| = 13$

ii) $2|x - 1| = 8$

iii) $\frac{1}{2}|x + 3| - 4 = 1$

iv) $|2x - 4| + 3 = 6$

GAME: Get the Points

This is a game for two players. You will need dice and grid paper.

• Each player creates a coordinate grid with axes that go from –6 to 6 and then rolls a pair of dice 10 times to generate 10 ordered pairs (odd numbers are negative, even numbers are positive). Each player then plots these points on his or her own grid.

• Players take turns. On each turn, the player creates a quadratic or absolute value function and then graphs it. The goal is to create a graph that passes through as many of the plotted points as possible.

• The game continues for a pre-set amount of time, such as 10 minutes. The winner is the player who has drawn graphs through the most plotted points.

UNIT 5 Revision

1. a) Which functions are equivalent?

A $f(x) = 3x + 2$

B $f(x) = 4x^2 + 8x + 16$

C $f(x) = 4(2 + x)^2$

D $f(x) = 4x^2 + 16x + 16$

b) Sketch the graph of the equivalent functions. How do the graphs show the functions are quadratic? How do they show the functions are equivalent?

c) How else could you show the functions are equivalent?

2. a) Sketch the graph of each function.

i) $f(x) = 0.4x^2 + 1.2x – 3$

ii) $f(x) = 4x^2 – 10x + 7$

b) Estimate the minimum value and x-intercepts of each function.

3. How could you predict that the graph of $f(x) = 4x^2 – x + 8$ has no x-intercepts without graphing it?

4. Kinley currently sells about 45 small baskets a month for Nu 50 each. He predicts that for every increase in price of Nu 10 per basket, he will sell two fewer baskets.

a) Write a function to represent his total sales in terms of the number of price increases of Nu 10.

b) Sketch a graph of the function.

c) What is the best price for Kinley to charge? How do you know?

5. a) What are the zeros, or x-intercepts of each?

i) $f(x) = (x – 4)(x + 3)$

ii) $f(x) = 3(x – 6)(x – 2)$

iii) $f(x) = (2x – 5)(3x + 3)$

iv) $f(x) = (x – 0.5)(x + 2.5)$

v) $f(x) = (3x – 1.8)(x + 5)$

b) What are the coordinates of the vertex for each function?

c) Sketch the graph of each function.

6. $f(x) = (x – 4)(x – h)$
For what values of h does the vertex of this parabola have a positive x-coordinate? Explain.

7. a) What geometric transformations, in what order, should be applied to $f(x) = x^2$ to result in each function?

i) $f(x) = x^2 – 30$

ii) $f(x) = (x + 30)^2$

iii) $f(x) = –2x^2 – 30$

iv) $f(x) = 3(2 – x)^2$

v) $f(x) = (x + 30)^2 – 7$

vi) $f(x) = 8 – 0.1x^2$

b) What are the coordinates of the vertex of each function?

c) Describe the transformations using mapping notation. For a composite transformation, use one mapping.

8. If you were to apply each composite transformation to the function $f(x) = x^2$, what would be the final function?

a) translate 4 right and translate 3 up

b) compress vertically by $\frac{1}{2}$ and translate 6 up

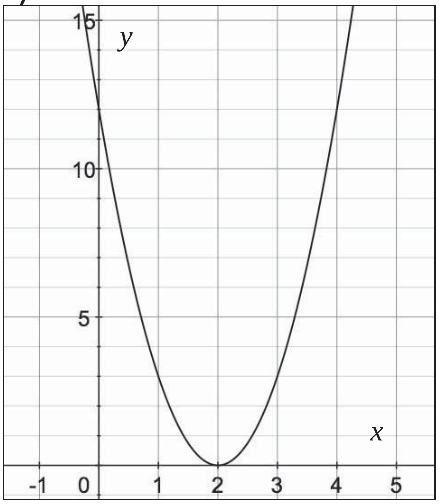
c) translate 3 left, reflect in the x-axis, and stretch vertically by 3

d) $(x, y) \rightarrow (x – 8, y + 2)$

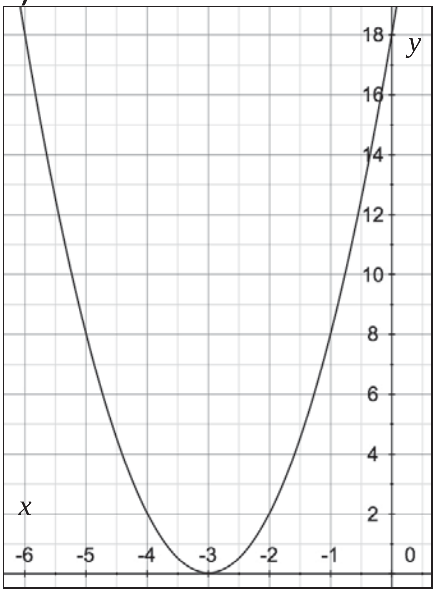
e) $(x, y) \rightarrow (x – 8, 3y + 2)$

9. What is the equation of each graph?

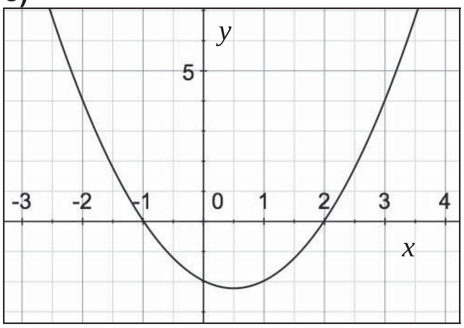
a)



b)



c)



10. Sketch each graph, if $f(x) = |x|$.

a) $f(x + 3)$ **b)** $f(2x + 4)$

c) $f(2x) + 4$ **d)** $-3f(2x - 3)$

11. Draw or use algebra tiles to factor.

a) $6x^2 - 8x$ **b)** $3x^2 + 12x + 12$

c) $x^2 + 6x + 8$ **d)** $x^2 - 2x - 3$

e) $2x^2 + x - 1$ **f)** $6x^2 + x - 1$

12. Factor each.

a) $16x^2 - 4$

b) $16x - 12x^2$

c) $25x^2 + 30x + 9$

d) $x^2 + 3x - 88$

13. Fill in the blanks.

a) x^2 + __ x + 25 = (x + __)^2

b) (3x – __)^2 = __x^2 – 24x + __

14. Solve each.

a) $2x^2 - 5x + 2 = 0$

b) $6x^2 - 7x = 3$

c) $2x + 14 = -x + 4 + x^2$

15. Write an equation in the form $ax^2 + bx + c = 0$ that would match each pair of roots or zeros.

a) 3 and 8 **b)** –2 and –7

c) $\frac{3}{4}$ and $\frac{4}{5}$ **d)** $\frac{3}{5}$ and $\frac{6}{5}$

16. A ladder is leaning against a wall. The top of the ladder is 9.4 m up the wall. The ladder is 6.6 m longer than the distance from the bottom of the wall to the bottom of the ladder. How far is it from the bottom of the wall to the bottom of the ladder?

17. The difference between the squares of two consecutive integers is 35. What are the numbers?

18. Solve each.

a) $6 + |x| = 8$

b) $|3x - 2| = 6$

c) $|3x + 1| + 4 = 10$

UNIT 6 DATA, STATISTICS, AND PROBABILITY

Getting Started

Use What You Know

Dema's classmates ran a 100 m race and recorded these times (in seconds):

13.9	14.3	14.4	13.7	15.2	15.4	13.9
13.9	14.5	14.7	14.4	13.8	13.1	13.8
12.4	13.8	12.7	13.4	13.9	14.0	14.4
14.3	14.5	11.8	12.9	12.3	12.8	13.7
13.1	15.0	14.8	14.2	14.4	14.8	15.2

A. Determine each value:
- the maximum value and minimum value
- the range
- the median
- the lower and upper quartiles

B. Create a box and whisker plot of the data using the values from **part A**.

C. Use your box and whisker plot to answer these questions.

i) What percentage of the data values lies below 13.9 s?

ii) What percentage of the data values lies between 13.4 s and 14.4 s?

iii) What percentage of the data values lies below 13.4 s?

D. Pema ran the race in a time of 12.4 s. Compare her time to the times of the other students using your box and whisker plot.

Skills You Will Need

1. The World Health Organization collected data about residents of Thailand who were over 60 years old. Each number represents the percentage of each gender that assists in daily household chores.

Gender	Food preparation	Cleaning house	Sewing/ mending	Washing dishes	Washing/ ironing clothes	Helping to garden
Male (%)	22.2	28.5	7.3	20.0	17.0	34.3
Female (%)	49.8	51.4	30.0	48.3	43.1	22.7

a) Construct a double bar graph.

b) What conclusions can you make about the data?

2. Here are the times of a group of runners in a 400 m race (in seconds):

66.0, 58.9, 61.1, 69.0, 53.5, 64.3, 53.5, 51.1, 55.2, 54.8, 66.0

Find the mean, median, and mode(s) for the data set.

3. Here are the daily maximum temperatures (^{o}C) for a summer day in various cities in Europe. Create a stem and leaf plot for the data.

21.1	18.7	30.0	23.5	24.9	25.1
27.2	25.4	19.6	26.7	24.8	30.3
27.1	19.4	22.9	29.1	26.6	19.7
23.5	28.4	26.8	21.2	23.1	25.0

4. This chart shows population characteristics for some countries.

Country	Life expectancy (years)	Per capita income (U.S. $)	Number of people per motor vehicle
Bangladesh	57	230	1200
Brazil	66	3,370	9
Canada	78	19,570	1.6
China	70	530	225
Ethiopia	50	130	800
Iceland	79	24,590	1.8
India	59	310	225
Japan	80	34,630	2.1
Mexico	73	4,010	9
Norway	78	26,480	2.2

a) Create a scatter plot of Life expectancy vs. Per capita income. Use Per capita income as the independent variable.

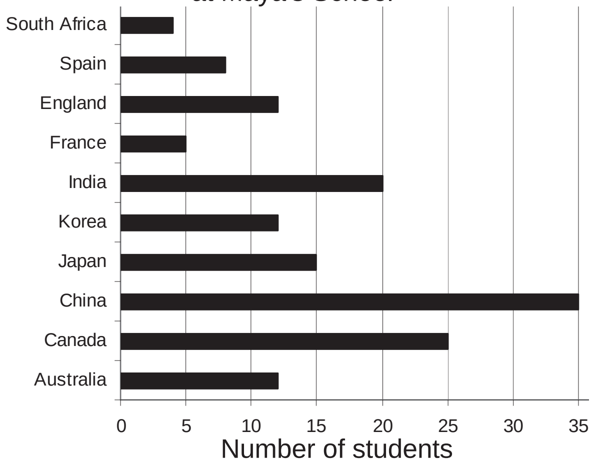
b) Create a scatter plot of Life expectancy vs. Number of people per motor vehicle. Use Number of people per motor vehicle as the independent variable.

c) For each graph, where appropriate, draw a line of best fit and comment on the type and strength of the correlation.

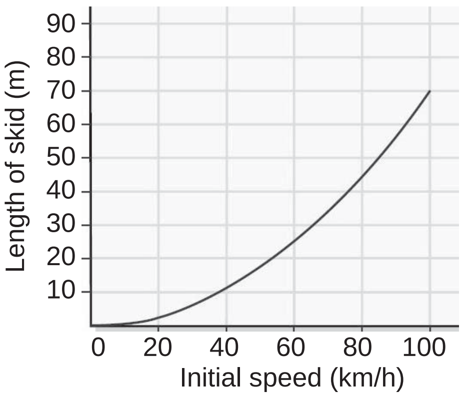
d) Do the correlations you described in **part c)** make sense? Explain.

5. For each graph indicate whether the data is discrete or continuous and whether the graph deals with one or two variables. If appropriate, identify the independent and dependent variables.

Number of Students from Different Countries at Maya's School



Length of a Car Skid When Brakes are Applied at Various Speeds



6.1.1 Histograms and Stem and Leaf Plots

Try This

Two groups of tomato plants were sampled to create the data set below. One group was grown in Southern Bhutan while the second group was grown in Central Bhutan. Twenty plants were selected at random from each group and the number of tomatoes each plant produced was listed.

Grown in Southern Bhutan										Grown in Central Bhutan									
9	10	15	12	13	11	7	12	8	14	6	8	9	11	12	8	9	7	10	12
11	16	17	12	10	8	12	9	6	13	5	9	11	7	11	6	7	9	8	11

A. What conclusion might you draw from the data?

B. i) Complete the frequency table for each set of data.

Number of tomatoes/plant	Frequency (Southern Bhutan)	Frequency (Central Bhutan)
1 – 5		
6 – 10		
11 – 15		
16 – 20		

ii) Does your frequency table support your conclusion in **part A**?

C. From which organization of the data is it easier to draw conclusions — from the original list of numbers or from the frequency table? Explain.

- When you have large quantities of data, it is convenient to organize the data into **intervals**, sometimes called **bins**, to make it easier to draw conclusions.
- **Stem and leaf plots** and **histograms** can be used to graph the organized data. In both cases, you create bins of equal width. The number of pieces of data in each bin, called the **frequency**, is represented by the number of leaves.
- The bins for stem and leaf plots are based on place value. The bin sizes are always powers of ten, but you can usually choose how many digits to use for the "stems" and "leaves."

For example, in the stem and leaf plot to the right, data about heights of Class X girls (in cm) has been grouped into bins of size 10, with two digits in the stem and one digit in the leaves. The bins are 120 to 129, 130 to 139, 140 to 149, and so on.

With the 20 data values organized in seven bins, the data set is manageable for drawing conclusions about how it is distributed.

Heights of Class X Girls

Stems	Leaves
12	7
13	5 8 9
14	1 2 6 8
15	0 5 6 6 7 8
16	3 3 6 7 8
17	
18	1

Heights of Class 10 Girls

12	7
13	5 8 9
14	1 2 6 8
15	0 5 6 6 7 8
16	3 3 6 7 8
17	
18	1

This stem represents the interval 140 – 149.

The frequency for this stem is 4.

The modes are 156 and 163.

155 and 156 are the middle values so the median is 155.5.

The minimum value is 127. The maximum value is 181. The range is 181 – 127 = 54.

• Every number in the data set is visible in a stem and leaf plot, so it is easy to determine the **minimum and maximum values**, **range**, **median**, and **mode**(s). The median is the middle value in the data set when it is arranged in order. Since the values in a stem and leaf plot are in order and there are 20 values in this set of data, the median is the mean of the 10th and 11th values.

• Sometimes the range of a set of data is either so great or so small that you might want to set up the stems and leaves using different place value arrangements.

For example:

1	25 27 53
2	15 24
3	45
4	67 99

For the data set 125, 127, 345, 467, 499, 224, 215, 153, you might use a stem that represents the hundreds digit and leaves that represent the tens and ones.

0	.49
1	.24 .78
2	.30 .92
3	.78

For the data set 1.24, 1.78, 2.30, 3.78, 2.92, 0.49, you might use a stem that represents the ones digit and leaves that represent the hundredths.

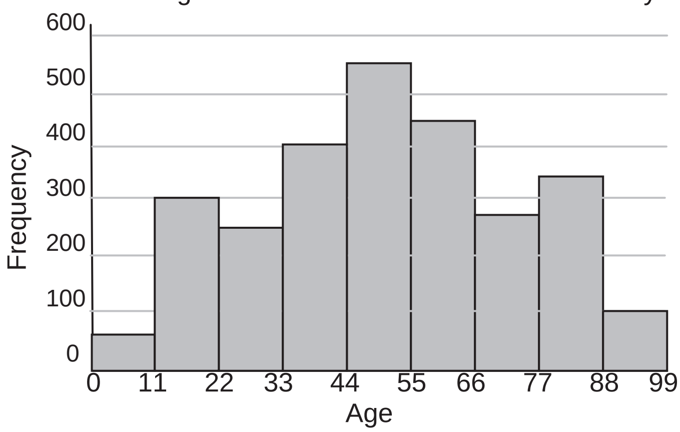
• Histograms use a series of connected bars to represent the frequency of **continuous** data, often measurements, organized into equal bins. You usually create a **frequency table** first and then construct the histogram.

For example, in Sonam's histogram below, the bars are connected to show the data is continuous.

Ages of Residents in Sonam's Community

Age	Frequency
0 – 11	50
11 – 22	300
22 – 33	250
33 – 44	400
44 – 55	550
55 – 66	430
66 – 77	270
77 – 88	340
88 – 99	100

Ages of Residents in Sonam's Community



• The bars in a histogram are of equal width and the height of each bar corresponds to the frequency of the bin it represents. Histograms look like bar graphs, but bar graphs have spaces between the bars because they represent discrete data and histograms do not because they represent continuous data.

• If you look at the horizontal axis, you will see values like 11, 22, 33, ... appearing both as the end point of one bar and as the beginning of the next bar. The actual data values for 11, 22, 33, ... are always plotted in the next bar up. For example, an age of 11 would go in the second bar, and an age of 22 in the third bar.

• The individual pieces of data are not visible in a frequency table or histogram. For this reason, the minimum and maximum values and the median cannot be determined directly from the table or histogram.

• Typically, the size of the bin for a histogram is chosen so that there is a minimum of five bins and a maximum of 12 bins, although any number of bins is possible. Too many or too few bins results in a histogram that may not effectively show how the data values are distributed.

• The bin width or size, can be determined by dividing the range of the data by the desired number of bins and then rounding to a suitable bin width.

$$\text{Bin width} \approx \frac{\text{Range}}{\text{Desired number of bins}}$$

For example, for the 2690 data values in Sonam's histogram on **page 168**, the minimum value was 1 and the maximum was 98, so the range was 98 – 1 = 97. Sonam wanted nine bins, so she divided the range by 9: 97 ÷ 9 = 10.8. Because rounded bin widths of 10, 20, 30, ... are usually used to make the data easier to work with, she chose bins of width 10.

• Note that the conclusions drawn from a histogram can be influenced by the bin width used because the bin width will affect the appearance of the **data distribution**. This will be investigated later in the unit.

D. i) Create a stem and leaf plot for each set of tomato data in **part A**.
ii) Do the stem and leaf plots support the conclusions you made earlier? Explain.

Examples

Example 1 Creating Graphs to Show How a Data Set is Distributed

Tshewang's teacher recorded the test scores of his students. He wanted to see how the results were distributed so he would know how well the class had performed.

Test Scores (%)

71	78	92	79	75	88	73
72	88	44	79	70	99	75
75	92	72	89	58	98	85
80	94	52				

a) Create a graph to show how the test scores were distributed. Find the minimum and maximum values, range, mode, and median.

b) Describe the distribution of the data. How do you think Tshewang's teacher will summarize the results? [Continued]

Example 1 Creating Graphs to Show How a Data Set is Distributed [Cont'd]

a) Solution 1

Test Scores (%)

4	4
5	2 8
6	
7	0 1 2 2 3 5 5 5 8 9 9
8	0 5 8 8 9
9	2 2 4 8 9

Maximum is 99

Minimum is 44

Range is 99 – 44 = 55

Mode is 75

Median is (78 + 79) ÷ 2 = 78.5

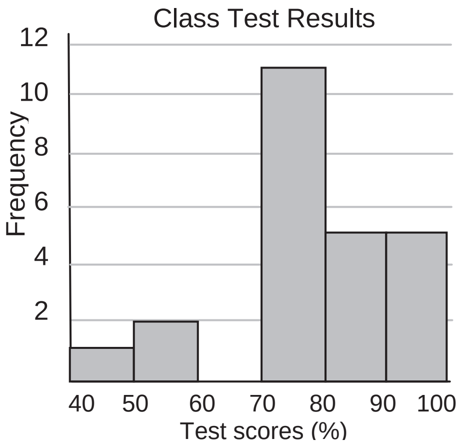
Thinking

- I used the tens digit for the stems and the ones digits for the leaves, since the data values were all two-digit numbers.
- I wrote the leaves for each stem in the right column, in increasing order.
- The minimum and maximum scores were easy to find because the values are in order.
- For the mode, I looked for numbers that repeated in the leaves and 75 repeated the most.
- There are 24 data values, so the median is the mean of 12th and 13th values.

a) Solution 2

Test Scores

Percent	Frequency
40 – 50	1
50 – 60	2
60 – 70	0
70 – 80	11
80 – 90	5
90 – 100	5



Thinking

- I created a frequency table using a bin width of 10 because I wanted 6 bins and a bin width of 10 was easy to work with.
- For each bin, I counted the number of data values in the set and then recorded the total in the table.
- I used the frequency table to create the histogram. I remembered to put 70 in the 70 to 80 bin and 80 in the 80 to 90 bin.

Original data

71	78	92	79	75	88	73
72	88	44	79	70	99	75
75	92	72	89	58	98	85
80	94	52				

In order from least to greatest

44, 52, 58, 70, 71, 72, 72, 73, 75, 75, 75, 78, 79, 79, 80, 85, 88, 88, 89, 92, 92, 94, 98, 99

Maximum is 99

Minimum is 44

Range is 99 – 44 = 55

Mode is 75

Median is (78 + 79) ÷ 2 = 78.5

• I knew I couldn't find the maximum, minimum, range, mode, and median directly from the histogram so I went back to the original data and ordered the values from least to greatest.

• Because there was an even number of values, 24, I knew the median was the mean of the two middle values, the 12th and 13th values.

• One of my classmates graphed the data in a stem and leaf plot with the same intervals. When we turned the stem and leaf plot, it showed the data distribution in the same way as the histogram.

4	4
5	2 8
6	
7	0 1 2 2 3 5 5 5 8 9 9
8	0 5 8 8 9
9	2 2 4 8 9

b) Solution

• 21 out of 24 students, which is most of the students, scored 70% or more on the test.

• 11 students earned marks between 70% and 79%.

His teacher would say the class as a whole performed well on the test.

Thinking

• I looked to see where most of the pieces of data were clustered. They were between 70% and 100%.

• I then looked for the bin with the greatest number of values, which was 70 to 80.

Example 2 Creating a Double Stem and Leaf Plot

The points scored for and against a basketball team are listed below.

Points for: 129, 128, 134, 125, 132, 107, 97, 127, 118, 135, 127, 114, 109, 118

Points against: 113, 127, 132, 109, 101, 90, 88, 105, 109, 122, 119, 102, 110, 97

a) Create a double stem and leaf plot for the data.

b) Why is a double stem and leaf plot a suitable graph for this set of data?

c) What conclusions can you draw from the data? Explain.

Solution

a) Points Scored

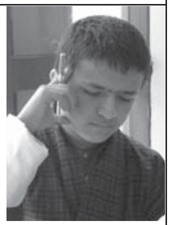
For Against

Leaves	Stems	Leaves
	8	8
7	9	0 7
9 7	10	1 2 5 9 9
8 8 4	11	0 3 9
9 8 7 7 5	12	2 7
5 4 2	13	2

b) A double stem and leaf plot is a suitable graph because it allows you to examine the distribution in each set of data and also to compare the distribution in both sets of data h.

c) The team seems to score more points than it allows in a game. This team likely has a winning record, because it appears that they score more points than are scored against them.

Thinking



a) The data values went from numbers in the 80s to the 130s, a range of about 60, so it made sense to make the stems the tens.

• Since I was creating a double stem and leaf plot, I created a table with three columns with the stems in the middle because both sets of data share the same stems.

• I put the leaves for the "points for" data in the left column, arranging these numbers in decreasing order, and the leaves for the "points against" in the right column, arranged in increasing order.

c) I looked first at the plot on the left (points for). The team scored under 120 points in six games and over 120 points in eight games.

• Then I looked at the plot on the right (points against). In 11 games, they allowed opposing teams to score over 120 points against them in only three games.

Example 3 Creating and Interpreting Histograms

This data set shows the value of purchases, in ngultrums, by 30 tourists at a market.

420	1295	875	600	780	1110
595	960	1030	775	680	1450
935	795	840	680	570	860
1150	945	855	670	990	1225
995	480	650	1000	1135	865

a) Create a frequency table and a histogram for the data.

b) What is the typical amount each tourist spends? Explain.

Solution 1

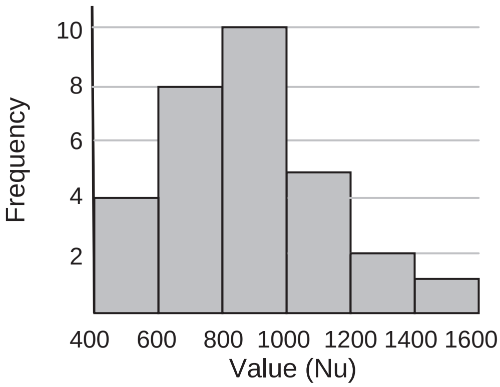
a) Range is 1450 – 420 = 1030

Bin width is 1030 ÷ 5 = 206, or about 200

Value of Tourist Purchases

Value of groceries (Nu)	Tally	Frequency
400 – 600	////	4
600 – 800	~~////~~ ///	8
800 – 1000	~~////~~ ~~////~~	10
1000 – 1200	~~////~~	5
1200 – 1400	//	2
1400 – 1600	/	1

Value of Tourist Purchases



b) The typical amount spent per tourist was between Nu 600 and Nu 1200 (23 out of 30, or about 75% of the purchases were between Nu 600 and Nu 1200).

Thinking

a) I wanted about five bins. Since the range was about 1000, I divided the range by 5 and that gave me a bin width of about 200.

• I realized while creating the frequency table, that I needed a sixth bin for the data value 1450, so I just added an extra row to the table.

• To complete the frequency table, I included a tally column to keep track as I counted the values in each bin.

• I constructed a histogram from the frequency table.

b) I looked for where the data clustered. The bars between 600 and 1200 were tallest so I added the frequencies for those three bars: 8 + 10 + 5 = 23.

Example 3 Creating and Interpreting Histograms [Continued]

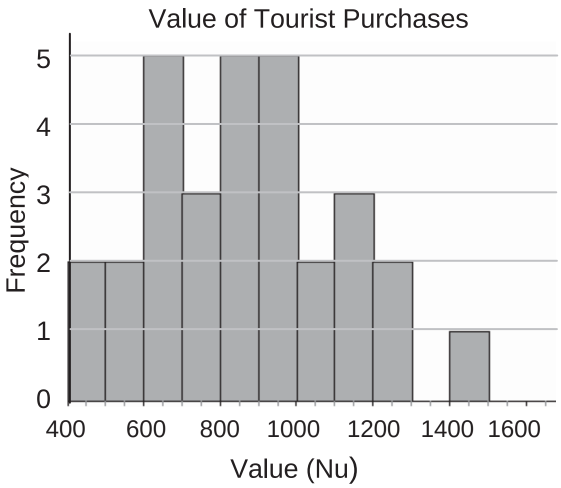
Solution 2

a) Range is 1450 – 420 = 1030

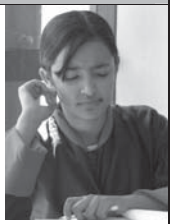
Bin width is 1030 ÷ 10 = 103, or about 100

Value of Tourist Purchases

Value of groceries (Nu)	Tally	Frequency
400 – 499	//	2
500 – 599	//	2
600 – 699	~~////~~	5
700 – 799	///	3
800 – 899	~~////~~	5
900 – 999	~~////~~	5
1000 – 1099	//	2
1100 – 1199	///	3
1200 – 1299	//	2
1300 – 1399		0
1400 – 1499	/	1



Thinking



a) I wanted about 10 bins. Since the range was about 1000, I divided the range by 10 and that gave me a bin width of about 100.

• I realized while creating the frequency table, that I needed an 11th bin for the data value 1450, so I just added an extra row to the table.

• To complete the frequency table, I included a tally column to keep track as I counted the values in each bin.

• I constructed a histogram from the frequency table.

b) The typical amount spent per tourist was between Nu 600 and Nu 1000 (18 out of 30, or about 60% of the purchases were between Nu 600 and Nu 1000).

b) I looked for where the data clustered. The bars between 600 and 1000 were tallest so I added the frequencies for those four bars: 5 + 3 + 5 + 5 = 18.

Practising and Applying

1. Is each statement true or false?

a) Histograms and stem and leaf plots display data grouped in intervals.

b) Bins in a histogram are always based on place value.

c) You can accurately determine the median, maximum, and minimum values of a data set directly from a histogram.

d) Histograms have no spaces between the bars because the data values are continuous.

e) Any stem and leaf plot of continuous data can be made into a histogram.

f) All histograms can be made into stem and leaf plots.

2. The quality control manager at a biscuit factory wants to ensure a consistent product, so the number of biscuits in every 100th tin is counted.

Number of Biscuits per Tin

135	154	188	137	123	151
123	119	108	119	143	150
129	144	123	145	127	126
127	132	133	127	142	117
122	137	96	99	122	134
132	128	107	150	108	125

a) Create a stem and leaf plot.

b) Use your intervals in the stem and leaf plot to create a histogram.

c) Why would these types of graphs be useful for the manager?

3. The following times, in seconds, were recorded for a 100 m sprint.

13.9	14.3	14.4	13.7	15.2	15.4
13.9	14.5	14.7	14.4	13.8	13.1
12.4	13.8	12.7	13.4	13.9	14.0
14.3	14.5	11.8	12.9	12.3	12.8
13.1	15.0	14.8	14.2	14.4	14.8
13.9	13.8	14.4	13.7		

3. a) Organize the data into a frequency table and create a histogram.

b) What conclusions can you make about how the runners compare?

4. Display the following data in a double stem and leaf plot.

Lengths of stories (number of words)

Group 1: 150, 183, 287, 599, 163, 298, 376, 422, 355, 246, 168, 478, 505, 355

Group 2: 324, 277, 256, 189, 324, 385, 412, 276, 299, 199, 310, 255, 290, 389

5. Use the stem and leaf plot to answer the questions below.

Vehicle Speeds on Canadian Highways (km/h)

Stems	Leaves
8	0 5 5 7 7 7 9 9 9 9 9
9	2 3 3 4 4 6 6 6 6 8 8 9 9 9 9 9
10	0 1 1 1 1 4 6 8 8 8 9

a) How many vehicles had their speeds measured?

b) How many vehicles were travelling under 90 km/h?

c) If the maximum speed limit was 90 km/h, how many vehicles were exceeding the limit?

d) What was the range of speeds?

e) What was the median speed?

6. Based on the data in the double stem and leaf plot below, Aparna concludes that females spend a lot more time watching television than males. Is her conclusion valid? Explain.

Hours Per Week Spent Watching TV

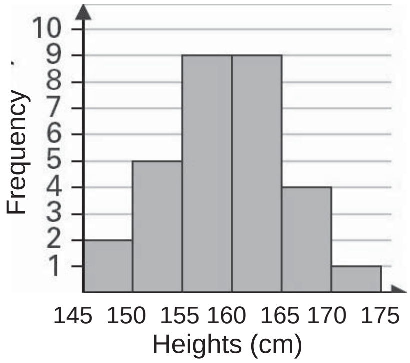
Female		Male
5	1	1 5 5 6 8 9
5 2	2	3 4 6
8	3	1
9 8 6	4	2

7. For each graph below

a) make three observations

b) discuss whether you think the bin widths or intervals used are appropriate

i) Heights of Students in Rishi's Class



ii) Number of Passengers on a Train During a Six-Week Period

Stems	Leaves
9	5 7 9
10	0 0 3 5 8 9
11	2 4 4 6 8 8 8
12	3 4 7 7 9 9 9 9
13	2 4 4 6 8
14	2 3 3 3 5
15	0 1

8. In a study of the longevity of a particular species of cat, biologists recorded the lifespans of 30 cats.

Lifespans of Cats (in years)

12.9	13.2	14.1	13.9	12.8
13.1	13.2	13.6	13.0	13.4
12.9	13.3	11.8	12.8	14.6
10.4	14.8	11.5	13.5	13.6
9.6	14.5	13.5	13.8	14.4
13.1	13.6	12.8	12.9	13.3

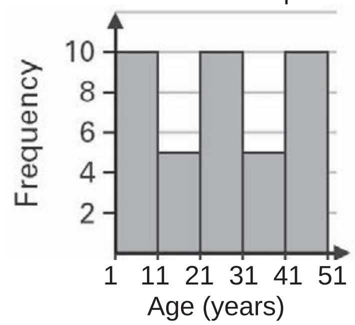
a) Create a stem and leaf plot and a histogram for the data above. Use the same bin width for both.

b) Which is more helpful for finding the median lifespan of a cat? Why?

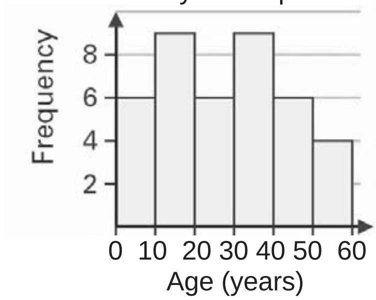
c) How are the displays the same? How are they different?

9. Two students drew histograms for the same set of data about the ages of 40 people.

Yuden's Graph



Maya's Graph



a) Make a conclusion based on Yuden's graph.

b) Make a conclusion based on Maya's graph.

c) Compare your conclusions above and discuss why they are different.

10. Why does it make sense to organize a set of data that contains a large number of values into equal intervals, or bins, in order to draw conclusions about the data?

6.1.2 EXPLORE: Investigating Bin Width in Histograms

It can be unhealthy for a person's body fat to exceed a certain percentage of his or her total body mass. This is why body fat is measured as part of an overall fitness assessment.

The chart below shows the body fat percentage of a random sample of 50 American men. This data was collected as part of a study to find out about the health of the population.

Body Fat (%) in 50 American Men

12.3	20.8	15.6	11.8	31.6
5.9	21.2	17.7	21.3	32.0
25.3	22.1	14.0	32.3	7.7
10.4	20.9	3.7	40.1	13.9
28.7	29.0	7.9	24.2	10.8
20.9	22.9	22.9	28.4	5.6
19.2	16.0	3.7	35.2	13.6
12.4	16.5	8.8	32.6	4.0
4.1	19.1	11.9	34.5	10.2
11.7	15.2	5.7	32.9	6.6

This data set can be used to explore how changing the bin width in a histogram affects our impressions about the distribution of the data and the nature of the conclusions we can draw from the graph.

A. i) Group the data using a bin width of 5 (%) and create a frequency table. Start the first bin at 0 (%).

ii) Create a histogram for the frequency table you created.

iii) Based on this histogram, what is the most common body fat percentage in American males?

B. i) Group the data using a bin width of 6 (%) and create a frequency table. Start the first bin at 0 (%).

ii) Create a histogram for the frequency table you created.

iii) Based on this histogram, what is the most common body fat percentage in American males?

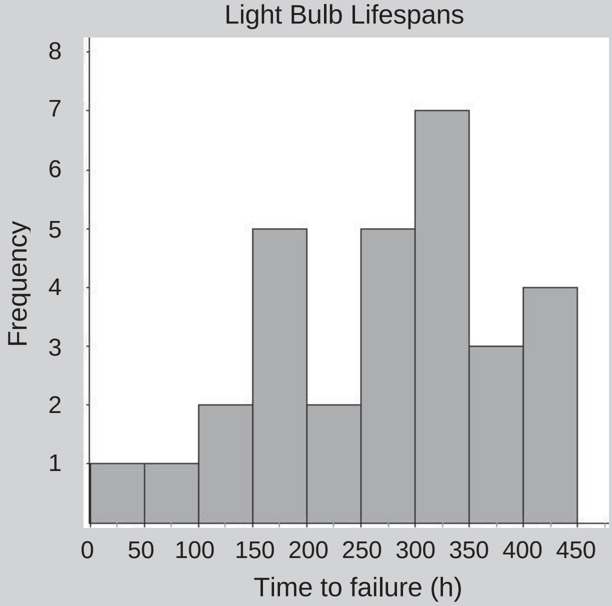
C. Discuss how the bin width used to create histograms affects the conclusions that can be drawn from these graphs.

6.1.3 Histograms and Box and Whisker Plots

Try This

A light bulb manufacturer was interested in the lifespan of the bulbs it produces. Thirty light bulbs were illuminated until they failed. The times to failure (in hours) are given below and graphed in the following histogram.

410	201	350	103	45	158
59	257	338	198	324	418
367	128	425	289	170	238
313	319	259	356	319	424
276	305	298	189	345	156



A. i) What is the bin width used for this histogram?

ii) Why do you think this bin width was used?

B. The manufacturer claims that over 60% of its light bulbs last between 150 h and 350 h. Describe how the graph shows whether or not this is a valid claim.

• Although histograms are useful to see how a data set is distributed, they do not directly show how the data is related to the median. It is often useful to see how specific pieces of data compare to the typical or average in the data set.

• **Box and whisker plots**, sometimes called **box plots**, are designed to show how the values in a data set relate to the median. The data is always grouped into four intervals, which are centred around the median. Although the intervals for stem and leaf plots and histograms are equal, the intervals for box plots are usually not equal. However, the number of data values in each interval is always the same.

• The four intervals in a box plot each contain 25% of the data and are separated by these three numbers:

- The **lower quartile**, or $\mathbf{Q_1}$, is calculated by determining the median of all the data from the minimum value to the median.

- The median, or $\mathbf{Q_2}$, is the middle value of the whole data set.

- The **upper quartile**, $\mathbf{Q_3}$, is calculated by determining the median of all the data from the median to the maximum value.

• For data sets with an odd number of values, some people do not include the median when calculating Q_1 and Q_3.

For example, consider the data set: 3, 4, 5, 7, 8, 10, 12

The median (Q_2) is 7. If you include the median when calculating Q_1 and Q_3, Q_1 is 4.5 and Q_3 is 9. If you do not include the median when calculating Q_1 and Q_3, Q_1 is 4 and Q_3 is 10. When the data set is large, it makes very little difference to the results whether or not you include the median. However, when the data set is small, your decision can influence the results. It is more mathematically sound to include the median when calculating Q_1 and Q_3 (and this is what has been done throughout this unit, unless otherwise indicated).

• The size of each interval is based on the data. The four intervals are:

- The first interval, which contains 25% of the data values — it starts at the minimum and ends at the lower quartile, Q_1.

- The second interval, which contains 25% of the data values — it starts at Q_1 and ends at the median, Q_2.

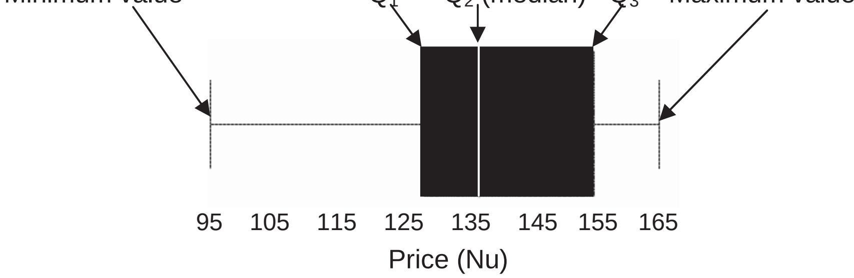
- The third interval, which contains 25% of the data values — it starts at the median, Q_2, and ends at the upper quartile, Q_3.

- The fourth interval, which contains 25% of the data values — it starts at Q_3 and ends at the maximum data value.

• The values for Q_1, Q_2, and Q_3, along with the minimum and maximum values, in order from least to greatest, is called the **5-number summary** for a data set and is used to frame the construction of the box and whisker plot.

For example, this box plot displays the prices of the same item in 15 stores:

124	135	158	110	128	131	158	95
137	165	152	144	129	143	162	



• To determine 5-number summary, the data values are arranged in order:

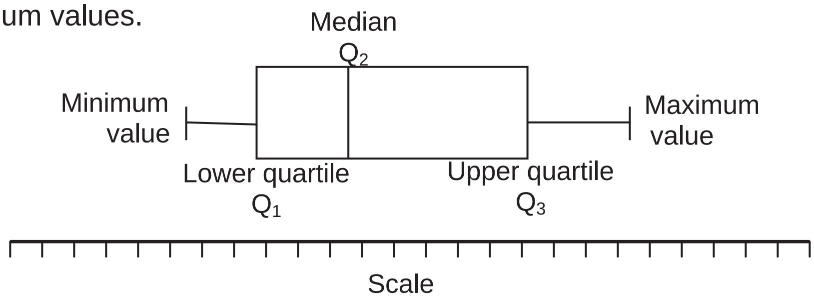
95, 110, 124, 128, 129, 131, 135, 137, 143, 144, 152, 158, 158, 162, 165

- The minimum value is 95 and the maximum value is 165.
- There are 15 scores, so the median, or Q_2, is the 8th score, 137.
- Q_1 is the median of the bottom 8 scores, the mean of the 4th and 5th scores, 128.5.
- Q_3 is the median of the top 8 scores, the mean of the 11th and 12th scores, 155.

The 5-number summary is 95, 128.5, 137, 155, and 165.

• To create a box and whisker plot, follow these steps:

- Draw a number scale that is appropriate for the 5-number summary.
- Draw dots above the scale to locate the lower quartile (Q_1), the median (Q_2), and the upper quartile (Q_3).
- Draw a box with left and right sides at the lower and upper quartiles.
- Draw a vertical line at the median equal to the height of the box.
- Draw whiskers, or lines, to connect the sides of the box to the maximum and minimum values.



• A box plot shows how the data is distributed relative to the median:

- The width of the box in relation to the median indicates how the middle 50% of the data is clustered around the median.
- A wider box indicates a greater spread in the middle 50% of the data.
- Longer whiskers indicate a greater spread between the extreme values and the upper and lower quartiles.

• A box and whisker plot is usually constructed from the original data set, but it can also be estimated from a frequency table or histogram (see **example 1**).

• Creating a box plot and histogram which use the same horizontal scale can be useful for finding out more information about the distribution of a set of data.

C. Create and use the 5-number summary to draw a box and whisker plot for the light bulb data in **parts A and B**. Use the same scale as the histogram.

D. What conclusions can you draw from the data displayed in these graphs?

Examples

Example 1 Creating a Box Plot and a Histogram

This data set shows the monthly savings of 65 people.

a) Construct a histogram and a box and whisker plot on the same scale. Show your work.

b) What observations can you make about the data from each graph?

Monthly Savings

Savings (Nu)	Number of people
0 – 200	16
200 – 400	12
400 – 600	8
600 – 800	12
800 – 1000	10
1000 – 1200	7

Solution

a) Monthly Savings

Savings (Nu)	Number of persons
0 – 200	16
200 – 400	12
400 – 600	8
600 – 800	12
800 – 1000	10
1000 – 1200	7

Estimate the median, Q_2

There are 65 data values so the median is the 33rd value. There are 28 data values in the first two bins (16 + 12) and 36 in the first three bins (16 + 12 + 8) so the median is in the third bin. Since the 33rd value is the 5th one in this bin of 8 values, use $\frac{5}{8}$ of the bin width (200) and add it to 400:

Median is $400 + \frac{5}{8} \times 200 = 400 + 125 = 525$

Estimate the lower quartile, Q_1

The lower quartile is the median of the first half of the data. Since there are 65 data values, it is the median of the first 33 data values, or the 17th number. There are 16 data values in the first bin and 28 values in the first two bins so the 17th value is in the second bin. Since the 17th value is the 1st one in this bin of 12 values, use $\frac{1}{12}$ of the bin width (200) and add it to 200:

Lower quartile is $200 + \frac{1}{12} \times 200 \approx 217$

[Continued]

Thinking

a) I knew that I could create the histogram from the frequency table using the same bins, but the box plot was more difficult because I didn't have the original data values.

• I had to estimate the 5-number summary for the box plot using the data in the frequency table.

Example 1 Creating a Box Plot and a Histogram [Continued]

Solution

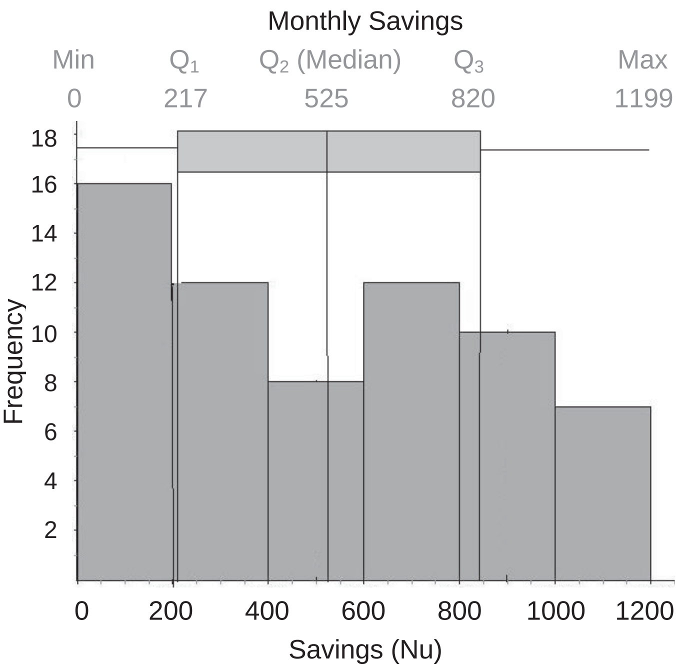
Estimate the upper quartile, Q_3

The upper quartile is the median of the last half of the data. Since there are 65 data values, it is the median of the last 33 data values, or the 49th value. There are 48 data values in the first four bins and 58 in the first five bins, so the 49th value is in the fifth bin. Since the 49th value is the 1st one in this bin of 10 numbers, use $\frac{1}{10}$ of the bin width (200) and add it to 800:

Upper quartile is $800 + \frac{1}{10} \times 200 = 800 + 20 = 820$

Estimate the extremes

Minimum is 0 and maximum is 1199



b) This histogram shows that the largest group of people saves between Nu 0 and Nu 200 a month. It also shows that, as the amount of savings increases, fewer and fewer people save greater amounts.

The box and whisker plot shows that about 50% of the people are able to save between about Nu 217 and Nu 820 a month. It also shows that the median amount saved is about Nu 525 a month.

Thinking

- I used the end points of the bins containing the least value (0 – 199) and greatest value (1000 – 1199) to estimate the extremes.

- I noticed that if I placed the box plot above the histogram so they shared the same scale, I could see how the data values in the histogram relate to the median.

b) It's easier to get frequency information from the histogram and percentage information from the box plot.

Example 2 Comparing Two Data Sets Using Box and Whisker Plots

The lifespans of 30 Brand A batteries and 30 Brand B batteries are shown below.

Measured Lifespans (years) of 30 Brand A Batteries

5.1	7.3	6.9	4.7	5.0
6.2	6.4	5.5	5.5	6.9
6.0	4.8	4.1	5.3	8.1
6.3	7.5	5.0	5.7	8.2
3.3	3.1	4.3	5.9	6.6
5.8	6.4	6.1	4.6	5.7

Measured Lifespans (years) of 30 Brand B Batteries

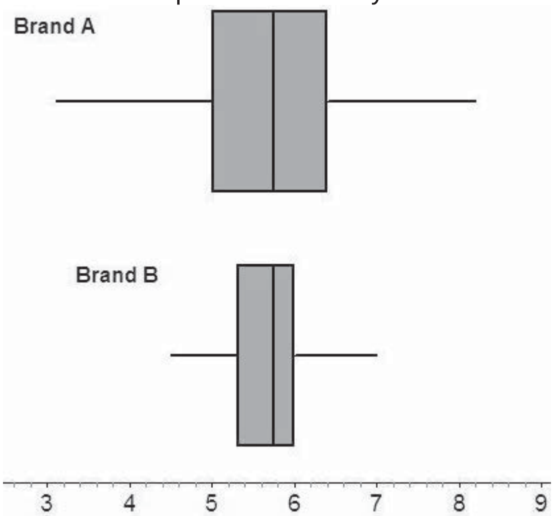
5.4	6.3	5.0	5.9	5.6
4.7	6.0	4.5	6.6	6.0
5.0	6.5	5.8	5.4	4.9
5.7	6.8	5.6	4.9	6.0
4.9	5.7	6.2	7.0	5.8
6.8	5.9	5.3	5.6	5.9

Which brand of battery appears to be more reliable? Explain.

Solution

5-Number Summary	Brand A	Brand B
Minimum	3.1	4.5
Maximum	8.2	7.0
Lower quartile, Q_1	5.0	5.3
Median, Q_2	5.75	5.75
Upper quartile, Q_3	6.4	6.0

Comparison of Battery Life



Battery life (years)

Both brands have the same median life. There is greater spread in the data for Brand A. The data set for Brand B has a smaller range and is more closely clustered around its median. This means the battery life for Brand B is more consistent and therefore more reliable.

Thinking

- I created two box plots on the same scale so I could compare shapes.
- I ordered the data values in each set from least to greatest to find the 5-number summaries.
- Each data set had 30 values so:
  - the median, or Q_2 was the mean of the 15th and 16th numbers
  - Q_1 was the 8th number
  - Q_3 was the 23rd number
- I used a scale from 3 to 9 years to allow for both brands' maximum and minimum values.
- I plotted the median (Q_2), minimum, maximum, and quartiles (Q_1 and Q_3) for each brand, and then drew the box and whiskers.
- 50% of Brand B batteries have a lifespan between 5.3 and 6 years, while 50% of Brand A batteries have a lifespan of between 5 and 6.4 years.

Practising and Applying

1. An oil refinery conducted 40 daily measurements of the carbon monoxide (CO) levels emitted by one of its stacks, measured in parts per million (ppm).

Daily Carbon Monoxide Measures (ppm)

4	30	52	85
12	34	55	86
15	36	58	86
15	37	58	99
20	38	58	102
20	40	59	102
20	42	63	110
21	43	63	125
25	43	71	132
30	45	75	141

a) What is the 5-number summary?

b) Create a box plot.

c) Create a frequency table. Use the intervals 0 – 15, 15 – 30, 30 – 45, ...

d) Create a histogram on the same scale as the box plot.

e) What does each graph tell you about the situation in the refinery?

2. This stem and leaf plot shows the number of days each member of a running club ran with the club in May.

Days Run

Stems	Leaves
0	4 6 6 7 8 8 8
1	0 1 3 5 5 8 9
2	0 1
3	0

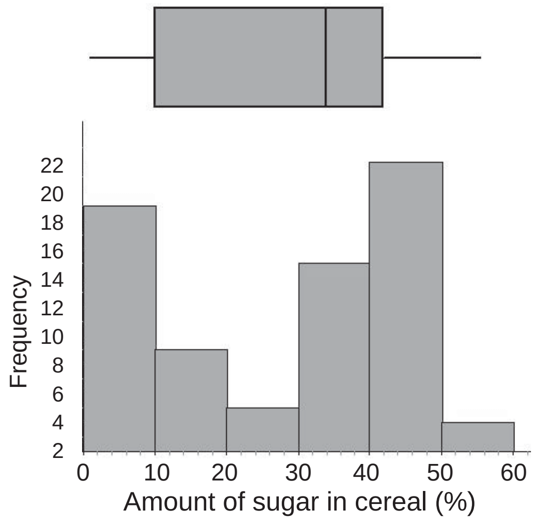
a) Construct a histogram.

b) What is the 5-number summary?

c) Construct a box plot above the histogram.

d) Compare how the two graphs show how the data is distributed.

3. Here are a histogram and box plot for the percentage of sugar found in 62 different brands of breakfast cereal sold in stores in North America.



a) Estimate the median sugar percentage.

b) Between about what two values do the middle 50% of the data values lie?

c) Give a possible reason for the shape of the histogram.

4. Rajesh and Meto work at a computer store. The data below shows the number of items each has sold every month over the last 18 months.

Rajesh

51	17	25	39	7	49
62	41	20	6	43	13
45	54	63	44	25	32

Meto

34	47	1	15	57	24
20	11	19	50	28	37
18	25	45	31	27	40

a) Make two box and whisker plots using the same scale.

b) Use the plots to compare their sales.

5. A group of Canadian students were surveyed about the amount of television they watch each week.

a) Estimate the 5-number summary.

b) Construct a box and whisker plot and a histogram on the same scale.

c) Discuss how each graph shows the distribution of the data.

Television Watched by Canadian Students

Hours	Number of students
10 – 15	2
15 – 20	12
20 – 25	23
25 – 30	97
30 – 35	40
35 – 40	38
40 – 45	8

6. The following data represents the points scored by Dema and Lemo in their last 20 basketball games.

Dema

25	19	12	8	11
22	20	18	15	13
19	24	14	9	13
26	22	20	18	15

Lemo

9	12	29	15	18
20	32	8	10	12
6	4	14	22	35
30	12	7	28	15

a) Create a pair of box plots on the same scale.

b) Create a pair of histograms on the same scale. Use the same scale as for the box plots.

c) What conclusions can you make using each pair of graphs?

7. Two groups of students wrote an exam. Both groups performed similarly on past exams. On this exam, Group 1 took the exam under difficult conditions (the room was cold and noisy), while Group 2 had better conditions.

7. Here are their exam marks.

Group 1

153	150	132	123	148
137	112	146	140	154
152	110	133	125	140
166	135	149	88	105

Group 2

184	69	129	150	157
167	141	179	124	166
168	159	165	180	148
133	175	160	155	125

a) Make two box plots using the same scale.

b) Do the conditions under which each group took the test appear to have affected the results? Explain.

8. In the frequency table below, the 50 states in the United States are grouped by the dates they joined the country.

Years	Number of states added
1775 – 1800	16
1800 – 1825	8
1825 – 1850	6
1850 – 1875	7
1875 – 1900	8
1900 – 1925	3
1925 – 1950	0
1950 – 1975	2

a) Estimate the 5-number summary and construct a box plot.

b) If you had the exact date that each state joined the United States and constructed a box plot for this data set, would it look exactly the same as the one you created in **part a)**? Explain.

9. Explain how you would estimate the value of the lower quartile for a set of data in a frequency table or histogram.

10. What information does a box plot provide that a histogram does not?

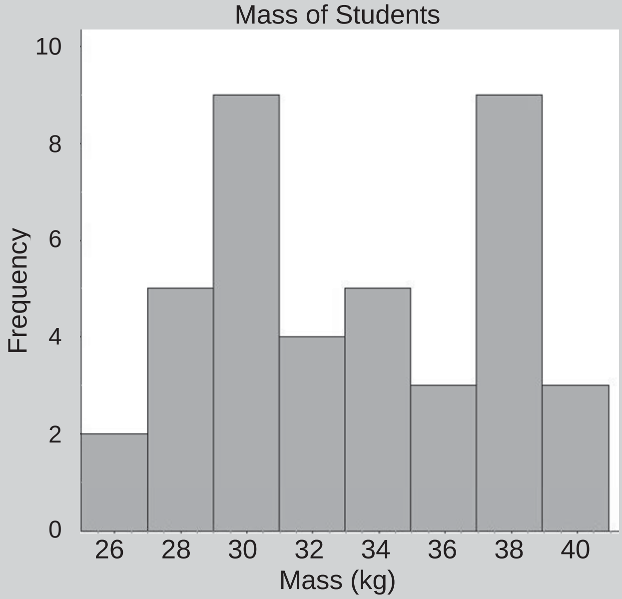
6.1.4 Data Distribution

Try This

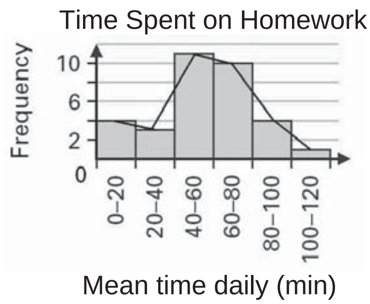
This histogram shows the distribution of the mass in kilograms of the students in Dorji's class.

A. i) Describe the shape of the graph.

ii) Suggest why the graph might have this shape.



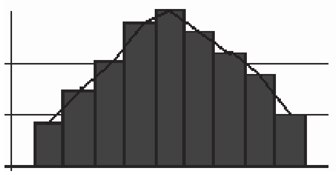
A **frequency polygon** is created by using line segments to join the midpoints of histogram bar tops, as shown in the graph to the right. It is used to smooth out the "steps" of the histogram and form a simpler representation of the histogram and make it easier to identify a curve of best fit and predict trends.



A histogram creates a picture that can help you interpret how a set of data is distributed. The distribution is classified by the shape associated with the histogram and corresponding frequency polygon. The shape of a histogram is affected by the size of the intervals chosen and how the data values are distributed among the intervals.

The following shapes, or distributions are described as follows

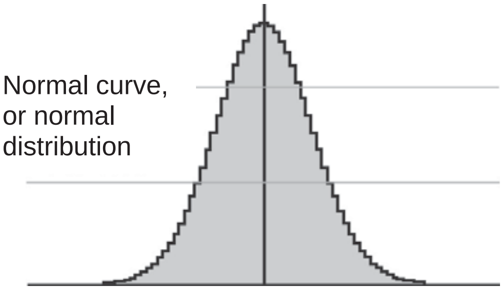
- A mound-shaped or **normal distribution** is almost symmetrical about a line passing through the interval with the greatest frequency. Its frequency polygon is a mound-like curve.



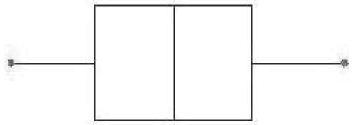
If you roll a pair of dice and record the sum, you could get a normal distribution. A sum of 7 should occur most often and would be in the middle. Sums of 2 and 12 would be the least frequent and would be represented by the outermost bars.

Many natural variables, such as mass or height of a large sample of humans, have a frequency polygon that forms a mound-shaped curve that looks like a bell, called the **normal curve**. Experience has shown that, as the sample size increases, the distribution of data will more closely resemble this curve.

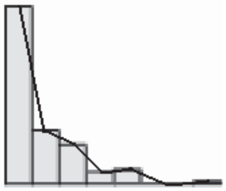
In a normal distribution,
- there are very few exceptionally large or small data values
- the mean is in the middle and is about the same as the median and mode

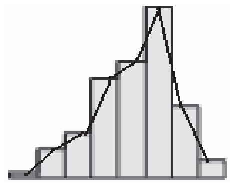


A box plot of the data will look symmetrical.

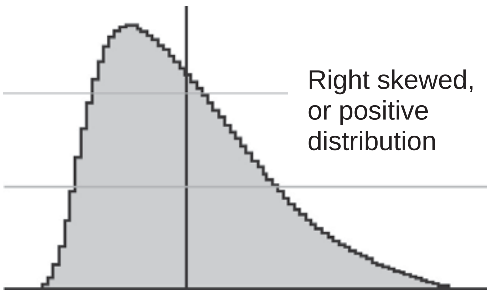


• In a **skewed distribution**, the intervals with the greatest frequency are near one end of the histogram.

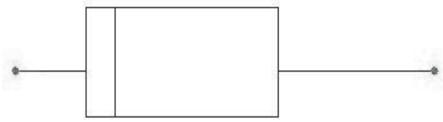




If the distribution is **right** or **positively skewed**, most large frequencies occur on the left side of the histogram, creating a frequency polygon with a tail that appears to be pulled to the right. These extreme values pull the mean to the right so it is greater than the median.



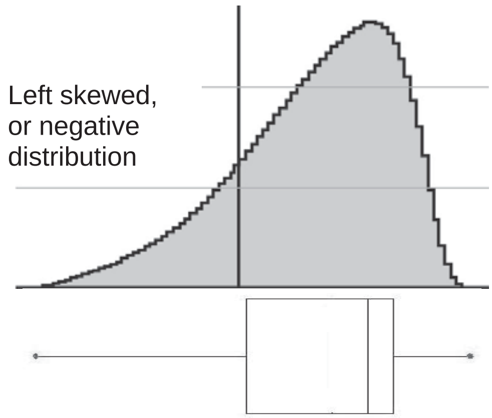
A box plot of the data looks as if the box was shifted to the left — the right whisker is longer, and the median is closer to the left side of the box.



A histogram of the number of children in a Canadian family would have a right, or positively skewed distribution because most families have one or two children, and fewer and fewer families have more children.

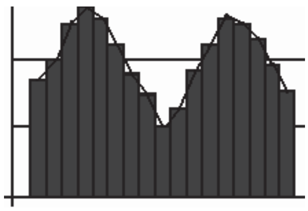
If the distribution is **left** or **negatively skewed**, most large frequencies occur on the right side of the histogram, creating a frequency polygon with a tail that appears to be pulled to the left. These extreme values pull the mean to the left so it is less than the median.

A box plot of the data looks as if the box was shifted to the right — the left whisker is longer, and the median is closer to the right side of the box.



If most students did well on a math test and only a few did poorly, the scores on the test would form a left or negatively skewed distribution.

• A **U-shaped distribution** peaks at both ends of the range and can be described as **bimodal**. The frequency polygon is a U-shaped curve and indicates that there may be two different groups within the data.



The distribution of the heights of a group of 18-year-old students could have a U-shaped distribution. The females would have a different average height than the males. The first mound would consist primarily of the heights of the female students and the second mound would consist primarily of male heights.

• In a **uniform distribution**, each bin has a similar frequency. The frequency polygon resembles a straight line that has very little slope.

The distribution that shows the frequency of each roll of a die rolled 100 times could have this shape. Each outcome (1 to 6) is equally likely, so the frequencies for each are likely to be about the same.

B. Which distribution describes the histogram, Mass of Students, from **part A**?

Examples

Example 1 Creating a Frequency Polygon

This frequency table shows the magnitude of earthquakes in Canada from 1960 to 1964.

a) Create a histogram and a frequency polygon.

b) Describe the shape of the frequency polygon.

c) Classify the type of distribution this represents.

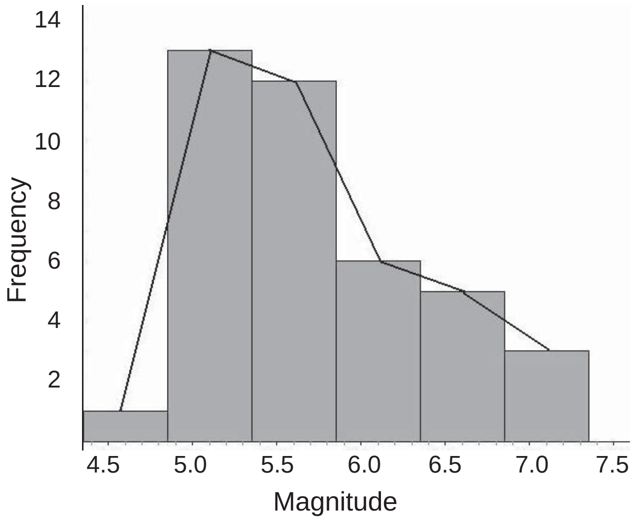
d) Why might this type of distribution occur with earthquake data?

Earthquakes in Canada

Magnitude	Frequency
4.35 – 4.85	1
4.85 – 5.35	13
5.35 – 5.85	12
5.85 – 6.35	6
6.35 – 6.85	5
6.85 – 7.35	3

Solution

a) Earthquakes in Canada 1960 to 1964



b) The frequency polygon rises quickly, then decreases more slowly to the right.

c) The distribution is right, or positively skewed.

d) It makes sense that violent earthquakes would happen less often than moderate ones.

Thinking

a) I created the histogram and then used a line to join the midpoints of the top of each bar to create the frequency polygon.

b) The intervals with the greatest frequencies are the 2nd and 3rd bars. The bars decrease in size as you move to the right.

Example 2 Identifying Type of Distribution

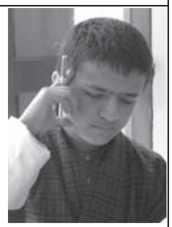
Identify the type of data distribution each situation will likely show.

a) The frequency of the possible outcomes when a spinner divided into four equal sections, each a different colour, is spun 50 times

b) The frequency of the heights of all the females in Class 10 in your school

c) The number of people walking near a school each hour from 7 am to 7 pm

d) The frequency of mothers by age, 20 years and older, who gave birth to their first child in Bhutan last year

Solution	Thinking
a) Uniform distribution	 **a)** The number of times each colour is spun should be about the same, since each sector is the same size, so the probability of spinning each would be the same. The bars in the histogram should all have about the same height.
b) Normal distribution	**b)** There should be a large group of female students whose heights will be about the same and close to the median height for a female in class 10. The number of females who are extremely short or tall will be few. The histogram will be mound-shaped.
[Continued]	

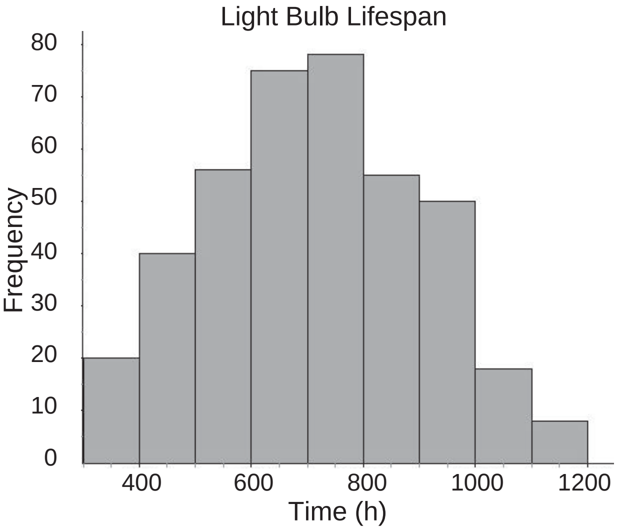
Example 2 Identifying Type of Distribution [Continued]

Solution	Thinking
c) U-shaped distribution	**c)** There will many students walking near the school in the morning and late afternoon as they go to and from school. These time intervals will have the tallest bars in the histogram, giving it a U-shape.
d) Right or positively skewed distribution	**d)** Most of the mothers who gave birth would have been in their early twenties. The frequency will decrease as age increases. The histogram will have bars decreasing in height, with the tallest bars on the left.

Example 3 Examining a Data Distribution

This frequency table and histogram show the lifespan in hours of 400 light bulbs that were tested at a light bulb manufacturing company.

Lifespan (hours)	Frequency (number of light bulbs)
300 – 400	20
400 – 500	40
500 – 600	56
600 – 700	75
700 – 800	78
800 – 900	55
900 – 1000	50
1000 – 1100	18
1100 – 1200	8



a) Identify the type of distribution.

b) Estimate the mean, median, and mode. Show your work.

c) How do the mean, median, and mode relate to each other in this distribution?

d) Describe how the distribution relates to the median.

Solution	Thinking
a) Normal distribution	**a)** The graph has an almost symmetrical mound shape, which indicates a normal distribution.

Light Bulb Lifespan

Lifespan (hours)	Frequency (number of light bulbs)
300 – 400	20
400 – 500	40
500 – 600	56
600 – 700	75
700 – 800	78
800 – 900	55
900 – 1000	50
1000 – 1100	18
1100 – 1200	8

b) *Estimate the mean*

[350 × 20 + 450 × 40 + 550 × 56 + 650 × 75 + 750 × 78 + 850 × 55 + 950 × 50 + 1050 × 18 + 1150 × 8] ÷ 400

= 285,400 ÷ 400 = 713.5

Estimate the median

$700 + \frac{9.5}{78} \times 100 = 700 + 12.2 = 712.2$

Estimate the mode

Midpoint of 700 to 800 is 750

c) All three averages: the mean, the median, and the mode, occur in the same interval, 700 to 800. The estimated mean and median suggest they all occur early in the interval. That might also be true for the mode, but there is no way to be sure.

d) The graph appears to be almost symmetric about a line through the median.

b) The set of data is grouped, so there is no way of knowing the individual times. I knew I would have to estimate.

• To estimate the mean:

- I used the midpoint of each bin to represent each data value in the bin and then multiplied by the number of data values in the bin. For example, the midpoint of the 300 to 400 bin was 350 and, since there were 20 values in the 300 to 400 bin, I multiplied 350 × 20. I did this for each bin.
- I found the total of all the estimated data values.
- I divided the total by 400, the total number of light bulbs tested.

• To estimate the median:

- I figured that with 400 pieces of data, the median would be the mean of the 200th and 201st values.
- I counted 191 data values in the first four bins (20 + 40 + 56 + 75) and 269 (191 + 78) in the first five bins, so I knew the median was the mean of the 9th and 10th values in the fifth bin (700 to 800).
- I assumed all 78 values were spread evenly over the bin of 100 values, so the 1st value would be $700 + \frac{1}{78} \times 100$, the 2nd value would be $700 + \frac{1}{78} \times 100$, and so on.

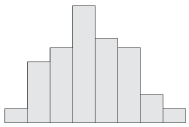
• To estimate the mode, I found the middle value in the most frequent interval, 700 to 800.

d) About the same number of data values lie on either side of the median with a bit more on the left.

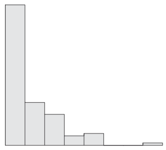
Practising and Applying

1. For each histogram, sketch the frequency polygon and identify the type of distribution.

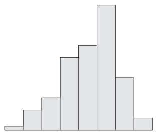
a)



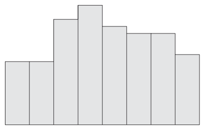
b)



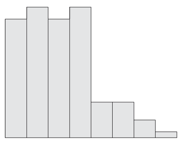
c)



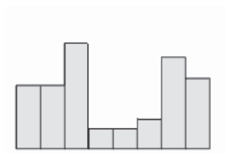
d)



e)



f)



2. For each histogram in **question 1**, sketch what the corresponding box plot might look like.

3. The set of data was collected from measurements of the diameters of a sample of trees in the forest.

Tree Diameters (cm)

10.6	14.3	14.7	15.3	15.5
16.2	16.4	16.9	17.0	17.4
17.9	18.0	18.7	18.7	19.1
19.2	19.5	19.5	19.6	20.2
20.3	20.5	20.7	21.2	21.3
21.8	22.3	22.7	23.0	23.5
24.2	25.1			

3. d) All of the trees in this sample are the same type and age. Why are they not all the same size?

4. To determine the health of a stand of trees, Rajesh, an employee who works in the National Forest, wants to know the typical height of a 5-year-old fir tree. He measured a sample of trees from a stand of 5-year-old trees.

Tree Heights (cm)

39	45	14	36	23
46	10	49	31	34
24	45	57	41	42
32	22	21	31	45
43	60	48	28	33
36	12	32	25	35
12	61	92	51	26
56	50	33	77	32
8	38	15	57	20
55	55	56	42	65

a) Create a histogram of the data and identify the type of distribution.

b) Find the mean height of a 5-year-old fir tree. How do you know the median height is close to the mean height without calculating?

5. In 1798, the English scientist Henry Cavendish repeatedly measured the density of the earth (tonnes/metre) relative to that of water in a careful experiment with a torsion balance. Here are his 23 measurements.

Density of the Earth (t/m)

5.36	5.62	5.27	5.46	5.53
5.57	5.29	5.29	5.39	5.30
5.10	5.79	5.58	5.44	5.42
5.75	5.34	5.63	5.65	5.34
5.47	5.68	5.85		

a) Determine the mean, median, and mode. How are they related?

b) Create a histogram of the data.

c) Describe the distribution.

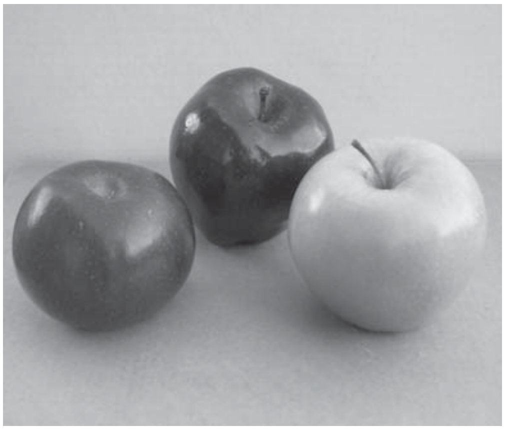
6. The following box and whisker plots represent the masses of three different varieties of apples grown in Canada. A sample of 80 apples was used for each.

a) Which sample has a close-to-normal distribution? Explain how you know.

b) Which sample is left skewed? Explain how you know.

c) Which sample is right skewed? Explain how you know.

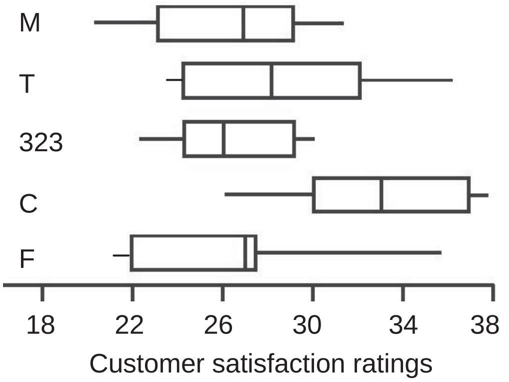
d) What does the distribution within each sample and the comparison of the distributions tell you about the apples in each sample?



A McIntosh, a Red Delicious, and a Granny Smith apple

7. The box plots at the top of the next column show consumer satisfaction ratings for five different types of cars that are popular in North America.

7. The cars in order from top to bottom are: Mirage, Tracer, 323, Civic, Festiva



a) Which car seems to have the most consistent ratings? Explain.

b) Are any of these data sets close to a normal distribution? Explain.

c) Identify the data sets that are right skewed. How do you know?

d) What conclusion can you draw from this set of data?

8. A fisheries researcher compiled the following data on lengths of 6-year-old goldfish in the same pond.

Goldfish Lengths (mm)

217	230	220	221	225
219	217	225	228	234
231	226	220	226	222
225	214	221	233	227
223	225	238	220	213
235	240	210	218	235
223	226	223	234	224
231				

a) Make a histogram and frequency polygon of the data.

b) Describe the distribution.

c) How likely would it be for a goldfish within each of the following ranges to be found in the same pond? Explain.

210 mm to 220 mm

220 mm to 230 mm

9. Identify the distribution of the data in each table. Explain your thinking.

Table A

10 – 19	3
20 – 29	5
30 – 39	17
40 – 49	20
50 – 59	11
60 – 69	4

Table B

2 – 5	7
6 – 9	8
10 – 13	9
14 – 17	3
18 – 21	1
22 – 25	1

CONNECTIONS: Normal Distribution and Sample Size

Suppose you measured the heights of all the students in your class and then graphed the data in a histogram.

• What type of distribution do you think the data would have?

• How do you think the distribution would compare with a histogram that included the heights of all the students in your school?

1. a) Measure the heights of all the students in your class and record the data.

b) Organize your data in a frequency table using an appropriate bin width. Create a histogram and frequency polygon.

c) Describe the shape of the frequency polygon and identify the distribution.

2. a) Have your teacher ask the teachers in your school to provide the height measurements of the students in their classes.

b) Collect the data and create a frequency table, histogram, and frequency polygon for the new data. Do not forget to include the data from your own class.

c) Compare the data from your class to the data from the whole school. Discuss the similarities and differences.

6.2.1 Correlation and Lines of Best Fit

Try This

On a spring day at Paro International Airport, a pilot flying for Druk Air noticed that the temperature gauge showed the following outside temperatures as the plane's altitude increased on takeoff.

Altitude (km)	6.9	7.0	7.5	8.1	8.7	9.0	9.5	10.8
Temperature (°C)	15	13	5	–5	–15	–26	–44	–60

A. Create a scatter plot of the data. Is a line of best fit appropriate? Explain.

B. Describe the correlation. Is it positive or negative? weak or strong?

• **Variables** are things that you measure, control, or manipulate in experiments and in research. A set of data involving two different variables is often collected and then graphed to determine whether a relationship exists between the independent and dependent variables.

• The independent variable, usually *x*, is the quantity for which you choose values. The dependent variable, often *y*, is the quantity whose values are measured as they are affected by different values of the independent variable.

• If you wanted to see whether there was a relationship between two variables, for example, a student's height and age, you would

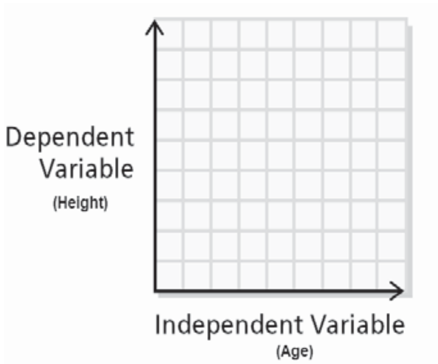
- Determine which is the independent variable and which was the dependent variable. In this case, age would be independent and height would be dependent.

- Choose students of different ages and then measure their heights.

- Record the students' ages and heights in a table of values.

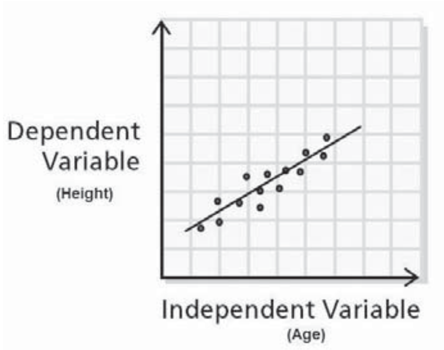
Independent variable (*x*) Age	Dependent variable (*y*) Height

- Plot the data in a **scatter plot**, with the vertical axis for the dependent variable, height, and the horizontal axis for the independent variable, age.



The resulting pattern of points indicates the type and strength of the relationship between the variables.

If the data is continuous and shows a linear relationship, as it does for this age and height data, a line of best fit can be drawn through the plotted points, as shown to the right. The line of best fit is used to create an equation to represent the relationship.



Note that a line of best fit is not appropriate if there is no apparent linear relationship or if the data values are discrete.

• **Correlation** is a measure of the strength of the relationship between two variables. If the relationship is strong, you can use one variable to predict values for the other. Note that this does not mean there is a cause-and-effect relationship, that is, that one causes the other. Both variables may be affected by something other than each other.

For example, in North America, an increase in outdoor temperature in warmer months causes an increase in ice-cream sales. At the same time, an increase in outdoor temperature also causes an increase in crime rates. So, if you graph ice-cream sales against crime rates, you will see a correlation and you can predict crime rates from ice-cream sales. However, you cannot conclude that higher ice-cream sales cause crime. It is the higher temperatures that are affecting both ice-cream sales and crime rates.

• You can estimate a correlation by examining the pattern of points in a scatter plot. A scatter plot of a relationship between two variables shows

- a *positive correlation* when the pattern of points slopes up and to the right, which means the two quantities increase together

- a *negative correlation* when the pattern slopes down and to the right, which means as one quantity increases the other decreases

- *no correlation* when no pattern appears

- a *strong correlation* when the points form a line or nearly form a line

- a *weak correlation* when the points are dispersed widely, but roughly form a line

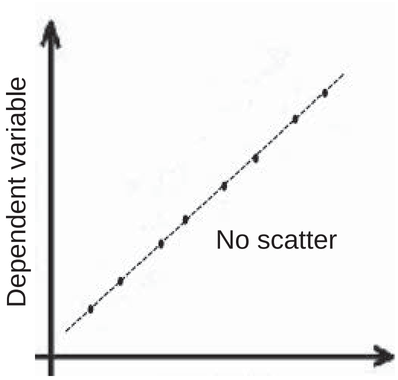
Note that you do not need a line of best fit to establish a correlation as correlations apply to both continuous and discrete data.

• The **correlation coefficient** is a numerical value, represented by *r*, which ranges from –1 to 1. You can use it to represent the extent to which two variables are linearly related. It can be calculated using a complex formula, but most graphing calculators and statistical computer programs can easily determine the value.

The closer the value of r to 1 or to –1, the stronger the correlation, positive or negative.

A *correlation coefficient of exactly 1* means the data is perfectly linear, and as one variable increases, the other variable also increases. We call this a perfect positive linear correlation.

If the data values are continuous, you can draw a line of best fit.

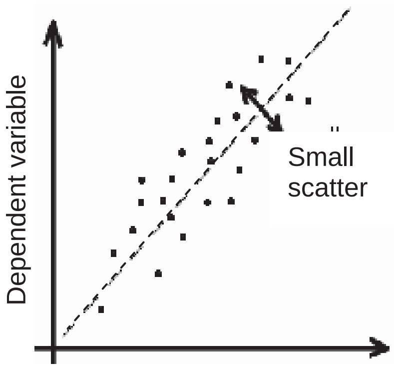


$r = 1$

A *correlation coefficient close to 1* means the data values are clustered close to a line stretching from the lower left to the upper right. We call this a strong positive linear correlation.

As one variable increases, the other variable also increases but the correlation is not as strong as with a correlation of exactly 1.

If the data values are continuous, you can draw a line of best fit.

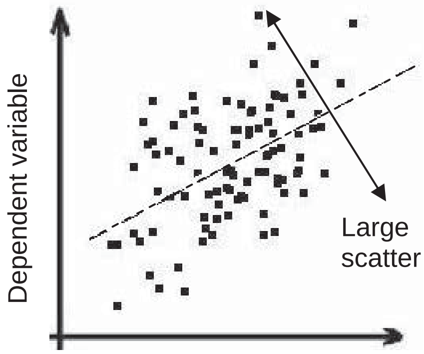


r is close to 1

A *correlation coefficient close to 0.5* means the data values are loosely clustered in a line from the lower left to the upper right. We call this a weak positive linear correlation.

As one variable increases, the other variable also increases, but the correlation is not as strong as with a correlation of close to 1.

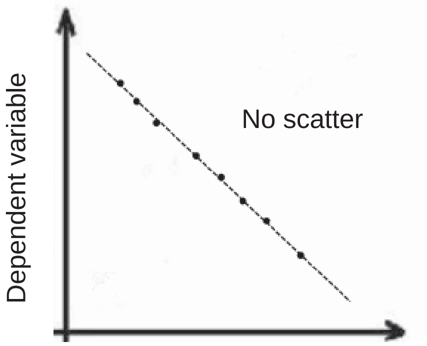
If the data values are continuous, you can draw a line of best fit.



r is close to 0.5

A *correlation coefficient of exactly –1* means the data values are perfectly linear, and as one variable increases, the other variable decreases. We call this a perfect negative linear correlation.

If the data values are continuous, you can draw a line of best fit.



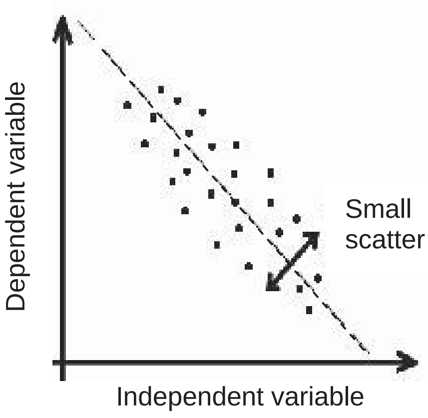
Independent variable

$r = -1$

A *correlation coefficient close to –1* means the data values are clustered close to a line stretching from the upper left to the lower right. We call this a strong negative linear correlation.

As one variable increases, the other variable decreases, but the correlation is not as strong as with a correlation of –1.

If the data values are continuous, you can draw a line of best fit.

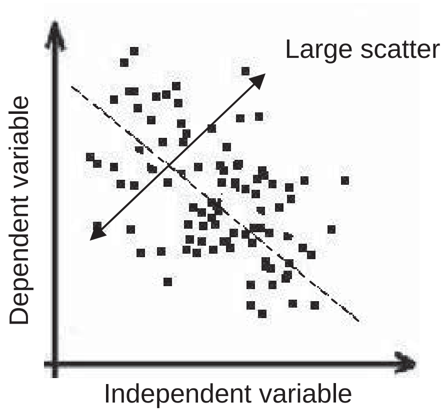


r is close to –1

A *correlation coefficient close to –0.5* means the data values appear to be somewhat linear and are loosely clustered in a line from the upper left to the lower right. We call this a weak negative linear correlation.

As one variable increases, the other variable decreases, but the correlation is not as strong as with a correlation of close to –1.

If the data values are continuous, you can draw a line of best fit.

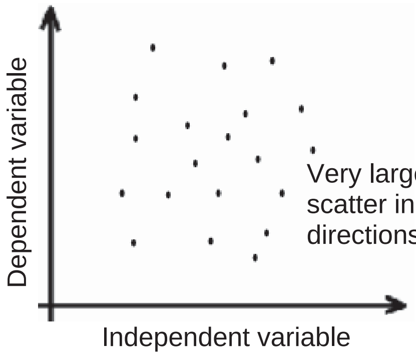


r is close to –0.5

A *correlation coefficient of 0* means the data values show no linear relationship. As values of one variable increase, some values of the other variable increase while others decrease.

We say this data set has no linear correlation.

Even if the data values are continuous, you cannot draw a line of best fit.



$r = 0$

C. Estimate the value of the correlation coefficient for the outside temperature data from **part A**. Explain your estimate.

Examples

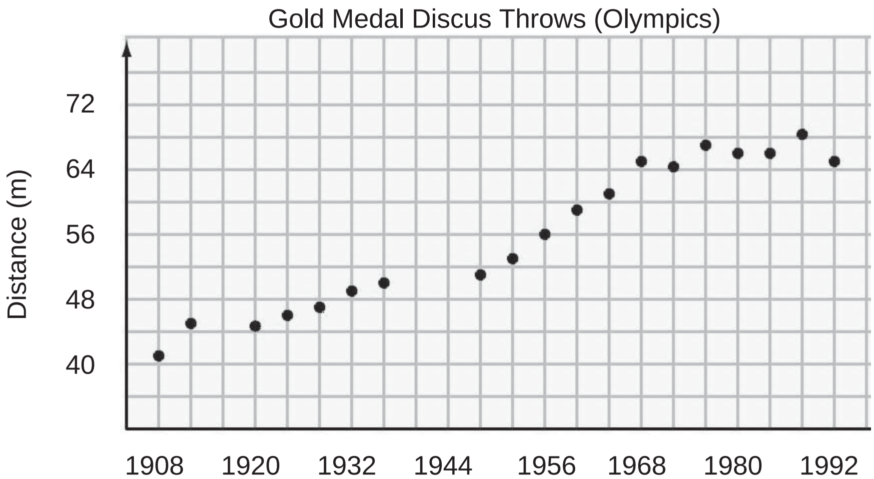
Example 1 Examining the Strength of a Relationship

This scatter plot shows gold medal throws in the men's discus competition for the Summer Olympics, 1908 to 1992.

a) Is a line of best fit appropriate for this data set? Explain.

b) Estimate the correlation coefficient.

c) The Olympics were not held in 1944 because of World War II. Use the graph to predict what the distance in 1944 might have been. How confident are you in your prediction?



[Continued]

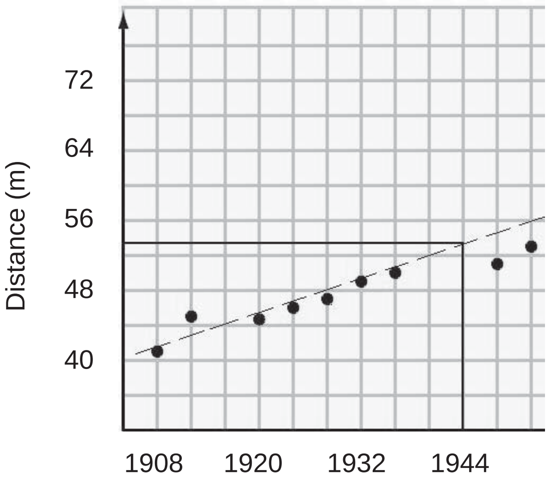
Example 1 Examining the Strength of a Relationship [Continued]

Solution

a) A line of best fit is not appropriate. The data values are discrete since the Olympics only happen every 4 years, so there are no data values between the plotted points.

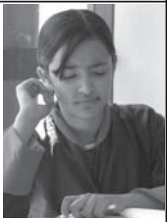
b) The correlation coefficient is close to 1.

c) Gold Medal Discus Throws (Olympics)



It looks like the winning distance in 1944 would have been about 53 m. Since the correlation is strong, I am fairly confident in my prediction.

Thinking



a) Even though the points form a linear pattern, you can't draw a line of best fit because Olympic years are discrete.

b) The data values are not perfectly linear, but there is definitely a linear relationship. The points rise from left to right indicating that as time progresses, the winning distances increase. This shows a strong positive correlation, which is close to 1.

c) To help interpolate, or predict for 1944, I drew a dashed line to represent the linear relationship.

Example 2 Estimating and Describing Correlation

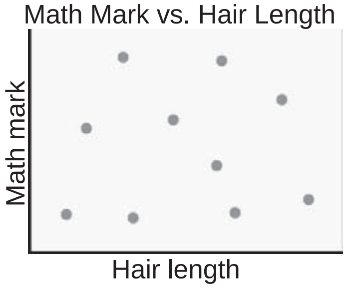
For each scatter plot

a) describe the type of correlation

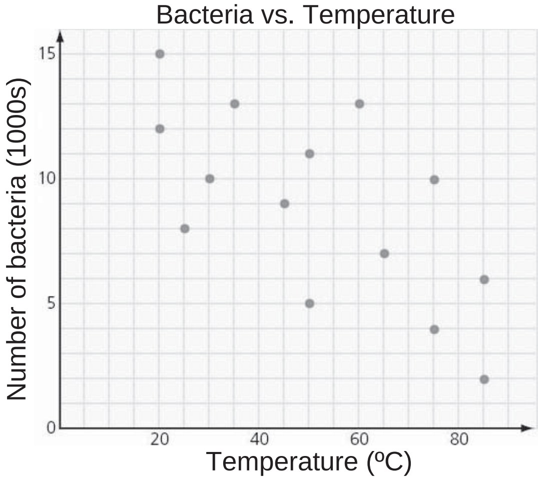
b) estimate the correlation coefficient

c) decide if there is a relationship between the variables and explain your thinking

Math Mark vs. Hair Length



Bacteria vs. Temperature



Solution	Thinking 
Math mark vs. Hair length	
a) no correlation	**a)** The points were widely scattered and with no linear pattern.
b) The correlation coefficient would be about 0.	**b)** With no correlation, I knew r was close to 0.
c) There appears to be no linear relationship between a student's math mark and hair length. That makes sense, because the length of your hair should not affect how well you learn at school.	**c)** Since there was no correlation, I knew there was no relationship.
Bacteria vs. Temperature	
a) a weak negative correlation	**a)** The points were scattered but clustered along a line with a negative slope.
b) The correlation coefficient would be negative and it would be closer to –0.5 than –1.	**b)** The correlation coefficient is negative because as one variable increases, the other decreases. Since the correlation was weak, I knew r was closer to –0.5 than –1.
c) There is a weak linear relationship between temperature and number of bacteria. That makes sense, since the bacteria cannot live in high temperatures.	**c)** As temperature rises, the number of bacteria decreases.

Example 3 Correlation and Cause and Effect

A data set was collected that compared life expectancy (LE) to the number of people per television in a number of countries.

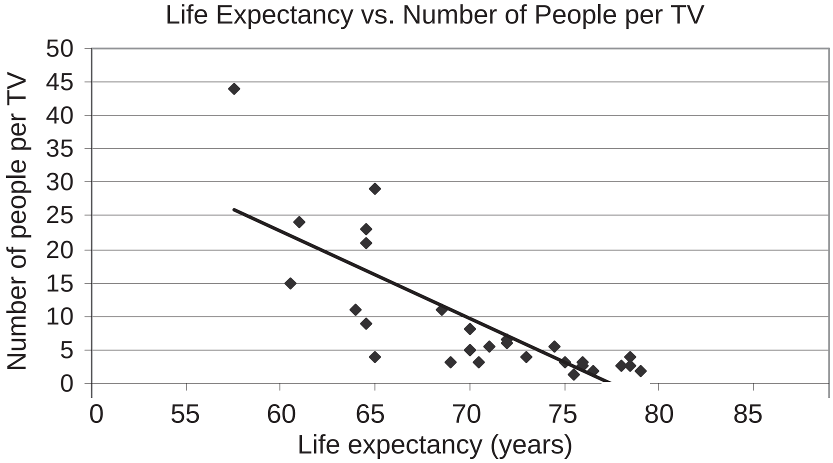
Country	LE	No. of people/TV
Argentina	70.5	4.0
Canada	76.5	1.7
China	70.0	8.0
Colombia	71.0	5.6
France	78.0	2.6
Germany	76.0	2.6
Indonesia	61.0	24.0
Japan	79.0	1.8
Mexico	72.0	6.6
Morocco	64.5	21.0

Country	LE	No. of people/TV
Philippines	64.5	8.8
Poland	73.0	3.9
Romania	72.0	6.0
S. Africa	64.0	11.0
Taiwan	75.0	3.2
Thailand	68.5	11.0
UK	76.0	3.0
US	75.5	1.3
Venezuela	74.5	5.6

[Continued]

Example 3 Correlation and Cause and Effect [Continued]

Rishi created a scatter plot and drew the line of best fit



a) What are the independent and dependent variables?

b) What does this scatter plot tell you about the relationship?

c) Explain why a line of best fit is appropriate for this situation.

d) Estimate the correlation coefficient and describe the correlation.

e) Rishi concluded that, in order to increase life expectancy, the number of televisions in the country must be increased. Is his conclusion valid? Explain.

Solution

a) The independent variable is life expectancy and the dependent variable is number of people per TV.

b) It shows that the number of people per TV in a country is related to life expectancy.

c) The data values are continuous and there appears to be a linear correlation so a line of best fit is appropriate.

d) The correlation coefficient is close to –1, which is a strong negative correlation.

e) No. The scatter plot shows that you can make a reasonable prediction of life expectancy if you know number of people per TVs or vice versa. But you cannot conclude that an increase in the number of TVs in a country will increase life expectancy. A correlation, even if it is strong, does not mean cause and effect.

Thinking

a) Life expectancy is on the *x*-axis and number of people/TV is on the *y*-axis.

c) Life expectancy can be any number, not just whole numbers, so it's continuous and the data set shows a linear trend.

d) The data points are close-together in a linear pattern that goes down from left to right in a negative slope.

e) A strong correlation only tells you that two data sets are correlated. It doesn't tell you that a change in the independent variable *causes* the change in the dependent variable.

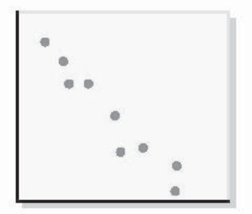
Maybe there is some other factor that influences both variables, such as prosperity, since that often leads both to more TVs and to longer life expectancy.

Practising and Applying

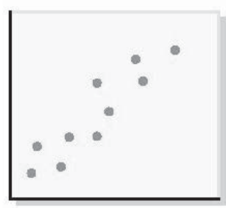
1. a) Examine each scatter plot. What type of correlation is shown — positive, negative, or no correlation? Explain.

b) Estimate the value of the correlation coefficient for each scatter plot.

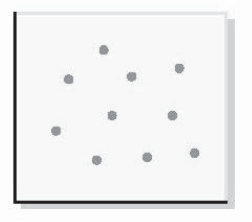
I



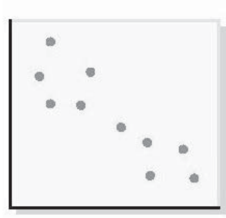
II



III



IV



2. Is each pair of variables described below positively correlated, negatively correlated, or not correlated? For those with a correlation, explain your thinking.

a) temperature and altitude

b) the size of a person's hand and the amount in that person's savings account

c) outside temperature and cold drink sales

d) the length of a student's fingernails and his or her final grade in English

e) the number of pages left to be read in a book and the number of pages already read

3. The rate at which crickets chirp is related to air temperature as shown in the table at the top of the next column.

3. a) What are the independent and the dependent variables? Explain.

Cricket Chirp Rate

Temp. (°C)	Chirps per second
15	20
17	27
16	22
18	30
15	19
16	21
16	20
16	24
15	22
14	24
16	25

b) Create a scatter plot for the data. Is a line of best fit appropriate? Explain.

c) Estimate the correlation coefficient. Describe the type of correlation.

d) If the temperature were 10°C, how many chirps per second would you expect? Explain your prediction.

e) How confident are you in your prediction for **part d)**? Explain.

4. A movie theatre monitors weekly attendance during the first 10 weeks of a movie's showing.

Movie Attendance

Week	Attendance
1	2246
2	2115
3	1935
4	1675
5	1440
6	1200
7	995
8	722
9	664
10	590

a) Create a scatter plot for the data. Is a line of best fit appropriate? Explain.

[Continued]

4. [Cont'd] **b)** Estimate the correlation coefficient. Justify your decision.

c) The movie will close when weekly attendance drops below 350. Predict when the movie will close. Explain your predicted.

5. The winning women's Olympic long-jump distances for the years 1948 to 2000 are shown in this table.

Women's Long-jump

Year	Distance (m)
1948	5.69
1952	6.24
1956	6.35
1960	6.37
1964	6.76
1968	6.82
1972	6.78
1976	6.72
1980	7.06
1984	6.96
1988	7.40
1992	7.14
1996	7.12
2000	6.99

a) Create a scatter plot. Is a line of best fit appropriate? Explain.

b) Estimate the correlation coefficient. Justify your decision.

c) If the Olympics had been held in 1944, what might the winning distance have been?

d) Why is it difficult to predict the distance of the winning jump in the 2016 Olympics?

6. One way runners are measured is by stride rate (number of steps per second). In a study of 21 top female runners, researchers measured the stride rate for different speeds. This table gives the average stride rate of in relation to the speed.

Female Stride Rates

Speed (miles/h)	Stride rate
4.83	3.05
5.15	3.12
5.33	3.17
5.68	3.25
6.09	3.36
6.42	3.46
6.74	3.55

a) Create a scatter plot and determine whether there is a linear relationship.

b) Is a line of best fit appropriate? Explain.

c) Would it make sense to use this graph to estimate the stride rate of a runner for a speed of 50 miles/h? Explain.

7. Compare the correlations shown in Figures A and B below.

a) Estimate the correlation coefficient for each scatter plot.

b) Compare the correlations.

Figure A

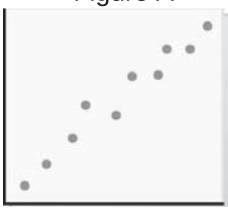


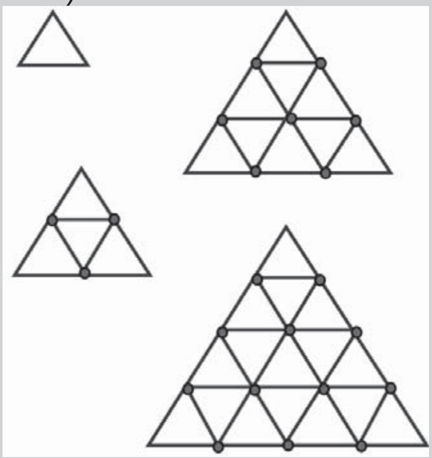
Figure B

6.2.2 Non-Linear Data and Curves of Best Fit

Try This

The table below relates the number of marked intersection points (dots) within or on each large triangle to the side length of the large triangle. (The side length of each small triangle is 1 unit.)

Side length (units)	Number of intersection points
1	0
2	3
3	7
4	
5	
6	



A. Complete the table above.

B. i) Sketch a scatter plot of the data.

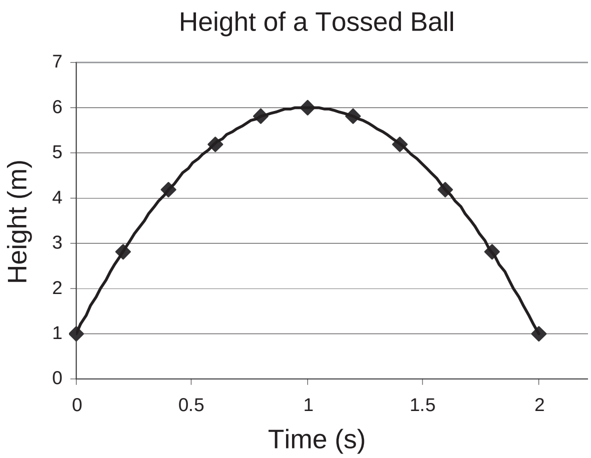
ii) Does there appear to be a linear relationship? Explain.

Not all relationships between variables are linear. Sometimes a curve fits the pattern of points in a scatter plot better than a straight line.

• A non-linear relationship can be modelled by a **curve of best fit**, when the data values are continuous. We often use only part of the curve to model the relationship. The types of curves are described below.

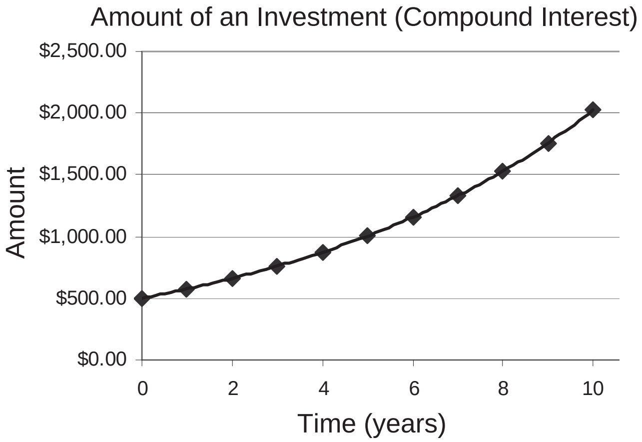
A quadratic curve or relationship

A quadratic curve, called a parabola, can be used to represent a relationship such as the relationship between time and the height of a ball that has been tossed into the air.



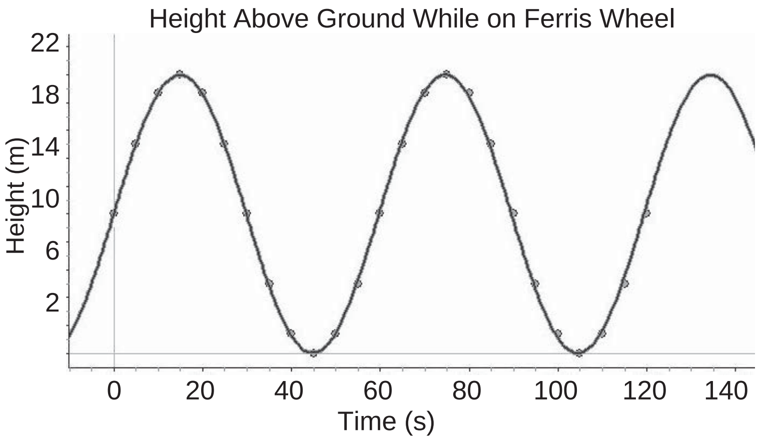
An **exponential curve** or relationship

An exponential curve can be used to represent a relationship such as the relationship between time and the amount of an investment earning compound interest.

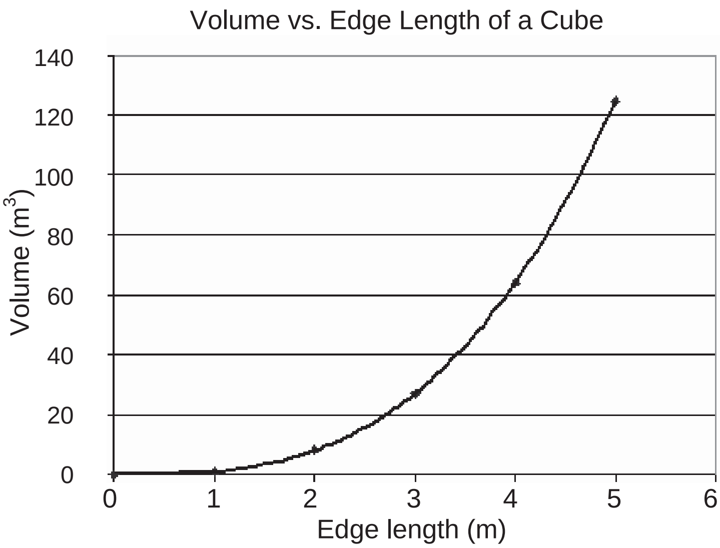


Other curves or non-linear relationships

A *repeating or periodic curve* can be used to represent a relationship such as the relationship between time and the height of a person above the ground while riding a Ferris wheel.

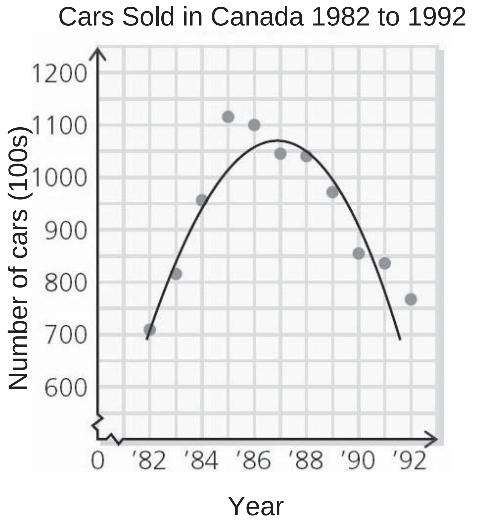


A *cubic curve* can be used to represent a relationship such as the relationship between the volume of a cube and its edge length.



• As with a line of best fit, you can draw a curve of best fit even if the points do not fall exactly along the curve, as long as the data values are continuous. You need to decide on the type of curve that best fits the pattern and find a position for it that relates to the points. To do this, draw a smooth curve that passes through or near as many points as possible. A piece of string may help you to decide the shape and location of the curve.

• You can estimate and make predictions from curves of best fit by interpolating and extrapolating, just as you do with lines of best fit.



C. Use the graph and the table from **part A** to predict the number of intersection points there will be when the side length of the large triangle is 7 units. Explain each prediction.

D. What do you think causes this type of curve? Explain.

Examples

Example 1 Working With Quadratic Relationships

A relief package of food, water, and medicine is released from an airplane to help people cope with the effects of an earthquake. The table below shows the container's height above ground for the first part of its freefall (up until the time its parachute is fully open).

Time (s)	Height above ground (m)
0	6000
4	5920
8	5680
12	5280
16	4720
20	4000

a) Create a scatter plot of the data.

b) Draw a curve of best fit. What kind of curve is this?

c) If the container continues to fall at this rate, about when will it be 3000 m above the ground?

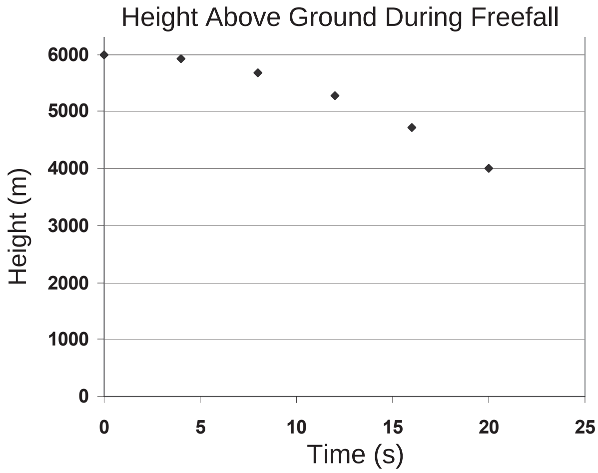
d) Does your estimate make sense? How do you know?

[Continued]

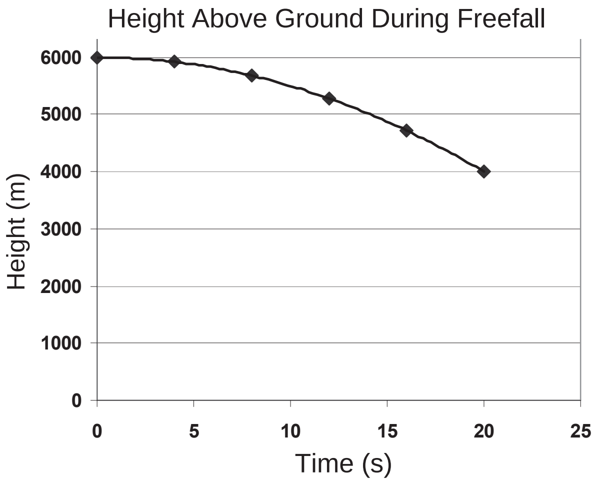
Example 1 Working With Quadratic Relationships [Continued]

Solution

a)

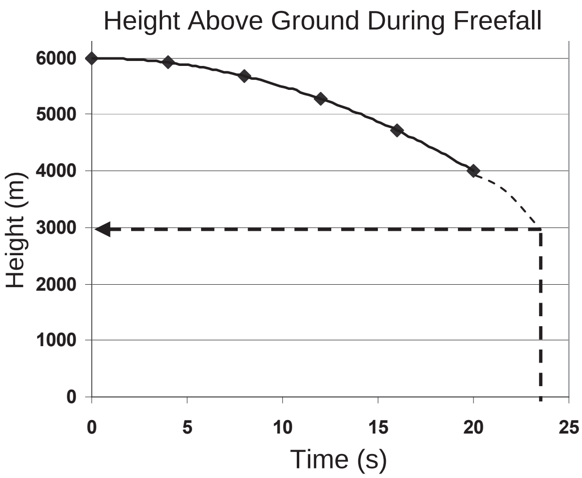


b)



I think it is a quadratic curve.

c)



The container will be at about 3000 m after 24 s.

Thinking

a) Time is the independent variable and height is the dependent variable because it makes sense that the container's height will depend on elapsed time. I put time on the horizontal axis and height on the vertical axis.

b) I drew a decreasing curve that passed through all the points.

• The curve looks like half a parabola.

c) I extended the curve past the last point to extrapolate.

d) My prediction makes sense because the next drop in height would be a drop of 880 m, which means it would be at 3120 m at 24 s. This is close to my prediction of 3000 m at 24 s.

d) After each 4 s interval it dropped 160 m more in height than the last time: it dropped 80 m, then 240 m, then 400 m, then 560, and then 720 m, so the next drop would be 880 m.

Example 2 Working With Exponential Relationships

The population of a bacteria colony is measured over a 6 hour period.

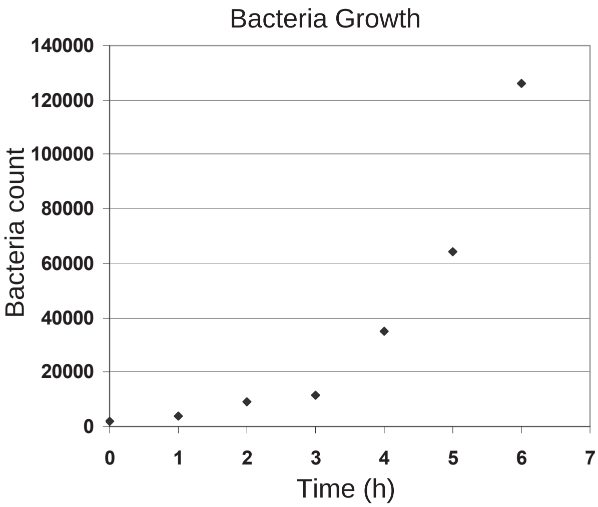
Time (h)	0	1	2	3	4	5	6
Bacteria count	1,800	3,700	9,100	11,500	35,100	64,100	125,900

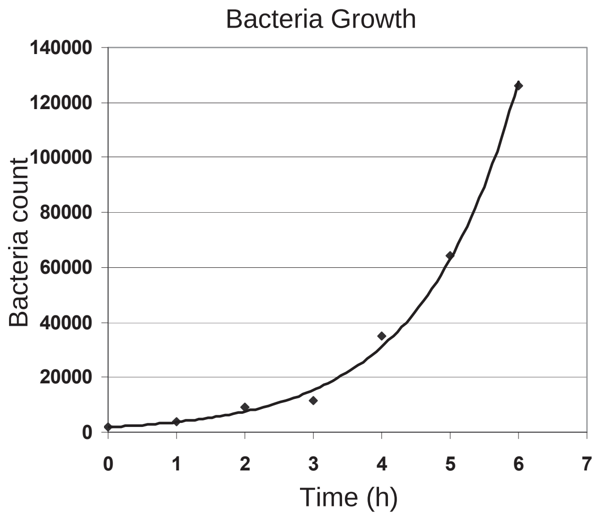
a) Create a scatter plot and draw a curve of best fit. What kind of curve is this?

b) Estimate the bacteria count at 3 h, 15 min.

Solution

a)

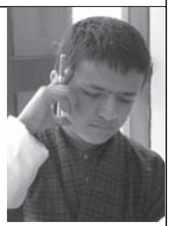




I think this is an exponential curve.

[Continued]

Thinking



a) Time is the independent variable and bacteria count is the dependent variable because the number of bacteria depends on elapsed time.

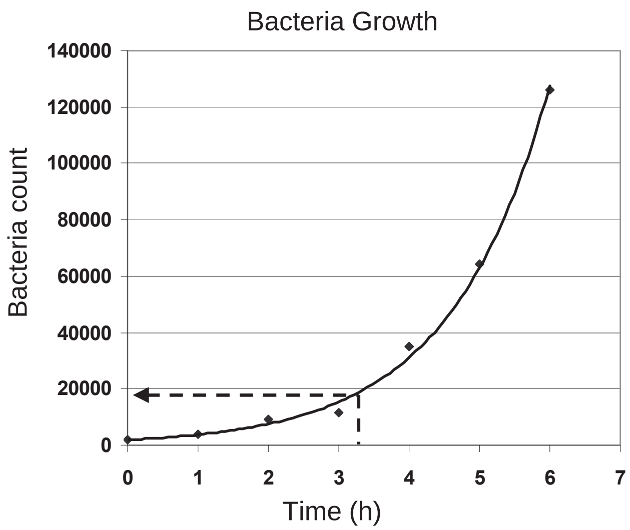
• I drew an increasing curve that passed through or near as many points as possible. I also tried to get an equal number of points on either side of the curve.

• The curve increased really fast so I figured it was probably exponential.

Example 2 Working With Exponential Relationships [Continued]

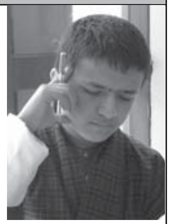
Solution

b)



The bacteria count was about 19,000 at 3 h, 15 min.

Thinking

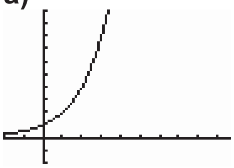


b) I estimated the count at 3 h, 15 min by interpolating at 3.25 h on the horizontal axis (3 h, 15 min = 3.25 h).

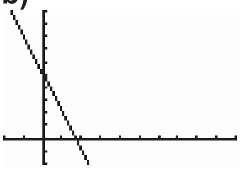
Practising and Applying

1. Identify the following as either linear, quadratic, exponential, or none of these.

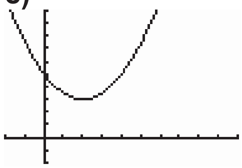
a)



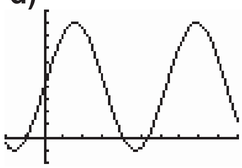
b)



c)



d)



2. AIDS or Acquired Immune Deficiency Syndrome is a disease that affects millions of people world-wide. This table shows the global estimates of cumulative AIDS cases from 1982 to 1996.

a) Create a scatter plot. Draw a curve of best fit.

b) What kind of relationship is it?

c) Why does this curve make sense for this set of data?

d) Estimate the number of AIDS cases in 2000. Does your estimate make sense? Explain.

Year	AIDS cases (millions)
1982	0
1983	0.1
1984	0.2
1985	0.4
1986	0.7
1987	1.1
1988	1.6
1989	2.3
1990	3.2
1991	4.2
1992	5.5
1993	6.9
1994	8.5
1995	10.4
1996	12.5

3. The data set below was collected from trees growing in a site with relatively poor soil quality.

Age (years)	Diameter (cm)	Age (years)	Diameter (cm)
4	2.03	23	11.94
5	2.03	25	16.51
8	2.54	28	15.24
8	5.08	29	11.43
8	7.62	30	15.24
10	5.08	30	17.78
10	8.89	33	20.32
12	12.45	34	16.51
13	8.89	35	17.78
14	6.35	38	12.70
16	11.43	38	17.78
18	11.68	40	19.05
20	13.97	42	19.05
22	14.73		

a) Use the data in the table to estimate the diameter of a 32-year-old tree.

b) How confident are you in your estimate? Explain.

c) Sketch a curve or line of best fit.

d) Use the sketch to answer **part a) and part b)** again.

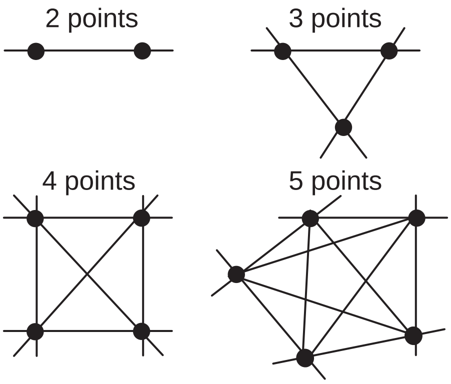
4. The data values in the table below show estimates of the world's population over the last 2000 years.

Year	Population (billions)
0	0.3
1000	0.32
1750	0.8
1800	1
1930	2
1960	3
1974	4
1987	5
1990	5.2
1992	5.4
1995	5.7

Create a graph and use it to estimate the world's population in these years:

a) 500 **b)** 1500 **c)** 2000

5. These diagrams show points joined by all possible line segments.



a) Extend the pattern to include a figure with six points.

b) Make a table of values that relates the number of points to the number of line segments.

c) Graph the data.

d) Is the relationship between the number of points and number of line segments a linear or nonlinear relationship? Explain.

e) Predict the number of line segments for seven points. Explain your prediction.

6. Examine this square dot pattern.

4 16 36

a) Extend the pattern to form the next two dot arrays and then complete the table below (n is the figure number and S_n is the number of dots in each figure).

n	1	2	3	4	5
S_n	4	16	36		

6. b) Draw the graph of S_n vs. n.

c) Is it a linear or nonlinear relation? Explain.

d) Predict S_n when n = 30. Explain your prediction.

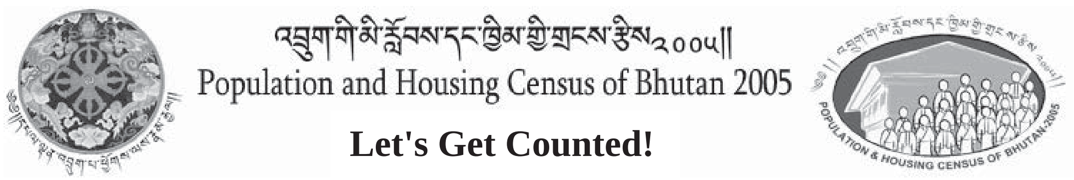
7. Suppose you are given a table of values representing the results of an experiment and you draw a graph of the data. What kinds of patterns would you look for to decide each?

a) if the relationship is linear

b) if the relationship is quadratic

c) if the relationship is exponential

CONNECTIONS: Data Collection by Census



On May 30 and 31, 2005, Bhutan conducted its first ever Population and Housing Census in accordance with international standards, as prescribed by the United Nations. The 2005 Census was undertaken on the basis of the command received from His Majesty the King and the resolution of the 82nd National Assembly of Bhutan. The Census gathered current data about the citizens of Bhutan and the lives they lead.

1. Research.

- Find out what the word Census means.
- Discover who was involved in creating the Bhutan Census.
- Find out how the data sets were collected.
- Find out how what kinds of data were collected.
- Explain why you think the government would like to know this information.

2. Most countries around the world conduct censuses regularly.

- Research to find out which country has the oldest census data on record.
- What is the most common interval of time between censuses in most countries?

Chapter 3 Probability

6.3.1 Dependent and Independent Events

Try This

A bag holds six bank notes — three are Nu 10 notes, two are Nu 20 notes, and one is a Nu 50 note.

Six notes are inside:
- three Nu 10 notes
- two Nu 20 notes
- one Nu 50 note

A. Suppose you were to reach in without looking and draw out a note. What is the probability that you will draw a Nu 20 note?

B. i) Suppose you draw a Nu 20 note and then replace it. What is the probability that you will draw a Nu 20 note the next time?

ii) Suppose you draw a Nu 20 note and do not replace it. What is the probability that you will draw a Nu 20 note the next time?

When you want to find the **probability** that two events will both occur, you must first consider if and how the events are related. Sometimes the outcome of one event will affect the outcome of another event.

• When the outcome of one event does not affect or depend on the other event, the two events are **independent events**.

For example, if you were to toss a coin that lands with Khorlo-Mebar (Burning Wheel) facing up and then roll a die, the probability of rolling a 2 is $\frac{1}{6}$. If you were to toss a coin and get Tashi Ta-gye (Eight Auspicious Signs) facing up and then roll a die, the probability of rolling a 2 is still $\frac{1}{6}$. The outcome of the coin toss does not affect the outcome of the roll of the die because they are independent events.

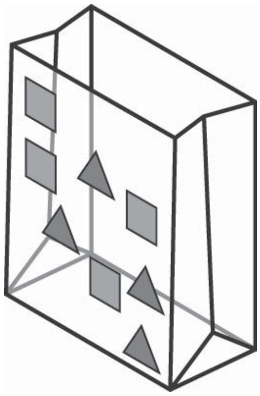
$P(2) = \frac{1}{6}$

$P(2) = \frac{1}{6}$

Tossing a coin and rolling a die are independent events — the result of the coin toss does not change the probability of rolling a 2.

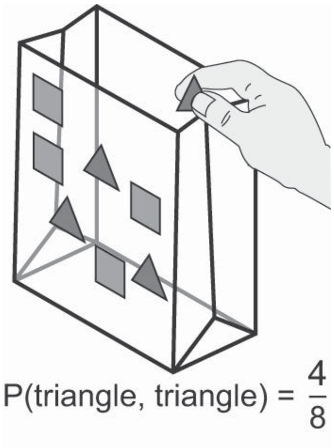
• When the outcome of one event affects or depends on the outcome of another event, the events are **dependent events**.

For example, suppose there are four triangular and four square blocks in a bag. If a triangular block is drawn and then replaced, the probability of drawing a triangular block the next time is $\frac{4}{8}$. These two events — drawing a triangular block followed by drawing another triangular block *after replacing the first block* — are independent because the outcome of the first event does not affect the outcome of the second event.



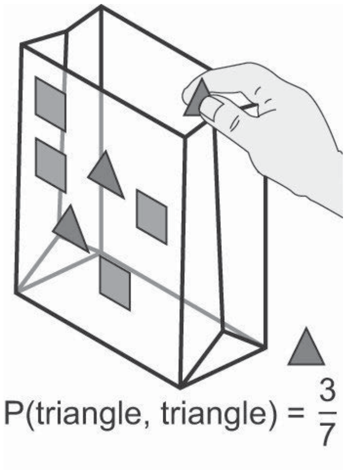
Suppose the triangular block were drawn and not replaced. The probability of drawing a triangular block the next time would change from $\frac{4}{8}$ to $\frac{3}{7}$. These two events — drawing a triangular block followed by drawing another triangular block *without replacing the first block* — are dependent events because the outcome of the second event was affected by the outcome of the first event. The outcome of the first event changed the **sample space** for the second from 8 possible outcomes to 7 possible outcomes because one block was removed.

Independent Events



Drawing a triangle block, replacing it, and then drawing another triangle block

Dependent Events



Drawing a triangle block and not replacing it, then drawing another triangle block

C. Look back at your answers to **part B** on the previous page.

i) Why are the probabilities different between **parts i) and ii)**?

ii) Which situation involves dependent events? How do you know?

Examples

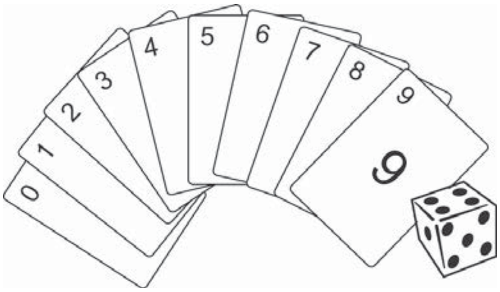
Example 1 Probability of Independent and Dependent Events

a) Which pair describes two independent events? Which are dependent events?

Pair A

• drawing a 4 from a deck of number cards (0 – 9)

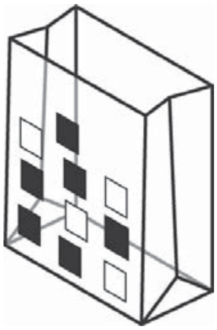
• rolling a 4 on a die



Pair B

• drawing a black tile and not replacing it

• drawing a white tile on the second draw



b) For pair A, what is the probability of rolling a four on the die?

c) For pair B, what is the probability of drawing a white tile in the second draw if the first tile drawn was black?

Solution	Thinking
a) *Pair A* independent events	**a)** I know the two events in pair A are independent because, no matter what number card you draw, it doesn't affect the outcome of rolling the die.
Pair B dependent events	• The two events in pair B are dependent. The probability of drawing a particular colour on the second draw is affected by what you draw the first time because the first tile drawn isn't replaced. This affects the sample space — instead of 10 tiles to drawn from, there are now 9 tiles. So the probability of drawing a second white tile is $\frac{4}{9}$ if a black tile is drawn the first time or $\frac{3}{9}$ if a white tile is drawn the first time.
b) *Pair A* $P(4) = \frac{1}{6}$	**b)** When two events are independent, you can figure out the probability of the second event as if it were a separate event because it isn't affected by the other event. Since there are 6 equally possible outcomes when you roll a die and 1 of the outcomes is four, the probability is $\frac{1}{6}$.
c) *Pair B* $P(\text{white}) = \frac{4}{9}$	**c)** If there are 10 tiles to start with and 1 black tile is drawn and not replaced, there are 9 tiles left, or 9 possible outcomes. Since 4 white tiles are still in the bag, there's a $\frac{4}{9}$ chance that a white tile will be drawn next.

Example 2 Deciding if Events are Independent or Dependent

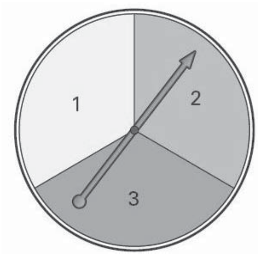
Dawa spins this spinner twice.

Are these two events independent?

Event A Spinning a 1 on the first spin

Event B Spinning a number on the second spin such that the sum of both spins is 5

Show your work.



Solution 1

Create an outcome chart

Spin 2

Spin 1	**1**	**2**	**3**
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

If A happens …

Spin 2

Spin 1	**1**	**2**	**3**
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

P(B) = 0

If A does not happen …

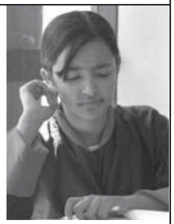
Spin 2

Spin 1	**1**	**2**	**3**
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

P(B) = $\frac{2}{6}$

Since P(B) = 0 if Event A happens, and P(B) = $\frac{1}{3}$ if Event A does not happen, Events A and B are dependent (and therefore, they are not independent).

Thinking



- I knew if the probability of Event B was not affected by Event A, they were independent.
- I created an outcome chart to find all possible outcomes of two spins.
- Event A (spinning a 1 on the first spin) was represented by the first row of the chart. That meant the sample space for Event B was only 3 possible outcomes because there are only three possible outcomes if the first spin has to be 1. Since 0 of them result in a sum of 5, the probability of Event B happening if Event A happens is $\frac{0}{3}$, or 0.
- Then I thought about what would happen if Event A didn't happen (if the first spin was a 2 or a 3). This was represented by the second and third rows of the chart, so the sample space for Event B was now 6 possible events (since there are only six possible outcomes if the first spin was 2 or 3). Since only 2 of these result in a sum of 5, (2, 3) and (3, 2), the probability of Event B happening, if Event A didn't happen, is $\frac{2}{6}$ or $\frac{1}{3}$.
- Since the probability of Event B is affected by the outcome of Event A (whether Event A happens or not), I knew the two events are dependent.

Solution 2	**Thinking**
If my first spin is 2 or 3, I can get a sum of 5 on my second spin. If my first spin is 1, a sum of 5 on my second spin is impossible. That means the outcome of Event A affects Event B so they cannot be independent events.	I used reasoning to figure it out. 

Practising and Applying

1. The 11 letters in the word PROBABILITY are written on 11 cards and placed in a bag. Two cards are drawn from the bag, one after the other.

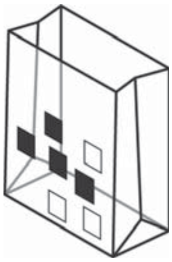
a) You draw a card with the letter B and then replace it. What is the probability that you will draw a card with the letter B on the second draw?

b) You draw a card with the letter B and do not replace it. What is the probability that you will draw a card with the letter B on the second draw?

c) Are the probabilities in **part a) and part b)** the same or different? Why?

d) Which are dependent, the two events in **part a)** or the two events in **part b)**? How do you know?

2. A bag contains three white tiles and four black tiles. You reach in and draw one tile and then you draw another tile.



What is the probability of drawing each?

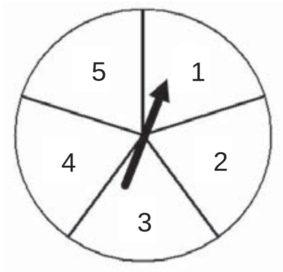
a) a second white tile if the first tile drawn is white and you replace it before drawing again

b) a second white tile if the first tile drawn is white and you do not replace it

c) a white tile on the second draw if the first tile drawn is black and you replace it

d) a white tile on the second draw if the first tile drawn is black and you do not replace it

3. You spin this spinner twice. Tell whether the events in each pair below are dependent or independent. Explain how you know.



a) *Event A* First spin is 4
Event B Second spin is 4

b) *Event A* First spin is even
Event B Second spin is odd

c) *Event A* First spin is 3
Event B The sum of the first and second spin is 3

d) *Event A* first spin is 2
Event B the difference between the first and second spins is 1

4. Is each pair of events described below independent or dependent? How do you know?

Pair A
• Rolling a 2 on a die
• Rolling a second time and getting a number that results in a total of 5 or more for both rolls

Pair B
• Rolling a 3 or a 4 on the first roll of a die
• Rolling a number less than 5 on the second roll

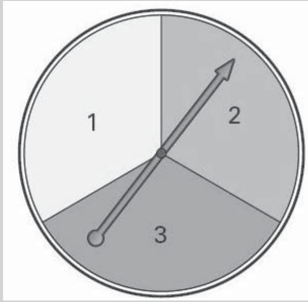
5. a) Describe two events that are dependent. How do you know they are dependent?

b) Describe two events that are independent. How do you know they are independent?

6.3.2 Calculating Probabilities

Try This

Dechen is going to spin this spinner twice.



A. i) What is the probability that she will spin a 3 on the first spin?

ii) What is the probability that on her second spin she will spin a number that is greater than the number she spun the first time?

iii) What is the probability that she will spin a sum of 4 on the two spins?

• To calculate the probability of two independent events happening, you can calculate the product of each event happening.

For example, consider these two independent events involving rolling a die twice:

Event A	*Event B*
Rolling a 1 on the first roll	Rolling an even number on the second roll
P(A: 1 on first roll) = $\frac{1}{6}$	P(B: even on second roll) = $\frac{1}{2}$

The probability of rolling a 1 on the first roll and an even number on the second roll is P(A, B) = $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

The tree diagram below shows why $\frac{1}{12}$ makes sense — there are 12 possible outcomes in the sample space when you roll two dice and only one of them is a roll of 1 followed by a roll of an even number, (1, E).

First Roll	**1**		2		3		4		5		6	
Second roll	**E**	O	E	O	E	O	E	O	E	O	E	O
	(1, E)	(1, O)	(2, E)	(2, O)	(3, E)	(3, O)	(4, E)	(4, O)	(5, E)	(5, O)	(6, E)	(6, O)

The probability of rolling a 6 and then rolling an even number is 1 in 12, or $\frac{1}{12}$.

• If two events are dependent, the separate probabilities of each event happening cannot be multiplied to calculate the probability of both events happening because one outcome is related to the other. This fact can be used to determine in two events are dependent or independent.

For example, consider these two events involving rolling a die twice:

Event A Rolling a 1 in the first roll *Event B* Rolling a sum of 5 in two rolls

To figure out whether these two events are dependent or independent, you can determine the probability of both events happening as if they were separate independent events and then compare that probability to the probability of both events happening as combined events. If the probabilities are the same, the events are independent. If the probabilities are different, the events are dependent

For example:

Determine the probability of both events happening as separate events

P(A: rolling a 1 in the first roll) = $\frac{1}{6}$

P(A) = $\frac{1}{6}$ because, out of six possible outcomes in the sample space (1, 2, 3, 4, 5, and 6), one of them is 1.

P(B: rolling a sum of 5 in two rolls) = $\frac{4}{36}$

+	1	2	3	4	5	6
1	2	3	4	**5**	6	7
2	3	4	**5**	6	7	8
3	4	**5**	6	7	8	9
4	**5**	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

P(B) = $\frac{4}{36}$ because, out of 36 possible outcomes in the sample space, four of them have a sum of 5 (1 + 4, 2 + 3, 3 + 2, and 4 + 1).

P(A) × P(B) = $\frac{1}{6} \times \frac{4}{36} = \frac{1}{6} \times \frac{1}{9} = \frac{1}{54}$

Determine the probability of both events happening as combined events

P(A and B) = $\frac{1}{36}$

+	1	2	3	4	5	6
1	2	3	4	**5**	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

P(A and B) = $\frac{1}{36}$ because, for a sum of 5 in two rolls if the first roll is a 1, the second roll has to be 4. There is only one favourable outcome out of a sample space of 36 possible outcomes.

Compare the probabilities Since $\frac{1}{54} \neq \frac{1}{36}$, the events must be dependent.

- In the example above, Event B happened after Event A, but it is possible to have two events happen at the same time. You can still determine if the events are dependent or independent in the same way. For example:

You roll a die *once* and the following two events can happen in the same roll:

Event A Rolling an even number *Event B* Rolling a 6

Probability as separate events

P(A: even) = $\frac{3}{6}$ P(B: 6) = $\frac{1}{6}$

Probability as combined events

P(even and 6) = $\frac{1}{6}$

P(A) × P(B) = $\frac{3}{6} \times \frac{1}{6} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Compare the probabilities Since $\frac{1}{12} \neq \frac{1}{6}$, the events must be dependent.

• The following steps summarize one way to determine if two events are independent or dependent:

Step 1: Determine the probability of both events happening as separate events.

Step 2: Determine the probability of both events happening as combined events.

Step 3: Compare the probabilities from Steps 1 and 2 — if the probabilities are the same, the events are independent; if they differ, the events are dependent.

• Depending on the complexity of the situation, you might use reasoning instead to figure out if two events are dependent. For example, since rolling a 6 makes it certain that you have also rolled an even number, the two events are dependent. The event of rolling a 6 clearly affects the event of rolling an even number.

• The probability of Event B happening if you assume Event A has already happened is called a **conditional probability**. It is written as P(B|A) and read as "*the probability of B given A*." Since P(A and B) = P(A) × P(B|A), you can use P(A and B) to calculate the conditional probability of two dependent events. You can also use the conditional probability, P(B|A), to calculate P(A and B). (See **Example 2**.)

B. Look at your answers to **part A** on the previous page.

i) Are the events in **parts i) and ii)** independent or dependent? Explain.

ii) Are the events in **parts i) and iii)** independent or dependent? Explain.

Examples

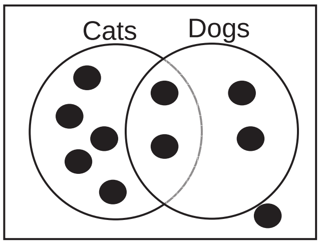
Example 1 Calculating to Show That Two Events are Dependent

In a group of ten families, two families have both cats and dogs as pets, five have only cats, two have only dogs, and one has no pets. You select one family at random from the group. Show that the two events below are dependent.

Event A	*Event B*
Selecting a family that has a dog	Selecting a family that has a cat

Solution

Probability of both events happening as combined events



$$P(A \text{ and } B) = \frac{2}{10}$$

Thinking

• I drew a Venn diagram to sort the families into the different subgroups: those with only cats, those with cats and dogs, those with only dogs, and those with no pets. Each dot represented one family — I placed each dot according to what pets each family had. The dot outside the circles is the family with no pets.

• To determine the probability of both events happening at the same time, I created a fraction with the number of families with cats and dogs, 2, as the numerator and the total number of families, 10, as the denominator.

Probability of both events happening as separate events

$P(A) = \frac{4}{10}$ $\quad$ $P(B) = \frac{7}{10}$

$P(A \text{ and } B) = \frac{4}{10} \times \frac{7}{10} = \frac{28}{100}$

$\frac{2}{10} \neq \frac{28}{100}$, so Events A and B are dependent.

• 4 out of 10 families have dogs and 7 out of 10 families have cats.

• I compared the product of the two probabilities to $\frac{2}{10}$. Since they were different, I knew the events were dependent.

Example 2 Determining Conditional Probabilities

You draw two cards, one at a time, from a deck of playing cards.

a) Use reasoning to explain why Events A and B are dependent if the first card is not replaced before drawing the second card.

b) What is P(B|A)?

c) Show that P(A and B) = P(A) × P(B|A).

Event A	*Event B*
Draw a 7 in the first draw	Draw a 7 in the second draw

Solution

a) Events A and B are dependent since the probability of Event B is affected by the outcome of Event A.

b) $P(B|A) = \frac{3}{51}$

c) $P(A \text{ and } B) = \frac{12}{2652}$

$P(A) = \frac{4}{52}$

$P(B|A) = \frac{3}{51}$

$$P(A) \times P(B|A) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}$$

Therefore,
P(A and B) = P(A) × P(B|A)

Thinking

a) If I draw a card and do not replace it, the sample space changes from 52 for the first draw to 51 for the second draw, which means the probability of Event B is affected by Event A.

b) P(B|A) is the probability of Event B, if Event A happens. If Event A happens, the first card drawn is a 7, which leaves 51 cards and three of them are 7s. That means there is a 3 in 51 chance of Event B happening (drawing a 7 the second time).

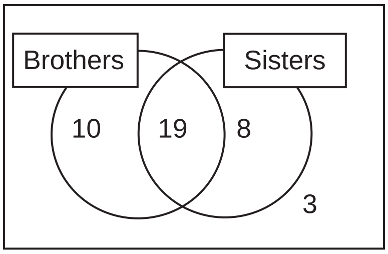
c) To find the probability of both Events A and B happening, I thought about all the possible arrangements of two cards I can draw. There would be 2652 possibilities (52 for the first card × 51 for the second). Of these, there would be 12 with two 7s:

7♥, 7♠	7♦, 7♠	7♣, 7♠	7♠, 7♥
7♥, 7♣	7♦, 7♣	7♣, 7♦	7♠, 7♦
7♥, 7♦	7♦, 7♥	7♣, 7♥	7♠, 7♣

• The probability of drawing a 7 the first time is $\frac{4}{52}$ and the probability of drawing a 7 the second time is $\frac{3}{51}$ (from **part b)**).

Practising and Applying

1. The Venn diagram below shows the number of students in a class of 40 who have only brothers, only sisters, brothers and sisters, and no siblings.



a) A student is randomly selected. What is the probability that the student

i) has brothers?

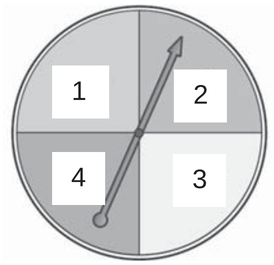
ii) has sisters?

iii) has brothers and sisters?

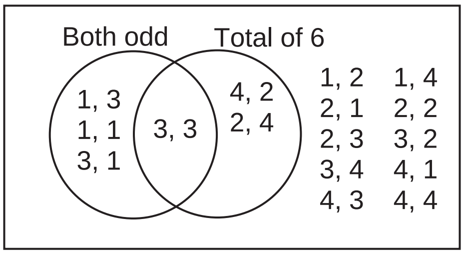
iv) has no siblings?

b) Are the events in **part i) and part ii)** dependent or independent? Explain.

2. This spinner is spun twice.



Explain how you can you use the Venn diagram below to determine each. (Hint: The pairs of numbers represent the numbers spun, for example, 2, 3 is a spin of 2 and then 3.)



a) probability that both spins will be odd

b) probability that the sum of both spins will be 6

c) whether the two events in **part a) and part b)** are dependent

3. Dechen randomly chooses an integer from 1 to 50.

Event A The number is a multiple of 3

Event B The number is a multiple of 5

a) What is the probability of each?

i) Event A happening

ii) Event B happening

iii) Events A and B both happening

b) Are Events A and B dependent or independent? Explain.

4. Indra randomly chooses an integer from 1 to 100

Event A The integer is even

Event B The integer is a multiple of 4

a) What is the probability of each?

i) Event A happening

ii) Event B happening

iii) Events A and B both happening

b) Are Events A and B dependent or independent? Explain.

c) Show each:

i) P(A and B) = P(A|B) × P(B)

ii) P(A and B) = P(B|A) × P(A)

5. A bag contains black and white marbles. Two marbles are pulled out one at a time without replacing the first one. The probability of selecting a black marble first is 0.6. The probability of selecting a black and then a white marble is 0.25. What is the probability of selecting a white marble if a black is selected first?

6. Show that each pair of events is dependent. You can use calculating or reasoning or both.

a) Rolling a 4, and then rolling again so that the sum is 10

b) Selecting a red marble from a bag of five red and five blue marbles, and then selecting a blue marble (without replacing the red marble first)

UNIT 6 Revision

1. Describe a situation that has not already been presented in this unit where you might use each graph.

a) a stem and leaf plot

b) a double stem and leaf plot

c) a histogram

d) two box and whisker plots

2. You are given a set of data that contains 100 numbers. What are some advantages and disadvantages of displaying the data using each graph?

a) a stem and leaf plot

b) a histogram

c) a box and whisker plot

3. One week after planting, the heights of 30 bean plants were measured (cm).

5	25	22	32	10	25
21	27	29	39	37	18
10	36	11	33	48	16
24	36	45	38	7	26
15	37	17	22	25	13

a) Create a stem and leaf plot.

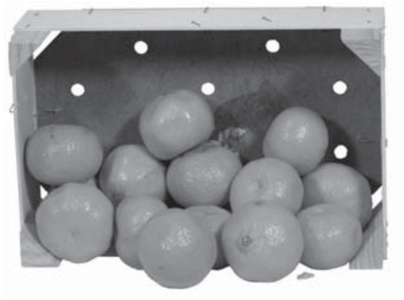
b) Use the stem and leaf plot to create a histogram with the same interval size.

c) Identify the type of distribution.

d) Create a box and whisker plot.

e) Which of the three types of graph do you prefer? Explain why.

4. Crates of oranges were packed in two different shipments. The mean diameter (in cm) of the oranges in each crate was recorded.



Shipment 1 (mean diameter per crate in cm)

9.1	9.0	9.1	9.0	10.0
8.9	8.3	8.8	9.4	8.5
8.9	9.8	8.7	8.4	9.6
9.2	9.0	8.3	9.2	7.0
9.5	8.5	8.7	8.4	9.6
11.0	9.5	9.3	9.7	10.0

Shipment 2 (mean diameter per crate in cm)

8.9	9.9	8.4	9.7	6.0
6.1	9.7	7.2	6.8	8.1
6.0	9.3	9.3	8.5	6.8
7.1	8.5	8.9	6.7	8.8
6.4	9.8	7.1	9.6	6.4
8.3	6.2	10.0	7.7	9.1

The shipment of larger, more uniformly sized oranges will be sold as oranges and the other shipment will be made into juice.

a) Create a box and whisker plot for each shipment using the same scale.

b) Which shipment should be used for juice? Justify your decision.

5. Ten batteries from each of three brands (A, B, and C) were tested to determine their lifespans in hours.

a) Create three box plots on the same scale to compare them.

b) Which brand appears to be the best? Explain.

Brand A (Lifespan in h)

41	189	204	102	28
94	179	87	116	155

Brand B (Lifespan in h)

39	65	22	64	22
171	99	32	142	70

Brand C (Lifespan in h)

24	95	159	122	41
72	75	44	43	18

6. This frequency table shows the results of a math exam in Pema's class.

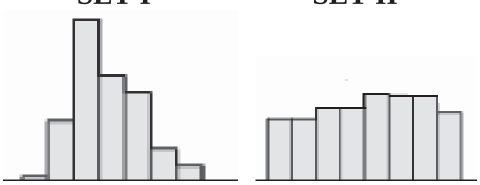
Score (%)	Frequency
40 – 49	2
50 – 59	3
60 – 69	5
70 – 79	12
80 – 89	10
90 – 99	8

a) Construct a histogram and a box plot on the same scale.

b) Use your graphs to summarize how the data set is distributed.

7. Two sets of data were graphed.

SET I SET II



a) Sketch a frequency polygon for each.

b) Describe the shape of each frequency polygon and then identify the distribution.

c) Describe a situation that each histogram might represent.

8. The list below gives the age in months of 30 snow leopards tagged in Jigme Dorji National Park.

47	28	31	41	39	25
21	29	26	23	34	25
33	37	28	45	18	36
54	40	33	47	42	29
37	22	42	37	48	64

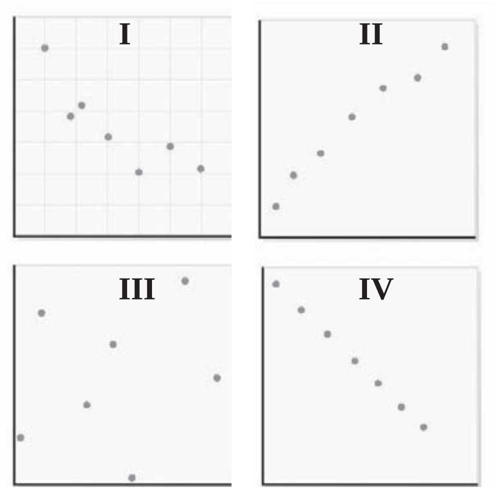
a) Create a histogram for this data set.

b) Describe the shape of your graph and identify the distribution.

c) Find the mean, median, and mode.

d) How likely is it that a snow leopard that is caught and tagged will be less than 50 months old?

9. Examine scatter plots I to IV.



a) Which scatter plot shows the strongest correlation?

b) Which scatter plots show a negative correlation?

9. c) Which scatter plot shows no correlation?

d) Which scatter plot shows a weak negative correlation?

e) Which scatter plot shows a strong positive correlation?

f) Estimate the correlation coefficient for each scatter plot.

10. For each relationship below, identify the type of correlation you would expect to find and estimate the correlation coefficient

a) *Relationship A*
the relationship between the volume of petrol remaining in a gas tank and the distance driven since the tank was filled

b) *Relationship B*
the relationship between the average amount of monthly rainfall and the number of umbrellas sold by a store

11. Give an example of two quantities, or variables that would show each.

a) a positive correlation

b) a negative correlation

c) no correlation

12. Statistics Canada produces reports based on Canadian census data. The table below shows how the average life expectancy for Canadians has changed every 10 years since 1920.

Year	Male	Female
1920	58.8	60.6
1930	60.0	62.1
1940	63.0	66.3
1950	66.4	70.9
1960	68.4	74.2
1970	69.4	76.5
1980	71.9	79.1
1990	74.6	81.0

12. a) Sketch two scatter plots on the same grid, one for the male life expectancy data and one for the female data.

b) Describe each correlation.

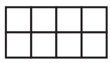
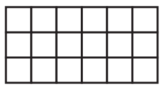
c) Estimate each correlation coefficient.

d) Is it appropriate to draw lines of best fit for this set of data? Explain.

e) Use the graphs to predict the average life expectancy both for Canadian males and for Canadian females in the year 2021.

13. A set of rectangles is made up of 1 cm squares. The first rectangle measures 1 cm × 2 cm. The next is 2 cm × 4 cm, then 3 cm × 6 cm.
The pattern continues with the length always being twice the width.

a) Draw the patterns of squares to help you visualize the pattern. Use your diagrams to complete the table.

Shape	Width (cm)	Length (cm)	Area (cm^2)
	1	2	2
	2		
	3		
	4		
	5		

b) Create a scatter plot to show each.

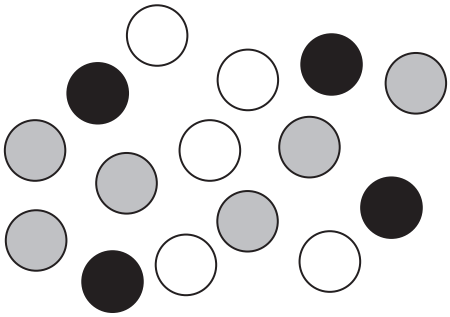
- the relationship between length and width
- the relationship between area and width

c) Is it suitable to draw a line or a curve of best fit for each relationship in **part b)**? Explain.

14. Explain how not replacing an item when you draw items from a bag affects the probability of drawing two items of the same kind, one after the other.

15. Novin rolled a pair of dice. The first roll was 6 and the sum of both rolls was 10. He calculated the probability of both events happening by multiplying the probability of getting a 6 on the first roll by the probability of rolling a sum of 10 on both rolls. What did he do wrong?

16. A bag contains 4 black, 5 white, and 6 grey balls.



a) You draw a black ball and do not replace it. What is the probability of drawing a white ball next?

b) You draw a grey ball and do not replace it. What is the probability of drawing a black ball next?

c) You draw a grey ball and do not replace it. What is the probability of drawing another grey ball?

d) You draw a white ball and do not replace it. What is the probability of drawing another white ball?

17. You randomly select one integer from 1 to 19.

a) Create a Venn diagram to help you determine the probability of selecting a number that is both greater than 10 and even.

b) How do you know the two events in **part a)** are dependent?

18. You have a bag of blue and red marbles. You select two marbles one after another without replacing the first one. The probability of selecting a blue marble is 0.6. The probability of selecting two blue marbles is 0.3. What is the probability of selecting a blue marble second if the first marble was blue? Explain your thinking.

19. You roll a die twice.

a) What is the probability that you roll a number greater than 4 the first time?

b) What is the probability that the total of the two rolls is 6?

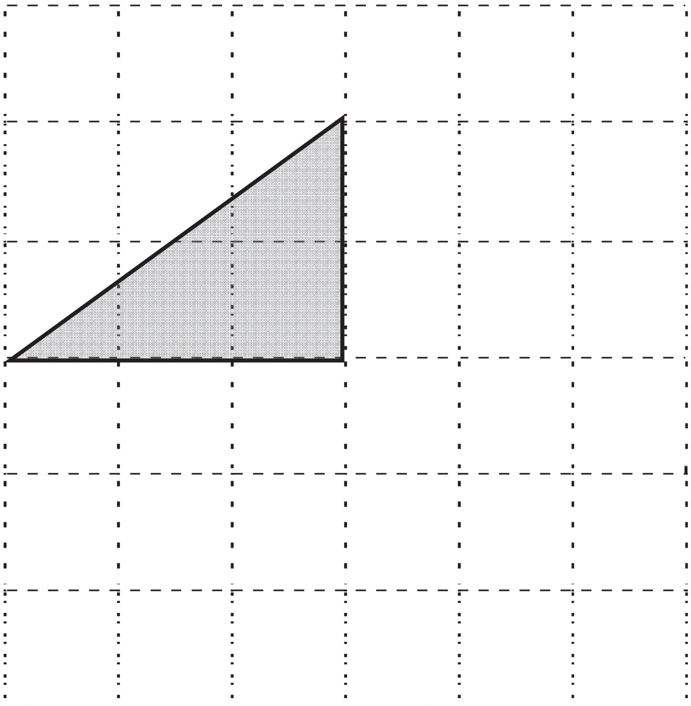
c) Are the events in **parts a) and b)** independent? Explain.

UNIT 7 TRIGONOMETRY

Getting Started

Use What You Know

A. Create a 6-by-6 grid and then copy the triangle as shown below.



B. On the grid, draw all triangles that are similar but not congruent to the triangle. Each triangle must have at least two side lengths with whole number values and each vertex of each triangle must be located at a grid intersection. If triangles are in different positions, they are considered to be different.

C. How do you know each triangle you drew is similar to the first triangle?

D. How do you know there are no other such triangles on the grid?

Skills You Will Need

1. Calculate the value of *x* for each.

a) $\frac{x}{10} = \frac{30}{40}$

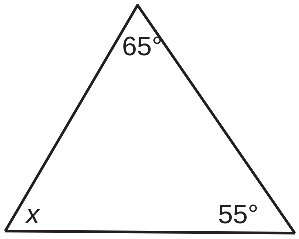
b) $\frac{12}{x} = \frac{8}{15}$

c) $\frac{7}{36} = \frac{x}{12}$

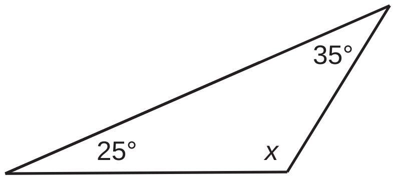
d) $\frac{15}{x} = \frac{16.5}{1.21}$

2. Without measuring, calculate the missing angle in each triangle.

a)

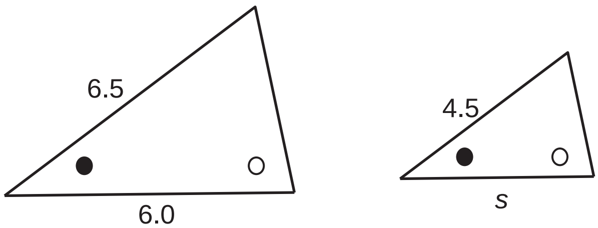


b)



3. Without measuring, calculate the length of the missing side *s*.
(Hint: The triangles are similar.)

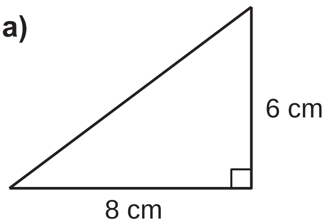
a)



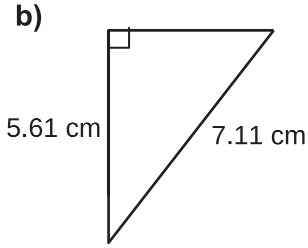
b) Draw another triangle similar to the two above.

4. Without measuring, calculate the length of each missing side.

a)

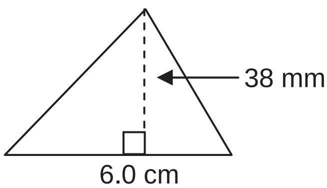


b)

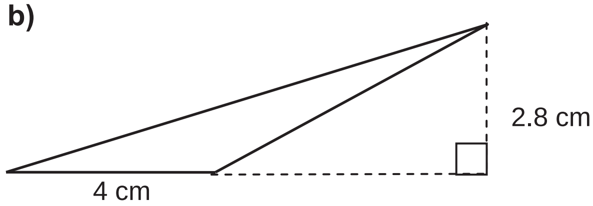


5. Calculate the area of each triangle.

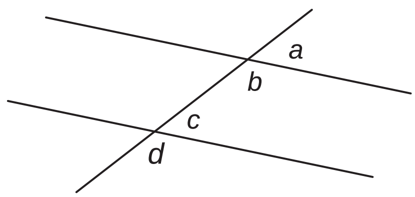
a)



b)



6. Two of the lines below are parallel.
Which pairs of angles in the diagram
are equal? How do you know?

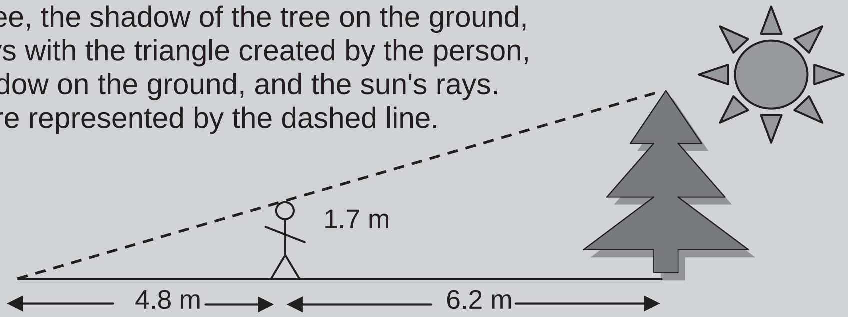


Chapter 1 Defining Trigonometric Ratios

7.1.1 Using Similarity Properties to Solve Problems

Try This

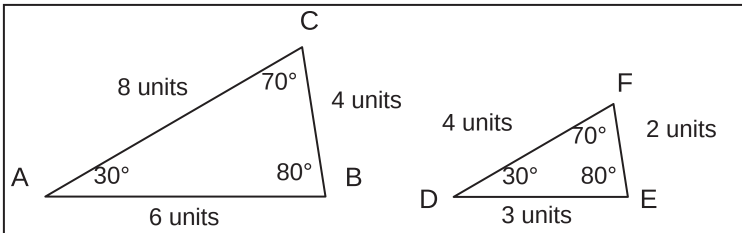
To find the height of the tree, you can compare the triangle created by the tree, the shadow of the tree on the ground, and the sun's rays with the triangle created by the person, the person’s shadow on the ground, and the sun's rays. The sun's rays are represented by the dashed line.



A. What is the height of the tree?

Recall that two triangles are similar when they have equal angles and when the corresponding sides are proportional. The ratio of the corresponding sides is called the **scale factor**.

For example, if side AB in ΔABC is twice as long as its corresponding side DE in ΔDEF, the scale factor that relates ΔDEF to ΔABC is 2. That means BC is also twice as long as EF and AC is also twice as long as DF.



ΔABC ~ ΔDEF because

$\angle A = \angle D$

$\angle B = \angle E$

$\angle C = \angle F$

and

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{CB}{FE} = 2$$

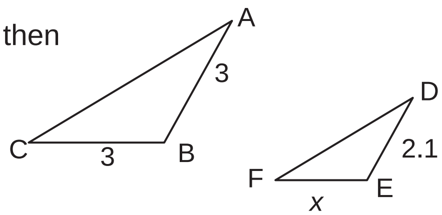
• When you indicate that two triangles are **similar**, it is important to list corresponding vertices in the same order. In the two triangles above, ∠A = ∠D, so the vertices A and D are written in the same position (first) in the similarity statement, Δ**A**BC ~ Δ**D**EF. ∠B = ∠E, so they are both listed next, ΔA**B**C ~ ΔD**E**F, and ∠C = ∠F, so they both come last, ΔAB**C** ~ ΔDE**F**.

• When you know two triangles are similar, you can set up proportions to determine the missing side lengths in the triangles.

For example:

To find *x* in ΔDEF, because ΔABC ~ ΔDEF, then

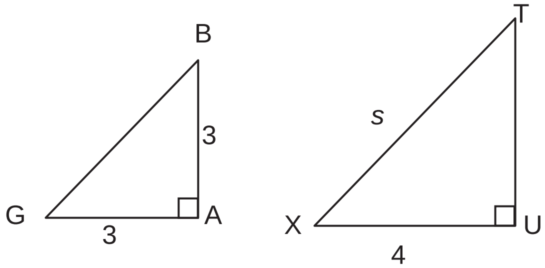
$\frac{AB}{BC} = \frac{DE}{EF} \rightarrow \frac{3}{3} = \frac{2.1}{x}$, and $x = 2.1$ units.



• Sometimes you have to use the Pythagorean theorem to determine the side lengths to be used in the proportions.

For example,

To find s in ΔTUX, because ΔBAG ~ ΔTUX, then $\frac{AG}{UX} = \frac{BG}{TX} \rightarrow \frac{3}{4} = \frac{BG}{s}$.



You cannot solve a proportion with two unknown values, so you use the Pythagorean theorem to find BG.

$$3^2 + 3^2 = BG^2$$
$$18 = BG^2$$
$$BG = \sqrt{18} \approx 4.24$$

Now you can complete the proportion and solve it: $\frac{3}{4} = \frac{4.24}{s}$, so s = 5.65 units

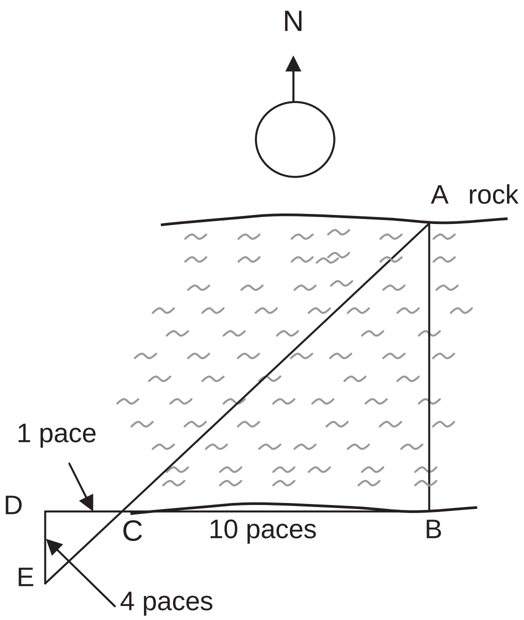
• Sometimes, to solve a missing length problem you have to create the similar triangles yourself.

For example, suppose you want to know the distance across a river, but you cannot measure it directly. You can create two similar triangles following these steps (refer to the diagram on the next page as you read these instructions):

- Find a marker such as a rock on the other side of the river and call it point A.
- Mark point B on your side of the river directly across from point A.
- Walk ten paces west from point B and mark point C.
- Walk one more pace west and mark point D.
- Turn south and count the paces to the point (marked E) so that point E, point C, and the rock (point A) all line up. In this example, it takes four paces to reach point E.

ΔABC ~ ΔEDC because of AAA ($\angle$ACB = $\angle$ECD and $\angle$B = $\angle$D = 90º)

Now you can use similar triangles and the Pythagorean theorem to determine the distance across the river, AB.



$$\frac{BC}{DC} = \frac{AB}{ED}$$

$$\frac{10}{1} = \frac{AB}{4}$$

AB = 40

The distance across the river (AB) is 40 paces.

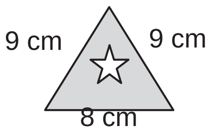
B. i) Name the similar triangles in the diagram in **part A**.

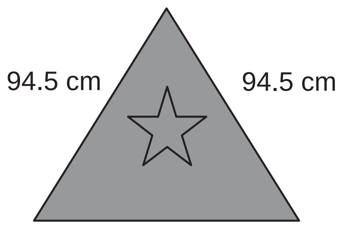
ii) What proportion could you set up to find the height of the tree?

Examples

Example 1 Solving for a Side Length When Triangles are Similar

Sonam wants to enlarge an isosceles triangle design to make a large copy for her poster. She uses a dilatation. How long is the base of the larger triangle?

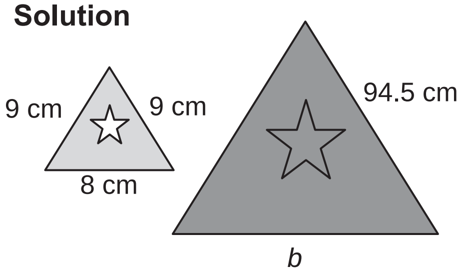




[Continued]

Example 1 Solving for a Side Length When Triangles are Similar [Continued]

Solution



$$\frac{9}{94.5} = \frac{8}{b}$$

$$9b = 756$$

$$b = 84$$

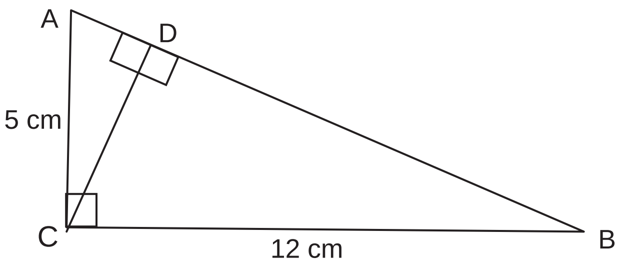
The base is 84 cm.

Thinking

- I knew the triangles were similar because a dilatation results in similar triangles.
- I knew two side lengths in one triangle and one side length in the other triangle that corresponded to one of them. I knew it corresponded because of the star design.
- That meant I could set up a proportion using ratios of corresponding sides and solve it to determine the missing base length.

Example 2 Determining Similarity

How long is CD?



Solution 1

$$\frac{AB}{AC} = \frac{BC}{CD}$$

$$\frac{AB}{5} = \frac{12}{CD}$$

$$5^2 + 12^2 = AB^2$$

$$25 + 144 = AB^2$$

$$AB = 13$$

$$\frac{13}{5} = \frac{12}{CD}$$

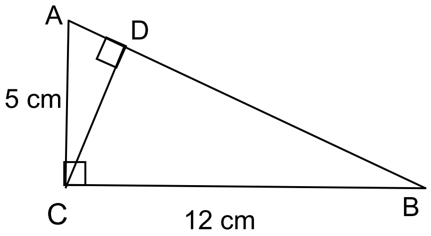
$$CD = \frac{12 \times 5}{13}$$

$$CD \approx 4.62 \text{ cm}$$

Thinking

- I knew $\Delta ABC \sim \Delta ACD$ since each has a 90° angle and they share $\angle A$ (AAA).
- I set up a proportion based on the similarity.
- To find CD, I first needed to calculate AB. I knew I could use the Pythagorean theorem.
- I solved the proportion for CD.

Solution 2



$$\text{Area of } \triangle ABC = \frac{12 \times 5}{2}$$
$$= 30$$

$$30 = \frac{AB \times CD}{2}$$

$$5^2 + 12^2 = AB^2$$
$$25 + 144 = AB^2$$
$$AB = 13$$

$$30 = \frac{13 \times CD}{2}$$
$$CD = 60 \div 13$$
$$\approx 4.62 \text{ cm}$$

Thinking

- When I looked at the diagram, I realized that CD was a height of $\triangle ABC$ if AB was the base. So, if I could figure out the area of $\triangle ABC$ and the length of AB, then I could use the area formula, $A = bh$, to figure out CD (h).

- I was able to figure out the area of $\triangle ABC$ by thinking about CB as the height and AC as the base.

- I created an equation using the area formula, with AB as the base and CD as the height. Then realized I needed to find AB before I could solve it.

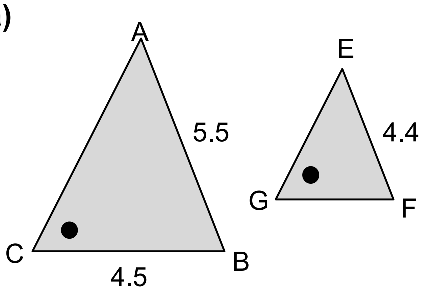
- I used the Pythagorean theorem to find AB.

- I substituted 13 for AB in my equation and solved for CD.

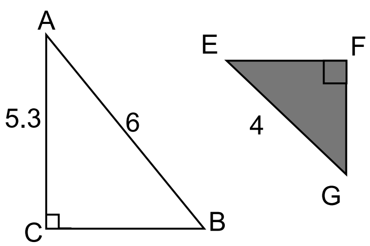
Practising and Applying

1. Calculate the length of side FG without measuring. The triangles in each pair are similar.

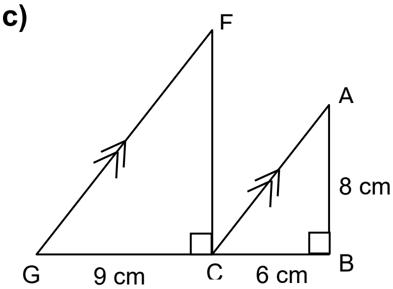
a)



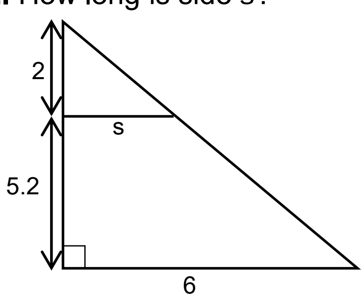
b)



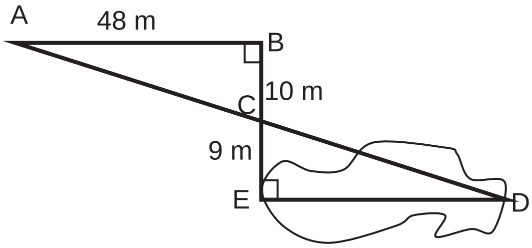
c)



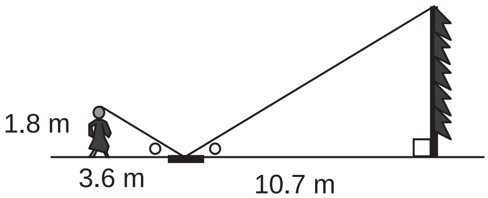
2. How long is side s?



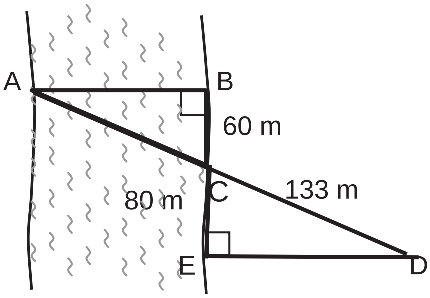
3. What is the distance across the lake from D to E?



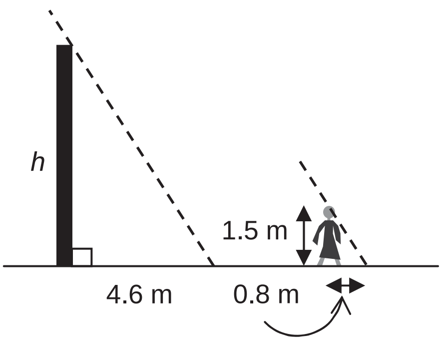
4. The woman on the left is looking at the reflection of the top of the prayer flag pole in the mirror on the ground. How tall is the prayer flag pole?



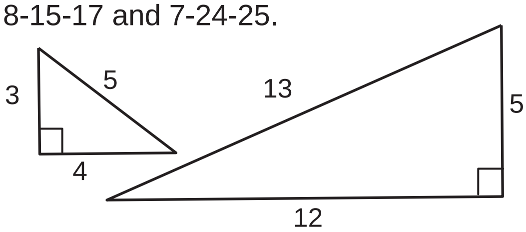
5. How wide is the river?



6. At the same time of day, the shadow of a pole and the shadow of a girl were measured. How tall is the pole? (Hint: The sun's rays are parallel.)

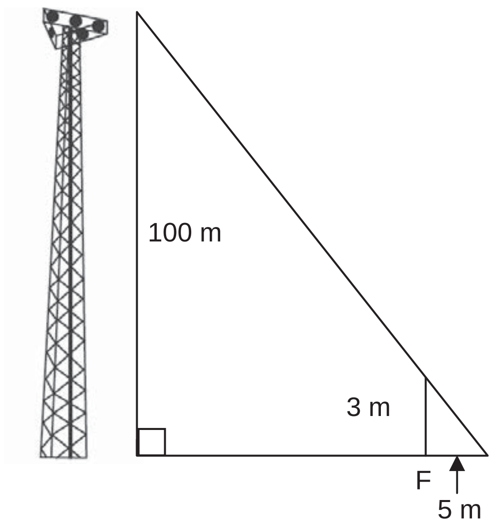


7. Pythagorean triples are whole number values that could be the three sides of a right triangle. Some examples are 3-4-5 and 5-12-13, as shown below. Other examples are 8-15-17 and 7-24-25.



Use triangles similar to either of those above to determine which Pythagorean triples could include the number 24 as one of the side lengths. Find all three possibilities.

8. Pema positioned a 3 m pole at point F. Then he stood 5 m east of the pole to sight the top of a 100 m radio tower. How far was he from the tower (assuming his eye is at ground level)?



9. Explain how knowing the concept of similar triangles can help you measure distances that are otherwise hard to measure.

7.1.2 EXPLORE: Special Ratios in Similar Triangles

This diagram includes three triangles:

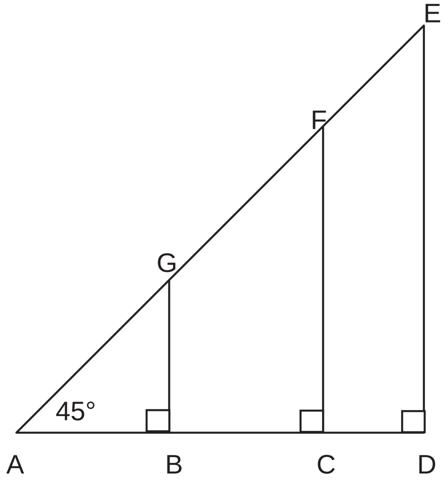
ΔABG, ΔACF, and ΔADE

Each triangle has

• a vertical side (BG, CF, and DE)

• a horizontal side (AB, AC, and AD)

• an hypotenuse (AG, AF, and AE)



Use these three triangles to answer the questions below.

A. Copy the chart below.

B. For each triangle, measure the lengths of all three sides (vertical, horizontal, and hypotenuse) to the nearest tenth of a centimetre. Record the values in the first three columns of your chart.

C. For each triangle, calculate these three ratios:

$$\frac{\text{vertical}}{\text{hypotenuse}} \qquad \frac{\text{horizontal}}{\text{hypotenuse}} \qquad \frac{\text{vertical}}{\text{horizontal}}$$

Round each value to three decimal places. Record the values in the last

h l f h

$\angle$ **A is 45º**	**vertical**	**horizontal**	**hypotenuse**	$\frac{\textbf{vertical}}{\textbf{hypotenuse}}$	$\frac{\textbf{horizontal}}{\textbf{hypotenuse}}$	$\frac{\textbf{vertical}}{\textbf{horizontal}}$
ΔABG						
ΔACF						
ΔADE						

D. What do you notice about the values in each of the last three columns?

[Continued]

E. Copy the chart below. Record the values from the last three columns of your chart from **part C** in the 45° column of your new chart.

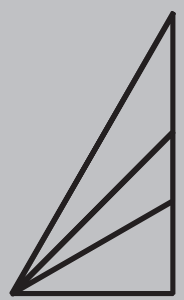
F. Create sets of three right triangles like the ones on the previous page, but change $\angle A$ to one of the other values in the chart below. Complete the chart for the three ratios.

Angle A	**30°**	**45°**	**60°**
$\frac{\text{vertical}}{\text{hypotenuse}}$			
$\frac{\text{horizontal}}{\text{hypotenuse}}$			
$\frac{\text{vertical}}{\text{horizontal}}$			

G. Why was it not necessary to put in three values for the three different triangles you used for each angle?

H. i) Use the picture on the left below to help explain what happened to the values in the first and second rows of the chart.

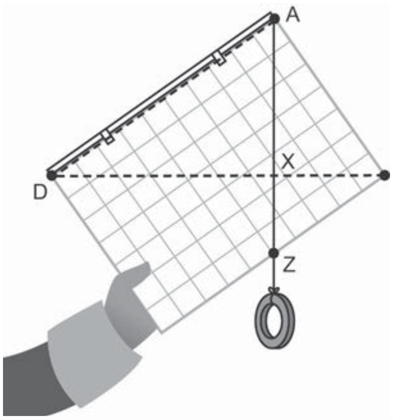
ii) Use the picture on the right below to help explain what happened to the values in the third row of the chart.



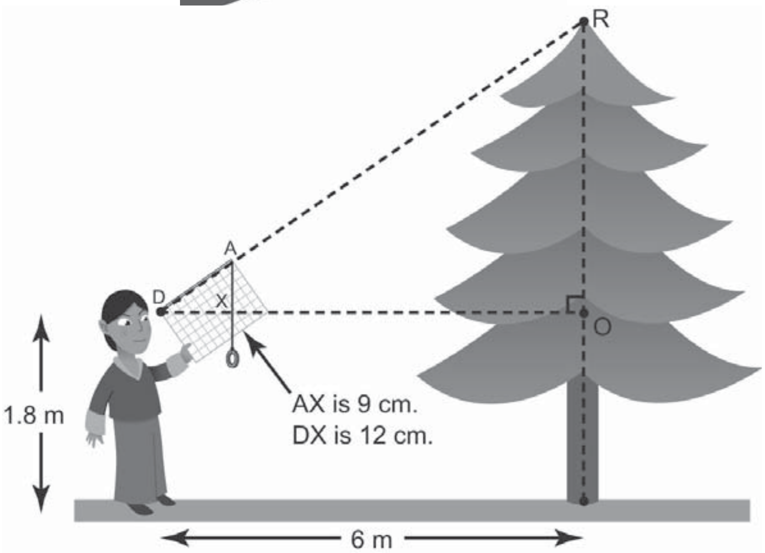
I. What do you notice about the values in the first and second rows of the chart in **part F**?

CONNECTIONS: Using a Clinometer

A clinometer is a device you can use to estimate the heights of very tall objects indirectly using similar triangles.



You can construct a very simple clinometer with a drinking straw, a piece of graph paper glued to a rigid piece of cardboard, a length of string, and a weight, as shown.



You sight the top of the tall object through the straw and someone marks a point X on the graph paper where the weighted string crosses the horizontal line from your eye.

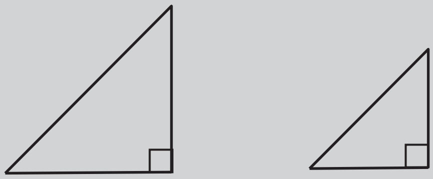
1. Why is $\triangle ADX$ on the clinometer similar to $\triangle RDO$ (the triangle formed by your eye at D, the top of the tree at R, and the middle of the tree directly across from your eye, at O)?

2. Use $\triangle ADX$ and $\triangle RDO$ to estimate the height of the tree.

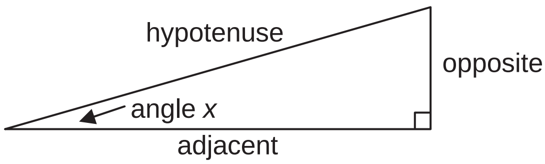
7.1.3 The Sine, Cosine, and Tangent Ratios

Try This

A. Carefully construct two different right triangles, each with a 45° angle.
i) What is the ratio of the leg length to the hypotenuse in each triangle?
ii) How do you know?



In any right triangle, the two legs of the triangle are given special names with respect to each non-right, or acute angle. The leg that is part of the angle is called the **adjacent** side and the leg that is not part of the angle is called the **opposite** side. The side opposite the right angle is called the **hypotenuse**.

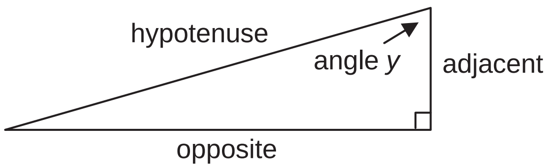


• The ratios of the side lengths in any right triangle have special names, according to the acute angle being considered. For angle *x* in the triangle above, the special ratios are:

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos x = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan x = \frac{\text{opposite}}{\text{adjacent}}$$

The ratios are pronounced "sine of *x*," "cosine of *x*," and "tangent of *x*," if *x* is the angle. They are called **trigonometric**, or **trig ratios**.

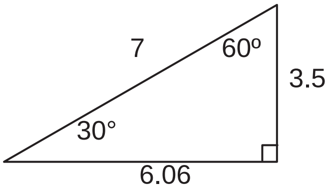
• For the other acute angle in the right triangle, angle *y*, the side lengths that are adjacent and opposite change and the trigonometric ratios are:



$$\sin y = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos y = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan y = \frac{\text{opposite}}{\text{adjacent}}$$

• These ratios only apply to the acute angles in a right triangle. They do not apply to the right angle.

For example, in this triangle, these ratios can be calculated:

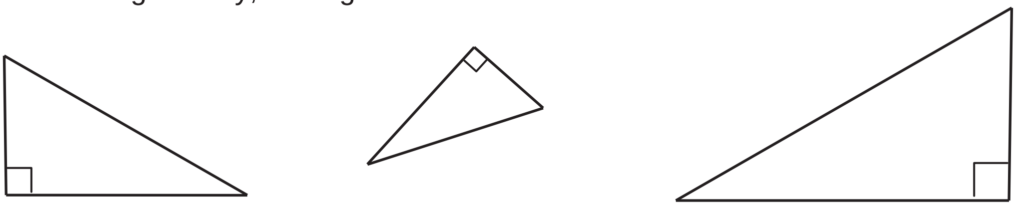


$\cos 30° = \frac{6.06}{7} \approx 0.866$ $\qquad$ $\cos 60° = \frac{3.5}{7} = 0.5$

$\tan 30° = \frac{3.5}{6.06} \approx 0.577$ $\qquad$ $\tan 60° = \frac{6.06}{3.5} \approx 1.732$

• Regardless of the dimensions of the right triangle, the sine, cosine, and tangent of a particular angle size do not change. This is because all right triangles with a particular pair of acute angles are similar, so the ratios of the sides do not vary.

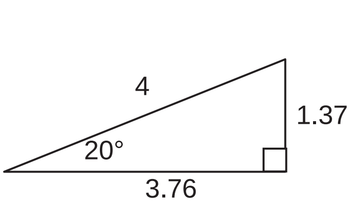
For example, all these right triangles have acute angles of 30° and 60°. Although their side lengths vary, the trig ratios are the same.

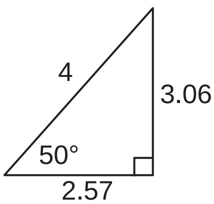


$\sin 30° = 0.5$	$\cos 30° \approx 0.866$	$\tan 30° \approx 0.577$
$\sin 60° \approx 0.866$	$\cos 60° = 0.5$	$\tan 60° \approx 1.732$

• The sine and cosine ratios for any angle between 0° and 90° are always less than 1 because the opposite and adjacent sides used in the ratios are the legs of a right triangle, which are always shorter than the hypotenuse. Therefore, the numerator for the sine or cosine ratio is always smaller than the denominator, the hypotenuse.

• The value of the sine ratio increases as the angle increases from 0° to 90° while the cosine ratio decreases. This is because the opposite side becomes longer and the adjacent side becomes shorter as the angle increases, while the hypotenuse stays the same length.





$\sin 20° = \frac{1.37}{4} = 0.342$ $\qquad$ $\sin 50° = \frac{3.06}{4} = 0.766$

$\cos 20° = \frac{3.76}{4} = 0.940$ $\qquad$ $\cos 50° = \frac{2.57}{4} = 0.643$

• The value of the tangent ratio depends on the relative sizes of the sides that are opposite and adjacent to the angle. In the example on the previous page, the tangent for the 30º angle is less than 1 because the opposite side is shorter than the adjacent side. However, the tangent of the 60° angle is greater than 1 because the opposite side is longer than the adjacent side. Tan 45° is 1 since the opposite side is the same length as the adjacent side. Tangent is not defined for 90° since the denominator (representing the adjacent side) is 0. Similarly, tan 0° is 0 since the numerator (representing the opposite side) is 0.

• You can use your calculator to calculate trig ratios, given the angles, or to calculate the angles, given the trig ratios.

For example:

First make sure your calculator is in degree mode (look for DEG in the display),

- To calculate the sine of 45º, press [sin] 45 [=].
- To calculate the angle for cosine 0.7, press [$\cos^{-1}$] 0.7 [=].

(Note that $\cos^{-1}$, $\sin^{-1}$, and $\tan^{-1}$ are called inverse functions.)

• The chart below can help you estimate trig ratios.
For example, an angle of 62º would have a sine of about 0.88, a cosine of about 0.47, and a tangent of about 1.95.

Angle	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.174	0.342	0.500	0.643	0.766	0.866	0.940	0.985
Cosine	0.985	0.940	0.866	0.766	0.643	0.500	0.342	0.174
Tangent	0.176	0.364	0.577	0.839	1.192	1.732	2.747	5.671

NOTE: A trig table for multiples of 5º is found at the back of this book on **page 380**.

B. Which trigonometric ratios were you working with in **part A**?

Examples

Example 1 Comparing Sine and Cosine

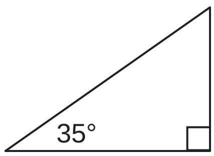
A right triangle has an angle of 35°.

a) Without measuring, predict which ratio will be greater: the sine of 35° or the cosine of 35°. Explain your prediction.

b) Estimate each ratio and then measure and calculate to check.

Solution

a)



Predict

cos 35° > sin 35°

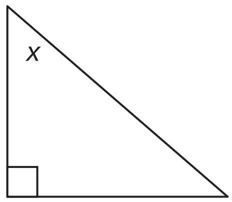
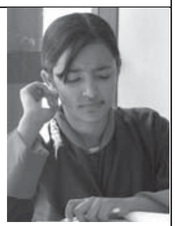
because adjacent > opposite

Thinking

a) I drew a right triangle with an angle of 35°. I knew it didn't matter what the dimensions were as long as it was a right triangle with an angle of 35° because the trig ratios are the same.

• I could see that the side adjacent to 35° was a bit longer than the opposite side, so I predicted the cosine would be greater than the sine.

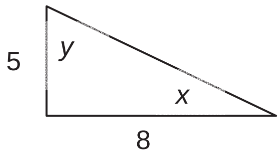
Solution	Thinking
b) *Estimate* sin 35° is about 0.6 cos 35° is about 0.8 *Measure and calculate* sin 35° = $\frac{2.15}{3.75}$ = 0.574 cos 35° = $\frac{3.07}{3.75}$ = 0.819 cos 35° > sin 35°	**b)** I know sin 30° = 0.5 and sin 40° = 0.643, so I estimated sin 35° to be about 0.6. • I know cos 30° = 0.866 and cos 40° = 0.766, so I estimated cos 35° to be about 0.8. • I measured to check. My prediction was correct — the cosine ratio was greater than the sine ratio. • My estimate for the cosine was about right but my estimate for the sine was a bit high.

Example 2 Determining an Angle with a Particular Tangent	
The tangent of an acute angle is 1.2. **a)** Is the angle greater than or less than 45º? How do you know? **b)** Estimate the angle. Use a calculator to check your estimate.	
Solution	**Thinking**
a)  x > 45º because opposite side > adjacent side	 **a)** I sketched a right triangle, with one leg a bit longer than the other because the tangent (the ratio of the two legs) was just a bit greater than 1. • I located angle x so that the longer leg was opposite to it, because $\frac{\text{opposite}}{\text{adjacent}} > 1$. • I knew angle x was bigger than the other acute angle because its opposite side was longer than its adjacent side. That meant it had to be greater than 45° because both acute angles in a right triangle add to 90°, so if one is bigger, it must be greater than 45°.
b) *Estimate* $\tan^{-1}$ (1.2) is about 51º. *Calculate to check* $\tan^{-1}$ (1.2) = 50.19º	**b)** I knew tan 50° = 1.192 and tan 60° = 1.732, so I estimated that a tan of 1.2 would have an angle of about 51°. • I used a calculator to get the actual angle. My estimate was close.

Example 3 Determining an Angle when Side Lengths are Known

The legs of a right triangle are 5.0 cm and 8.0 cm. What are the angle measures in the triangle?

Solution



$\tan x = \frac{5}{8} = 0.625$

If $\tan x$ is 0.625, then $x = 32°$.

$y = 90 - 32 = 58°$

Thinking

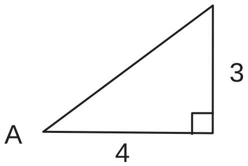
• I sketched the triangle and labelled both acute angles.

• If I had wanted to use the sine or cosine ratios, I would have to have used the Pythagorean theorem to calculate the hypotenuse. I used the tangent ratio instead because I already had the values I needed.

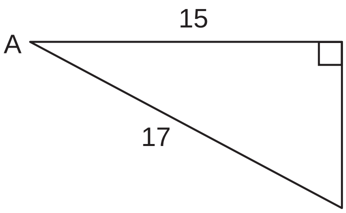
Practising and Applying

1. Determine the sine, cosine, and tangent for each ∠A using the given measurements. Round to two decimal places.

a)

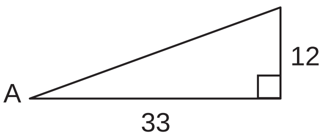


b)

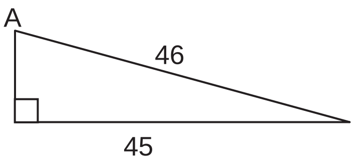


2. Determine the sine, cosine, and tangent for each ∠A. Round to two decimal places.

a)



b)



3. Determine the sine, cosine, and tangent for each angle.

a) 35° **b)** 55°

c) 12° **d)** 80°

4. For which acute angle is each true?

a) $\sin x = 0.85$

b) $\cos x = 0.24$

c) $\tan x = 0.2$

5. Draw three triangles, one to fit each description.

a) The sine of one angle is 0.7.

b) The cosine of one angle is 0.5.

c) The tangent of one angle is 2.4.

6. a) The sine of ∠A is greater than the sine of ∠B and both angles are acute. What do you know about the sizes of the two angles?

b) If sin ∠A > cos ∠B and both angles are acute angles, can you be certain about which angle is greater? Explain.

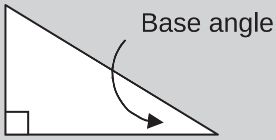
7. How does knowing about similarity help you understand how trigonometric ratios work?

7.1.4 Trigonometric Identities

Try This

A. i) Draw two right triangles so that the cosine of the base angle in one triangle is the same as the sine of the base angle in the other triangle.

ii) What do you notice about the two triangles?



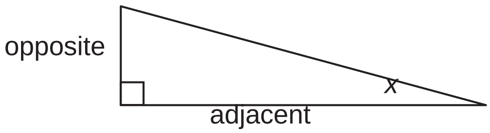
Some equations are true for only certain values of a variable, for example, 2x = 6 is only true if $x = 3$. But other equations are true for all values of the variable in the equation, for example, $2x + 2x = 4x$ is true for any value of x. Equations that are true for all values of the variable are called **identities**.

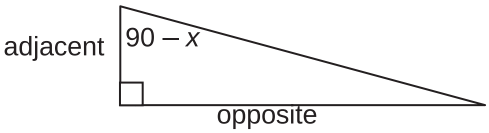
• Some equations involving trig ratios are **trigonometric identities**. These identities represent relationships among the trig ratios and are useful for finding other trig ratios if you know one trig ratio and for solving problems.

- One trigonometric identity involves the relationship between the sine and cosine of complementary angles, that is, angles that have a sum of 90°.

$$\boxed{\cos (90^o - x) = \sin x \qquad \sin (90^o - x) = \cos x}$$

This identity makes sense because, in any right triangle, the two acute angles add to 90°, so if one angle is x, the other angle must be $90^o - x$ and, because of the placement of the acute angles in the right triangle, the opposite side for one angle is the adjacent side for the other, and vice versa.

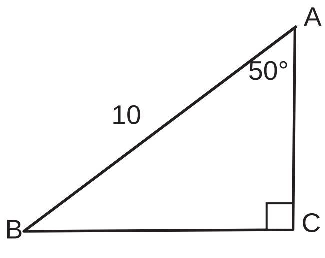




You can use this identity to find the cosine ratio if you know the sine ratio, or the sine ratio if you know the cosine ratio.

For example, suppose you were asked to calculate the cosine of $\angle$B in ΔABC.

You could follow the steps on the left below to find the length of BC and then calculate the cosine ratio. Or you could use $\sin (90^o - x) = \cos x$, as as shown on the right.



$$\sin 50° = \frac{BC}{10} \rightarrow 0.766 = \frac{BC}{10}$$

$$BC = 0.766 \times 10$$

$$BC = 7.66$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{7.66}{10} = 0.766$$

If $\cos x = \sin (90^o - x)$, then

$\cos 40^o = \sin 50^o = 0.766$

So $\cos 40^o = 0.766$

As you can see, using the identity $\sin (90^o - x) = \cos x$ was simpler.

- Another identity is sometimes called the Pythagorean identity, since it is based on the Pythagorean theorem.

$$\boxed{\cos^2 x + \sin^2 x = 1 \qquad \sin^2 x + \cos^2 x = 1}$$

The following explains why this identity works:

Since the Pythagorean theorem applies to any right triangle, the equation to the right is true.

$$\text{adjacent}^2 + \text{opposite}^2 = \text{hypotenuse}^2$$

Divide both sides of the equation by hypotenuse^2.

$$\left\{\begin{array}{l} \dfrac{\text{adjacent}^2}{\text{hypotenuse}^2} + \dfrac{\text{opposite}^2}{\text{hypotenuse}^2} = \dfrac{\text{hypotenuse}^2}{\text{hypotenuse}^2} \\[2ex] \dfrac{\text{adjacent}^2}{\text{hypotenuse}^2} + \dfrac{\text{opposite}^2}{\text{hypotenuse}^2} = 1 \end{array}\right.$$

Substitute cos x for one ratio and sin x for the other.

$$\cos^2 x + \sin^2 x = 1$$

This identity is useful for finding the cosine of an angle if you know its sine, or vice versa. You can rearrange the equation to suit your needs.

$$\sin^2 x + \cos^2 x = 1 \rightarrow \sin^2 x = 1 - \cos^2 x \rightarrow \sin x = \sqrt{1 - \cos^2 x}$$

For example, suppose you know that cos x is 0.5 and you want to know sin x.

If $\cos x = 0.5$ and $\sin x = \sqrt{1-\cos^2 x}$, $\sin x = \sqrt{1 - 0.5^2} = \sqrt{1 - 0.25} \approx 0.866$

• Sometimes identities are true because they are definitions that relate trig ratios.

For example, some mathematicians define tan x as $\dfrac{\sin x}{\cos x}$, if x is not 90°.

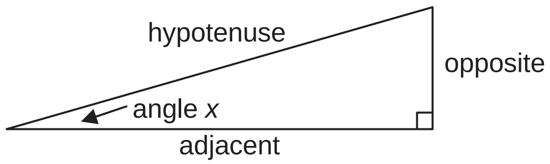
It is clear that this must be true since

$$\frac{\sin x}{\cos x} = \frac{\text{opposite}}{\text{hypotenuse}} \div \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{adjacent}} = \tan x$$

• Mathematicians have defined three other trigonometric ratios based on the primary trig ratios: sine, cosine, and tangent. Each one is the reciprocal of a primary trigonometric ratio. These are read as *secant*, *cosecant*, and *cotangent*.

$$\boxed{\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}}$$

They can also be described in terms of the sides of the right triangle:



$$\sec x = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc x = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cot x = \frac{\text{adjacent}}{\text{opposite}}$$

- The secant, cosecant, and cotangent ratios can be considered identities because, as defined, they represent relationships among the trig ratios.

- Since sine and cosine are always between 0 and 1, the cosecant and secant are always 1 or more. If cos x = 0, then sec x is not defined (you cannot divide by 0). Similarly, if sin x = 0 (when x = 0°), csc x is not defined. If tan x = 0 (when x = 0°) or tan x is not defined (when x = 90°), cot x is not defined.

• If you know the sine or cosine of an angle, you can use the Pythagorean identity, and the identities defining tangent, secant, cosecant, and cotangent to calculate the values of all of the other five trig ratios for that angle.

• Certain trig ratios are basic and should be memorized. They are shown here:

	0°	**30°**	**45°**	60°	90°
sin x	0	0.5	$\frac{\sqrt{2}}{2}$ or 0.707	$\frac{\sqrt{3}}{2}$ or 0.866	1
cos x	1	$\frac{\sqrt{3}}{2}$ or 0.866	$\frac{\sqrt{2}}{2}$ or 0.707	0.5	0

- Note that the values $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$ are exact while their decimal equivalents, 0.866 and 0.707, are approximations. By knowing these approximate values, you can estimate angle sizes for other situations.

For example, if you know sin x is 0.75; you know x is between 45° and 60°.

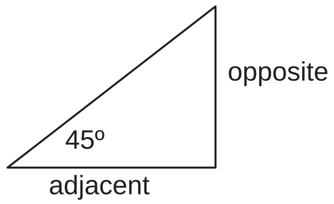
- These basic ratio values are easy to reconstruct using the Pythagorean theorem and the trig identities. The explanations that will allow you to recall these are described below.

Reconstructing the ratios for 45º

In a 45°-45°-90° right triangle, the opposite and adjacent sides are equal, so

$$\text{opposite}^2 + \text{opposite}^2 = \text{hypotenuse}^2$$

$$\frac{\text{opposite}^2}{\text{hypotenuse}^2} + \frac{\text{opposite}^2}{\text{hypotenuse}^2} = \frac{\text{hypotenuse}^2}{\text{hypotenuse}^2}$$



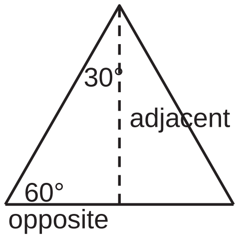
$$2 \times \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)^2 = 1$$

$$\frac{\text{opposite}}{\text{hypotenuse}} = \sqrt{\frac{1}{2}}$$

$$\sin 45^\circ = \frac{\sqrt{2}}{2} \approx 0.707$$

Reconstructing the ratios for 30º and 60º

A 30°-60°-90° right triangle is half of an equilateral triangle. Therefore, the side opposite the 30° angle is half the hypotenuse. The side adjacent to the 30° angle can be calculated using the Pythagorean theorem.



$$\text{hypotenuse}^2 = \left(\frac{\text{hypotenuse}}{2}\right)^2 + \text{adjacent}^2$$

$$\text{hypotenuse}^2 - \left(\frac{\text{hypotenuse}}{2}\right)^2 = \text{adjacent}^2$$

$$\text{adjacent}^2 = \frac{3}{4}\,\text{hypotenuse}^2$$

$$\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2 = \frac{3}{4}$$

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{\frac{3}{2}} \text{ or } \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \approx 0.866$$

Since cos 30° = sin 60° and sin 30° = cos 60° (using cos (90° – x) = sin x), the values for 60° angles can also be reconstructed.

Reconstructing the ratios for 0º and 90º

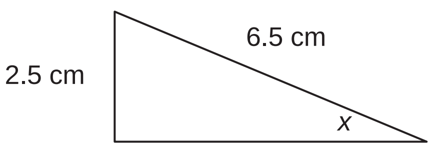
- sin 0° must be 0 since there is no opposite side in the triangle (the length is 0) so that means cos 90° is also 0.
- sin 90° and cos 0° must each be 1 since $\sin^2 x + \cos^2 x = 1$.

B. Which identity were you really using in **part A**?

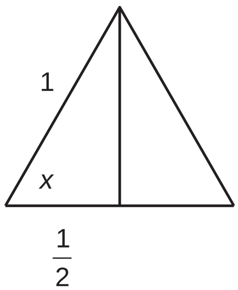
Examples

Example 1 Calculating cos *x* When sin *x* is Known

Calculate cos x for the right triangle shown.



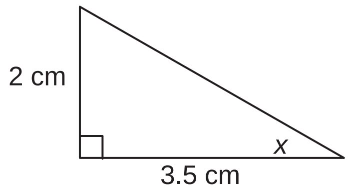
Solution 1	Thinking
The missing side length is $\sqrt{6.5^2 - 2.5^2} = \sqrt{36} = 6$, so $\cos x = \frac{6}{6.5} \approx 0.923$	I used the Pythagorean theorem to calculate the other side length so I could determine the cosine ratio. 
Solution 2	**Thinking**
$\sin^2 x + \cos^2 x = 1$ $0.385^2 + \cos^2 x = 1$ $\cos^2 x = 1 - 0.385^2$ $\cos^2 x \approx 0.852$ $\cos x \approx 0.923$	I knew from the given measurements of the triangle that sin x was $\frac{2.5}{6.5}$ or 0.385, so I used the relationship between the squares of the sines and cosines to find cosine. 

Example 2 Using the Reciprocal Ratios	
In a right triangle, sec x = 2. What is the value of x?	
Solution 1	**Thinking**
If sec x = 2, then cos x = $\frac{1}{2}$.  The angle must be 60°.	• I knew that sec $x = \frac{1}{\cos x}$, so I used the secant ratio to find cosine. • I knew that cosine was $\frac{1}{2}$ for one of the angles in a 30°-60°-90° triangle. I drew that triangle as half of an equilateral triangle to help me figure out whether x was 30° or 60°. • Cosine is based on the adjacent side, so the angle with cosine of $\frac{1}{2}$ must be the 60° angle. 
Solution 2	**Thinking**
If sec x = 2, then cos x = $\frac{1}{2}$. $\cos^{-1} \frac{1}{2} = 60°$	My calculator doesn't have a button for $\sec^{-1}$ that gives the angle if you enter the secant, but it does have a button for $\cos^{-1}$. That's why I used the relationship between secant and cosine to find the value of the cosine ratio. 

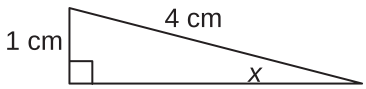
Practising and Applying

1. For each angle *x* below, use the given measurements to determine the secant, cosecant, and cotangent.

a)



b)



2. Calculate the other five trigonometric ratios based on the given one.

a) sin $x = 0.8$

b) cos $x = 0.4$

c) tan $x = 1.0$

d) sec $x = 5$

3. Determine the acute angle for which each is true.

a) csc $x = 2$

b) cot $x = 1.4$

c) sec $x = 3.4$

4. Draw a picture and indicate the necessary calculations that show why $\tan 60° = \sqrt{3}$.

5. In each case, which acute angle is greater, *x* or *y*? How do you know?

a) sec *x* > sec y

b) csc *x* > csc y

c) cot *x* < cot y

6. Complete each with an acute angle.

a) sin 37° = cos ___

b) cos 42° = sin ___

c) sin ___ = 0.8

d) cos ___ = 0.8

e) tan ___ = 0.8

7. Explain how you know that an acute angle with a sine of 0.7 must be close to 45°.

8. Tell whether each statement is true or false. Explain your thinking.

a) cos *x* > sin *x*, if *x* is less than 45°

b) cos *x* < tan *x*, if *x* is less than 45° but greater than 0°

c) sin *x* = (tan *x*)(cos *x*), if *x* is not 90°

d) cos *x* = (tan *x*)(sin *x*), if *x* is not 90°

9. The tangent of an angle is 0.5. Explain how to calculate each of the other five trig ratios.

Chapter 2 Applying Trigonometric Ratios

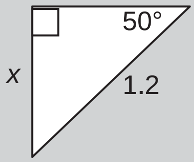
7.2.1 Calculating Side Lengths and Angles

Try This

A. Describe how to calculate the value of x in this right triangle

ii) using the sine ratio

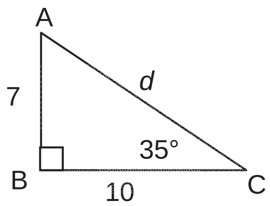
ii) using the cosine ratio



The properties of similar triangles can be used to determine unknown lengths and angles in one triangle using information about known lengths and angles in another similar triangle. Trigonometric ratios can be used to determine unknown lengths or angles in any right triangle.

- In the example below, an unknown side length is found using trig ratios and known side lengths.

ΔABC is a right triangle. You can determine the length of d using the known side lengths and the sine or cosine ratio for the known angle 35°:



• $\sin 35° = \frac{7}{d}$, so $d = \frac{7}{\sin 35°} = \frac{7}{0.574} \approx 12.20$

• $\cos 35° = \frac{10}{d}$, so $d = \frac{10}{\cos 35°} = \frac{10}{0.819} \approx 12.21$

Notice that the value for d varies slightly when it is calculated using a different trig ratio. This is because the side measurements are approximations.

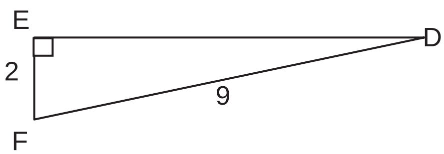
• You cannot use tan to find d, because you end up with an equation without d as an unknown: $\tan 35° = \frac{7}{10}$.

Note that you could also have used the Pythagorean theorem to determine d:

$$d^2 = \sqrt{7^2 + 10^2} = \sqrt{149} \approx 12.21$$

- In the next example, an unknown angle measure is found using trig ratios and known side lengths.

ΔDEF is a right triangle. You can determine the measure of angle D using the sine ratio and the known side lengths.



$\sin D = \frac{2}{9} \approx 0.222$

If sin D is 0.222, $\angle D \approx 12.83°$.

If you were to find the length of DE using the Pythagorean theorem, you could instead have used the cosine or tangent ratio to find $\angle$D:

$DE^2 + 2^2 = 9^2 \rightarrow DE^2 = \sqrt{9^2 - 2^2} = \sqrt{77} \approx 8.77$

$\cos D = \frac{8.77}{9} \approx 0.974$, so $\angle D \approx 13.09°$

$\tan D = \frac{2}{8.77} \approx 0.228$, so $\angle D \approx 12.85°$

Notice that the value of the angle varied again.

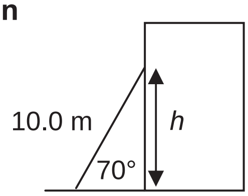
B. Could you have calculated the value of x in **part A** directly using the tangent ratio? Explain.

Examples

Example 1 Solving a Problem using Trigonometric Ratios

Vishnu leaned a 10 m ladder against a building at an angle of 70°.
How high up the building does the ladder go?

Solution



$$\sin 70° = \frac{\text{opposite}}{\text{hypotenuse}}$$

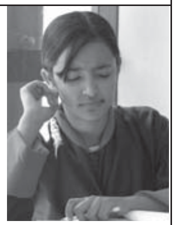
$$\sin 70° = \frac{h}{10.0}$$

$$0.940 = \frac{h}{10.0}$$

$$h = 9.4$$

The ladder goes about 9.4 m up the wall.

Thinking

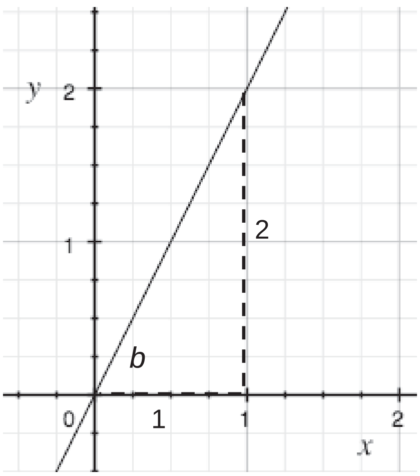


- I drew a picture and labelled it with the information I knew to help me visualize.
- I knew that I wanted to find h, the side length opposite the 70° angle.
- Since I knew the hypotenuse, 10 m, and wanted to know the opposite side, I used the sine ratio.
- I used my calculator to get the value of sin 70° and substituted it into the equation. Then I solved the equation.

Example 2 Relating Trig Ratios to Equations of Lines on a Graph

At what acute angle does the graph of $y = 2x$ cross the x-axis?

Solution



$\tan b = \frac{2}{1} = 2$, so $b \approx 63.4°$

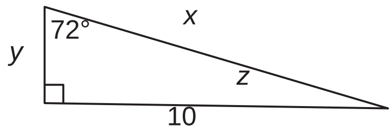
Thinking

- First I graphed $y = 2x$ and then I located the angle where $y = 2x$ crossed the x-axis. I labelled the angle b.
- I could see that angle b could be one of the acute angles in a right triangle with vertices at (0, 0), (1, 0), and (1, 2). So I drew a right triangle and used the coordinates of the vertices to figure out the lengths of its legs.
- I saw that the tangent of angle b was $\frac{2}{1}$.
- I used my calculator to find the angle that has a tangent of 2.

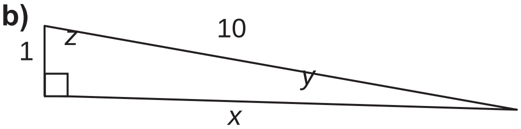
Practising and Applying

1. Calculate the values of x, y, and z for each right triangle, without measuring.

a)



b)



2. A ladder leaning against a wall forms a 25° angle with the wall at the top of the ladder. If the ladder reaches 2.8 m up the wall, how long is the ladder?

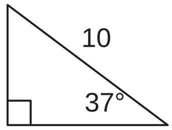
3. A tree is 6.2 m tall. It casts a shadow that is 13.4 m long. At what angle do the rays from the sun meet the top of the tree?

4. For safety reasons the maximum angle at which a wheelchair ramp can be sloped is 5°. Could a ramp with a slope of $\frac{1}{7}$ be used for a wheelchair? Explain.

Wheelchair ramp at Paro Hospital

5. On the path to a school, the rise is 80 m for a run of 500 m. What is the angle of the path from the horizontal?

6. Calculate the area of this triangle.

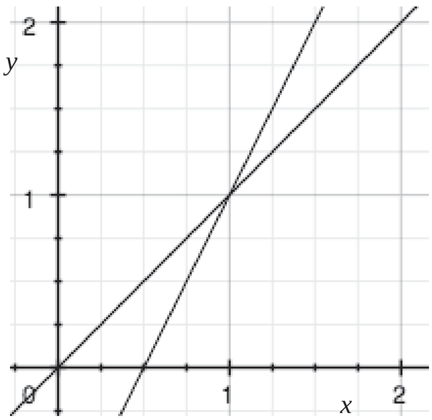


7. a) Graph $y = 3x + 3$.

b) At what acute angle does $y = 3x + 3$ cross the x-axis?

c) What would the acute angle be if the graph were $y = 3x + 5$? Explain your answer.

8. Determine the acute angle formed where the graph of $y = x$ and the graph of $y = 2x - 1$ intersect.



9. A ladder is safest when it forms an angle of 60° to 75° with the ground.

a) If you are using an 8 m ladder, what are the minimum and maximum distances you should place it from the wall?

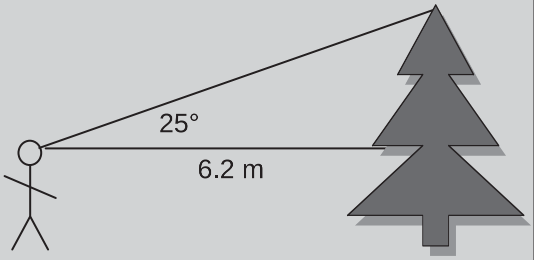
b) Is the ladder closer to the wall when the angle is 60° or when it is 75°?

10. A right triangle has five components besides the right angle: two acute angles and three side lengths. How many of these components do you need to know in order to use trigonometric ratios to calculate all five components? Explain.

7.2.2 Angles of Elevation and Angles of Depression

Try This

Karma is looking up at the top of a tree. He measured his distance from the tree, 6.2 m, and measured the angle up from his eye, 25°. He drew this picture to help him visualize the situation. He plans to determine the height of the tree by multiplying tan 25° by 6.2.

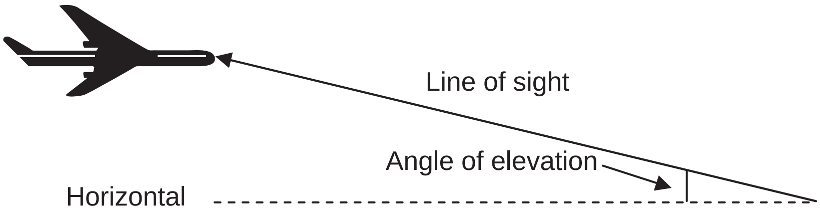


A. i) Explain the thinking behind his plan.

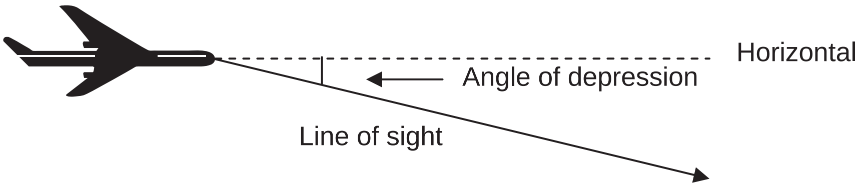
ii) What has he forgotten to consider?

Two special angles are often used to solve trigonometry problems.

- If someone is looking at something up high, the acute angle formed by the horizontal and the line of sight is called the **angle of elevation**.

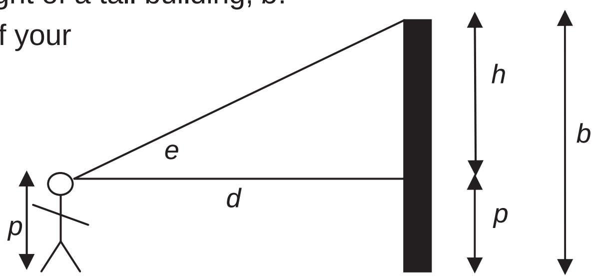


- If a person is looking down at something low, the acute angle formed by the horizontal and the line of sight is called the **angle of depression**.



For example, to determine the height of a tall building, b:

- measure the angle of elevation of your line of sight, e
- measure your distance from the building, d
- use the tangent ratio of $\angle e$ to calculate the height of the building above your eye, h
- add h to your height up to your eye, p



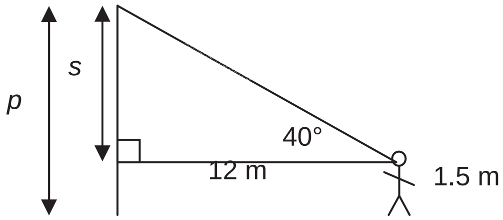
B. Is the angle in **part A** an angle of elevation or depression? Explain.

Examples

Example 1 Using an Angle of Elevation

Dawa stood 12.0 m from a prayer flag pole and looked up to the top of the pole at an angle of 40°. Dawa's eyes are 1.5 m above the ground. How tall is the pole?

Solution



$$\tan 40° = \frac{s}{12}$$

$$0.839 = \frac{s}{12}$$

$$s \approx 10.1 \text{ m}$$

10.1 + 1.5 = 11.6

The pole is 11.6 m tall.

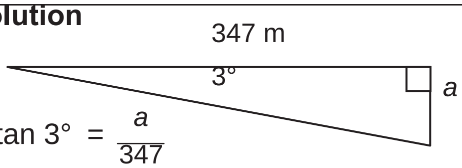
Thinking

- I drew a diagram and labelled it with the information I knew.
- I saw that I could use the tangent ratio for 40° to figure out the height of the pole above Dawa's eyes, *s*.
- I remembered to add *s* to the height of Dawa's eyes (1.5 m) to find the height of the pole, *p*.

Example 2 Using an Angle of Depression

A plane is landing. It is a horizontal distance of 347 m from where it will touch down. The angle of depression is 3°. At what altitude is the plane currently flying?

Solution



$$\tan 3° = \frac{a}{347}$$

$$0.052 = \frac{a}{347}$$

$$a \approx 18.0$$

The plane is flying at an altitude of 18 m.

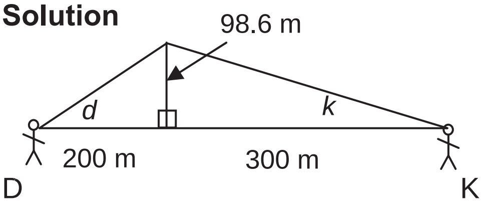
Thinking

- I drew a diagram and labelled it with the information I knew.
- I saw that I could use the tangent ratio for the 3° angle to figure out the altitude, *a*.

Example 3 Solving a Problem Using Two Angles of Elevation

Dendup and Karma are both looking at the top of a 100 m high hill. Both boys are standing on flat ground. Their eyes are 1.4 m above ground level. Dendup is 200 m from the centre of the hill and Karma is 300 m from the centre of the hill, but on the opposite side. At what angle of elevation is each boy looking up at the hill?

Solution



$\tan d = \frac{98.6}{200} = 0.493$, so $d \approx 26.24°$

$\tan k = \frac{98.6}{300} = 0.329$, so $k \approx 18.19°$

Dendup is looking up at an angle of 26°.

Karma is looking up at an angle of 18°.

Thinking

- I made a sketch of the situation and labelled it with what I knew — I subtracted 1.4 m from 100 m for the height of the hill above the boys' eyes.
- I used tan to figure out angles *d* and *k*, since I knew the opposite and adjacent side lengths.
- It made sense that Karma's angle of elevation was smaller than Dendup's, since Karma is farther away.

Practising and Applying

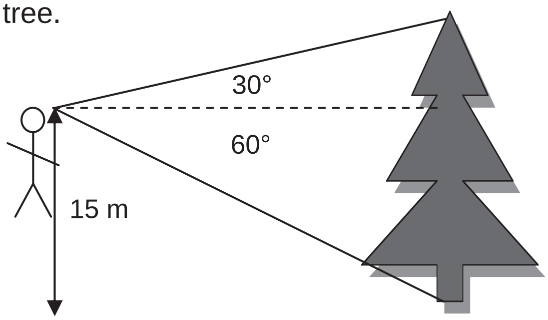
1. Pema's eyes are 1.5 m above ground. He is looking at the top of an 18.6 m high tree that is 27.5 m away. He and the tree are on level ground. At what angle is he looking up?

2. Maya is standing at the window of a tall building. Her eyes are 12.6 m above the ground and she is looking down at an object that is a horizontal distance of 59 m away from the base of the building. What is the angle of depression?

3. Dechen sights a car from a tower. Her eyes are 22 m above the ground. The angle of depression of her sight line is 26°. How far is the car from the base of the tower?

4. Points A and B are 15 m apart on a flat road on opposite sides of a prayer flag. The angle of elevation is 22° from point A and 31° from point B. How far above road level is the top of the prayer flag?

5. A boy looks at the top of a tree from a window 15 m above ground at an angle of elevation of 30°. He also looks at the base of the tree at an angle of depression of 60°. Find the height of the tree.



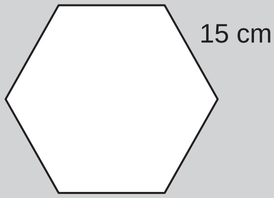
6. Two students, whose eyes are about 140 cm above ground, are standing 150.2 m apart, looking at the top of a hill from opposite sides. One is looking at an angle of 30° and the other at an angle of 35°. How high is the hill?

7. Tell how the angle of depression changes as you get closer to an object. Explain why that happens. Use diagrams to support your explanation.

7.2.3 Areas of Polygons

Try This

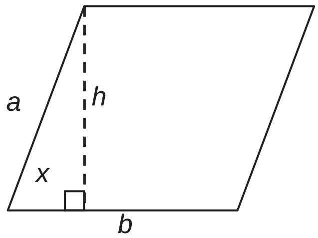
Nima found a regular hexagon tile. He wanted to know its area, but all he could easily measure was a side length of 15 cm.



A. i) Draw the smallest square you can that encloses the polygon.

ii) Estimate the area of the hexagon.

Trigonometric ratios are useful for calculating the areas of triangles, parallelograms, and regular polygons.

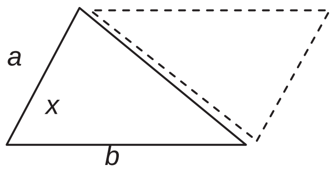


• The formula for the area of a parallelogram is $A = bh$, where b is the base and h is the height. But what if you only know the side lengths of the parallelogram, a and b? You can still calculate the area by using the sine ratio.

Since the height and side length form a right triangle with part of the base, $\sin x = \frac{h}{a}$ and $h = a \sin x$. That means the formula for the area of a parallelogram can be written like this:

$A = a \times b \times \sin x$, or $A = ab \sin x$, where x is the measure of the angle less than 90°.

$$A = ab \sin x$$

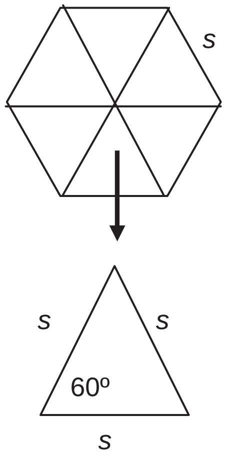


• Since you can always combine two congruent triangles to form a parallelogram, the area of one triangle is half the area of the related parallelogram.

Since the area of a parallelogram is $A = ab \sin x$, $A = \frac{1}{2}ab \sin x$, where a and b are two side lengths and x is the angle where they meet.

$$A = \frac{1}{2}ab \sin x$$

Each regular polygon can be divided into congruent triangles that are always isosceles and sometimes equilateral. The area of the polygon is the sum of the areas of the triangles. The area of each triangle can be found using trigonometry.



• A regular hexagon is made up of six equilateral triangles with side lengths *s* and angles of 60°.

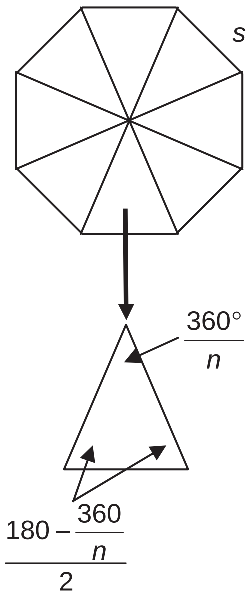
The area of each triangle can be found with $A = \frac{1}{2}ab \sin x$.

Since $a = b = s$ and $\sin 60° = 0.866$, the area of each triangle is $A = \frac{1}{2} \times s \times s \times 0.866$ or $A = \frac{1}{2} \times 0.866s^2$.

So the area of a regular hexagon is $6 \times \frac{1}{2} \times 0.866s^2$, or

$$A = 3 \times 0.866s^2$$

For example, for a regular hexagon with a side length of 1 cm, $A = 3 \times 0.866\,(1)^2 = 2.60$ cm^2.

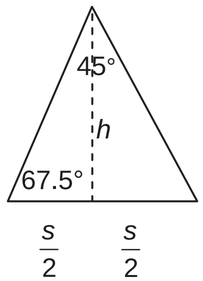


• For other regular polygons, the area is still the total of the areas of the congruent triangles. The number of triangles in each polygon and their angles vary.

If you know the side length of a regular polygon, you know one of the side lengths of each triangle. You can use the tangent ratio to find the height. You can then find the area of each triangle using $A = \frac{1}{2}bh$.

The size of each triangle's angle at the centre of the polygon is $\frac{360}{n}$, where *n* is the number of sides of the polygon. This is because the 360° angle in the centre is distributed equally among the *n* angles that meet there.

Since the sum of the angles in any triangle is 180° and the two other angles are equal (isosceles triangle), each of those angles is $\frac{180 - \frac{360}{n}}{2}$ degrees.



For example, for a regular octagon, each central angle is $\frac{360}{8}$ or 45° and the other two angles in each triangle are each half of 135°, or 67.5°.

$\tan 67.5° = \frac{h}{s \div 2} \rightarrow 2.414 \approx \frac{h}{s \div 2} \rightarrow h \approx 1.21s$

Using $A = \frac{1}{2}bh$, the area of each triangle is

$A = \frac{1}{2} \times s \times 1.21s \rightarrow A = \frac{1}{2} \times 1.21 \times s^2 \rightarrow A = 0.605s^2$

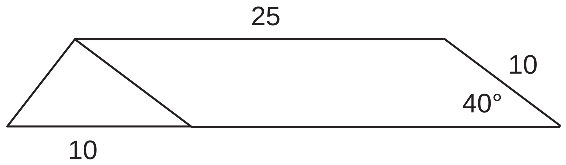
The area of the octagon is $8 \times 0.605s^2$, or $4.84s^2$.

B. i) Use one of the methods shown to calculate the area of the hexagon in **part A**.

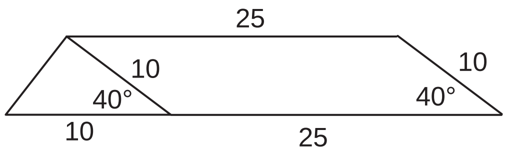
ii) Was your estimate in **part A ii)** reasonable?

Example 1 Using Trig to Find the Area of a Composite Shape

The trapezoid below is made up of a triangle and a parallelogram.
Find the area of the trapezoid.



Solution



Area of the parallelogram

$A = ab \sin x$ [a = 25, b = 10, x is 40º]

$= 250 \times 0.643$

≈ 160.75 square units

Area of the triangle

$A = \frac{1}{2}ab \sin x$ [a = 10, b = 10, x is 40º]

$= 100 \times 0.643 \div 2$

≈ 32.15 square units

Area of the trapezoid

160.75 + 32.15 = 192.90 square units

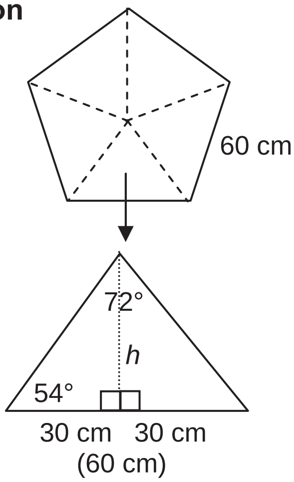
Thinking

- I sketched the trapezoid and added more information that I knew:
  - I knew the opposite sides of the parallelogram were the same length as the sides I knew.
  - I knew the triangle had a 40° angle. The sides of the parallelogram are parallel so the two 40° angles correspond.
- I calculated the area of the parallelogram using the sine of the 40° angle and the two side lengths.
- I calculated the area of the triangle using the sine of the angle contained by the two known side lengths.
- I combined the area of the parallelogram and the area of the triangle to get the area of the trapezoid.

Example 2 Using Trig Ratios to Find the Area of a Regular Pentagon

A regular pentagon has a side length of 60 cm. What is its area?

Solution



The height of the triangle

$\tan 54° = \frac{h}{30} \rightarrow 1.376 = \frac{h}{30}$

$h = 30 \times 1.376$

$h = 41.28$ cm

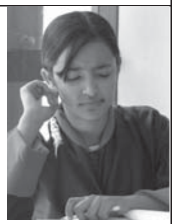
The area of the triangle

$A = \frac{bh}{2} = \frac{60 \times 41.28}{2} \approx 1238.4 \text{ cm}^2$

The area of the pentagon

$5 \times 1238.4 \approx 6192 \text{ cm}^2$

Thinking



• A regular pentagon is made up of five congruent isosceles triangles. I planned to find the area of one triangle and then multiply by 5.

• I knew the triangle's base was 60 cm because it's the same as the side of the pentagon, so all I needed was its height.

• To calculate the height, I wanted to use the tan ratio, so I needed to know the angle opposite the height. I calculated all the angles:

- The top angle was 72° (360° ÷ 5).

- Each other base angle is 54° since they're equal, and the sum of the angles in a triangle is 180° [(180 – 72) ÷ 2 = 54],

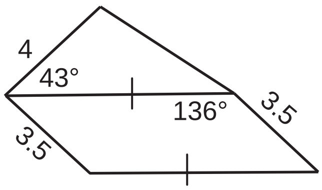
• I used tan 54° to find the height, and then the formula $A = \frac{1}{2}bh$ to find the area.

• I multiplied by 5 since there are five of these triangles in the pentagon.

Practising and Applying

1. Calculate the area of a rhombus with a side length of 20 cm and angles of 60° and 120°.

2. Calculate the area of this shape.



3. Calculate the area of a regular 10-sided shape (a decagon) with a side length of 10 cm.

4. Each of the shapes below has a perimeter of 60 cm:

• a square

• a regular pentagon

• a regular hexagon

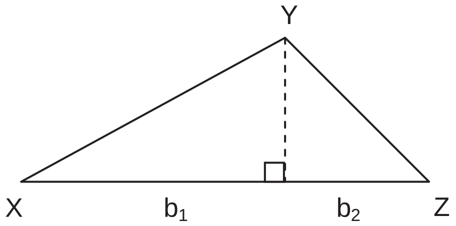
• a regular decagon

a) Calculate the area of each shape. You will find it helpful to sketch each.

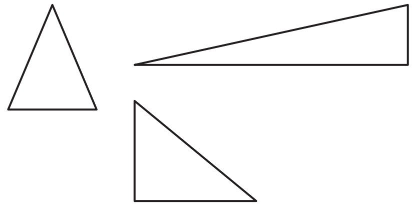
b) What do you observe?

c) Why do you think a circle with circumference 60 cm might have a greater area than any of the four shapes listed above?

5. Express the area of this triangle in terms of b_1, b_2, $\angle X$, and $\angle Z$.



6. How is trigonometry useful in determining the areas of any triangle? Explain.



GAME: Race to Five

Race to Five is a game for two players. You will need two dice and two calculators.

Players take turns.

• Roll two dice and combine the digits of the two values rolled to create an angle measure.

• Calculate the sine, cosine, or tangent of the angle.

Players keep a running total of the trigonometric ratio values they get each time.

The first person to get the highest value less than or equal to 5.1 wins the game.

For example:

Suppose you roll a 3 and a 5.

You can calculate the sine, cosine or tangent of either 35° or 53°. Your decision about which ratio and which angle will depend on where you are in the game.

If you already have 4.214 points, you do not want more than 0.886 or you will go over 5.1.

sin 35° = 0.574	cos 35° = 0.819	tan 35° = 0.777
sin 53° = 0.799	cos 53° = 0.602	tan 53° = 1.327

It looks like the best choice is cos 35°.

You will win the game because your total will be 4.214 + 0.819 = 5.013.

7.2.4 Vectors and Bearings

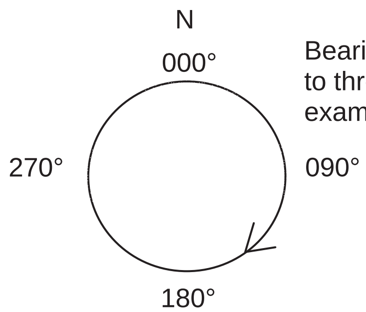
Try This

Here is a set of walking directions

- Face south and go 2 km.
- Turn toward the east and walk another 2 km.

A. After following these directions, about how far are you from your starting position? How do you know?

Directions are often given using words like north, south, northeast, and so on, but it is sometimes important to be more precise. In these cases, it is common to use **bearings**. Bearings are angles measured in degrees clockwise from the north.

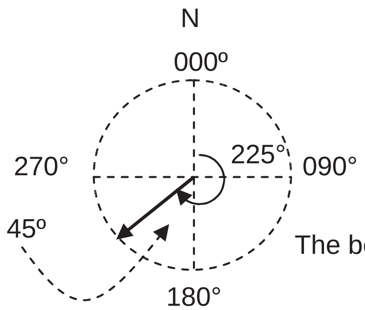


Bearings are reported to three digits, for example, 45º is 045 º.

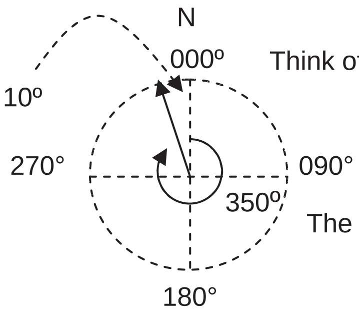
- If you are given directions to walk 1 km at a bearing of 180° and then 2 km at a bearing of 090°, you would move like this.

The arrows showing the components of the trip are called **vectors**. Vectors show movement in terms of direction and distance. A longer vector represents a trip of greater distance. A distance that is twice as long as another will be represented with a vector that is twice as long, as shown above. The direction (arrow) of the vector tells the bearing.

- You can use the benchmark degrees of 0° or 360° (north), 90° (east), 180°(south), and 270° (west) to help you determine the bearing of a vector.



The bearing of this vector is 180 + 45 = 225º.



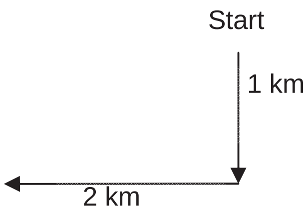
Think of 000º as 360º.

The bearing of this vector is 360 – 10 = 350º.

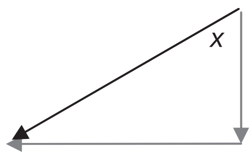
• If you have taken a trip that involves several components shown by several vectors, you can describe the final result using a single vector with a bearing.

For example, if someone walks 1.0 km at a bearing of 180° and then 2.0 km at a bearing of 270°, the trip can be shown two ways, as two vectors, as shown on the left, or as one vector, as shown on the right below.

Showing all components | Showing the final result using one vector



OR



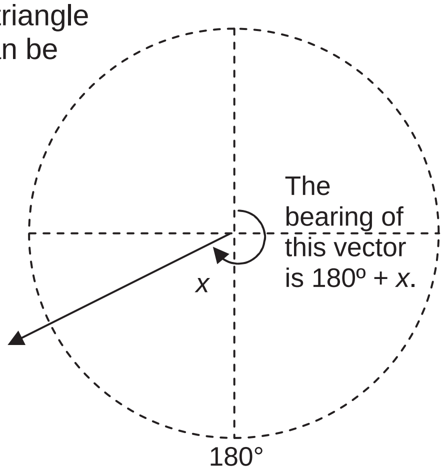
The trip can be represented by the single black vector.

To determine the bearing of the single vector, you need to know the value of angle x as indicated in the diagram below. The bearing will be 180º + x°.

Because the two original vectors formed a right triangle and their lengths are known, the tangent ratio can be used to find x.

$\tan x = 2 \div 1 = 2$

If $\tan x$ is 2, then $x \approx 63.43°$.



If $x \approx 63.43°$,

then the bearing is $180 + 63.43 \approx 243°$.

To determine the distance of the single vector, you can use the Pythagorean theorem and the known values of the two original vectors:

$$1^2 + 2^2 = v^2$$
$$5 = v^2$$
$$v \approx 2.24$$

The single vector that describes the trip is about 2.24 km long at a bearing of 243°.

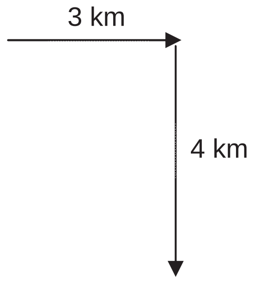
B. Use a single vector to describe the trip in **part A**. Use a distance and a bearing to describe the vector.

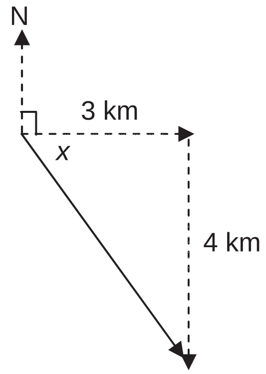
Examples

Example 1 Combining Two Vectors

You walk 3 km at a bearing of 90°. Then you turn and walk 4 km at a bearing of 180°. Use a single vector to describe the trip. Use a distance and a bearing to describe the vector.

Solution





The bearing is

$\tan x = \frac{4}{3}$ or 1.333

If $\tan x$ is 1.333, then x is about 53.12°.

$90 + x = 90 + 53.12 \approx 143^o$

The vector distance, v

$v = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ km

The single vector is 5 km at a bearing of 143° from the north.

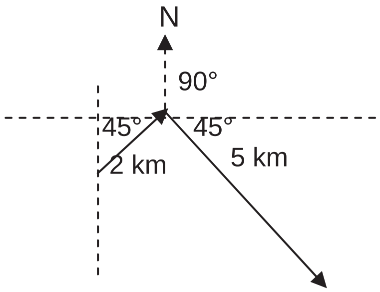
Thinking

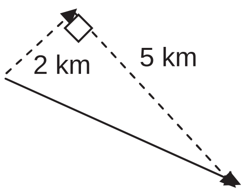
- I drew and labelled a diagram using two vectors to describe the trip.
- I connected the start and end points to create the single vector that would also describe the trip.
- I noticed that the single vector completed a triangle with the original two vectors.
- I knew that if I could find angle x, I could add it to 90° to calculate the bearing of the single vector from the north.
- I used the tangent ratio for x because I knew the opposite and adjacent side lengths of the triangle.
- To calculate the vector distance, I used the Pythagorean theorem, because the vector was the hypotenuse of a right triangle with the original vectors as the legs.

Example 2 Combining More Than Two Vectors

Lungten walked 2 km at a bearing of 45° and then 5 km at a bearing of 135°. Use a single vector to describe his trip. Use a distance and a bearing to describe the vector.

Solution





The vector distance

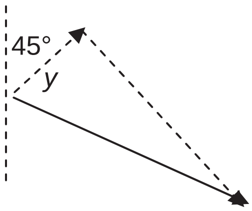
$2^2 + 5^2 = v^2$

$4 + 25 = v^2$

$v = \sqrt{29} \approx 5.4$ km

The vector bearing

$\tan y = \frac{5}{2} = 2.5$, so $y \approx 68.20°$



$45 + 68.20 \approx 113°$

The single vector is about 5.4 km at a bearing of 113° from the north.

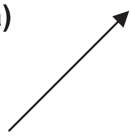
Thinking

- I drew and labelled a diagram using two vectors to describe the trip.
  I broke the 135° bearing from North into two parts: a 90° part and a 45° part.
- Then I connected the start and end points to create a single vector to describe the trip.
- I noticed that the triangle was a right triangle since the angle where the vectors met was 180 – (45 + 45) = 90°.
- To calculate the length of the vector, I used the Pythagorean theorem.
- To calculate angle *y*, I used the tan ratio because I knew the opposite side length and the adjacent side length of angle *y*.
- I added the value of angle *y*, 68°, to 45° (the bearing of the first original vector) to get the bearing of the single vector.

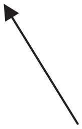
Practising and Applying

1. Estimate the bearing for each vector.

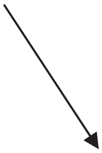
a)



b)



c)



2. Draw a vector to show each bearing.

a) 225°

b) 135°

c) 315°

d) 240°

3. a) Draw a single vector to represent each two-part trip. What is its bearing and distance?

i) 2 m at a bearing of 225° and 1 m at a bearing of 45°

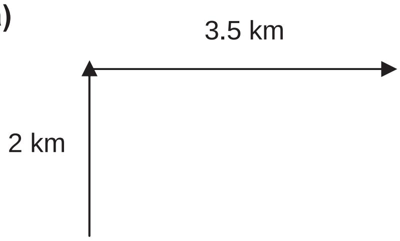
ii) 1 m at a bearing of 160° and 1 m at a bearing of 340°

iii) 2 m at a bearing of 050° and 1 m at a bearing of 230°

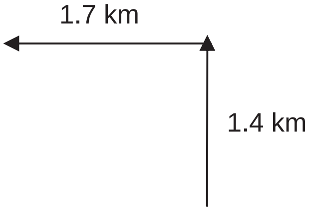
b) Why was each two-part trip in **part a)** fairly easy to describe?

4. Represent each trip as a single vector. What is its bearing and distance?

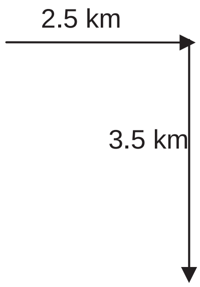
a)



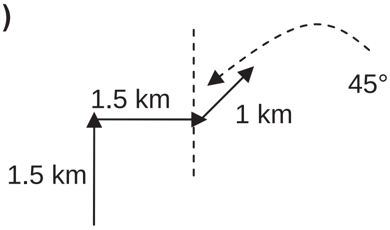
b)



4. c)



d)



5. Tshewang walked 3 km at a bearing of 135° and 4 km at a bearing of 45°. Represent his two-part trip as a single vector. What is its bearing and distance?

6. A trip can be described as 10 km at a bearing of 120°. What two-part trips could have resulted in this single trip? List three possibilities using different pairs of bearings each time.

7. You are representing a two-part trip with a single vector. The trip includes a 2 km part at a bearing of 90° and a 4 km part at a bearing of 180°, but you are not sure which part happened first.

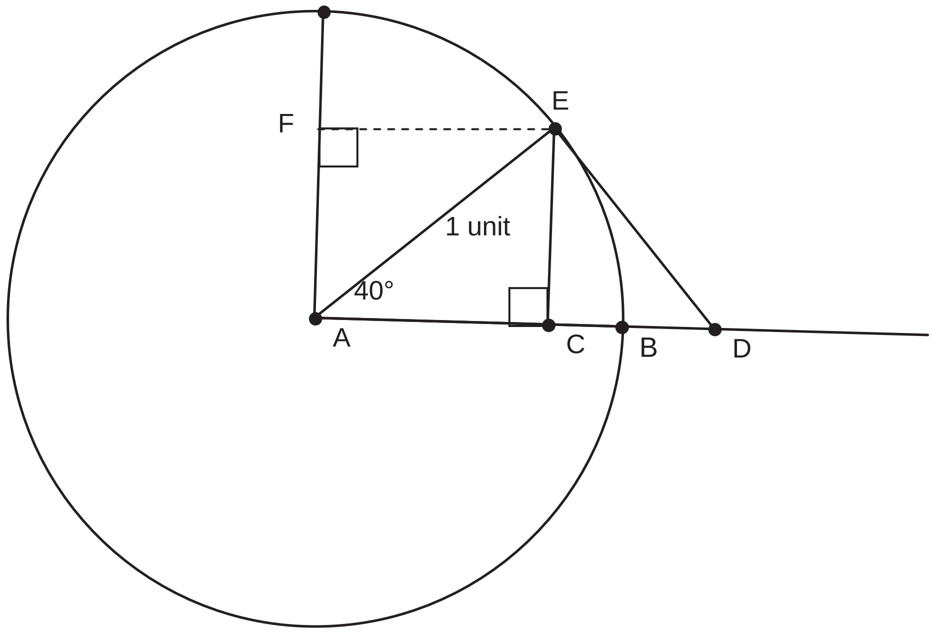
a) Does the length of the vector depend on which part happened first? Explain.

b) Does the bearing of the vector depend on what part happened first? Explain.

CONNECTIONS: Relating Trigonometric Ratios to Circles

Mathematicians often think about the trigonometric ratios in terms of a circle.

Examine the circle below with radius 1 unit. The angle where ED meets the radius of the circle is a right angle.



1. Measure each length on the diagram, based on AE as 1 unit.

a) EC **b)** AC **c)** DE

2. Look up the values for sine, cosine, and tangent of 40°. What do you notice?

3. Show that $\Delta ACE \cong \Delta EFA$.

4. Show that $\Delta EFA \sim \Delta ECD$.

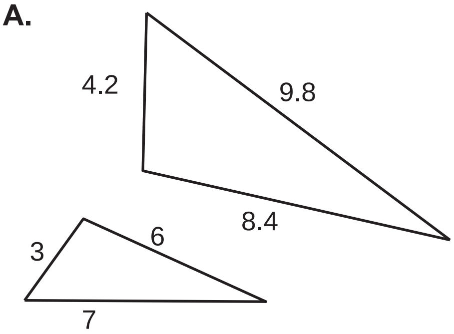
5. Use the relationship in **question 4** to express the length of ED in terms of a trig ratio.

You might find it interesting to know that the word *tangent* also means a line that touches a circle at exactly one point.

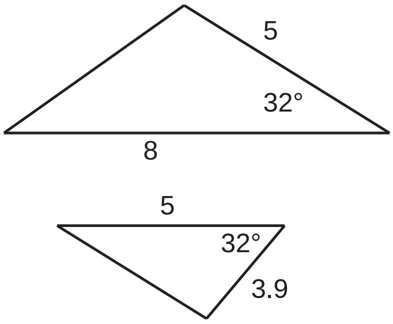
UNIT 7 Revision

1. Which pairs of triangles are similar? Explain how you know.

A.

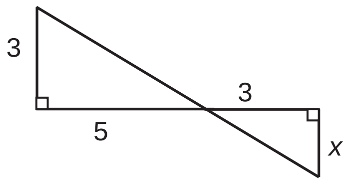


B.

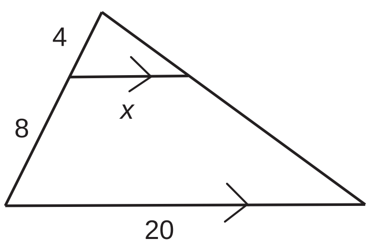


2. What is the value of *x*?

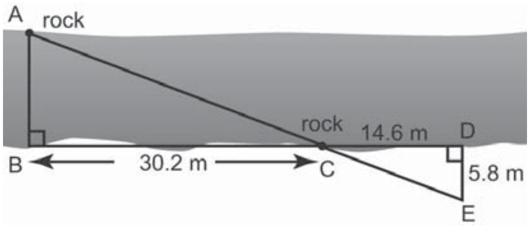
a)



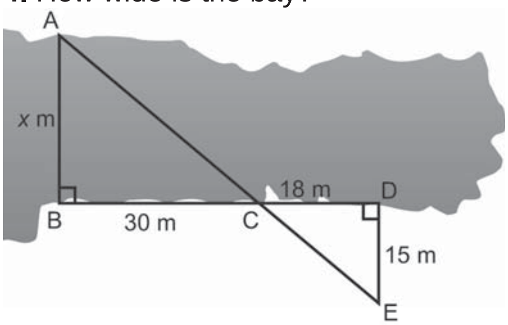
b)



3. How wide is the river?

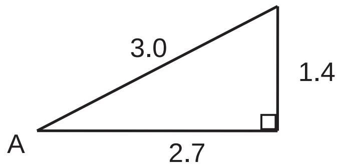


4. How wide is the bay?

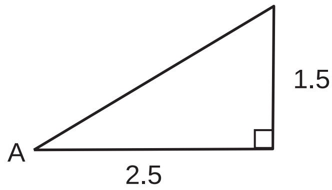


5. Determine the sine, cosine and tangent of $\angle A$ for each. Then calculate the size of $\angle A$ to the nearest degree.

a)



b)



6. For each triangle in **question 5**, order the following ratios from least to greatest: sec $\angle A$, csc $\angle A$, cot $\angle A$. What do you notice? Why does this happen?

7. Fill in each blank with an acute angle.

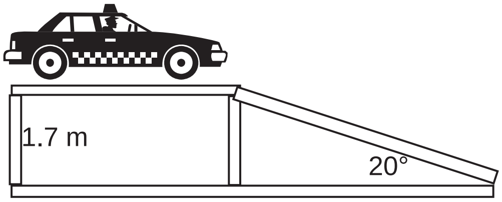
a) cos ___ = 0.2

b) sin ___ = 0.2

c) tan ___ = 1.5

d) sin ___ = cos 85°

8. A garage is building a ramp so that workers can stand below a car to work on it. If the platform for the car must be 1.7 m above the ground and the angle at which the ramp meets the ground is 20°, how long should the ramp be?



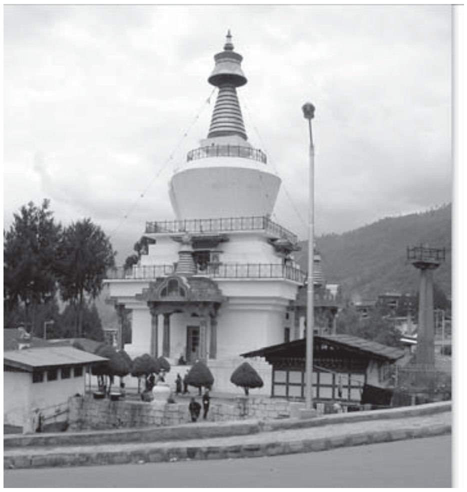
9. Graph the lines $y = 2x$ and $y = 3x + 1$. Calculate the acute angle where the two lines intersect.

10. A carpenter leans a 4 m ladder against a wall. It reaches 3.5 m up the wall. Find the angle the ladder makes with the wall.

11. A ramp rises 2.5 m for every 5.5 m of run. What is the slope angle of the ramp at its base?

12. From the top of a cliff, the angle of depression from the horizontal down toward a car is 30°. If the cliff is 60 m high, how far is the car from the base of the cliff?

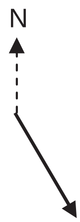
13. Dawa is looking up at the top of the structure from 120 m away. His eyes are 155 cm above ground level. The angle of elevation is 5°. Estimate the height of the structure.



14. Calculate the area of a regular nonagon (9-sided shape) with a side length of 20 cm.

15. What bearing describes each vector?

a)



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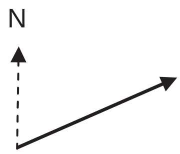
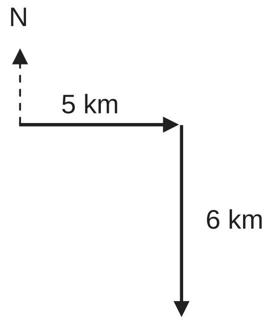


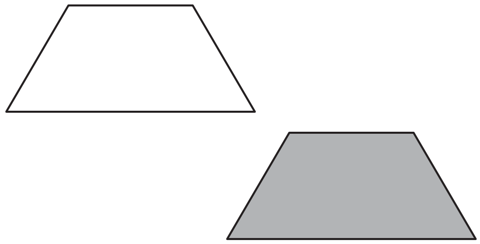
16. Draw a single vector to represent this two-part trip. Describe its bearing and distance.



UNIT 8 GEOMETRY

Getting Started

Use What You Know



A. How can you use each method below to determine whether the white trapezoid is symmetrical?

i) folding

ii) measuring

iii) using coordinates

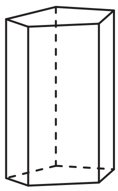
B. What measurements would you need to take to determine whether the white and gray trapezoids are congruent? Explain.

C. How could you use the two trapezoids to create a shape that has rotational symmetry? How do you know it has rotational symmetry?

Skills You Will Need

1. What is the name of each shape?

a)



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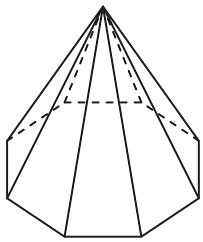
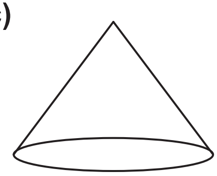
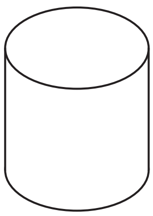


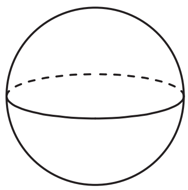
c)



d)



e)



2. Use a ruler, protractor, and compass to draw each triangle.

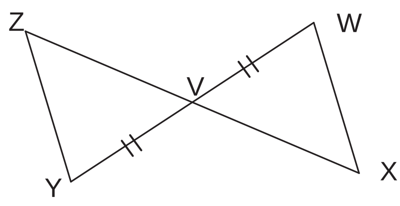
a) Draw ΔABC: $AB = 7.5$ cm, $BC = 6.3$ cm, and $\angle B = 24^{\circ}$

b) Draw ΔPQR: $PQ = 4.7$ cm, $QR = 6.6$ cm, and $PR = 7.5$ cm

c) Draw ΔLMN: $LM = 8$ cm, $\angle L = 12^{\circ}$, and $\angle N = 47^{\circ}$

3. a) What information do you need to determine if two triangles are congruent?

b) What further information could you use to prove that $\Delta VZY \cong \Delta VXW$? Explain how this information would help. Find more than one answer.

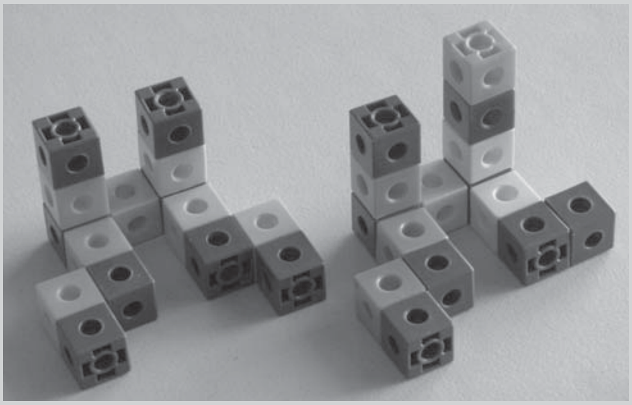


Chapter 1 Symmetry and Reasoning

8.1.1 2-D and 3-D Reflectional Symmetry

Try This

Gembo made these two structures using 15 cubes for each.



A. Why might someone say that the structure on the left is more balanced than the structure on the right?

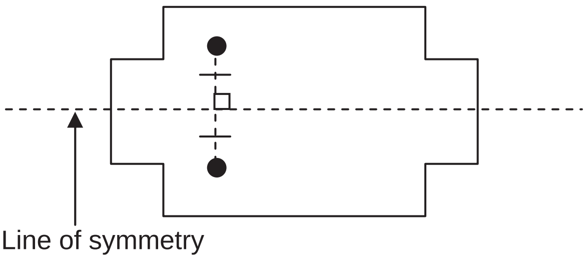
B. Why might the structure on the left more likely be a model for a building than the structure on the right?

• Research has shown that humans like balance in the form of **symmetry**. You have probably noticed this about your own preferences. A face looks pleasant when it is the same on both sides. We build things like windows, for example, to have symmetrical properties.

• When people say that a shape is symmetrical, they usually mean that, from a certain point of view, the shape looks the same on the left or top as it does on the right or bottom. This is called **mirror symmetry**, or **reflectional symmetry**. If you could fold the shape along a line, one half would match the other **congruent** half.

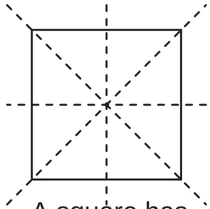
• The line that you imagine to be the fold line is called the **line of reflection** or **line of symmetry**. It has this name because if a mirror were placed on this line, the mirror image of one half of the shape would reflect exactly on top of the other half.

• One of the properties of reflectional symmetry is that for each point on one side of the shape, the line segment joining it to its matching point on the other side of the shape is perpendicular to the line of symmetry. The line of symmetry bisects the line segment.



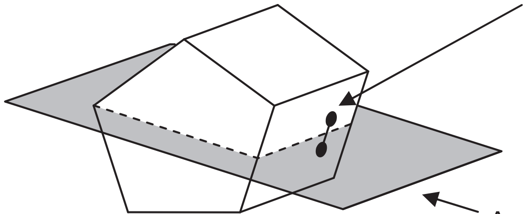
• Many shapes have more than one line of symmetry.

For example, any square has four lines of symmetry, as shown on the right. This means that when you fold the square along any of these lines, each half will match its opposite half exactly.



A square has four lines of symmetry.

• 3-D shapes can also have reflectional symmetry. The imaginary surface that cuts, or divides a 3-D shape into congruent and matching halves is called the **plane of symmetry**. Reflectional symmetry of 3-D shapes is sometimes called **plane symmetry**.



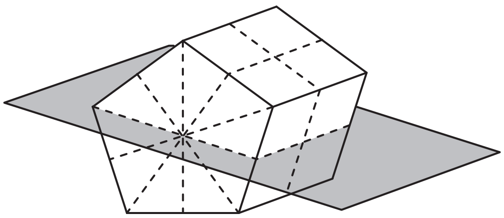
As with reflectional symmetry of 2-D shapes, a line segment joining any two matching points on either side of the plane of symmetry is bisected by the plane and perpendicular to the plane.

A plane of symmetry

• You cannot fold 3-D shapes to determine whether they have reflectional symmetry, but you can visualize to estimate whether the shape is the same on both sides of a plane of symmetry.

For example, if you look at a classmate's head, you can visualize a vertical plane of symmetry that passes through the chin, mouth, nose, and between the eyes. This indicates that the head is symmetrical.

• The regular pentagon-based prism shown below has six planes of symmetry; one of them is shown. The other five planes of symmetry are indicated by the dashed lines. The planes of symmetry are related to the lines of symmetry of the prisms' faces. Five of the six planes, including the one shown, are extensions of the lines of symmetry in the 2-D pentagon base. One plane of symmetry divides the lateral faces in half; that plane is an extension of one of the lines of symmetry of each rectangle face.



In addition to the plane of symmetry shown in grey, this regular pentagon-based prism has five other planes of symmetry, each shown by dashed lines. The planes of symmetry relate to the lines of symmetry of the prism's faces.

• If you were to cut the prism along any plane of symmetry, the resulting two pieces would be congruent halves, each half the mirror image of the other.

C. i) Which structure in **part A** has a plane of symmetry? How do you know?

ii) How many planes of symmetry are there and where are they?

Examples

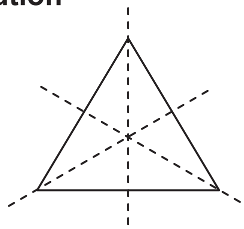
Example 1 Creating a Shape with Reflectional Symmetry

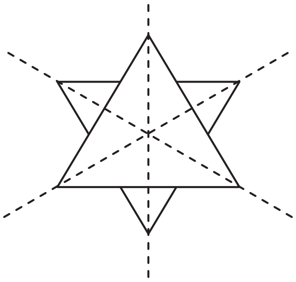
a) Create two shapes with three lines of symmetry.

b) Create two shapes with four lines of symmetry.

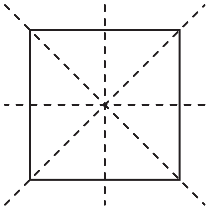
Solution

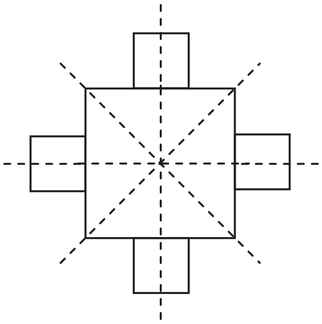
a)





b)





Thinking

a) I know that an equilateral triangle has three lines of symmetry. Each line joins a vertex to the midpoint of the opposite side.

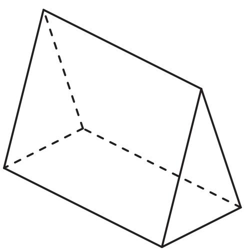
• To get another shape was harder, I decided to modify the shape I already had by adding something to each part. I cut out the shape and folded it to make sure the folded halves matched for each of the three lines of symmetry.

b) I used the same idea for four lines of symmetry by starting with a square for the first shape and then modifying it to create the second shape. I cut out the shape and folded it to make sure the folded halves matched for each of the three lines of symmetry.

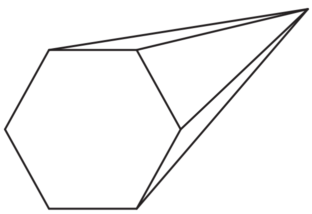
Example 2 Locating Planes of Symmetry

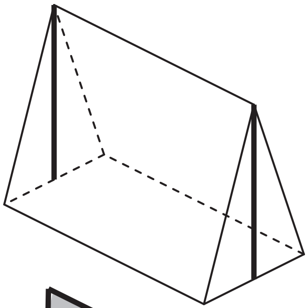
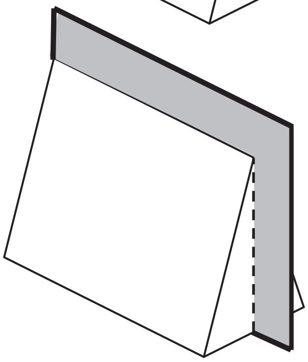
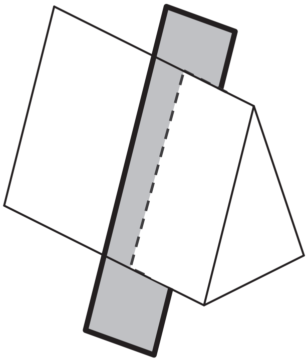
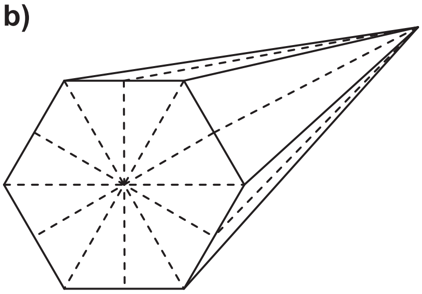
Show the planes of symmetry of each shape.

a) prism with an isosceles triangle base



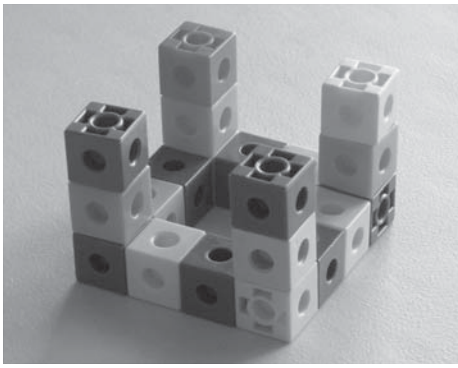
b) regular hexagon-based pyramid



Solution	**Thinking**
a)    There are two planes of symmetry.	 **a)** I knew that if a plane is going to divide the prism into mirror halves, then one of the faces has to be divided into mirror halves as well. So I first located the lines of symmetry on all the faces. • One plane of symmetry goes through the line of symmetry on both triangle bases, as well as through one of the lines of symmetry on one lateral rectangular face and the edge opposite to it. • A second plane of symmetry goes through the other lines of symmetry on each of the lateral rectangular faces.
b)  There are six planes of symmetry.	**b)** I located the six lines of symmetry for the regular hexagon base. I knew that any plane that went through one of those lines and the apex (top) of the pyramid would be a plane of symmetry because it would cut the pyramid into matching halves.

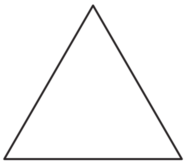
Practising and Applying

1. How many planes of symmetry does this structure have?

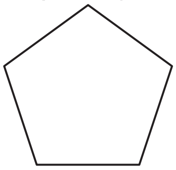


2. How many lines of symmetry does each shape have? Sketch each and show the lines.

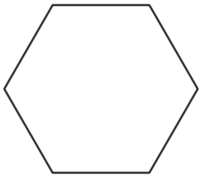
a) equilateral triangle



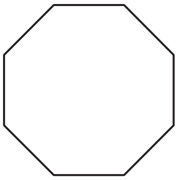
b) regular pentagon



c) regular hexagon

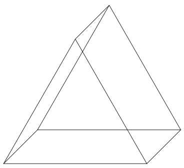


d) regular octagon

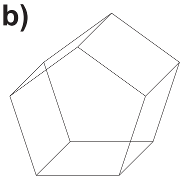


3. Use your results from **question 2** to predict the number of planes of symmetry for each shape below.

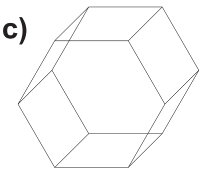
a)



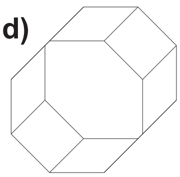
b)



c)



d)



4. a) What do you notice about the number of lines of symmetry in a regular polygon? Explain why this property applies to all regular polygons.

b) How does this property relate to the number of planes of symmetry of a prism with a regular polygon base?

c) How does this property relate to the number of planes of symmetry of a pyramid with at regular polygon base?

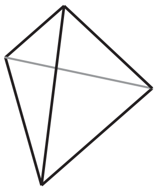
5. Create two 2-D shapes, each with six lines of symmetry.

6. Is it possible to create a triangle with two lines of symmetry?

7. a) How many different planes of symmetry are there in a square-based prism that is not a cube? Show this in a sketch.

b) How many different planes of symmetry are there in a cube? Show this in a sketch.

8. How many different planes of symmetry are there in a regular tetrahedron (a triangular pyramid with each face an equilateral triangle)? Show this in a sketch.



9. a) How many lines of symmetry does a circle have?

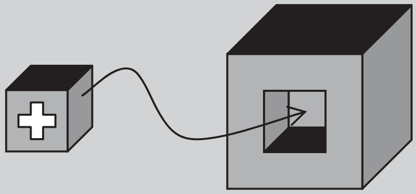
b) How many planes of symmetry does a cylinder have? Explain.

c) How many planes of symmetry does a cone have? Explain.

8.1.2 2-D and 3-D Rotational Symmetry

Try This

A baby is playing with a small cube block that fits into a square hole in a bigger cube block.



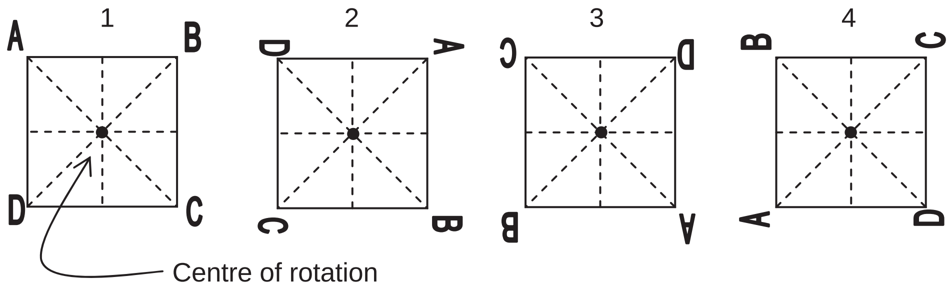
A. How many ways can the baby fit the small cube block into the hole? (The cross design on the front of the block has to face forward.)

There are various kinds of symmetry, even though in normal conversation the word 'symmetry' usually refers to reflectional symmetry.

• One type of symmetry in mathematics is **turn symmetry**, which is also called **rotational symmetry**. A shape has turn symmetry when it is turned or rotated around a fixed point called the **centre of rotation** and it looks the same more than once during one complete rotation.

• The number of times the shape looks the same during a complete rotation is the **order of turn symmetry**.

For example, the square below has turn symmetry of order 4, since it looks the same 4 times as it is turned 360º around its centre.



• When you describe turn symmetry, you must indicate

- where the centre of the rotation is, and
- how many times the shape looks the same (the order of turn symmetry) during a complete rotation of 360º around that centre of rotation.

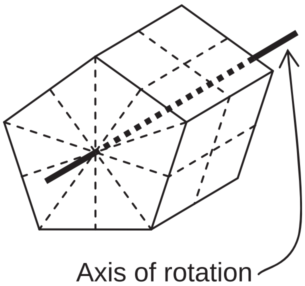
• If a shape has no turn symmetry, this means it has to be turned a full 360° before it looks the same as it does in its initial position. That means the order of turn symmetry for a shape with no turn symmetry is 1 (because it looks the same once in a full rotation).

• A 3-D shape has turn symmetry when it can be turned or rotated around a line called the **axis of rotation** and it looks the same more than once within a full rotation. As with 2-D turn symmetry, the number of times the shape looks the same is the order of turn symmetry.

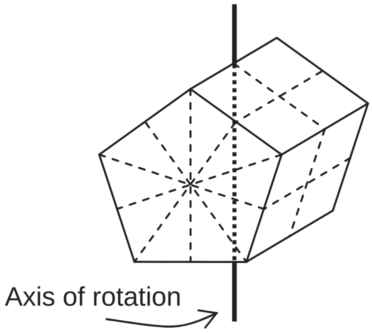
For example:

When it is rotated around the axis that passes through the centres of the bases, the regular pentagon-based prism below has turn symmetry of order 5.

When it is rotated around the axis that passes through the lateral faces, the regular pentagon-based prism has turn symmetry of order 2.



Turn symmetry of order 5



Turn symmetry of order 2

• An axis of rotation connects or passes through two of these parts of a 3-D shape:

- a vertex
- the centre of a face or base
- the midpoint of an edge

In the first example above, the axis passes through the centres of both bases. In the second example, the axis passes through the midpoint of an edge and the centre of a lateral face.

B. i) How does the question in **part A** relate to rotational symmetry?

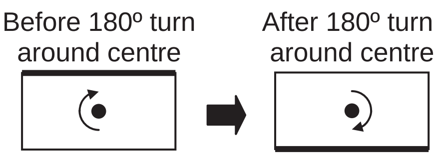
ii) What is the order of turn symmetry of the small cube? Explain.

Examples

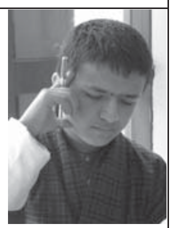
Example 1 Describing Turn Symmetry in 2-D

Create three shapes with rotational symmetry of order 2.

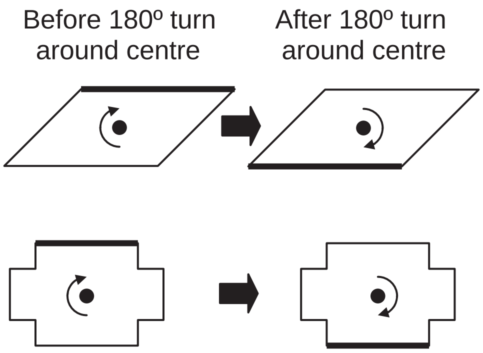
Solution



Thinking



• I know that a rectangle has rotational symmetry of order 2 because when I turn it through a full rotation around the centre, it looks the same twice.



• I know that a parallelogram has rotational symmetry of order 2 because when I turn it through a full rotation around the centre, it looks the same twice.

• For the third shape, I used a rectangle that I knew had turn symmetry of order 2 and added two small congruent rectangles to two sides, directly opposite each other. That way, it didn't change the symmetry.

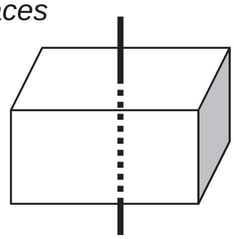
• The new shape has turn symmetry of order 2 because it looks the same twice in a full rotation.

Example 2 Axes of Rotation for a Rectangular Prism

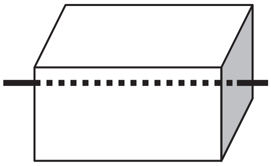
Find all the axes of rotation of a rectangular prism that is not a cube.

Solution

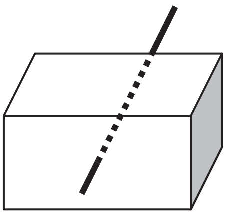
Through the centres of the top and bottom faces



Through the centres of the side faces



Through the centres of the front and back faces



Thinking

• I knew that if the shape had turn symmetry, I could find axes of rotation by visualizing lines that pass through its vertices, the centres of its faces, and the midpoints of its edges.

• I visualized a line that passed through the centres of the top and bottom faces. I knew it was an axis of rotation because if I turned the prism around the axis, it would look exactly the same after half a turn.

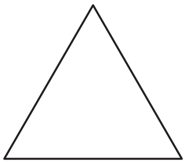
• I realized that I could visualize an axis of rotation through any two opposite faces — through the two side faces and through the front and back faces.

• I tried visualizing lines that pass through vertices and through midpoints of edges, but none of them would work as axes of rotation.

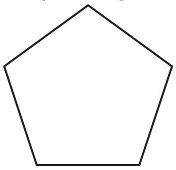
Practising and Applying

1. Use the centre of each regular polygon as the centre of rotation. What is the order of turn symmetry for each?

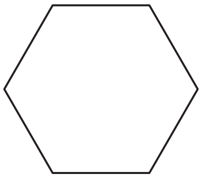
a) equilateral triangle



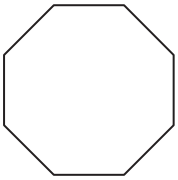
b) regular pentagon



c) regular hexagon



d) regular octagon



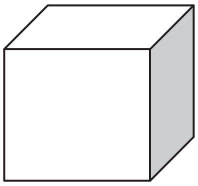
2. What do you notice about the order of turn symmetry in a regular polygon? Explain why this property applies to all regular polygons.

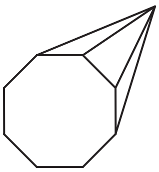
3. Is it possible for a triangle to have turn symmetry of order 2? Explain.

4. Create a 2-D shape with turn symmetry of order 4 that is not a square.

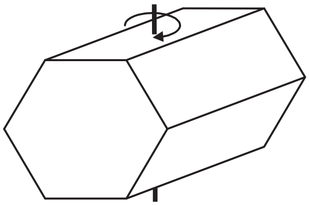
5. a) Which pairs of points on this cube can you connect to create axes of rotation?

b) How many axes of rotation are there for a cube?





7. Passang noticed that in a regular hexagon-based prism, there is turn symmetry of order 2 using an axis of rotation that passes through the centres of two opposite lateral faces.



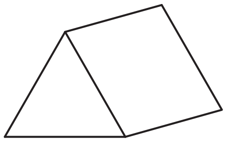
There are other axes of rotation, with order 2 or more.

a) How many axes of rotation are there? Use a diagram to show the axes.

b) Describe the turn symmetry for each axis of rotation in **part a)**.

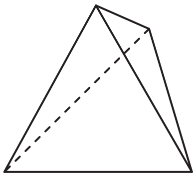
8. a) How many axes of rotation are there in a triangular prism with an equilateral triangle base? Use a diagram to show the axes.

b) Describe the turn symmetry for each axis.

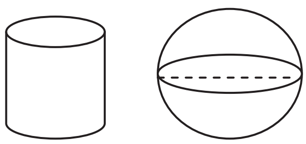


9. a) How many axes of rotation are there in a regular tetrahedron? Use a diagram to show the axes.

b) Describe the turn symmetry for each axis.



10. Which has more axes of rotation — a sphere or a cylinder? Explain.



8.1.3 Reasoning

Try This

At a very young age, Buthri noticed that only women get pregnant, not men. So when her parents told her she would soon be a big sister, she knew that her mother would be the one who would give birth. After she explained her reasoning to her cousin Mindu, he explained to her that the reason her mother was the one that would give birth was that only women have babies because their bodies are designed to give birth.

A. Compare the reasoning of Buthri and her cousin.

i) What information is Buthri using to conclude that her mother will give birth?

ii) What information is Mindu using to conclude that Buthri's mother will give birth?

B. Think of something that you know to be true.
What information has convinced you that it is true?

• There are many ways of explaining things. One way to categorize explanations is to distinguish between explanations that are based on inductive reasoning and those that are based on deductive reasoning.

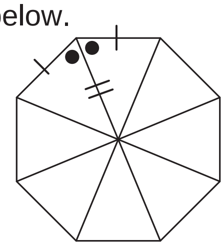
- If an explanation uses **inductive reasoning**, it is based on examples that suggest something might be always true.

- If an explanation uses **deductive reasoning**, it is based on knowledge and information.

For example, if for several regular polygons you compare their number of lines of symmetry to their order of turn symmetry (using the centre of the polygon as the centre of rotation), you will notice that the numbers are the same. You might predict they would also be the same for other regular polygons. In this case, you are using inductive reasoning. You used the particular examples to decide that something was probably true more generally. In order to give a deductive explanation for why the number of lines of symmetry is always equal to the order of rotational symmetry, you would have to use knowledge to explain why the numbers are the same.

• An example of deductive reasoning for why the number of lines of symmetry is always equal to the order of rotational symmetry is described below.

- Every regular polygon with *n* sides can be divided into *n* congruent triangles by the lines of symmetry (because the lines are lines of symmetry, the triangles on either side are congruent because of SAS and since each pair of triangles is congruent, then all the triangles are congruent).



[Continued]

- If the triangles are congruent, then each triangle can be moved $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, … and $\frac{n-1}{n}$ way around the shape before it comes back to where it started. This means the order of turn symmetry is n.

• When you have an idea or prediction that something is true, it is called a **conjecture**. After you have proven that it is true based on deductive reasoning, it is called a **theorem**. For this reason, deductive explanations (deductions) are often called **proofs**.

• If you show something to be true using inductive reasoning, mathematicians would say that you have **verified** it, but you have not proven it. Many young students use inductive reasoning because they do not have the mathematical knowledge to allow them to prove something deductively. Inductive reasoning is still valuable and is often the first step toward realizing something might be true.

• To prove something is not true, you only need to show one example for which it is not true, called a **counterexample**.

For example, if you conjectured that the order of turn symmetry is the same as the number of lines of symmetry for all polygons, you can prove that your conjecture is not true when by using a counterexample such as a parallelogram.

C. i) Was Buthri's explanation an example of inductive reasoning or deductive reasoning? How do you know?

ii) Was Mindu's explanation an example of inductive reasoning or deductive reasoning? How do you know?

Examples

Example 1 Distinguishing between Inductive and Deductive Reasoning

Identify each statement as an example of inductive or deductive reasoning.

a) "I have noticed that when a ball is released in the air it falls to the ground, so I know that when Mindu releases that ball it will fall."

b) "Gravity pulls all objects to the earth, so when Mindu releases that ball I know it will fall to the earth."

c) "Whenever I add the measures of the degrees in the non-right angles in a right triangle, I get 90°."

d) "The angles in a triangle always total 180°, so the non-right angles in a right triangle also total 90°."

Solution	Thinking
a) Inductive	**a)** The person is relying on examples to make a prediction.
b) Deductive	**b)** The person is using known information about gravity to make a prediction related to that information.

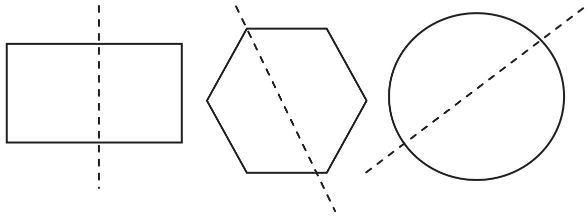
c) Inductive	**c)** The person is relying on experience to make a prediction.
d) Deductive	**d)** The prediction is a logical result of the first statement, which provides already proven information about triangles — if all the angles in a triangle add to 180° and a right triangle has one 90° angle, then it follows that the other two angles have to add to 90°.

Example 2 Disproving a Conjecture using a Counterexample

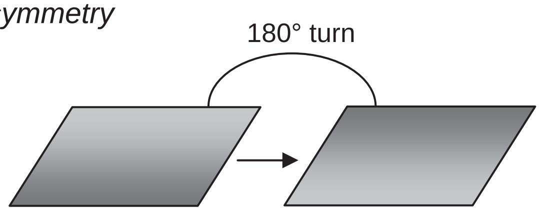
Does a 2-D shape with rotational symmetry always have reflectional symmetry?

Solution

Examples — all have both rotational and reflectional symmetry



A counterexample — an example that has rotational symmetry but not reflectional symmetry



No, a 2-D shape with rotational symmetry does not always have reflectional symmetry.

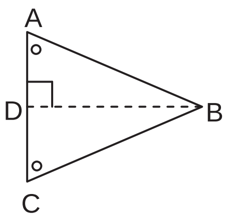
Thinking

- I drew a few 2-D shapes that had rotational symmetry and found that they all had lines of symmetry too. I started to think that the answer was yes — whenever there is rotational symmetry, there is also reflectional symmetry.
- Then I drew a parallelogram and saw that it has rotational symmetry, but no line of symmetry.
- Just one counterexample was enough to prove that the conjecture was not true.

Example 3 Proving a Conjecture Using Deductive Reasoning

Use deductive reasoning to prove that an isosceles triangle that is not equilateral has exactly one line of symmetry.

Solution



$\Delta ABD \cong \Delta CBD$

[Continued]

Thinking

- I drew a diagram to help me visualize my plan.
- I thought I should first prove that there is one line of symmetry, and then prove that there can't be more than one.

Example 3 Proving a Conjecture Using Deductive Reasoning [Continued]

Solution

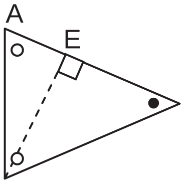
Proving there is one line of symmetry

ΔABD ≅ ΔCBD because of ASA:

- DB = DB because it is the same segment.
- ∠A = ∠C because ΔABC is isosceles
- ∠BDA = ∠BDC = 90º

Therefore DB is a mirror line.

Proving there is no other line of symmetry



ΔAEC is not congruent to ΔBEC because ∠B ≠ ∠A:

- ∠B = 180º – ∠A – ∠C = 180º – 2 × ∠A
- For ∠B to be equal to ∠A, ∠A must be 60º.
- ∠A can only be equal to 60º if ΔABC is equilateral, and ΔABC is not an equilateral triangle.

Because ΔAEC is not congruent to ΔBEC, CE is not a line of symmetry.

The same reasoning can be used for the third possible line of refection from vertex A to side CB.

Therefore there is exactly one line of symmetry in an isosceles triangle that is not equilateral.

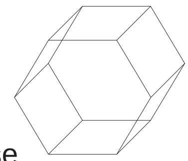
Thinking

- I showed that the triangles formed by the predicted line of symmetry are congruent, proving that there is one line of symmetry.
- To show that another line, CE, is not a line of symmetry, I proved that angles A and B are not equal in an isosceles triangle that is not equilateral:
- I knew that
  180º – ∠A – ∠C
  = 180º – 2 × ∠A
  because the sum of the angles in any triangle is 180º and ∠A = ∠C in ΔAEC.

Example 4 Proving a Conjecture Using Deductive Reasoning

Use deductive reasoning to prove that a prism always has more edges than vertices.

Solution



n edges on one base
n edges on the other base
n edges to connect the bases
That is 3*n* edges altogether.

n vertices on one base
n vertices on the other base
That is 2*n* vertices altogether.

Since 3*n* > 2*n*, there are always more edges than vertices on a prism.

Thinking

- The bases of a prism are congruent polygons and each base has *n* edges and *n* vertices.
- The other edges come from connecting each vertex on one base to its corresponding vertex on the other base. Since there are *n* vertices on each base, there are *n* connecting edges.
- I know that 3*n* > 2*n* when *n* is positive, like in this situation.

Practising and Applying

1. Identify each statement as an example of inductive or deductive reasoning.

a) "Dogs have sharp teeth, so it would hurt if you were bitten by a dog."

b) "It has rained every Friday this month, so I know that next Friday will be rainy."

c) "The sun will rise between those two hills. It always does."

d) "When I ride my bicycle downhill, I feel cool because the increased speed makes more wind to cool me."

2. a) Complete this chart describing plane symmetry in regular polygon-based prisms.

Number of sides in the prism's base	Number of planes of symmetry
3	
4	
5	
6	

b) Use the chart to make a conjecture about symmetry in a regular polygon-based prism.

c) Why is your conjecture the result of inductive reasoning?

d) Use deductive reasoning to explain why your conjecture must be true.

3. Lemo conjectured that pyramids with a certain base have one fewer face than prisms with that same base.

a) Verify her conjecture to be true using inductive reasoning.

b) Prove her conjecture to be true using deductive reasoning.

4. Drakpa conjectured that pyramids have the same number of planes of symmetry as the number of lines of symmetry in the base.

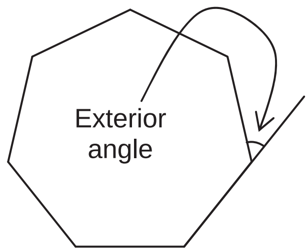
a) Verify that her conjecture is true using inductive reasoning.

b) Prove her conjecture to be true for pyramids using deductive reasoning.

5. Dechen conjectured that 2-D shapes with two lines of symmetry at right angles have to have rotational symmetry. Prove that her conjecture is true.

6. a) Measure the exterior angles in several polygons for which all angles are less than 180º and find their sum.

An exterior angle is the angle created by a side length and a line extending from the adjacent side length, for example:



b) Use what you noticed to develop a conjecture.

c) Prove your conjecture to be true using deductive reasoning.

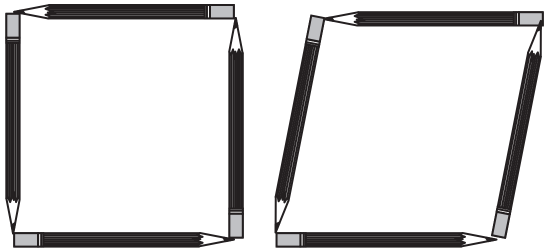
7. A student conjectured that triangular prisms with scalene right triangle bases cannot have symmetry. Use a counterexample to prove that this conjecture is false.

8. Why might some people think that deductive reasoning is more powerful than inductive reasoning?

8.2.1 EXPLORE: Rigidity

• Tshering once heard someone describe a rhombus as "a square that has been pushed over." He thought that was a good description.

He tried it out by creating a skeleton of a square using tape and four pencils of equal length. He then pushed on one of the vertices and the square became a rhombus.



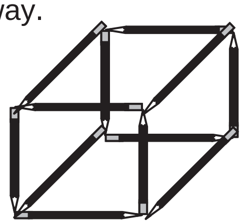
"Pushing" a square to make a rhombus

Tshering then wondered what other shapes could be pushed over. He and Devika decided to explore different shapes.

• Tshering and Devika were exploring something that mathematicians and engineers call **rigidity**. A shape that holds its shape when it is pushed is called a rigid shape. A square is not considered rigid because its shape changes when it is pushed.

• You can also test the rigidity of 3-D shapes in the same way.

For example, you can make the skeleton of a cube using pencils, sticks, or drinking straws and then push on one of its vertices. You will discover that it will not hold its shape.



A cube skeleton

A. Explore the rigidity of triangles by taping sticks together to make skeleton models. What kinds of triangles are rigid? Consider all types of triangles: scalene (including right scalene), isosceles (including right isosceles), and equilateral triangles.

B. i) You can make a square rigid by adding a diagonal to the skeleton. Explain why this works. Is your explanation an example of inductive or deductive reasoning? How do you know?

ii) What could you do to a regular pentagon to make it rigid? Try to do this with as little material as possible. Explain what you did and why it worked.

iii) Describe what you would do with other 2-D shapes to make them rigid.

C. Explore 3-D shapes by taping sticks together to make skeleton models. Predict which 3-D shapes are rigid and explain your prediction. Test your predictions by creating the skeleton models and testing their rigidity.

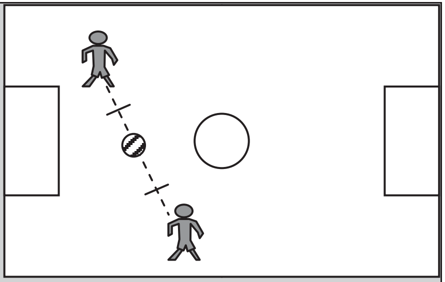
D. i) Why do you think people consider a triangle to be a unique shape?

ii) Why do you think triangles are commonly used in structures such as bridges and towers?

8.2.2 Perpendiculars and Bisectors

Try This

Two players are doing football drills. The players take their positions and the ball is placed an equal distance from each player. Upon hearing the whistle, the players must run to the ball and compete to possess it. One obvious location for the ball is at the midpoint of a line segment that goes from one player to the other. What other locations are possible?



A. i) Draw two points, P and Q, on a piece of paper to represent the two players. Then locate five points that are the same distance from P as from Q.

ii) How did you locate the points?

iii) How did you know they were the same distance from P as from Q?

B. i) Draw two intersecting line segments on a piece of paper. Locate five points that are the same distance from each line segment.

ii) How did you locate the points?

iii) How did you know they were the same distance from each line segment?

• People have been doing **constructions** with a straight-edge (a ruler with no markings) and a compass for thousands of years. Two of the most basic constructions are the **perpendicular bisector** of a line segment, and the **angle bisector**:

- A **perpendicular** is a line that intersects a line segment at 90º.
- A **bisector** divides something exactly in half (the prefix *bi* means two).
- A perpendicular bisector does both — it divides a line segment in half at 90º.
- An angle bisector divides an angle in half.

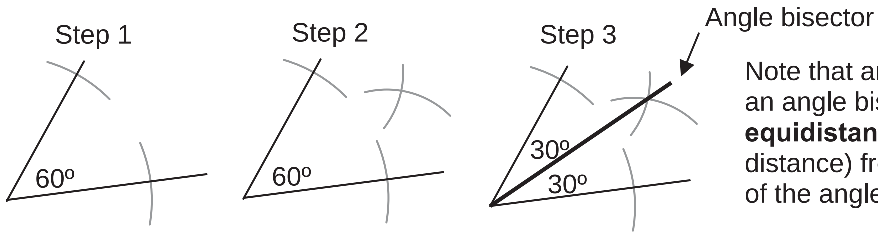
• Recall the following constructions you have learned in previous classes:

To construct an angle bisector

Step 1: Use a compass to make an arc on each arm to mark equal distances from the vertex.

Step 2: Use a compass to make intersecting arcs of equal radius from the intersection of the arcs and the arms.

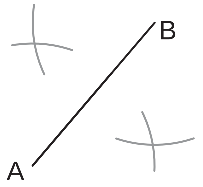
Step 3: Use a straight-edge to draw a line from the vertex through the intersection of these arcs to bisect the angle.



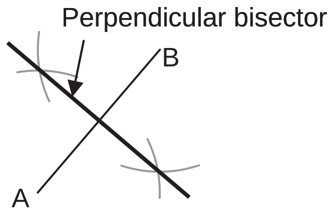
Note that any point along an angle bisector is **equidistant** (the same distance) from the arms of the angle it bisects.

To construct a perpendicular bisector of a line segment

Step 1
Use a compass to draw arcs of equal radius from each endpoint, A and B, on both sides of the line segment.



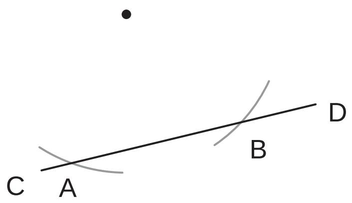
Step 2
Use a straight-edge to draw a line through the intersections of the arcs.



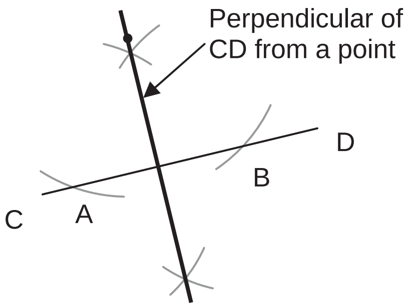
Any point along a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.

To construct a perpendicular from a point to a line segment

Step 1
Use a compass to draw an arc from the point so that it intersects CD in two places. The intersection points will be the endpoints of line segment AB.



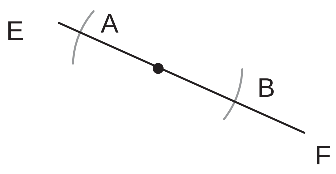
Step 2
Construct the perpendicular bisector of line segment AB (see *To construct a perpendicular bisector* above).



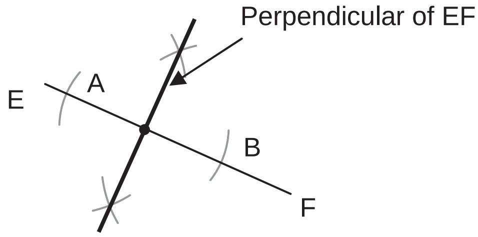
The perpendicular bisector of AB is a perpendicular from a point to CD.

To construct a perpendicular from a point on a line segment

Step 1
Use a compass to draw an arc from the point so that it intersects EF in two places. The intersection points will be the endpoints of line segment AB.



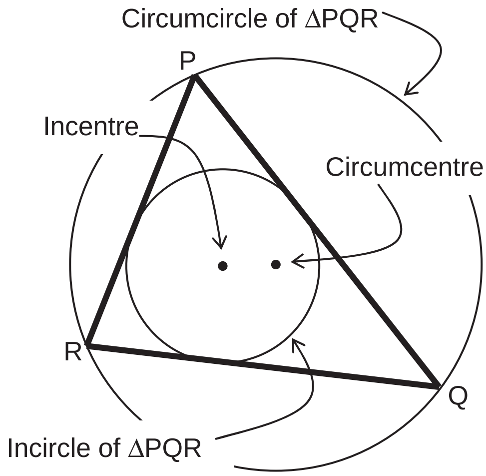
Step 2
Construct the perpendicular bisector of line segment AB (see *To construct a perpendicular bisector* above).



The perpendicular bisector of AB is a perpendicular through a given point on EF.

These constructions will help you construct these features of triangles:

- The **circumcircle** — the circle that passes through all three vertices
- The **circumcentre** — the centre of the circumcircle, which is found at the intersection of any two side length perpendicular bisectors
- The **incircle** — the circle inside the triangle that touches each side of the triangle once
- The **incentre** — the centre of the incircle, which is found at the intersection of any two angle bisectors



C. i) Connect points P and Q from **part A** to create line segment PQ and then construct the perpendicular bisector of PQ. What do you notice about this perpendicular bisector in relation to the five points?

ii) Construct the bisectors of the angles formed by the intersecting lines in **part B**. What do you notice about the angle bisectors in relation to the five points?

Examples

Example 1 Constructing a Triangle With a Straight-edge and Compass

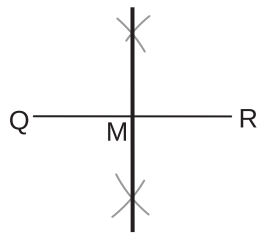
Without using a ruler or protractor, construct ∆PQR with

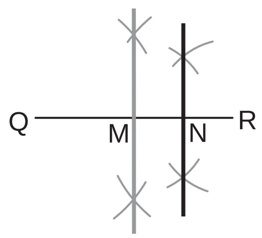
PQ = $\frac{3}{4}$QR and ∠PQR = 60°.

Solution

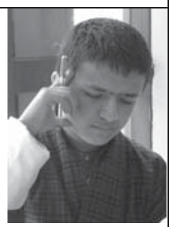
Step 1

Step 2





Thinking



Step 1: I drew a line segment and labelled it QR to create side length QR of ∆PQR.

Step 2: I needed to find $\frac{3}{4}$ of the length of QR so I could create side length PQ. To do this

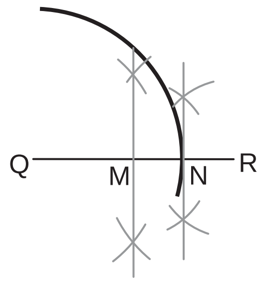
- I constructed QR's perpendicular bisector and labelled the intersection point M.
- I constructed MR's perpendicular bisector and labelled the intersection point N.

I knew QN was $\frac{3}{4}$ QR so I could use this information in Step 3 to begin locating vertex P.

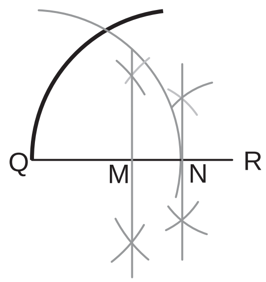
Example 1 Constructing a Triangle With a Straight-edge and Compass [Cont'd]

Solution

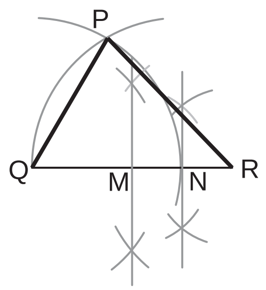
Step 3



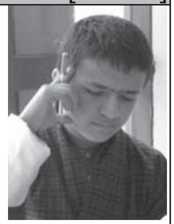
Step 4



Step 5



Thinking



Step 3: I put my compass point on Q and extended it to the length of QN. I then drew a large arc with a radius equal to QN, which I knew was $\frac{3}{4}$ QR. Vertex P could be anywhere on that arc to make PQ = $\frac{3}{4}$ QR

Step 4: I needed $\angle$ PQR = 60°. To do this,

- I visualized ΔPQN, an equilateral triangle with all angles 60°, including $\angle$ PQN. I knew that if $\angle$ PQN = 60°, then $\angle$ PQR = 60° because they are the same angles.

- I put my compass point on N and set it to the length of QN. I then drew a large arc with a radius equal to QN. Vertex P could be anywhere on that arc and $\angle$ PQR would be 60°.

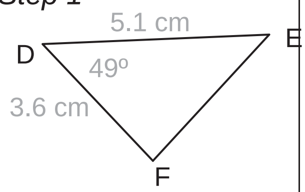
Step 5: I knew that vertex P had to be on both large arcs for $\angle$ PQR to be 60° and PQ to be $\frac{3}{4}$ QR so I located vertex P where the two large arcs intersected. I then drew QP and PR to complete the triangle.

Example 2 Constructing Perpendicular Bisectors to Construct the Circumcircle

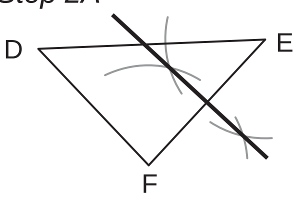
Construct the circumcircle of ΔDEF with DF = 3.6 cm, ED = 5.1 cm, and $\angle$ D = 49°.

Solution

Step 1



Step 2A



Thinking

Step 1: I drew ΔDEF by

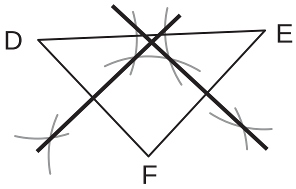
- using a ruler to draw a 5.1 cm line segment labelled ED

- using a protractor to make $\angle$ D = 49°

- marking point F 3.6 cm away from point D such that $\angle$ D = 49° and then drawing DF and FE

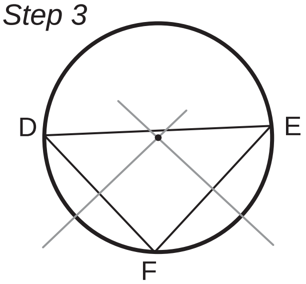
Step 2: To draw the circumcircle of ΔDEF, I needed to find the circumcentre — the centre of the circle that travels through D, E, and F. I knew the circumcentre had to be the same distance from E as from F, so I constructed the perpendicular bisector of EF. (Any point on a perpendicular bisector is an equal distance from the end points of the line segment it bisects.)

Step 2B



The circumcentre also had to be the same distance from D as from F, so I constructed the perpendicular bisector of DF. The only point the two perpendicular bisectors had in common was the intersection point, so that had to be the circumcentre.

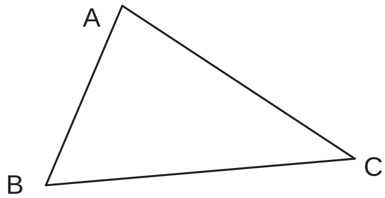
Step 3



Step 3: I placed my compass point on the circumcentre, set its radius at vertex E, and then drew the circumcircle. I knew it would also pass through vertices D and F because I already knew that its circumcentre was just as far from F as from E and just as far from D as from E.

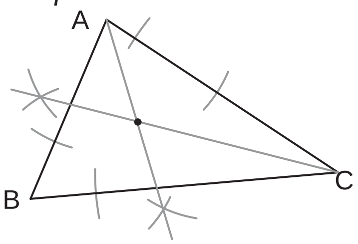
Example 3 Using Angle and Perpendicular Bisectors to Construct the Incircle

Construct the incircle of ΔABC using only a straight-edge and a compass.

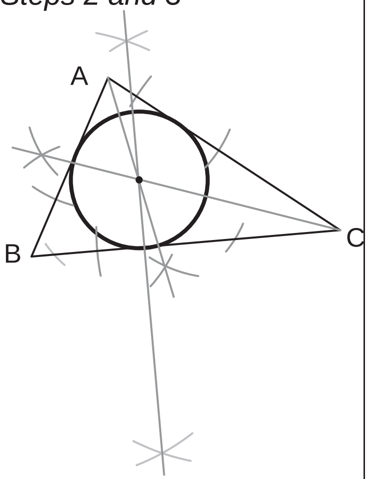


Solution

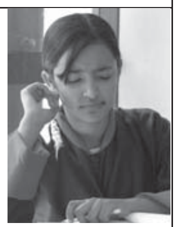
Step 1



Steps 2 and 3



Thinking



Step 1: To draw the incircle of ΔABC, I needed to find its incentre.

- The incentre had to be the same distance from AC as from BC, so I constructed $\angle C$'s bisector. (Any point along an angle bisector is the same distance from the arms of the angle it bisects.)
- The incentre also had to be the same distance from AC as from BC, so I bisected $\angle A$.
- The only point the angle bisectors had in common was the intersection, so that had to be the incentre.

Step 2: To find the point on BC where the incircle touched, I constructed a perpendicular from the incentre to BC.

Step 3: I placed my compass point on the incentre and set its radius at the intersection of the perpendicular bisector and BC. Then I drew the incircle. I knew the incircle would also just touch AB and AC because its incentre was just as far from AC as from BC and just as far from AB as from BC.

Practising and Applying

1. Construct the circumcircle for each.

a) Acute ΔABC: AB = 6.5 cm, BC = 4.3 cm, and $\angle B = 65^\circ$

b) Right ΔDEF: DE = 6.5 cm, EF = 4.3 cm, and $\angle E = 90^\circ$

c) Obtuse ΔJKH: JK = 6.5 cm, KH = 4.3 cm, and $\angle K = 115^\circ$

2. a) What do you notice about the location of the circumcentre in each type of triangle in **question 1**?

b) For which types of triangles do you predict the circumcentre will be outside the triangle?

c) How could you determine whether your conjecture from **part b)** is true?

3. Construct the incircle for each.

a) ΔLMN: LM = 8.1 cm, $\angle L = 81^\circ$, and $\angle M = 35^\circ$

b) ΔPQR: PQ = 7.4 cm, QR = 8.9 cm, and PR = 5.3 cm

4. a) Construct an equilateral triangle, its circumcircle, and its incircle.

b) Compare your constructions with a classmate's. What do you conclude about the circumcentre and incentre of an equilateral triangle?

5. Construct each, using only a straight-edge and a compass.

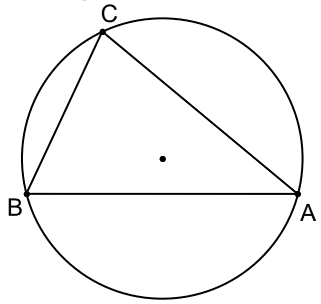
a) ΔABC: $AB = \frac{1}{2}AC$ and $\angle A = 45^\circ$

b) ΔABC: $AB = \frac{1}{2}AC$ and $\angle A = 30^\circ$

6. a) Draw a quadrilateral and then construct the circle that passes through three of its vertices (the circumcircle of the triangle formed by these vertices). Does the circle also pass through the fourth vertex of the quadrilateral?

b) Repeat **part a)** with different-shaped quadrilaterals until you feel ready to conclude whether all quadrilaterals have circumcircles.

7. Lobzang started with a large circle and randomly chose three points on it to form ΔABC.

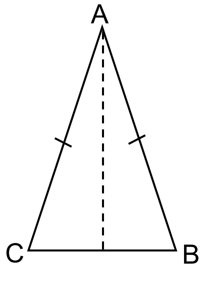


He constructed ΔABC's incircle. Then he randomly chose three points on ΔABC's incircle to form ΔDEF and constructed ΔDEF's incircle. He repeated this until he had constructed four circles altogether. Follow his procedure to see if there is a pattern.

8. a) Use trigonometry to prove that any point P on the bisector of an angle is equidistant from the arms of the angle.

b) Use your proof from **part a)** to prove that point P, the intersection of two angle bisectors, is the incentre of a triangle.

9. Andu noticed that for isosceles triangle ΔABC, the perpendicular bisector of BC (the non-equal side length) and the angle bisector of $\angle A$ (the angle opposite to BC) are the same line.



He conjectured that this was the case for any isosceles triangle.

a) To prove Andu's conjecture deductively, show how

• the triangles formed by the perpendicular bisector are congruent.

• the triangles formed by the angle bisector of the angle opposite to the perpendicular bisector are congruent.

b) Andu's conjecture works for any side length and its opposite angle in an equilateral triangle. Explain.

8.2.3 Medians and Altitudes

Try This

A family wants to divide its land equally between two children. Their triangular plot of land measures 58 m, 89 m, and 93 m along its boundaries.

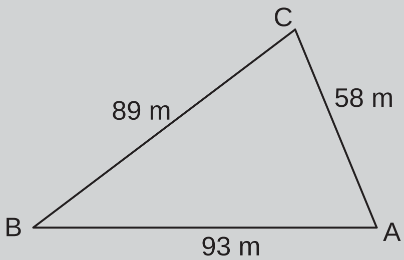
A. i) Draw a scale model of the plot of land, ΔABC, with sides AB = 93 mm, BC = 89 mm, and AC = 58 mm.

ii) Measure the height of ΔABC and then calculate ΔABC's area.

iii) Locate the midpoint of AB using a perpendicular bisector. Label it point M and then draw a line from C to M to create two triangles, ΔBMC and ΔAMC.

iv) Calculate the areas of ΔBMC and ΔAMC.

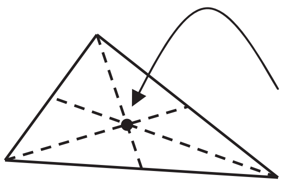
v) What do you notice about the areas of ΔABC, ΔBMC, and ΔAMC?



In the previous lesson, you learned how to construct the circumcentre and the incentre of a triangle. These are two of its centres. There other ways to think about the centre of a triangle. This lesson shows two more.

Medians and the centre of gravity

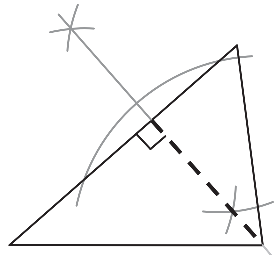
- The **centre of gravity** of a triangle is sometimes called the **centroid**.
- The centre of gravity is the intersection point of the triangle's **medians**. It is the point on which the triangle would balance under the influence of gravity (see the Balancing Triangles game on **page 295**).



- A median is a line segment that joins a vertex to the midpoint of the side opposite to it. A triangle has three medians.
- To construct a median of a triangle, locate the midpoint of a side length using a perpendicular bisector. Draw a line to connect it to the opposite vertex.
- You need to construct only two medians to locate the centroid, because the third median would automatically travel through the same intersection point.

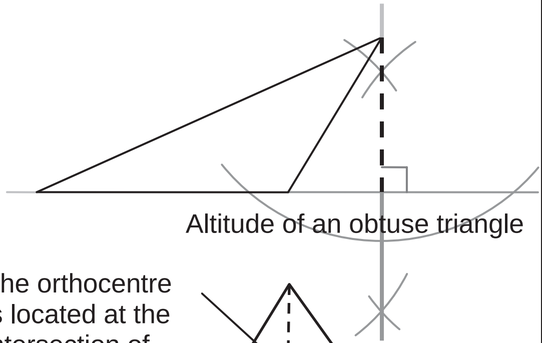
Altitudes and the orthocentre

- An **altitude** is a perpendicular line segment from a vertex of a triangle to its opposite side. A triangle has three altitudes.
- This is a useful construction because the altitude is the height of the triangle, which is used to determine the triangle's area.
- To construct an altitude for an acute triangle, construct a perpendicular from a vertex to the opposite side.



Altitude of an acute triangle

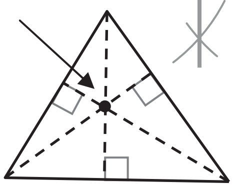
- To construct an altitude for an obtuse triangle, you may need to extend the side you will be using and then construct the perpendicular from the opposite vertex. In obtuse triangles, two of the altitudes are outside the triangle.



Altitude of an obtuse triangle

- The point of intersection of the altitudes is called the **orthocentre**.
- You need to construct only two altitudes to locate the orthocentre, because the third altitude would automatically travel through the same intersection point.

The orthocentre is located at the intersection of the altitudes.



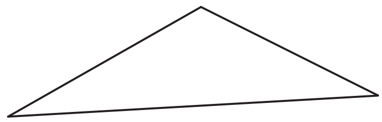
B. Which construction was used in **part A** to divide the triangular field in half? How else could the field be divided in half?

C. Construct and measure the altitude from C to AB to make sure you measured the height of the triangle properly when you calculated the area in **part A ii)**.

Examples

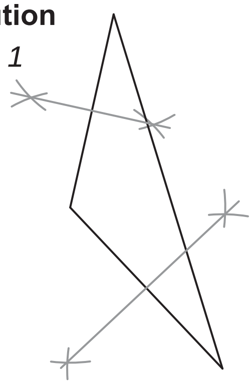
Example 1 Constructing Medians to Locate the Centroid of a Triangle

Locate the centroid of this triangle, using only a straight-edge and a compass.



Solution

Step 1

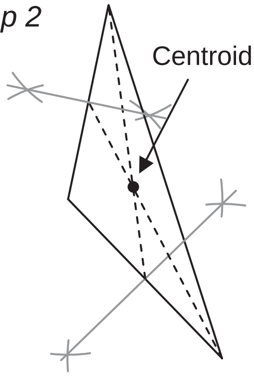


Thinking

- To locate the triangle's centroid, I had to construct two of its medians. I knew the point at the intersection of the medians would be the centroid.

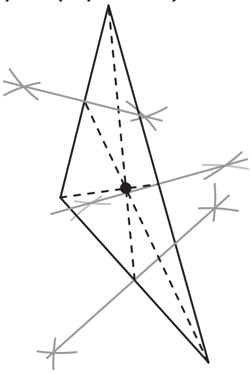
Step 1: A median is a line segment that goes from a vertex to the midpoint of the opposite side, so I constructed the perpendicular bisectors of two of the sides in order to locate their midpoints.

Step 2



Step 2: I drew the medians by connecting the midpoints of the two sides to the vertices opposite them. The intersection point was the centroid, or centre of gravity.

Step 3 (Optional)

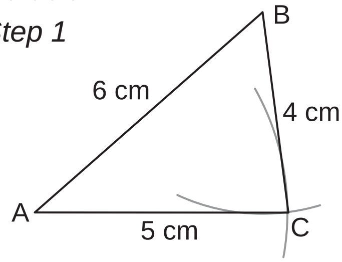


Step 3 (Optional): Just to double-check, I constructed the third median. I saw that it went through the intersection point of the other two medians.

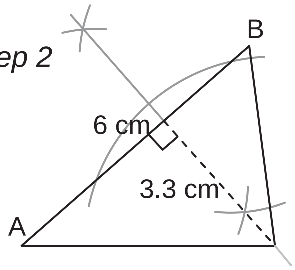
Example 2 Constructing the Altitude of a Triangle to Determine its Area
Draw $\triangle ABC$ with AB = 6 cm, BC = 4 cm, and AC = 5 cm. Determine its area.

Solution

Step 1



Step 2



Step 3

$A = bh \div 2 = (6)(3.3) \div 2$

$= 9.9 \text{ cm}^2$

Thinking

Step 1: I drew $\triangle ABC$ using my ruler to draw a 6 cm line segment, AB. To locate vertex C, I used my ruler to set my compass at 4 cm and then I made a 4 cm arc from B. I then re-set my compass to 5 cm and made a 5 cm arc from A — the intersection of the arcs was C.

Step 2: To determine $\triangle ABC$'s area, I needed its height. I knew that the altitude was the same as the height. I decided to construct the altitude from vertex C to side AB so I constructed a line perpendicular to AB that travelled through C. I measured the altitude to be 3.3 cm.

Step 3: I used the formula for the area of a triangle and substituted the values for height (3.3 cm) and base (6 cm).

Practising and Applying

1. Locate the centroid of each.

a) ΔABC: AB = 6.5 cm, BC = 4.3 cm, and $\angle B = 65°$

b) ΔDEF: DE = 6.5 cm, EF = 4.3 cm, and $\angle E = 90°$

c) ΔJKH: JK = 6.5 cm, KH = 4.3 cm, and $\angle K = 115°$

2. Do you think it is possible for the centroid, or centre of gravity of a triangle to be outside the triangle? Explain.

3. Determine the area of each triangle.

a) ΔPQR: PQ = 7.4 cm, QR = 8.9 cm, and PR = 5.3 cm

b) ΔLMN: LM = 4.1 cm, $\angle L = 23°$, and $\angle M = 122°$

4. a) Construct any triangle ΔABC, its altitude from A, and the median joining A to the midpoint M of the side BC.

b) What do you notice about the altitude of ΔABC compared to the altitudes of ΔAMB and ΔAMC?

c) Calculate the areas of ΔAMB and ΔAMC. What do you notice?

d) Compare your results with those of several classmates.

e) What can you conclude about medians and triangles? Are you using inductive or deductive reasoning?

5. a) Construct any triangle and its incircle.

b) Estimate whether the incircle covers more than half or less than half of the triangle's area.

c) Determine the areas of the triangle and incircle and compare them to check your estimate in **part b)**.

d) Compare your results with those of several classmates.

e) What can you conclude about the area of triangles and their incircles?

6. a) Draw or construct any equilateral triangle.

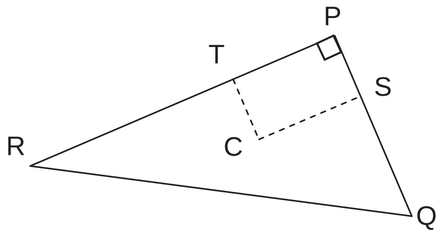
b) Locate its circumcentre, incentre, centroid, and orthocentre.

c) What do you notice about the location of these points?

7. a) Construct any right triangle ΔPQR with $\angle P = 90°$.

b) Locate its centroid. Label it point C.

c) Construct perpendiculars from C to PQ and from C to PR. Label the points of intersection of the side lengths and the perpendiculars S and T, as shown.



d) Measure to calculate the ratio of the lengths PQ to PS and PR to PT.

e) Compare your results with those of several classmates. What do you notice?

8. Draw several right triangles and construct their altitudes. Why is it easier to construct the altitudes of a right triangle than any other type of triangle?

9. Explain why it makes sense that the intersection of the medians of a triangle is the centre of gravity, or balance point of a triangle.

CONNECTIONS: Paper Folding Constructions

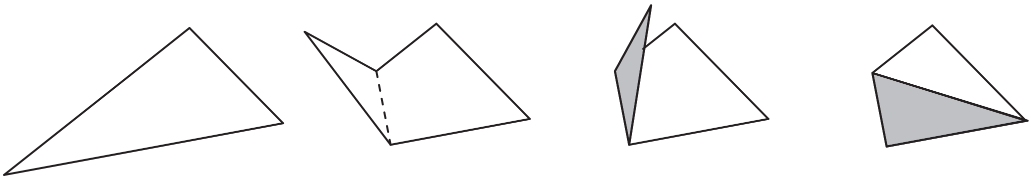
Although technically mathematical constructions require the use of a straight-edge and compass, it can be interesting to do things differently. Some mathematicians investigate constructions that can be done with a compass only (and no straight-edge). Others explore what can be done with paper folding.

For example, here is a very basic paper-folding "construction":

To construct the perpendicular bisector of a line segment, draw a line segment on a piece of paper and then fold the paper so that the two endpoints of the line segment match. When you unfold it, the fold line is the perpendicular bisector.

Below are some other paper-folding constructions for you to try.

1. Cut out an acute triangle and use folding to find the intersection of the three perpendicular bisectors. Measure to check that this point is equidistant from each vertex. What special point are you finding?



Folding an acute triangle to find the perpendicular bisector of one of its sides

2. Explore to find ways of using paper folding to construct the following:

a) an angle bisector

b) the incentre of a triangle

c) an altitude of a triangle

d) the orthocentre of a triangle

e) a median of a triangle

f) the centroid of a triangle

GAME: Balancing Triangles

In this game or contest, groups of players try to make a perfectly-balanced triangle out of stiff paper. They then compete against each other to see whose triangle balances the longest.

• Each group cuts a triangle out of stiff paper and then uses constructions to locate the centroid.

• When everyone is ready, a representative of each group balances his or her group's triangle by placing the centroid on the point of a compass.

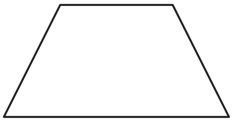
• The group whose triangle balances the longest wins.

Balancing a triangle on its centroid

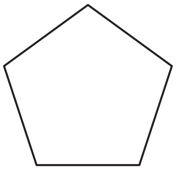
UNIT 8 Revision

1. How many lines of symmetry does each shape have? Sketch them.

a) isosceles trapezoid



b) regular pentagon



2. Create a shape that is not a rectangle but that has exactly two lines of symmetry. Show the lines of symmetry.

3. Describe a 3-D shape with exactly five planes of symmetry.

4. Describe the order of turn symmetry for each shape in **question 1**.

5. a) Describe the turn symmetry of a regular pentagon-based pyramid.

b) Describe the turn symmetry of a regular pentagon-based prism.

6. a) Sketch a 2-D shape that has no rotational symmetry.

b) Sketch a 3-D shape that has no rotational symmetry.

7. a) Describe the reflectional symmetry of a cone.

a) Describe the rotational symmetry of a cone.

8. There is a relationship between the number of edges of a pyramid and the number of sides on its base. Use inductive reasoning to develop and verify a conjecture about this relationship.

9. Use deductive reasoning to prove that an equilateral triangle has an order of turn symmetry equal to its number of sides.

10. Dema is making the skeleton of a square-based prism using sticks for its edges. Explain how she could make her cube rigid.

11. a) Construct ΔABC:

$AB = \frac{3}{4}AC$ and $\angle A = 22.5^\circ$

b) Construct the circumcentre and circumcircle of ΔABC.

12. a) Construct ΔPQR: PQ = 4.7 cm, QR = 6.6 cm, and PR = 7.5 cm

b) Construct the incentre and incircle of ΔPQR.

13. a) Draw ΔLMN: LM = 8.0 cm, $\angle L = 12^\circ$, and $\angle N = 47^\circ$

b) Explain how to use constructions to locate the centroid of ΔLMN.

c) Use constructions to locate the centroid.

14. a) Construct an altitude for ΔLMN from **question 13** and use it to calculate the area of ΔLMN.

b) Construct a different altitude for ΔLMN from **question 13** and use it to calculate the area ofΔLMN.

c) Why might your answers in **parts a) and b)** be slightly different?

15. Use deductive reasoning to explain why a median divides the area of a triangle in half.

16. The circumcentre, incentre, and centroid of any isosceles triangle are collinear (all in a straight line). Explain how you might use inductive reasoning to verify this conjecture.

Instructional Terms

calculate: Figure out the number that answers a question; compute

classify: Put things into groups according to a rule and label the groups; organize into categories

compare: Look at two or more objects or numbers and identify how they are the same and how they are different (e.g., compare 6.5 and 5.6; compare the size of the students' feet; compare two shapes)

conclude: Judge or decide after reflection or after considering data

construct: Make or build a model; the term construct is sometimes reserved for drawings that use a compass and straight-edge only.

create: Make your own example or problem

describe: Tell, draw, or write about what something is or what something looks like; tell about a process in a step-by-step way

determine: Decide with certainty as a result of calculation, experiment, or exploration

draw: **1.** Show something in diagram form **2.** Pull or select an object (e.g., draw a card from a deck; draw a tile from a bag)

estimate: Make a reasonable guess (e.g., estimate how long it takes to walk from your home to school; estimate how many leaves are on a tree; estimate 3210 ÷ 789)

evaluate: **1.** Determine whether something makes sense; judge **2.** Calculate the value as a number (e.g., evaluate $m^2 + 3$ for $m = 5$)

explain: Tell what you did; show how you know

explore: Investigate a problem by questioning, brainstorming, and trying new ideas

extend: **1.** In patterning, continue the pattern

justify: Give convincing reasons for a prediction, an estimate, or a solution; tell why you think your answer is correct

measure: Use a tool to describe an object or determine an amount (e.g., use a protractor to measure an angle; use balance scales to measure mass; use a measuring cup to measure capacity)

model: Show or demonstrate an idea using objects, pictures, words, and/or numbers (e.g., model addition of integers using red and blue counters, model a relationship using an equation)

predict: Use what you know to work out what is going to happen (e.g., predict the tenth number in the number pattern 1, 2, 4, 7, …)

relate: Describe how two or more objects, drawings, ideas, or numbers are connected or similar

represent: Show information or an idea in a different way (e.g., draw a graph of an equation; make a model from a word description; create an expression to model a situation)

show your work: Record all calculations, drawings, numbers, words, or symbols that make up the solution

simplify: Write a number or expression in a simpler form (e.g., combine like terms of a polynomial)

sketch: Make a rough drawing not necessarily to scale, (e.g., sketch a picture of the field with given dimensions)

solve: **1.** Develop and carry out a process for finding an answer to a problem **2.** Find the value of a variable in an equation or inequality

sort: Separate a set of objects, drawings, ideas, or numbers according to an attribute (e.g., sort 2-D shapes by number of sides)

visualize: Form a picture in your head of what something is like (e.g., visualize the number six as two rows of three dots; visualize the equation $y = x$ as a diagonal line at a 45º angle)

Definitions

A

absolute value: The value of a number without regard to its sign (e.g., |5| = 5 and |–5| = 5)

absolute value equation: An equation that represents an absolute value function when the function has a specific value (e.g., the absolute value function $f(x) = |x - 5| + 2$ can be used to solve the absolute value equation $|x - 5| + 2 = 7$)

absolute value function: A function that uses absolute values

accuracy: A measurement is accurate when the measurement is taken correctly. Accuracy is affected by the use and interpretation of the measuring instrument.

acute angle: An angle less than 90º

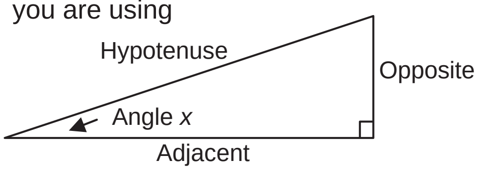
acute triangle: A triangle in which all interior angles are acute angles

address of a matrix: The location of an element in a matrix, usually described as an ordered pair with the row number first and the column number second (e.g., in this matrix, the element 9 is at (3, 2))

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 7 & 3 & 1 & 8 \\ 5 & 9 & 1 & 4 \end{bmatrix}$$

adjacency matrix: A square matrix that describes the connections in a network

adjacent side: In trigonometry, the leg of a right triangle that is next to the angle you are using

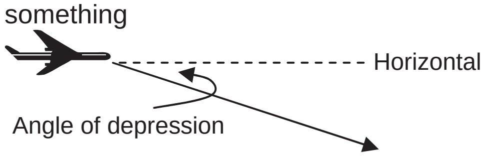


algebraic expression: A combination of one or more variables that may include numbers and operation signs (e.g., $8x + 2y^2 - 9$)

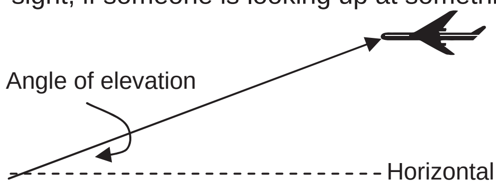
altitude: The line segment that represents the height of a 2-D or 3-D shape. In a triangle, it is a perpendicular segment from a vertex to the opposite side.

angle bisector: A line that separates an angle into two equal parts

angle of depression: The acute angle formed by the horizontal and the line of sight, if someone is looking down at something



angle of elevation: The acute angle formed by the horizontal and the line of sight, if someone is looking up at something



angle of rotation: The angle through which a shape has moved after a rotation

annually: Once a year, or yearly

apex: The highest point or vertex of a cone or pyramid when resting on its base

area: The measure of the surface of a 2-D shape, expressed in terms of the number of square units needed to cover the shape; the number of square units needed to cover a surface

array: A rectangular arrangement of items, usually numbers (e.g., a matrix is an array of items)

average: In common use, average is the same as mean. See *mean*

axis: A line drawn for reference when locating points in a coordinate system

axis of rotation: A line around which a 3-D shape may be turned; a shape has turn symmetry if it looks the same more than once within a full rotation.

axis of symmetry (of a parabola): A vertical line that is perpendicular to the x-axis and travels through the vertex of a parabola. The axis of symmetry is the line of symmetry of the parabola.

B

balance of a loan: The amount still owing on a loan at any point in time, taking into account all interest charged and all payments made up until that point

base: **1.** The face that determines the name of a prism or pyramid
2. In a 2-D shape, the line segment that is perpendicular to the height
3. The number that is repeatedly multiplied in a power (e.g., in the power 5^3, the base is 5)

bearings: Angles measured in degrees clockwise from the north, used with vectors as part of a set of directions

bimodal: See *U-shaped distribution*

binomial: A polynomial with two terms (e.g., $4x - 7y$ and $5x^2 + 3$ are binomials)

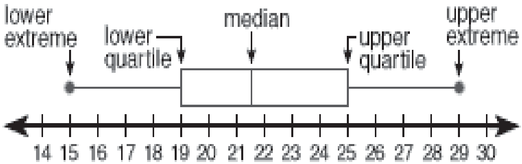
bin: An interval of data values used in frequency tables and histograms; bins are always of equal size, called the bin width

bin width: See *bin*

bisect: Divide in half (e.g., an angle bisector divides an angle in half, a perpendicular bisector passes through the midpoint of a line segment at a right angle)

bisector: See *angle bisector* and *perpendicular bisector*

box and whisker plot: A graph that uses the median (Q2) and extremes as well as the lower and upper quartiles (Q1 and Q3) to organize a set of data into four intervals that each contain an equal number of data values.



box plot: See *box and whisker plot*

C

capacity: The amount that a container can hold. Common units are millilitres (mL) or litres (L). Capacity can be related to volume (e.g.,1 mL = 1 cm^3).

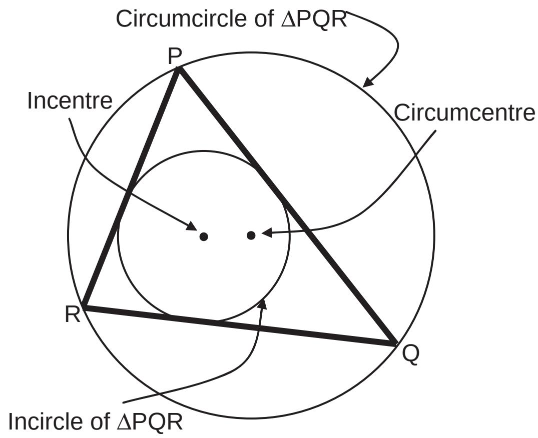
centre of gravity of a triangle: Sometimes called centroid; the intersection point of the medians of a triangle; the point on which the triangle would balance under the influence of gravity.

centre of rotation: A fixed point around which the points in a shape rotate in a clockwise (cw) or counter-clockwise (ccw) direction. The centre of rotation may be inside or outside the shape.

centroid: See *centre of gravity*

circumcircle: The circle that passes through all three vertices of a triangle

circumcentre: The centre of the circumcircle of a triangle, which is found at the intersection of any two side length perpendicular bisectors



circumference: **1.** The boundary of a circle **2.** the length of the boundary of a circle calculated using the formula $C = 2\pi r$, where r is the radius, or $C = \pi d$, where d is the diameter; also called perimeter

coefficient: The numerical coefficient is the number by which a variable is multiplied; the literal coefficient is the variable by which another variable is multiplied (e.g., in the term $3z$, the numerical coefficient of z is 3; in the term by^2, the literal coefficient of y^2 is b)

column matrix: A matrix with only one column and any number of rows

commission (amount): The amount of money a salesperson earns based on how much he or she sells

common factor: A number or algebraic expression that divides into two or more other numbers or expressions with no remainder

composite transformation: A transformation that can be described by two or more single transformations

compound interest: Interest charged on a loan (or earned in the case of an investment) for which the interest is calculated as a percentage of the outstanding balance (or growing balance in the case of an investment) at the end of each interest period

compounding frequency: How often compound interest is calculated on a loan or investment (e.g., annually or monthly)

conditional probability: The probability of Event B happening if you assume Event A has already happened, written as P(B|A) and read as "*the probability of B given A*"

cone: A 3-D shape that has a circular base and a curved surface from the boundary of the base to a vertex or apex

congruence: The property shared by geometric shapes that are identical in shape and size. Shapes can be congruent. Line segments and angles can also be congruent.

conjecture: An idea or prediction that something is always true

construction: Geometrical diagrams drawn using only a straight-edge and compass

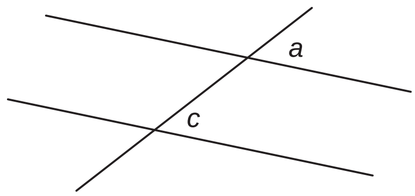
continuous data: A set of data with no gaps, represented by a solid line or curve on a graph; also represented by bars in a histogram with no spaces in between them

coordinates on a plane: A pair of numbers used to define a position in the *x*-*y*-plane; in the form of ordered pairs (*x*, *y*)

correlation: A description of the relationship between two variables as indicated by the pattern in the plotted points of a scatter plot. In a strong correlation the points form a line or nearly form a line and in a weak correlation the points roughly form a line but are dispersed. In a positive correlation the variables increase together and in a negative correlation one variable increases as the other variable decreases.

correlation coefficient: A measure of how well the points in a scatter plot fit a linear model. A value close to 1 or –1 indicates a good fit and a value close to 0 indicates a poor fit.

corresponding angles: Congruent angles that are formed by a transversal and two parallel lines (e.g., in the diagram below, a and c are corresponding angles)



cost price: The price that the seller or storeowner pays for an item. A seller will mark up this price in order to sell the item at a higher price and make a profit.

counterexample: One example that proves something is not true

cube: **1.** To raise a number to a power or exponent of 3 **2.** A polyhedron that has six congruent square faces

cube root: One of three equal factors of a number (e.g., the cube root of 8 is 2 because $2^3 = 8$)

curve of best fit: A non-linear relationship can be graphed in a scatter plot and then reasonably represented by a curve of best fit; curves can be quadratic (parabolas), exponential, or another shape such as a cubic

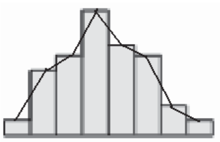
cylinder: A 3-D shape with two congruent, parallel, planar, circular faces joined by one curved lateral surface

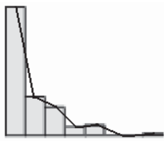
D

daily: Every day, or once a day

data: Information gathered in a survey, in an experiment, or by observing (e.g., data can be in words like a list of students' names, in numbers like quiz marks, or in pictures like drawings of favourite pets). The word data is plural, not singular

data distribution: The overall appearance of a set of data as a whole (e.g., a set of data might have a normal distribution or mound shape or it might be left or right skewed with the data values clustered at the right end or at the left end). The distribution of a data set is best observed when the data set is graphed in a histogram along with a frequency polygon, although box plots and stem and leaf plots also show data distribution.





Normal distribution   Right skewed distribution

deductive reasoning: An explanation that is based on reasoning using known or assumed facts

degree of a polynomial: The greatest exponent that appears in any term of a single-variable polynomial; the greatest sum of exponents in any term of a multi-variable polynomial (e.g., $3x + 2xy$ is degree 2 because of the term xy)

dependent event: In probability, an event that is affected by the outcome of another event

dependent variable: In a function, a variable whose output values are determined by the input values of the other (independent) variable; often represented by *y* and plotted on the vertical axis

digraph: See *directed graph*

dilatation: A transformation that enlarges or reduces a figure by a scale factor. Lines joining corresponding points on the original and the image meet at the centre of dilatation.

dilatation of a quadratic function: A vertical dilatation of a function transforms every point in the function using the mapping $(x, y) \rightarrow (x, ay)$. If $|a| > 1$, the dilatation is called a vertical stretch and if $0 < a < 1$, the dilatation is called a vertical compression. If $a < 0$, the dilatation is a negative dilatation and it appears as a refection in the x-axis.

dimension: The size or measure of an object (e.g., the width and length of a rectangle are its dimensions)

dimensions of a matrix: The size of a matrix described as the number of rows by the number of columns (e.g., this matrix has dimensions 3-by-4 or 3 × 4)

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 7 & 3 & 1 & 8 \\ 5 & 9 & 1 & 4 \end{bmatrix}$$

directed edge: An edge or path in a network that has a specified direction See *edges of a network*

directed graph: A network for which the edges have given directions, usually called digraphs See *edges of a network*

discount amount: The amount by which a marked price is decreased, usually to encourage a purchase

discrete data: A set of data that consists of isolated points that is sometimes represented by a dashed line or dashed curve on a graph

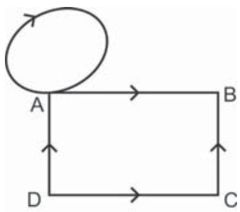
dividend amount: The amount of money an investor earns from holding shares in a company, usually annually. A dividend paid as a percentage is called the dividend rate.

dividend rate: See *dividend (amount)*

down payment: An initial partial amount of money paid at the time an item is purchased; often stated as a percentage of the purchase price (e.g., a 20% down payment is required on the purchase of a car worth Nu 200,000)

E

edge of a network: A path in a network that begins and ends at a vertex (e.g., the network below has five edges: between A and B, between B and C, between C and D, between D and A, and between A and A, the loop). Edges can be directed or not (e.g., the edges in the network below are all directed edges).



element of a matrix: Each item in a matrix

entire radical: A number that is entirely under the radical sign (e.g., $\sqrt{5}$ is an entire radical in its simplified form, whereas the entire radical $\sqrt{45}$ can be simplified to $3\sqrt{5}$, a mixed radical)

equation: A mathematical statement in which the value on the left of the equals sign is the same as the value on the right of the equals sign (e.g., the equation $5n + 4 = 39$ means that 4 more than the product of 5 and a number equals 39)

equidistant: The same distance (e.g., all points on the circumference of a circle are equidistant from the centre)

equilateral triangle: A triangle with three sides of equal length and with all angles equal and 60º

equitable shapes: Shapes that have the same area, but may not be congruent

equivalent functions: Functions that have the same output value for every input value. If two quadratic functions have the same output value for three different input values, they must be equivalent (e.g., $f(x) = 4(x + 2)(x – 6)$ is equivalent to $f(x) = 4x^2 – 16x – 48$).

equivalent interest rates: Interest rates that may have different compounding frequencies or different per annum interest rates but result in the same interest amount earned or charged (e.g., 14.5% p.a. compounded annually is essentially equivalent to 14% p.a. compounded monthly; 14% p.a. compounded monthly is equivalent to 14.9% p.a. compounded annually)

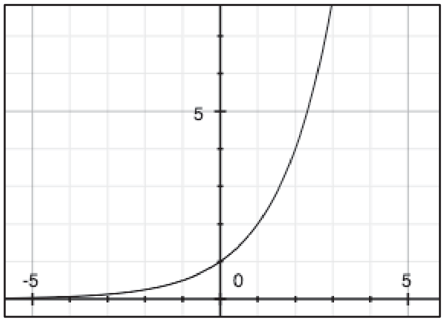
event: A set of outcomes for a probability experiment (e.g., if you roll a die with the numbers 1 to 6, the event of rolling an even number has the outcomes 2, 4, or 6); a subset of the sample space

expand: Write the full product of an algebraic or numerical (including radical) expression (e.g., $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 4x + 4$, $(\sqrt{3} + 2\sqrt{5})(\sqrt{5} + \sqrt{3}) = 13 + 3\sqrt{15}$)

exponent: A superscript in mathematics that denotes repeated multiplication (e.g., 4^3 means $4 \times 4 \times 4$; the exponent is 3); sometimes referred to as a power or an index

exponential function or relation: A relation between two variables that can be represented by an equation with an exponential expression (e.g., $y = 2^x – 5$)

exponential curve: The shape of a graph of an exponential relationship



extrapolate: To estimate a value that is beyond the range of given data by following a pattern or trend

extreme: The greatest or least value in a set of data

F

face value of a share: The printed price on a share or stock; sometimes called nominal value or issue price. Share-holders can sell their stocks at a market price that is lower than the face value (sold at a discount), equal to the face value (sold at par), or more than the f ace value (sold at a premium).

factor or factorise: To express a number or algebraic expression as the product of two or more numbers or algebraic expressions. The numbers or algebraic expressions in such a product are also called factors (e.g., $a^2 – b^2$ can be factored as $(a + b)(a – b)$, the factors are $(a + b)$ and $(a – b)$; 24 can be factored as 8×3 or $2 \times 2 \times 2 \times 3$; the factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24)

factored form of a quadratic relation: A degree 2 polynomial that can be written as the product of two linear factors and a number (e.g., $(x – 1)(x + 2)$, $2x(x – 3)$, and $5(x + 2)(x – 1)$)

favourable outcome: The desired result when calculating a probability (e.g., that a spinner will stop on green instead of red)

finite differences: In a table of values where the x-coordinates are evenly spaced, the first differences are the differences between consecutive y-coordinates. The second differences are the differences between consecutive first differences, and so on. For a linear function the first differences are constant, while for a quadratic function they are not. The second differences are constant for a quadratic function.

first-degree polynomial: A polynomial in which the exponent of the variable is 1 (e.g., $8x – 17$)

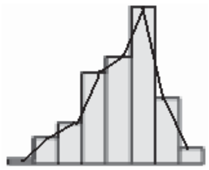
first differences: See *finite differences*

five-number summary (or **5-number summary**): The minimum value, the lower quartile (Q1), the median (Q2), the upper quartile (Q3), and the maximum value of a set of data used to divide a data set into four intervals, each containing 25% of the data values; the 5-number summary is used to frame the construction of a box plot

frequency: The number of times a data value or range of data values occurs in a data set

frequency of compounding: How often compound interest is calculated and applied to an investment or loan (e.g. daily, monthly, quarterly, semi-annually, and annually)

frequency polygon: A frequency polygon is created by using line segments to join the midpoints of the tops of the bars in a histogram



frequency table: A table that organizes a set of data into intervals and indicates the number of times data values occur in each interval

Age	Frequency
0 – 11	50
11 – 22	300
22 – 33	250
33 – 44	400
44 – 55	550

function: A relation between two variables such that for any value of the independent variable (usually x), there is only one corresponding value of the dependent variable (usually y)

function notation: A function rule or equation written using a form such as $f(x)$ for the dependent variable (e.g., $f(x) = 6x – 7$, $f(w) = 3w – 2$, and $g(a) – 2a^2 + 5$ are in function notation)

function rule: An equation that describes how the values of the dependent variable (output values) are calculated from the values of the independent variable (input values)

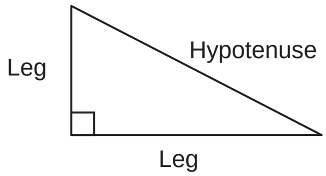
H

height: The perpendicular distance from the base of a geometric shape to its highest point

hexagon: A six-sided polygon

histogram: A graph with vertical or horizontal bars that show frequencies of data organized into intervals, sometimes called bins. The intervals line up side by side without gaps on the scale because the data values are continuous.

hypotenuse: The side opposite the right angle in a right triangle



identities: Equations that are true for all values of a variable

identity matrix: A matrix by which you can multiply another matrix, A, and the product will be A; it is like multiplying a number by 1

image: A new shape that is created when a shape undergoes a geometric transformation

incircle: The circle inside the triangle that touches each side of the triangle only once See *circumcentre* for diagram

incentre: The centre of the incircle of a triangle, found at the intersection of any two angle bisectors See *circumcentre* for diagram

independent events: In probability, when the outcome of one event does not affect or depend on the other event, the two events are independent events

independent variable: In a function, the input values upon which the output values of the other variable depend; often represented by x and plotted on the horizontal axis

inductive reasoning: An explanation that is based on examples that suggest something might be always true

inequality: A mathematical statement in which the value on the left side is compared with the value on the right side (greater than, greater than or equal to, less than, or less than or equal to) (e.g., the linear inequality $5n + 4 > 39$ means that 4 more than the product of 5 and a number is greater than 39)

integers: The set of whole numbers and their opposites (…, –2, –1, 0, 1, 2, …)

intercept: The distance from the origin of the x-y-plane to a point at which the graph meets or crosses the x- or y-axis (the x-intercept or y-intercept); the value of the y- or x-coordinate where the graph meets or crosses the axis

interest period: The period of time during which compound interest is calculated, also called the compounding period (e.g., if 5% p.a. interest compounded monthly is applied to a loan, the interest period is monthly and the interest rate for the interest period will be 5% ÷ 12 = 0.4%)

interior angle: One of the angles inside a polygon at a vertex (e.g., a square has four interior angles)

interpolate: To estimate or predict a value between two given elements of data by following a pattern or trend

intersection point: The point shared by two or more graphs. The values of the coordinates make the equation of each graph true.

interval: A range of values, often used in creating histograms (e.g., 0–10 is the interval from 0 to 10). Stem and leaf plots have intervals based on place value. In stem and leaf plots and histograms, the intervals are equal. In box and whisker plots, each interval contains 25% of the data values and the intervals are usually not equal.

inverse operation: An operation that undoes another operation (e.g., addition is the inverse of subtraction)

irrational number: A number that cannot be written as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$ (e.g., π, $\sqrt{5}$)

isosceles triangle: A triangle with at least two sides of equal length; some people define an isosceles triangle as having exactly two equal sides.

L

lateral surface: The surface of a prism, cylinder, pyramid, or cone that does not include the base(s)

left skewed: See *skewed distribution*

legs of a right triangle: The two side lengths of a right triangle that are perpendicular to each other and that are not the hypotenuse See *hypotenuse*

like terms: Terms of a polynomial that have the same variables and exponents but may have different numerical coefficients (e.g., in $3x^2 + 2x + 6x + 5$, the like terms are $2x$ and $6x$, which means that their sum or difference can be simplified); with radicals, like terms have the same radical (e.g., $3\sqrt{5} + 4\sqrt{5}$ can be simplified to $7\sqrt{5}$ because they are like terms, while $3\sqrt{5} + 3\sqrt{2}$ cannot)

line of best fit: The straight line that best models the relationship between two variables in a scatter plot of data

line of reflection: See *line of symmetry*

line of symmetry: A line that goes through a shape such that, if a shape were folded on the line, one half of the shape would match the other

linear equation: A degree 1 equation representing a linear relation between two variables, often in the form $y = \text{m}x + \text{b}$

linear inequality: See *inequality*

linear function or relation: A relation between two variables that appears as a straight line when graphed and is represented by a first-degree equation involving two variables (e.g., $y = 2x + 1$ and $f(x) = 3x – 4$)

linear system: See *system of linear equations*

loop of a network: An edge that begins and ends at the same vertex. A loop can be a directed edge or not

lower quartile: Also referred to as Q1 See *box and whisker plot*

M

mapping notation: A way to show a transformation rule by showing an original point and its image, connected by an arrow (e.g., $(x, y) \rightarrow (x + 2, y + 3)$)

marked price: The original selling price of an item. A discount is often applied to a marked price to bring the price down to a lower selling price to encourage a purchase.

markup amount: The amount the price of an item is increased above the cost price to allow the seller to make a profit

market price of a share: The value at which a share or stock is sold by the shareholder See *share* and *face value*

matrix (plural is matrices): A rectangular array of elements, usually numbers

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 7 & 3 & 1 & 8 \\ 5 & 9 & 1 & 4 \end{bmatrix}$$

maximum: The greatest value taken by a dependent variable. In a data set, the maximum value is the last value when the data values are in order from least to greatest. In a graph of a quadratic relation, the maximum is at the vertex of a parabola that opens downward.

mean: The sum of a set of numbers divided by the number of elements in the set; often called the average

median: **1.** A line that joins a vertex of a triangle to the midpoint of the opposite side **2.** The middle number of a set of data arranged in order. If there is an even number of numbers in the set, the median is the mean of the two middle numbers. In a box plot, the median is sometimes referred to as Q2.

midpoint: The point that divides the line segment into two equal parts

minimum: The least value taken by a dependent variable. In a data set, the minimum value is the first value when the data values are in order from least to greatest. In a graph of a quadratic relation the minimum is at the vertex of a parabola that opens upward.

mirror symmetry: Also called reflectional symmetry or line symmetry See *symmetry*

mixed radical: A number that is the product of an integer and a radical (e.g., $3\sqrt{5}$ is a mixed radical) See *entire radical*

mode: The data value(s) that occurs most often in a set of data; there can be one, more than one, or no mode.

monthly: Once a month

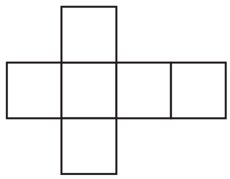
N

negative correlation: In a relationship between variables, as one variable increases, the other variable decreases

negative dilatation of a quadratic function: See *dilatation of a quadratic function*

negatively skewed: See *skewed distribution*

net: A 2-D pattern you can fold to create a 3-D shape. This is a net for a cube:



network: A set of connected "paths" consisting of edges (paths) and vertices (where paths meet) See *edges of a network*

nonlinear relation: A relationship between two variables that does not fit a straight line when graphed

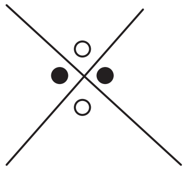
normal curve: See *normal distribution*

normal distribution: A set of data that is symmetrical about a line passing through the interval with the greatest frequency; sometimes described as a mound-shaped distribution See *data distribution*

numerical coefficient: See *coefficient*

O

opposite angles: Non-adjacent congruent angles that are formed by two intersecting lines as indicated by the symbols below:



opposite side: In trigonometry, the leg of a right triangle that is across from the angle being used See *adjacent side*

obtuse triangle: A triangle in which one angle is obtuse, that is, an angle greater than 90º and less than 180º

order of operations: Rules describing the sequence to use when evaluating an expression:

1 Evaluate within brackets

2 Calculate exponents and square roots

3 Divide and multiply from left to right

4 Add and subtract from left to right

order of turn symmetry: The number of times a shape looks the same during a complete rotation (e.g., a square has turn symmetry of order 4)

ordered pair: A pair of numbers in which the order is important. The coordinates of a point in the *x*-*y*-plane form an ordered pair (e.g., the ordered pairs (3, 5) and (5, 3) represent points in different positions)

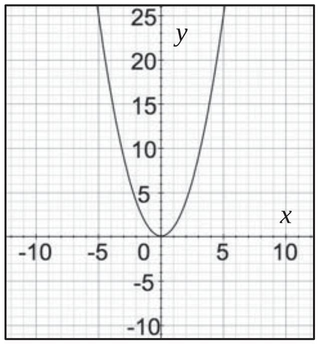
origin: The intersection of the axes on a coordinate grid represented by the ordered pair (0, 0)

orthocentre: The point of intersection of the altitudes of a triangle

outcome: A single event that is proposed as the result of a probability experiment (e.g., One outcome of rolling a die is the number 1. The five other possible outcomes are 2, 3, 4, 5 and 6.)

P

parabola: An open curve shaped like the graph of $y = x^2$; the graph of a quadratic relation. A parabola can open downward, as shown below, or upward



A parabola — the graph of $y = x^2$

parallel: Always the same distance apart (e.g., railway tracks are parallel)

parallelogram: A quadrilateral with pairs of opposite sides that are parallel

per annum (**p.a.**): Annual interest rate. Simple and compound interest percentages are always stated on an annual basis regardless of how frequently they are calculated. The per annum interest rate used to calculate the interest rate that is applied to each interest period (e.g., if an investment earns 5% p.a. compounded quarterly, it earns 1.25% interest (5% ÷ 4 = 1.25%) on the growing balance every three months).

percent: A special ratio that compares a number to 100 using the symbol %; also called percentage

percent commission: The percentage of sales that a salesperson earns on what he or she sells; also called rate of commission

percent discount: The percentage by which the marked price of an item is decreased to reach a lower selling price. A discount is applied to bring the price down to encourage a purchase.

percent markup: The percentage by which the cost price of an item is increased to reach a higher marked price and allow the seller to make a profit See *markup* and *cost price*

perfect cube: A whole number whose cube root is an integer (e.g., 8 is a perfect cube because $\sqrt{8}$ = 2 or –2)

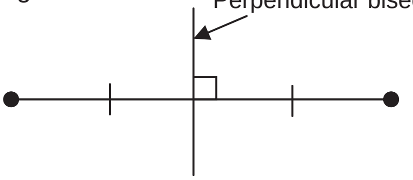
perfect power: A whole number whose root is an integer. Perfect squares and perfect cubes are perfect powers.

perfect square: A whole number whose square root is a whole number (e.g., 16 is a perfect square because $\sqrt{16}$ = 4)

perimeter: **1.** The boundary of a 2-D shape **2.** The length of the boundary. The circumference of a circle is a special perimeter.

perpendicular: At a right angle (e.g., the base of a triangle is perpendicular to the height of the triangle); or, a line that intersects a line segment at 90º

perpendicular bisector: A line segment that is perpendicular to another line segment and that bisects, or passes through the midpoint of, that line segment



π (pi): The value of the circumference of any circle divided by its diameter. It is an irrational number with a value of 3.141592654 … or about 3.14, rounded to two decimal places

plane of symmetry: The imaginary surface that cuts or divides a 3-D shape into congruent and matching opposite halves

plane symmetry: Reflectional or mirror symmetry of 3-D shapes

polygon: A closed 2-D shape formed by three or more line segments in a plane. Examples include quadrilaterals, hexagons, and decagons

polyhedron (plural is polyhedra): A 3-D shape that has faces that are all polygons

polynomial: An algebraic expression consisting of one or more terms with variables raised to whole-number powers, usually in the form a + bx + cx^2, where a, b, c are numbers

positive correlation: In a relationship between variables, as one variable increases, the other variable also increases

positively skewed: See *skewed distribution*

possible outcome: A single result that can occur in a probability experiment (e.g., when tossing a coin, getting Tashi Ta-gye is a possible outcome)

power: A numerical expression that shows repeated multiplication (e.g., the power 5^3 is a shorter way of writing 5 × 5 × 5). A power has a base and an exponent: the exponent tells the number of equal factors there are in a power. Sometimes the exponent is also called the power or index.

3 is the exponent of the power

$$5^3 = 125$$

5 is the base of the power

precision: The level of exactness in measuring. The more significant digits used or the smaller the unit, the more precise the measurement (e.g., 31 mm or 3.1 cm is more precise than 3 cm)

prime number: A whole number with exactly two different factors, 1 and itself (e.g., 17 is a prime number since its only factors are 1 and 17; 1 is not prime since there is only one factor)

principal: The original amount that is invested in the case of an investment, or borrowed in the case of a loan

prism: A polyhedron with two parallel and opposite congruent bases. The other faces are parallelograms or rectangles in the case of a right prism. The shape of the base of the prism determines the name of the prism (e.g., pentagon-based prism).

probability: A number from 0 (will never happen) to 1 (certain to happen) that represents how likely it is that an event will happen

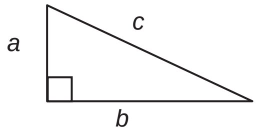
proof: A logical procedure using deductive reasoning to show or prove that a conjecture or statement is always true

prove: Show that a conjecture or statement is true using deductive reasoning

proportion: An equation of two equivalent ratios (e.g., $\frac{1}{2} = \frac{x}{50}$)

pyramid: A polyhedron with a polygon for a base; the other faces are triangles that meet at a single vertex (the apex)

Pythagorean theorem: The square of the length of the hypotenuse of a right triangle (the longest side) is equal to the sum of the squares of the lengths of the other two sides: $a^2 + b^2 = c^2$



Q

Q1, Q2, Q3: Values used to create a box plot See *box and whisker plot*

quadratic function or relation: A relation between two variables that can be represented by the quadratic equation $y = ax^2 + bx + c$. The graph of a quadratic relation is a parabola

quadratic equation: An equation that represents a quadratic function when the function has a specific value. The quadratic function $f(x) = ax^2 + bx + c$ can be used to solve a quadratic equation in the form $ax^2 + bx + c = d$.

quarterly: Four times a year, or every three months

quartile: **1.** One of three points (Q1 or lower quartile, Q2 or median, and Q3 or upper quartile) that divide a set of data into four equal parts **2.** One of four groups of a set of data that each contain an equal number of data values See *box and whisker plot*

R

radical: The root of a number written using a radical or root sign $\sqrt{\ }$ (e.g., $\sqrt{5}$ is a radical that means the square root of 5, $\sqrt[3]{10}$ is a radical that means the cube root of 10)

radical expression: A combination of one or more radical terms that may involve numbers and/or variables (e.g., $2\sqrt{5} + \sqrt[3]{10}$ or $\sqrt{b^8} - 2\sqrt{b^8} + \sqrt{b}$)

radius (plural is radii): A line segment that joins the centre of a circle to any point on its circumference; also the length of the line segment

random: A number that has been chosen randomly has been selected without any particular pattern or rule

range: The difference between the extremes (minimum and maximum) of a set of data

rate of commission: See *percent commission*

rational number: A number that can be expressed as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$

reflection: A transformation of a shape that produces a mirror image of the shape with respect to a line called the reflection line (in 2-D) or a plane called the plane of symmetry (in 3-D); also called a flip

reflection of a quadratic function: A negative dilatation See *dilatation of a quadratic function*

reflection line: See *reflection*

reflectional symmetry: Also called mirror, line, or plane symmetry See *symmetry*

regular polygon: A polygon with all sides and all angles congruent (e.g., a regular pentagon has all five side lengths equal and all five angles equal)

relation: A property that connects two sets of numbers or two variables. A relation can be expressed mathematically as a table of values, a graph, or an equation.

rhombus: A parallelogram with all sides congruent. A square is a special rhombus that has interior right angles.

right skewed: See *skewed distribution*

right triangle: A triangle with one right angle

rigidity: The inherent ability of a shape to maintain its shape when pushed

rise: The vertical distance between two points used with run to calculate slope See *slope*

roots of a quadratic equation: See *zeros*

rotation: A transformation in which each point in a shape moves about a fixed point (the centre of rotation) through the angle of rotation

rotational symmetry: Sometimes called turn symmetry. A shape has turn symmetry when it is turned or rotated around a fixed point called the centre of rotation (in 2-D) or a line called the axis of symmetry (in 3-D) and it looks the same more than once during one complete rotation

row matrix: A matrix with only one row and any number of columns

run: The horizontal distance between two points used with rise to calculate slope See *slope*

S

sample space: All possible outcomes in an experiment or probability situation (e.g., the sample space for rolling a die contains six possible outcomes: 1, 2, 3, 4, 5, and 6)

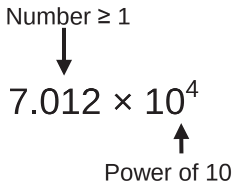
scalar: A magnitude or amount not associated with any direction. When a matrix is multiplied by a scalar, every element in the matrix is multiplied by the scalar.

scale factor: If two triangles are similar, corresponding side lengths relate by the same ratio, or scale factor

scalene triangle: A triangle with no congruent sides

scatter plot: A graph of isolated points on a coordinate grid. The graph can be used to try to determine a relationship between the two variables that are being graphed.

scientific notation: A way of writing a number as a decimal greater than or equal to 1 and less than 10, multiplied by a power of 10 (e.g., 70,120 is written as 7.012×10^4)



second-degree polynomial: A polynomial for which the sum of powers of at least one variable is 2 for at least one of the terms (e.g., $-3x^2 - 2x + 5$, $6y^2$, and $3xy + 5x$ are all second-degree polynomials)

second differences: See *finite differences*

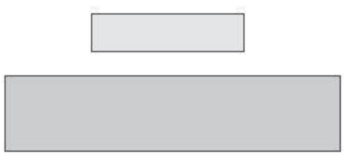
selling price: The price for which an item is sold, taking into account any discounts that might apply

semi-annually: Twice a year, or every six months

share: Partial ownership and investment in a company, sometimes called stock. People who buy shares or stocks in a company are called shareholders. The company uses the money invested to support the business.

significant figures: The number of digits in a number that are considered to play a role in determining the precision of a measurement (e.g.,
$\underline{2}0$ cm has one significant figure,
$\underline{22}$ cm has two significant figures, and
$\underline{202}$ m has three significant figures); sometimes called significant digits

similar: Identical in shape, but not necessarily the same size. The symbol for similarity is ~ (tilde), as in $\triangle ABC \sim \triangle DEF$. All congruent shapes are similar but not all similar shapes are congruent.



These rectangles are similar

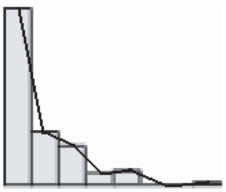
similar triangles: Triangles in which the pairs of corresponding sides are proportional

similarity: The property shared by geometric shapes that are identical in shape but not necessarily the same size

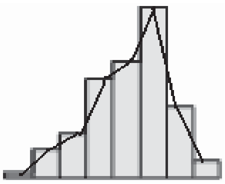
simple interest: Interest charged on a loan (or earned on an investment) for which the interest is always calculated as a percentage of the original principal, or original amount borrowed (or invested)

simplify a radical: Express a radical in an equivalent form such that the least possible value is left under the radical sign (as a mixed radical) or the root sign is gone altogether (as a whole number) (e.g., $\sqrt{300}$ can be simplified to $10\sqrt{3}$ and $\sqrt{100}$ can be simplified to 10)

skewed distribution: The intervals that have the greatest frequency are near one end of the histogram

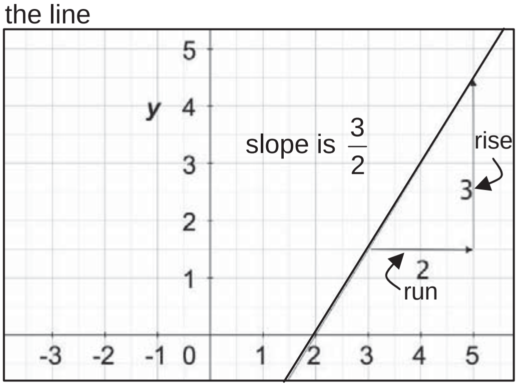


Right, or positively skewed



Left or negatively skewed

slope: A measure of the steepness of a line, expressed as the rise (vertical distance) divided by the run (horizontal distance) between any two points on the line



slope and *y*-intercept form: The equation of a straight line written as $y = mx + b$, where m is the slope of the line and b is the *y*-intercept; sometimes expressed as $y = mx + c$

solution: The values of variables that make an equation or inequality true (e.g., in the equation $5n + 4 = 39$, the solution is $n = 7$ because $5(7) + 4 = 39$)

solution to a quadratic equation: See *zeros*

solution of a system of linear equations: The values of the variables that make all equations in the system true; when all the equations in the system are graphed, the solution is the intersection point of the graphs

speed: The rate at which a moving object changes position with time (e.g., a sprinter who runs 100 m in 10 s has an average speed of 100 m/10 s or 10 m/s)

sphere: A 3-D shape in which every point on the surface is the same distance from a single point, the centre

square: **1.** A rectangle with equal sides **2.** To multiply a number or an expression by itself

square matrix: A matrix with the same number of rows as columns (e.g., a 3-by-3 matrix is a square matrix)

square number: A whole number that is a perfect square of another whole number (e.g., 1 is the square of 1, 4 is the square of 2, 9 is the square of 3, and so on)

square root: A number that multiplies by itself to result in another number (e.g., the square roots of 49 are 7 ($7 \times 7 = 49$) and –7 ($-7 \times (-7) = 49$)); in symbols, $\sqrt{49} = 7$; the positive square root is called the principal square root

standard form (of a linear relation): The equation of a straight line when written as $Ax + By = C$; sometimes written as $ax + by = c$ or $ax + by + c = 0$

standard form of a quadratic relation: A degree 2 polynomial with one, two, or three terms (e.g., $2x^2 + 4x + 1$, $x^2 + x - 2$, or $3x^2$)

stem and leaf plot: An organization of numerical data into intervals based on place values (e.g., the circled leaf in this stem-and-leaf plot represents the number 258)

Stem	Leaves
24	1 5 8
25	2 2 3 4 7 8 9
26	0 3
27	
28	8

stock: See *share*

supplementary angles: Two angles whose sum is 180º

surface area: The sum of the areas of all the faces and curved surfaces of a 3-D shape; sometimes referred to as total surface area

symmetry: **1.** Mirror or reflectional symmetry: when a 2-D shape is folded or reflected across a line (the reflection line), the two sides of the shape match **2.** Plane symmetry: when two halves of a 3-D shape are reflections of each other in the plane of symmetry **3.** Turn or rotational symmetry: when a 2-D or 3-D shape resembles the original shape more than once in a full turn

system of linear equations: A set of two or more equations that represent linear relations between the same two variables

T

table of values: An arrangement of numerical values in a chart or table that represents a relationship between two variables

term: Part of a numerical or algebraic expression, separated by addition or subtraction signs (e.g., $4x - 7y + 2z$ has three terms)

theorem: Something that has been proven by deductive reasoning to always be true

three-dimensional shape: A shape that occupies space in three dimensions; also called 3-D (e.g., a triangle-based prism is a 3-D shape)

transformation: Changing a shape according to a rule. Transformations include translations, rotations, reflections, and dilatations.

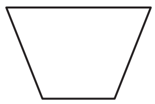
translation: A transformation of a shape in which each point moves the same distance and in the same direction; also called a *slide*

translation of a quadratic function: A vertical translation of a function transforms every point in the function's graph according to the mapping $(x, y) \rightarrow (x, y + v)$. If v > 1, the parabola moves up and if v < 1, the parabola moves down. A horizontal translation transforms every point using the mapping $(x, y) \rightarrow (x + h)$. If h > 1, the parabola moves to the right and if h < 1, the parabola moves to the left.

translation rule: A rule that determines or describes the effect of a transformation on any shape

trapezium: See *trapezoid*

trapezoid: A quadrilateral in which one pair of opposite sides are parallel



An isosceles trapezoid

trend: A pattern of general direction or movement, often for a variable that is measured against time; represented by the line or curve of best fit in a scatter plot

trigonometric or **trig ratios**: See *Trig Table and References* on **page 380**

trigonometric identities: See *Trig Table and References* on **page 380**

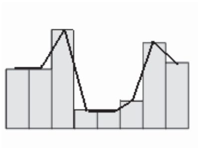
trinomial: A polynomial with three terms (e.g., $4x - 7y + 2z$ and $5x^2 + 2x - 3$ are trinomials)

turn symmetry: See *rotational symmetry*

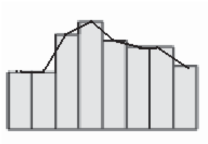
two-dimensional shape: A shape that lies in one plane and has only two dimensions; also called 2-D (e.g., a triangle is a 2-D shape)

U

U-shaped distribution: A distribution that peaks at both ends of the range and sometimes is described as bimodal; its frequency polygon creates a U-shaped curve



uniform distribution: A distribution where each bin has a similar frequency; its frequency polygon resembles a straight line that has very little slope



upper quartile: Also referred to as Q3 See *box and whisker plot*

V

variable: A letter or symbol, such as *a*, *b*, *x*, or *n*, that represents a number (e.g., in the formula for the area of a rectangle, $A = l \times w$, the variables *A*, *l*, and *w* represent the area, length, and width of the rectangle); in an experiment, you measure, control, or manipulate variables

vectors: Arrows that show movement in terms of direction and distance; used with bearings as a set of directions

verify: Use inductive reasoning to show that a conjecture is true

vertex form (of a quadratic relation): A degree 2 polynomial that can be written as the numerical multiple of the square of a linear term with an *x*-coefficient of 1 (e.g., $(x-2)^2 + 3$, $(x-1)^2 - 1$, and $3(x-2)^2$)

vertex of a parabola: The point where a parabola intersects its axis of symmetry; the maximum or minimum point

vertex of a network: A point in a network where two or more edges meet

volume: The amount of space occupied by an object

Y

yield percentage: The ratio, as a percentage, of the money received in dividends (from an investment in shares) to the money invested; also called percentage return

X, Y, and Z

***x*-intercept**: See *intercept*

***y*-intercept**: See *intercept*

zero product rule: If a product of factors is zero, then at least one of the factors must be zero (e.g., for a quadratic equation, if $(ax + b)(cx + d) = 0$, then $ax + b = 0$ or $cx + d = 0$)

zeros of a quadratic equation: Possible values of *x* when the quadratic function $f(x) = 0$; also called the solutions or roots. When the function is graphed as a parabola, the zeros are the *x*-intercepts (if the parabola crosses the x-axis). It is possible for there to be two, one, or no solutions to a quadratic equation.

ANSWERS

UNIT 1 MATRICES AND NETWORKS pp. 1–38

Getting Started — Skills You Will Need p. 1

1. a) 2 **b)** –6 **c)** –7 **d)** –24

2. a) –4 **b)** –2 **c)** 13 **d)** –12

3. a) –4 **b)** –12 **c)** 40 **d)** 72

4. a) 2.1 **b)** 0.5 **c)** 2 **d)** 40

5. a) 0.21 **b)** 36 **c)** 10.08 **d)** 18

1.1.1 Introducing Matrices p. 4

1. A

2. *Sample response*

a) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

3. a) 9-by-2
b) The only information needed is the plan number (column 1) and the year (column 2).

4. *Sample response*:

$$\begin{array}{c c} & \begin{array}{cccc} F1 & F2 & C1 & C2 \end{array} \\ \begin{array}{c} F1 \\ F2 \\ C1 \\ C2 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

5. a) 1-by-5
b) A 5-by-1 matrix could show the same information in 1 column with 5 rows instead of 1 row with 5 columns.

6. a) the number of items (ghos and kiras) each woman made
b) the total number of ghos or kiras that were made by all three women

7. It is a parallelogram.

8.

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 2 & 3 & 2 & 5 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

9. The numbers in a multiplication table are in a 10-by-10 square array.

1.1.2 Adding and Subtracting Matrices pp. 6–7

1. A

2. a) $\begin{bmatrix} 5 & 1.2 & 5.9 \\ 0 & 2 & 1.1 \end{bmatrix}$ **b)** $\begin{bmatrix} -7.7 & -12 \\ 6.5 & 2 \end{bmatrix}$ **c)** $\begin{bmatrix} 13 & -13.3 \\ 3.9 & 1.4 \\ -8.8 & 8.5 \end{bmatrix}$

3. a) $\begin{bmatrix} 15.2 & 12.9 & 10.6 \\ 11.9 & 12.4 & 11.6 \\ 9.9 & 9.3 & 7.5 \end{bmatrix}$

The numbers tell how much higher the higher average temperature was in each place for each month.

1.1.2 Adding and Subtracting Matrices [Cont'd] pp. 6–7

3. b) $\begin{bmatrix} -15.2 & -12.9 & -10.6 \\ -11.9 & -12.4 & -11.6 \\ -9.9 & -9.3 & -7.5 \end{bmatrix}$

The numbers tell how much the temperature went down in each place for each month.

4. $\begin{bmatrix} 3 & 3 & 2 & 2 \\ 1 & 5 & 4 & 1 \end{bmatrix}$

5. a) *Sample response*:

$A = \begin{bmatrix} 3 & 1 & 0 & 2 & 4 \\ 3 & 1 & 2 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & 1 & 0 \end{bmatrix}$

b) *Sample response*:

$A - B = \begin{bmatrix} 3 & 0 & -2 & 3 & 1 \\ 3 & -1 & -2 & 4 & 1 \end{bmatrix}$

c) *Sample response*:

$B - A = \begin{bmatrix} -3 & 0 & 2 & -3 & -1 \\ -3 & 1 & 2 & -4 & -1 \end{bmatrix}$

d) The signs for the corresponding elements are opposite; the opposite of $a - b$ is $b - a$, e.g., 6 – 4 = 2 but 4 – 6 = –2.

6. $\begin{bmatrix} 3 & 10 & 6 & 7 \\ 1 & 20 & 3 & 7 \end{bmatrix}$

7. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

8. *Sample response*:

a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 10 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -6 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 11 & 3 \\ 2 & 3 \\ 5 & 2 \end{bmatrix}$ **c)** $\begin{bmatrix} 12 & 3 \\ 4 & 8 \\ -1 & 2 \end{bmatrix}$

8. d) $\begin{bmatrix} 11 & 3 \\ 2 & 7 \\ -4 & 1 \end{bmatrix}$ **e)** $\begin{bmatrix} 12 & 3 \\ 4 & 8 \\ -1 & 2 \end{bmatrix}$

f) You get the same final matrix because the order does not matter when you add numbers or matrices.

9. *Sample response*:

a) $A = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} -9 & -1 & -10 & 5 \\ 14 & -4 & 28 & -4 \end{bmatrix}$;

$A = \begin{bmatrix} -8 & 0 & -12 & 1 \\ 8 & -8 & 15 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 4 & 0 & 4 & 4 \\ 9 & 5 & 13 & -2 \end{bmatrix}$

b) $A = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 9 & 1 & 10 & -5 \\ -14 & 4 & -28 & 4 \end{bmatrix}$;

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 8 & -5 \\ -15 & 4 & -27 & 4 \end{bmatrix}$

10. *Sample response*:
If you have two matrices showing the number of tourists from five different countries in two different years, you could subtract them to find the change in the number of visitors from each country over the two years.

11. To add or subtract matrices, you add or subtract elements that are in the same position in both matrices, so you need the matrices to be the same size.

1.1.3 Multiplying a Matrix by a Scalar pp. 9–10

1. a) $\begin{bmatrix} 6 & 1.2 \\ 2.4 & 3 \end{bmatrix}$ **b)** $\begin{bmatrix} 16 & 10\frac{2}{3} \\ 6\frac{2}{3} & 10 \\ 6 & 20\frac{2}{3} \\ 12 & 14 \end{bmatrix}$

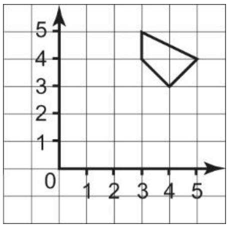
1. c) $\begin{bmatrix} 4.8 & 0.72 & 2.4 \\ 2.1 & 0.36 & 3 \end{bmatrix}$ **d)** $\begin{bmatrix} 6 & 15 \\ -1.2 & -2.7 \end{bmatrix}$

2. a) $\begin{bmatrix} 4 & 1 & -4 \\ 20 & 1 & -4 \\ -5 & 1 & 21 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & -2 & -7 \\ 5 & 3 & -2 \\ -5 & 8 & -2 \end{bmatrix}$

2. c) $\begin{bmatrix} 4 & 2 & -2 \\ 22 & 0 & -4 \\ -4 & -2 & 26 \end{bmatrix}$ **d)** $\begin{bmatrix} -4 & 1 & 8 \\ -16 & -3 & 4 \\ 7 & -7 & -11 \end{bmatrix}$

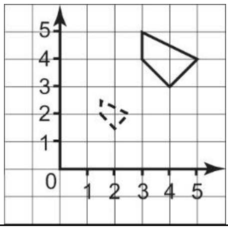
3. a) There are 4 columns in the matrix of coordinates so I know there are 4 vertices, and any shape with 4 vertices is a quadrilateral.

b)



c) $\begin{bmatrix} 2 & 1.5 & 1.5 & 2.5 \\ 1.5 & 2 & 2.5 & 2 \end{bmatrix}$

d) The new shape is smaller and similar.



4. $\begin{bmatrix} 2.94 & 5.15 & 0.74 & 0.44 \end{bmatrix}$

5. $\begin{bmatrix} 16{,}454 & 100{,}748 & 75{,}949 & 37{,}847 \end{bmatrix}$

$\begin{bmatrix} 16{,}800 & 10{,}2864 & 77{,}544 & 38{,}642 \end{bmatrix}$

6. a) 5

b) $\begin{bmatrix} -4 & 6 & 8 \\ 20 & -18 & 16 \\ 38 & 16 & -14 \end{bmatrix}$

c) *Sample response*: $\frac{1}{4} \times \begin{bmatrix} 3 & 14 \\ -5 & 8 \end{bmatrix}$

7. *Sample response*:
To find the total cost of phone use each day for each person, you would multiply each number of minutes by the rate per minute.

8. You multiply every element in the matrix by the scalar, so it does not matter how many elements there are or how they are arranged.

1.1.4 Multiplying Matrices p. 14

1. a) A × B, B × C, B × D, C × D, D × A, D × B

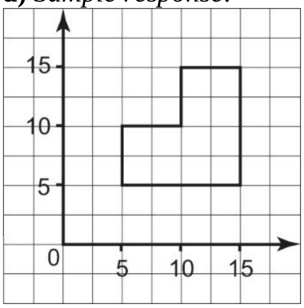
b) A × B = $\begin{bmatrix} 4 & 2 & 6 \\ 6 & 7 & 10 \end{bmatrix}$; B × C = $\begin{bmatrix} 10 & 1 & 6 \\ 2 & 4 & 5 \end{bmatrix}$

B × D = $\begin{bmatrix} 5 & 6 \\ 12 & 2 \end{bmatrix}$; C × D = $\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 2 & 2 \end{bmatrix}$

D × A = $\begin{bmatrix} 2 & 0 \\ 6 & 0 \\ 6 & 2 \end{bmatrix}$; D × B = $\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 0 & 8 & 2 \end{bmatrix}$

2. a) *Sample response*:
4-by-1 × 1-by-2, or 4-by-3 × 3-by-2, or 4-by-anything × anything-by-2 as long as both "anythings" are the same.
b) The number of columns in the first matrix has to match the number of rows in the second, and there must be 4 columns in the first matrix and 2 rows in the second matrix for the product matrix to be 4-by-2.

3. a) *Sample response*:



B = $\begin{bmatrix} 5 & 15 & 15 & 10 & 10 & 5 \\ 5 & 5 & 15 & 15 & 10 & 10 \end{bmatrix}$

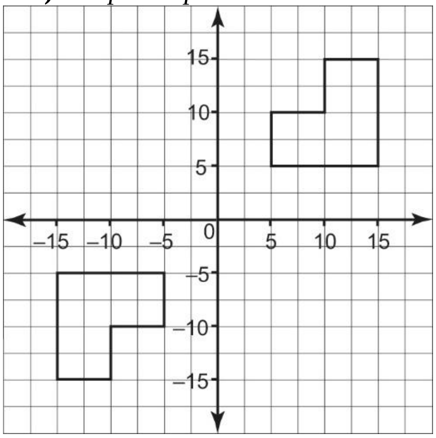
b) *Sample response*:

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 5 & 15 & 15 & 10 & 10 & 5 \\ 5 & 5 & 15 & 15 & 10 & 10 \end{bmatrix}$

$= \begin{bmatrix} -5 & -15 & -15 & -10 & -10 & -5 \\ -5 & -5 & -15 & -15 & -10 & -10 \end{bmatrix}$

1.1.4 Multiplying Matrices [Cont'd] p. 14

3. c) *Sample response*:



d) It turned, or rotated, a half turn about the origin.

4. $\begin{bmatrix} 50 & 100 & 500 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 6 \end{bmatrix} = \begin{bmatrix} 5500 \end{bmatrix}$;

Nu 5500 in notes

5. a) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

b) You get the same matrix you started with.

6. a) You cannot multiply a 3-by-2 matrix by a 1-by-3 matrix since the number of columns in the first matrix does not match the number of rows in the second one, but you can multiply a 1-by-3 matrix by a
3-by-2 matrix.

6. b) *Sample response*:

No; $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 16 \\ 2 & 10 \end{bmatrix}$

but $\begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 6 \end{bmatrix}$

7. a) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

b) A **×** B; A would be first so the number of columns in the first matrix, 3, matches the number of rows in the second matrix, 3.

8. a) *Sample response*:
If you had a number of prices in ngultrums to exchange for U.S. dollars, you would multiply a matrix with the prices by a scalar, the exchange rate.
b) *Sample response*:
If you were trying to figure out several students’ final grades, you could multiply a matrix with their marks for each part of their final grade by a matrix with the percent weighting for each part.

9. a) 3-by-4
b) Multiply the numbers in the 2nd row of the first matrix by the matching numbers in the 3rd column of the second matrix and add the products.

CONNECTIONS: The Seven Bridges of Konigsberg p. 16

1. It is not possible.

2. No; because there are 4 odd vertices and you need to have two odd vertices, so you enter on one of them and exit on the other.

1.2.2 Describing a Network With a Matrix p. 22

1. a) $\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$

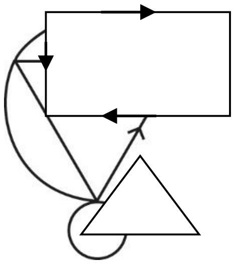
b) $\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$

c) $\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

2. *Sample response*:

a)

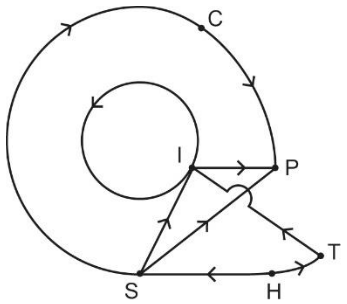
b) **c)**



d)

3. There are no loops.

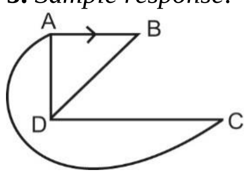
4. a) *Sample response*:



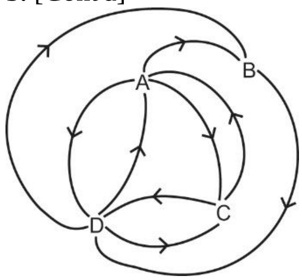
b)

$$\begin{array}{c|cccccc|} & C & I & P & S & H & T \\ C & 0 & 0 & 1 & 0 & 0 & 0 \\ I & 0 & 1 & 1 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 \\ S & 1 & 1 & 1 & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 1 & 0 & 1 \\ T & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

5. *Sample response*:

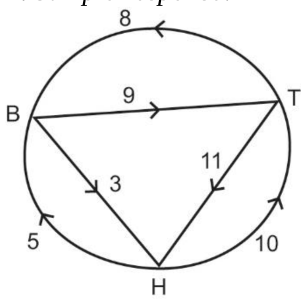


5. [Cont'd]



6. *Sample responses*:
- You can put your vertices in different locations.
- Wherever you have a curved line, you can make a straight line or vice versa.
- You can use two uni-directional arrows instead of one bi-directional path.

7. *Sample response*:



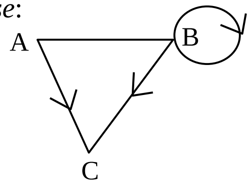
8. a) by the number of rows or columns
b) by numbers other than 0 where the row and column names are the same (along the principal diagonal)
c) wherever there is a 0 in both entries, from A to B and from B to A, where the row for one vertex intersects the column for the other
d) the total of the numbers in the row for that vertex
e) the total of the numbers in the column for that vertex

9. *Sample response*:
If there are many 0s, the graph is not as complicated as when there are many 1s and 2s. But if it was all 1s, it wouldn't be too complicated.

1.2.3 Solving Network Problems — pp. 26–27

1. a)
• 1 from A to A (A-C-A); 1 from A to B (A-C-B); 1 from A to C (A-B-C)
• 1 from B to A (B-C-A); 1 from B to B (B-C-B); 0 from B to C
• 0 from C to A; 1 from C to B (C-A-B);
2 from C to C (C-A-C or C-B-C)
b) Create an adjacency matrix for the digraph, $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and square it, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

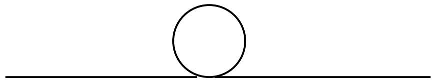
2. a) *Sample response*:



b)
• 1 from A to A (ABA); 1 from A to B (ABB); 1 from A to C (ABC)
• 1 from B to A (BBA); 2 from B to B (BBB, BAB); 2 from B to C (BBC, BAC)
• 0 from C to anywhere

6. a) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$;
There are no zeroes since there is always a way to get from a vertex to itself or to another vertex with one stopover:
from A to A through B (ABA),
from B to B through A (BAB),
from A to B through B (ABB),
from B to A through B (BBA).

6. c) *Sample response*:



The adjacency matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and if you square it, you get $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ with no zeroes.

7. a) When a ball is passed amongst the players, it is like a network — the people are the vertices and the edges are the paths of the ball.
b) It would tell about passes from each player to each other player (or passes from a player to him or herself) that went through another person.

8. The element in the 2nd row, 4th column tells the number of one-stopover trips from B to D. It is 0 because there cannot be any one-stopover trips from B to D — the only place that B can get to is D and that happens with no stopovers.

UNIT 1 Revision — pp. 28–29

1. *Sample response*:
$$\begin{bmatrix} -2 & -1 & -5 & -6 \\ 0 & 0 & -2 & -3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

2. *Sample responses*: to describe:
networks or paths, prices of many different items, an ecosystem, a set of student grades, coordinates of a shape on a grid

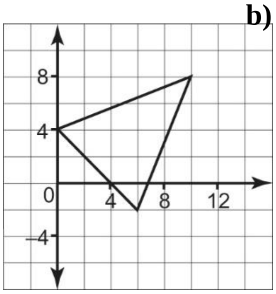
3. a) A and B; they have the same dimensions

3. b) $\begin{bmatrix} 3 & -2 & 0 & 3 \\ -2 & 3 & -2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

c) A and B

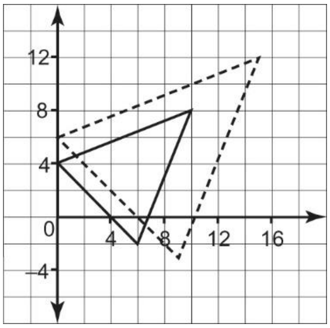
d) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & -1 & -8 & 2 \\ -1 & 2 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 & 0 & -3 \\ -2 & 1 & 8 & -2 \\ 1 & -2 & 0 & 0 \end{bmatrix}$;
you can subtract A – B or B – A

4. a)



b) $\begin{bmatrix} 0 & 15 & 9 \\ 6 & 12 & -3 \end{bmatrix}$

c)



d) It became bigger but it's similar.

5. 3 **×** 4

6. a) $\begin{bmatrix} 10 & -2 \\ -2 & 4 \\ -7 & 4 \end{bmatrix}$

b) They describe how much the numbers changed during the year; a negative element means the numbers went up and a positive element means they went down.
c) Yes, the matrices are the same size.
d) There is no real meaning to the total number of students in both months.

7. a) $A \times B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 6 \\ 4 & -2 & 0 \end{bmatrix}$; $B \times A = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}$

b) The product matrices have different dimensions.

8. 2 (first matrix), 1 (second matrix), 5 (product matrix)

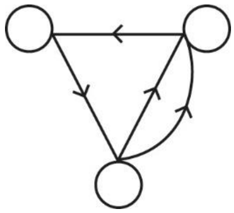
9. a) The number of rows in B is the same as the number of columns in A, so you can multiply A × B. The number of rows in A is the same as the number of columns in B, so you can multiply B × A.
b) 3-by-anything × the same anything-by-2, e.g., a 3-by-**5** matrix × **5**-by-2 matrix

10. a) B **b)** A × C

11. Start at left and end at right, or start at right and end at left.

12. a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

13. *Sample response*:



14. a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

b) There is 1 one-stopover path (ABC) and 1 two-stopover path (ABDC).

15. Some vertices are not connected, e.g., A to C, and A to F, so those teams have not played each other yet.

UNIT 2 COMMERCIAL MATH AND NUMBER pp. 31–63

Getting Started — Skills You Will Need p. 31

1. a) 13 cm **b)** 52 cm; ratio of perimeters = ratio of corresponding sides, so $\frac{5 + 12 + 13}{120} = \frac{13}{x}$ and $x = 52$ cm

2. $\sqrt{40} \times \sqrt{40} = 40$, but $(2\sqrt{10})(2\sqrt{10}) = 4 \times 10 = 40$, so $2\sqrt{10}$ must also be $\sqrt{40}$.

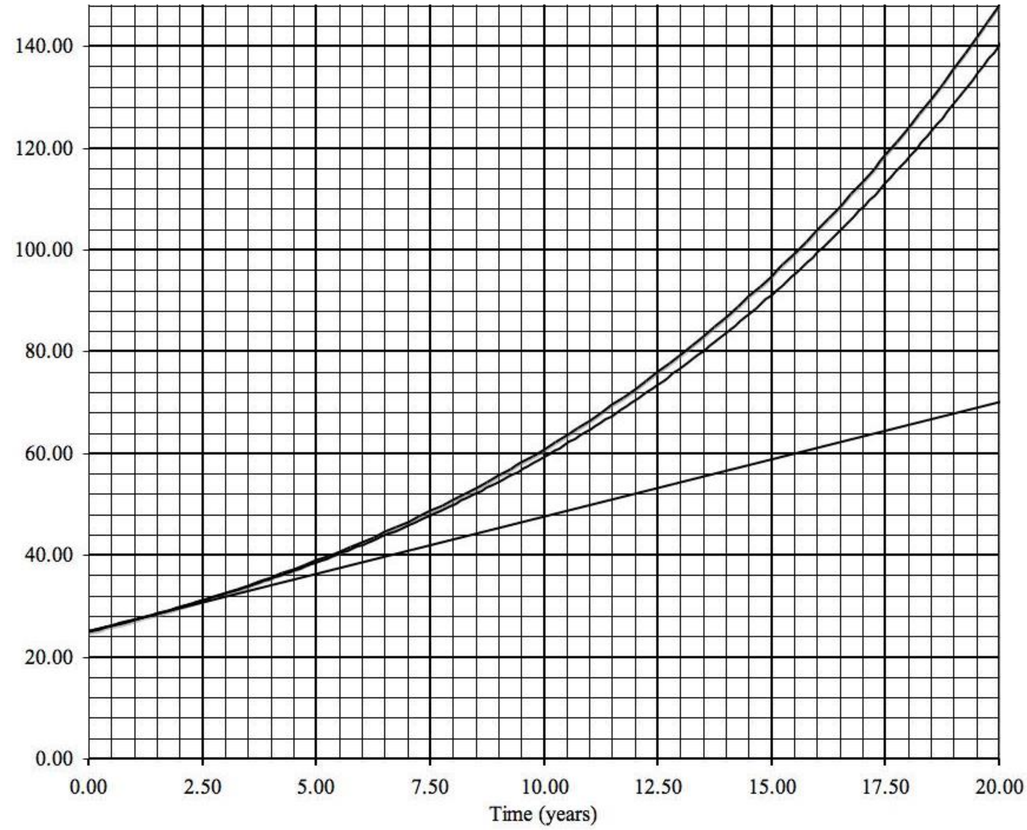
Getting Started — Skills You Will Need [Cont'd]	p. 31
3. Nu 2100; 0.07 × Nu 6000 × 5 years = Nu 2100 **4. a)** 160 **b)** 62.5 **c)** Equal; both are 10	**5.** Because of the power law of exponents, $x^6 = (x^2)^3$, which makes it a perfect cube and $x^6 = (x^3)^2$, which makes it a perfect square.

2.1.1 Purchasing Decisions	p. 35
1. a) Nu 20 **b)** 20% **2.** No. Jigme's calculation is incorrect. The discount percent is actually 20%, not 25%, because the percent discount is based on the marked price of Nu 140 not on the discounted selling price of Nu 112. **3.** Option A; Option A results in a price of 480 × 1.20 = Nu 576. Option B results in a price of 700 × 0.85 = Nu 595. **4.** 100%; *Sample response:* A cost price of Nu 30 would increase to a selling price of Nu 60 with a markup of 100%. A 50% discount on Nu 60 would reduce the price to the original cost, Nu 30. **5. a)** Nu 3100.00**b)** Nu 133,333.34 **6.** If a cost price is increased by more than 100%, the markup is more than the cost price. This is a reasonable sales practice to allow the seller to make a profit. If a marked price is reduced by a discount of more than 100%, the discount is more than the marked price, which would mean that the seller pays the buyer. This does not make sense. **7.** *Sample response:* A 10% commission on a sale of a Nu 3000 item would result in Nu 300 commission, which is less than a fixed commission of Nu 500 per item.	**8. a)** The percentage markup resulting from a markup of Nu 100 cannot be determined unless the cost price is known. Likewise, the amount of markup resulting from a 7% markup cannot be determined unless the cost price is known. Therefore, you cannot compare them. **b)** *Sample response*: If the cost price was Nu 100: Nu 100 vs. Nu 7. **c)** *Sample response*: If the cost price was Nu 10,000: Nu 100 vs. Nu 700. **9.** *Sample response:* If the same discount amount, e.g., Nu 100, were offered on an expensive item, e.g., Nu 10,000, and an inexpensive item, e.g., Nu 1000, it would make a significant difference to the selling price of the inexpensive item, making it attractive to buyers, but it would probably not affect the price of the expensive item significantly enough to make it attractive to buyers. If the same percent discount were offered on both items, the selling price of both items would be affected proportionally and there would be an equal incentive to buy both items. The same idea applies to markup and commission.

2.1.2 Compound Interest	p. 42
1. 6.25% **2. a)** Nu 624.36 **b)** Nu 649.71 **3. a)** Nu 624.00 **b)** Nu 648.00 The simple interest pays less than the compound interest. **4. a)** Nu 1141.43 **b)** Nu 1112.40 **c)** Nu 1080	**5. a)** Nu 9816.67 **b)** Nu 9631.20 **6. a)** Nu 15,970.88 **b)** Nu 10,970.88 **7.** 15% (compounded quarterly) **8.** 12.68%

9. a) I: Nu 27,327.08; Nu 60,879.73; Nu 148,253.63
II: Nu 27,250.00; Nu 59,184.09; Nu 140,110.27
III: Nu 27,250.00; Nu 47,500.00; Nu 70,000.00

b) Vertical axis is 1000s of ngultrums



c) *Sample response:*
The simple interest graph is linear so the investment grows at a constant rate.
Both compound interest graphs are exponential, which means the investments grow faster and faster over time. The graph of interest compounded quarterly curves a bit more sharply than the graph of interest compounded annually.

10. *Sample response:*
Compound interest calculates interest on interest already earned so it increases faster and faster over time, which is an exponential relationship. Simple interest calculates the same amount of interest each time since the interest is always based on the original principal, so the growth is constant and therefore the relationship is linear.

CONNECTIONS: The Rule of 72 **p. 43**

1. It appears to work.

2. Yes; *Sample response:*
I would expect it to at least double because the number of years to double if it were compounded annually would be about 72 ÷ 12 = 6 years, so the number of years to double if it were compounded monthly would have to be less than 6 years, because more frequent compounding results in more interest.

3. About 70 months

2.1.3 Dividends and Stocks — p. 46

1. A. Nu 22,000 B. Nu 40,000 C. Nu 42,500

2. A. 772 shares B. 212 shares C. 100 shares

3. a) A. Nu 3300 B. Nu 16,500 C. Nu 33,000
b) A. 20% B. 27.5% C. 25.88%
c) B; The discount to buy it was greatest, so it resulted in a higher yield percentage because the dividend rate was the same for each stock.

4. a) Nu 2500 **b)** 58.82%

5. a) 19.8% **b)** Nu 250 **c)** Nu 5500

6. Nu 2160 **7.** Nu 6250

8. *Sample response:* If a company earns more profit, it can afford to pay a higher dividend rate. A company might also pay a higher dividend to try to attract more investors.

9. a) The yield rate for stock investments has the potential to be much higher than the interest you would earn in a bank account.
b) Money in a savings account in a bank is more secure than money invested in stocks.

2.1.4 Using Commercial Math — p. 51

1. Option 3 is best and Option 2 is worst.

2. 4.1%

3. a) 7.73% (rounding would give 7.72% but the guarantee would not be met)
b) *Sample response:*
Yes; since Nu 100 increases by Nu 25 to Nu 125 in three years, it will increase by even more than Nu 25 in the next three years due to compounding. Therefore, the increase over six years would be more than 50%.

4. a) 8% **b)** 16%

5. a) *Sample response:*
Nu 10,000 monthly as I expect to live for more than 8.3 years.

5. b) *Sample response:*
An elderly person or a very ill person may choose to collect the lump sum prize because they might not live another 8.3 years. Or, someone who needs the money right away might choose the lump sum.

6. Nu 25,000,000

7. Nu 211,111.11 (or 2.11111 Lakhs)

8. a) 14.29% **b)** Nu 5625.35

9. *Sample response:*
The option with a payment of Nu 12,000 at the end of the year has a lower interest rate. Because both options require payment of Nu 12,000 in total, the plan with the later payment is charging the lower interest rate.

2.2.2 Simplifying Radicals — p. 55

1. a) $4\sqrt{3}$ **b)** $10\sqrt{10}$
c) $2\sqrt[3]{4}$ **d)** $\sqrt{110}$

2. $9\sqrt{2}$, 11, $3\sqrt{13}$, $4\sqrt{7}$, $6\sqrt{3}$

3. a) $\sqrt{18} = 3\sqrt{2}$; $\sqrt{45} = 3\sqrt{5}$ **b)** 6; 3

4. $\sqrt{920}$ because $13\sqrt{5}$ is the same as $\sqrt{13^2 \times 5}$, which is $\sqrt{169 \times 5}$, or $\sqrt{845}$.
Since $845 < 920$, $\sqrt{845}$ is less than $\sqrt{920}$

5. a) *Sample response:* 4

5. b) This is impossible because, for $\sqrt[3]{n}$ to be a whole number, *n* must have three identical factors. But $\sqrt{n}$ is an entire radical only if no pair of identical factors can be found.
c) *Sample response:* 27

6. a) No; *Sample response*: $\sqrt[4]{n}$ is a whole number only if *n* is a fourth power ($a \times a \times a \times a$), but any fourth power is a perfect square ($a^2 \times a^2$). E.g.,
$\sqrt[4]{16} = \sqrt[4]{2 \times 2 \times 2 \times 2} = 2$ and
$\sqrt{16} = \sqrt{2 \times 2 \times 2 \times 2} = \sqrt{4 \times 4} = 4$
b) Yes; *Sample response:*
$\sqrt[4]{9} = 1.732...$ and $\sqrt{9} = 3$

7. a) None because the 11^3 part is not a perfect square.
b) Any non-negative multiple of 3 such as 0, 3, 6, 9, ... since the cube root would be $3^2 \times 11 \times 5^{3m}$, where *m* is $\frac{1}{3}$ of the multiple of 3. The cube root means raising each power to $\frac{1}{3}$ its value ($\frac{1}{3}$ of the multiple of 3 is an integer).

8. The diagonal in the top left square is $\sqrt{2}$ units long and the full diagonal is 3 times as long, or $3\sqrt{2}$. Using the Pythagorean theorem, the full diagonal is
$\sqrt{3^2 + 3^2} = \sqrt{18}$, so $3\sqrt{2} = \sqrt{18}$.

9. a) *Sample response:*
Factor *n* to include as many pairs of identical factors as possible. For each identical pair under the square root symbol, create a term outside the square root symbol equal to the square root of the product of the identical factors;

$$\begin{aligned}\sqrt{56} &= \sqrt{2 \times 2 \times 2 \times 7}\\ &= \sqrt{2 \times 2} \times \sqrt{2 \times 7}\\ &= \sqrt{4} \times \sqrt{2 \times 7} = 2\sqrt{14}\end{aligned}$$

If no terms are left inside the root symbol, the square root symbol is removed, e.g.,

$$\begin{aligned}\sqrt{324} &= \sqrt{9 \times 9 \times 2 \times 2}\\ &= \sqrt{9 \times 9} \times \sqrt{2 \times 2} = 9 \times 2 = 18\end{aligned}$$

b) *Sample response:*
The term outside the radical must be moved inside the radical. This is done by multiplying the value inside by the square of the term outside; $m\sqrt{n}$

$$\begin{aligned}&= \sqrt{m \times m} \times \sqrt{n}\\ &= \sqrt{m \times m \times n} \text{ or } \sqrt{m^2 \times n}.\end{aligned}$$

2.2.3 Operations With Radicals — pp. 60–61

1. a) $6\sqrt{3}$ **b)** $(x-7)\sqrt{x}$
c) $10\sqrt{k}$ **d)** $2\sqrt{11} + \sqrt{55} - 11$

2. a) 6 **b)** $2x^4\sqrt{5}$ **c)** $2\sqrt{21}$

3. a) 2 **b)** $\frac{3}{\sqrt{2}}$ **c)** $3x^2$ **d)** $\frac{2\sqrt{10}}{\sqrt{3}}$

4. a) $12\sqrt{5} - 6\sqrt{55} - 4\sqrt{11} - 22$
b) $13 - 5x$

5. *Sample response:*
a) $\sqrt{7}$ **b)** $4\sqrt{5}$ **c)** $3\sqrt{14}$ **d)** 6 or 8

6. a) $m = 5$ **b)** $p = 6$ **c)** $k = 19$ **d)** $s = 128$

7. a) $\sqrt{6} + \sqrt{4}$ is the greatest.
b) The expression with the numbers that were closest together was the greatest.
c) $\sqrt{11} + \sqrt{8}$ is greatest because 11 and 8 are closest together.
d) $\sqrt{11} + \sqrt{8}$

8. a) $\frac{3\sqrt{11}}{11}$ **b)** $\frac{3}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}$ **c)** $\frac{\sqrt{13}}{\sqrt{13}}$
$= \frac{3}{\sqrt{11}} \times 1$

9. Look for ways to simplify terms and combine like terms, keeping in mind that $\sqrt{20} = 2\sqrt{5}$ and $\sqrt{x^3} = x\sqrt{x}$.

UNIT 2 Revision pp. 62–63

1. a) Nu 160 **b)** 25%
c) The item listed at Nu 300 with a 25% discount costs less.

2. a) Nu 1575 **b)** Nu 1575
c) No; the compounded rate is better because the interest from the first year will earn additional interest above the basic Nu 75 in the second year.

3. 10.93% p.a. compounded monthly

4. a) Simple: 4.42%; Compounded annually: 4.07%
b) Simple: 4.52%; Compounded annually: 3.80%
c) Simple: 5.68%; Compounded annually: 4.60%
d) Simple: 4.71%; Compounded annually: 3.94%

5. a) 347 shares **b)** Nu 5899 **c)** 23.61%

6. a) 158.82% **b)** Nu 3400

7. a) 63.64%
b) I would choose Option 3;
Option 2, 8% of 70,000, is Nu 5,600, and
Option 3, 3.5% of 180,000 is Nu 6,300.
Both options exceed Option 1.

8. 14% p.a. compounded quarterly

9. Nu 182,560.01 **10.** $\frac{21}{50}$

11. 15% compounded semi-annually; it would be equivalent to an annual interest rate of 15.56% instead of an annual rate of 15.79% in the first year.

12. A and B.
• A simplifies to 0.
• In B, all terms in the numerator simplify to something of the form $n\sqrt{2}$, and because the denominator is $\sqrt{2}$ the expression will have a value of *n*.
• I did not choose C because, although 49 and 64 are perfect squares, 108 is not, so the product will not simplify to an integer.

13. a) $3\sqrt{3}$ **b)** $10\sqrt{3}$
c) $5\sqrt[3]{2}$ **d)** $\sqrt{30}$

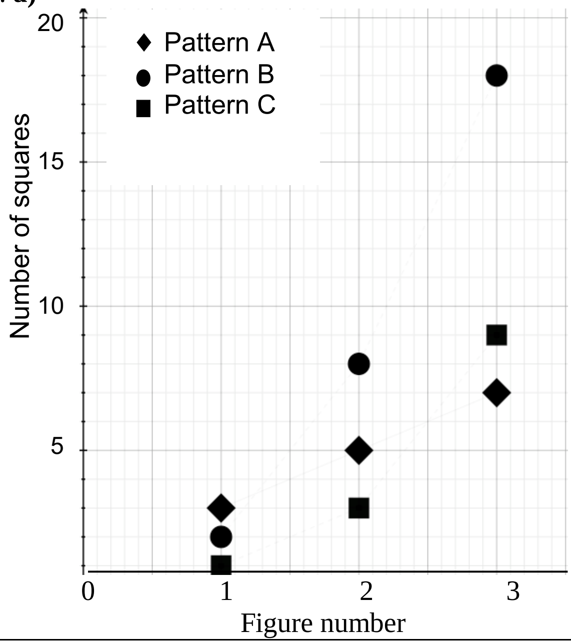
14. a) $9\sqrt{2}-\sqrt{7}$
b) $10\sqrt{6}+4\sqrt{3}-3\sqrt{2}-15$
c) 2 **d)** $1\frac{1}{3}$ **e)** $3x^2$

15) *Sample response:*
2007^6 is both a perfect square and a perfect cube (since its power is divisible by both 2 and 3). Therefore, their roots $\left(\sqrt{2007^6}\, and \sqrt[3]{2007^6}\right)$ are both integers. However, 6^{2007} is a perfect cube (as its power is divisible by 3), but not a perfect square (as its power is not divisible by 2). Therefore, it's a cube root is not an integer.

16. a) k = 4 **b)** p = 3 **c)** n = 28
d) m = 15

Getting Started — Skills You Will Need pp. 65–66r.

1. a)



b) *Pattern A* is linear because the number of squares increases by two for each figure and since this is a constant increase, the relationship is linear.

Pattern B appears to be quadratic because the plotted points form the right half of a parabola. For the three plotted points, the first differences are not constant but the second differences are constant.

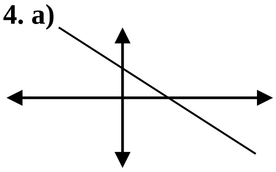
Pattern C is exponential because the plotted points form a curve with a steep increase. The number of squares in each figure is three times the number in the previous figure so neither first nor second differences are constant but the ratios of first differences are constant.

2. $a = 76$, $b = 68$, $c = 64$

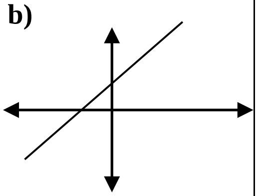
3.

	a)	b)	c)	d)
slope	2	$-\frac{3}{4}$	–3	0.75
y-intercept	3	–2	2	–1.65

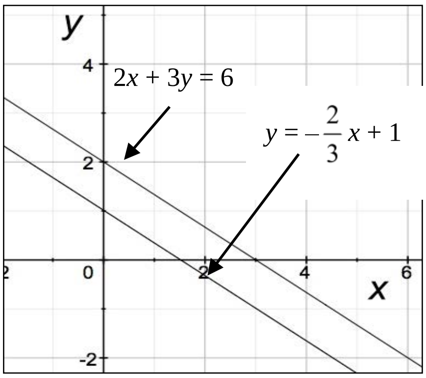
4. a)



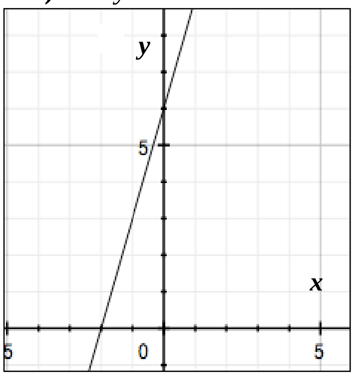
b)



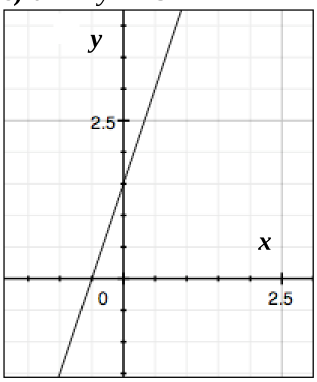
5.



6. b) $3x - y = -6$

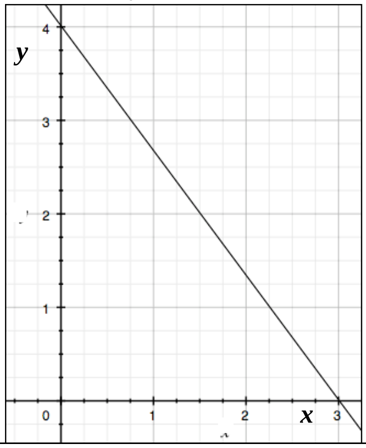


c) $6x - 2y = -3$



7. a) $x = -\frac{7}{5}$ **b)** $a = 4$

6. a) $4x + 3y = 12$



8. a) $a < -3\frac{1}{2}$ **b)** $x < 2$

9. 2

10. a) $200 + 2m$ **b)** $60s$
c) $1000n$ **d)** $0.04n$

3.1.1 Linear Functions p. 71

1. a) Not a function; (3, 2) and (3, 4) have the same x-coordinate, but different y-coordinates.
b) Function; no ordered pairs have the same x-coordinate and different y-coordinates.
c) Not a function; except for (0, 0), there are two ordered pairs with different y-coordinates for each x-coordinate.

2. a) Function; each student has only one age.
b) Usually not a function; there would likely be more than one student in the class with identical numbers of siblings.

3. a)

3	4	5
☆	☆	☆
☆	☆	☆
☆	☆	☆
	☆	☆
		☆

b) $f(n) = n$

4. a) *Sample response*:

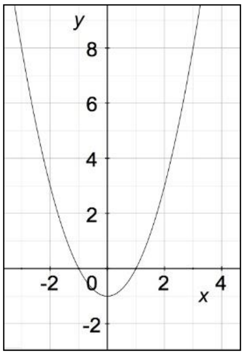
x	$f(x) = 4x - 3$
–1	–7
0	–3
1	1
2	5

b) For each x-value, there is only one y-value.

c) *Sample responses*:
You could draw a graph and extend it to see that the line has one y-value for each x-value.
OR
You can be sure it continues this way, since each next value of y is exactly 4 greater than the previous one.

5. a) *Sample response*:

x	$f(x) = x^2 - 1$
–1	0
0	–1
1	0
2	3
3	8



b) A vertical line will cross the graph in only one place.

6. Yes; for each value in the first column, there is only one possible value in the second column.

7. a) For each value in the first column, there is only one value in the second column.
b) There are no ordered pairs that have different y-coordinates for the same x-coordinate. This means that if you draw a vertical line, it will hit the graph at only one place no matter where you draw it.

3.1.2 Applications of Linear Functions pp. 76–77

1. a) $f(t) = \dfrac{4-2t}{5}$ **b)** $f(m) = \dfrac{4-5m}{2}$

2. a) $20t + 50f = 2000$, or any variation of it, e.g., $20t = 2000 - 50f$, where t represents the number of Nu 20 notes and f the number of Nu 50 notes

b) $f(f) = \dfrac{2000 - 50f}{20} = \dfrac{200 - 5f}{2}$

3. a) $f(l) = 5l + 91$ **b)** 24
c) $f(s) = (s - 91) \div 5$

4. a) $I = 0.042o + 0.045n$, where
I represents total interest
o represents the money invested at 4.2%
n represents the money invested at 4.5%

b) $f(o) = \dfrac{I - 0.042o}{0.045}$

8. a) Deki: $f(t) = 8t$; Thinley: $f(t) = 7t + 12$
b) Deki: 72 m; Thinley: 75 m
c) Deki; I solved the equations $8t = 100$ for Deki and $7t + 12 = 100$ for Thinley to get the time it would take each to finish the course. For Deki, t was 12.5 s and for Thinley, t was 12.57 s.
d) Because where you are on the course depends on how much time has passed.

5. a) $30r + 20s = 51$, where r is the number of hours at 30 km/h and s is the number of hours at 20 km/h

b) $f(s) = \dfrac{51 - 20s}{30}$ **c)** 0.9 h or 54 min

6. a) *Sample response*:
(1976, 45) and (2002, 60)

b) $e = \dfrac{15}{26}a - 1095$, or $f(a) = \dfrac{15}{26}a - 1095$, where a represents the year and e represents life expectancy
c) 64.6
d) 41.5; the actual plotted value is just a bit more than 40 so it is very close

7. a) $F = \dfrac{9}{5}C + 32$ **b)** $C = \dfrac{5}{9}(F - 32)$

9. No; If $y = mx + b$, there is one y-value for each x-value. That's why it's a function. But that also means that $x = \dfrac{y-b}{m}$. As long as m is not 0, there is only one x-value for each y-value, so this is also a function. If $m = 0$, the original line was horizontal and x did not appear in the equation, so you won't be able to write x in terms of y.

3.1.3 Graphs of Linear Inequalities pp. 82–83

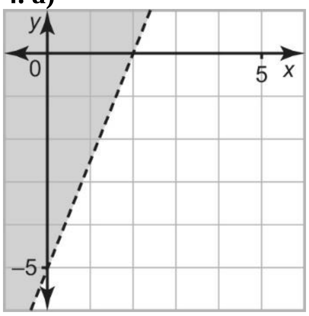
1. a) No **b)** Yes **c)** No

2. a) Below; *Sample response:*
I tested (0, 0). Since $0 < 0 + 6$, then (0, 0) belongs in the shaded region and it is below the boundary line.

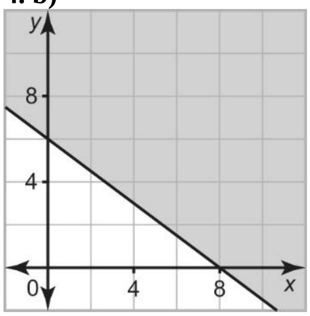
3. a) Both have dashed boundary lines and the lines are the same; the first one has shading on one side of the line and the second has it on the other side.

3. b) Both have the same boundary line and are shaded below the line; the first one has a solid line and the second has a dashed line.

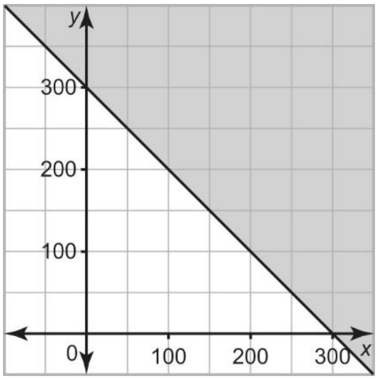
4. a)



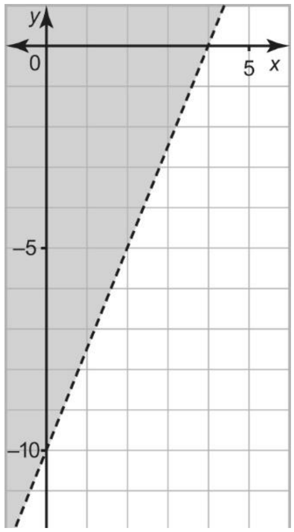
4. b)



4. c)



4. d)



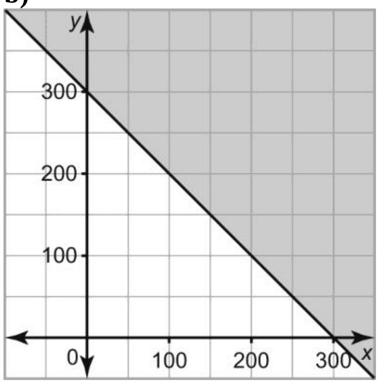
5. a) $y < -\frac{5}{2}x + 2$ **b)** $y \leq \frac{4}{3}x - 4$

c) $< \rightarrow \leq$; $y \leq -\frac{5}{2}x + 2$

d) $\leq \rightarrow \geq$; $y \geq \frac{4}{3}x - 4$

6. a) $x + y \geq 300$ (or any variation) where *x* and *y* represent the amount he earned at each job.

b)



7. a) A **b)** D

c) *Sample response:*
It could only be A or D since the line had to be solid. I tested the point (0, 0) and found that it should not be in the shaded region, so it had to be A.

d) *Sample response:*
It had to be the region opposite to my answer in **part a)** and the line had to be dashed.

8. If you choose any *x*-value, there are multiple *y*-values that go with it.

9. *Sample response:*
- Graph $2x - 4y = 10$ using the two intercepts (5, 0) and (0, –2.5).
- Draw a dashed line because the inequality sign is <.
- Test (0, 0) to see if it belongs in the shaded area. Since $2(0) - 4(0) < 10$, it does.
- Shade the side of the line that contains (0, 0).

3.2.1 Solving Algebraically — The Comparison Strategy pp. 89–90

1. a) The number of minutes when both plans cost the same;
If Plan A is $C = 1200 + 1.6m$ and Plan B is $C = 1400 + 1.4m$, where m is the number of minutes and C is the total cost, then
$1200 + 1.6m = 1400 + 1.4m$ because $C = C$.
If you solve $1200 + 1.6m = 1440 + 1.4m$ for m, you get the number of minutes for which both plans cost the same.
b) 1000 min; $1200 + 1.6m = 1440 + 1.4m \rightarrow 0.2m = 200 \rightarrow m = 1000$
c) Plan A for 600 min; Plan B for 1200 min

2. a) (–1, –9) **b)** (–2, –7) **c)** (1.2, 0.8)

3. 41 and 7

4. 38 and 57

5. 16 practice balls and 4 official balls

6. 60.2 L

7. 20 at Nu 4 and 40 at Nu 9

8. a) (5, 13) **b)** (2, 2) **c)** (–1, 3)

9. a) AC: $y = 2x - 2$; BD: $y = -2x + 10$
b) (3, 4)

10. Wind speed: 40 km/h;
Plane's speed with no wind: 280 km/h

11. a) (2, 2) **b)** (–1, 4) **c)** (6, 12)

12. So you can compare the expressions involving the other variable and then solve for that variable.

3.2.2 Solving Algebraically — The Substitution Strategy p. 94

1. a) $x = 1$; $y = 3$ **b)** $x = -2$; $y = -1$
c) $x = \frac{1}{2}$; $y = \frac{1}{2}$ **d)** $x = \frac{2}{3}$; $y = \frac{3}{4}$

2. a) $l = d + 5$ **b)** $l + d = 29$
c) Lhamo is 17 years old and Devika is 12 years old.

3. a) (–1, –2) **b)** (5, –2)
c) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ **d)** $\left(\frac{2}{3}, \frac{5}{6}\right)$

4. 150, 62

5. a) $x = 60$, $y = 60$ **b)** $x = 40$, $y = 50$

6. a) $x = 4$; $y = 6$ **b)** $x = 12$; $y = -8$
c) $x = 2.5$; $y = 3.1$ **d)** $x = -1.7$; $y = 2.3$

7. The comparison strategy requires that the same term be isolated in each equation so that the remaining parts of the equations can be compared. The substitution strategy requires that only one term be isolated in one equation so that it can be substituted into the other equation.

3.2.3 Solving Algebraically — The Elimination Strategy pp. 97–98

1. a) $x = -1$; $y = 2$
b) $x = 1$; $y = 1$
c) $x = 3$; $y = -5$
d) $x = -2$; $y = -4$

2. a) $a + b = 2500$, where a is the amount invested at 4% and b is the amount invested at 5%.
b) $0.04a + 0.05b = 115$, where a is the amount at 4% and b is the amount at 5%.
c) Nu 1000 at 4% and Nu 1500 at 5%.

3. a) $500t + 375c = 125{,}000$ for the amount of steel and $250t + 150c = 55{,}000$ for the amount of aluminium, where c represents the number of cars and t represents the number of trucks.
b) 100 trucks and 200 cars

4. 16 correct

5. a) $a = 100$, $b = 150$
b) $a = -1$, $b = 2$

7. a) (10, 12) **b)** (–16, 8) **c)** (–12, –18)

8. The company can make 1200 batches of light purple dye and 3600 batches of deep purple dye.

9. *Sample response:*
a) $2x + 3y = 10$ and $4x - 3y = 50$; You can add them and eliminate the $3y$ right away.

c) $a = 1200$, $b = 3000$

6. She can make two batches of deluxe grade and two batches of fine grade

b) $y = 2x + 5$ and $x + y = 15$; The expression for y $(2x + 5)$ is right there for you to substitute into $x + y = 15$.
c) $2y = 8x + 8$ and $2y = 4x + 15$; You can compare two equivalent expressions for $2y$.

CONNECTIONS: Matrix Solution of a Linear System — p. 99

1. $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8x+5y \\ 3x+2y \end{bmatrix}$ so

$\begin{bmatrix} 8x+5y \\ 3x+2y \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$;

The top elements in each matrix are equal so $8x + 5y = 11$ and the bottom elements in each matrix are equal so $3x + 2y = 4$.

2. a) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and

$\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \times \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; The top elements in each matrix are equal so $x = 2$ and the bottom elements in each matrix are equal so $y = -1$.

3. a) $\begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ or

$\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

3. b) $\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and

$\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} \times \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

c) $x = -4$ and $y = 3$

4. a) $\begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and

$\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

so $x = -1$ and $y = 1$.

UNIT 3 Revision — pp. 101–102

1. A; there is only one y-value for each x-value in A, but there are two y-values for some x-values in B.

2. a) No; for any x-value there is only one y-value, 3.
b) Yes; for $x = 3$, there are multiple y-values.

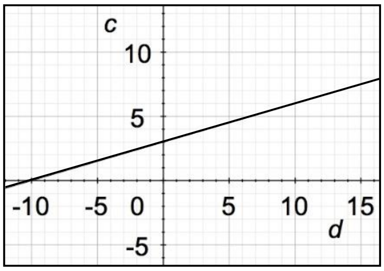
3. a) *Sample response*:

x	$f(x)$
1	7
2	4
3	1
4	–2
5	–5

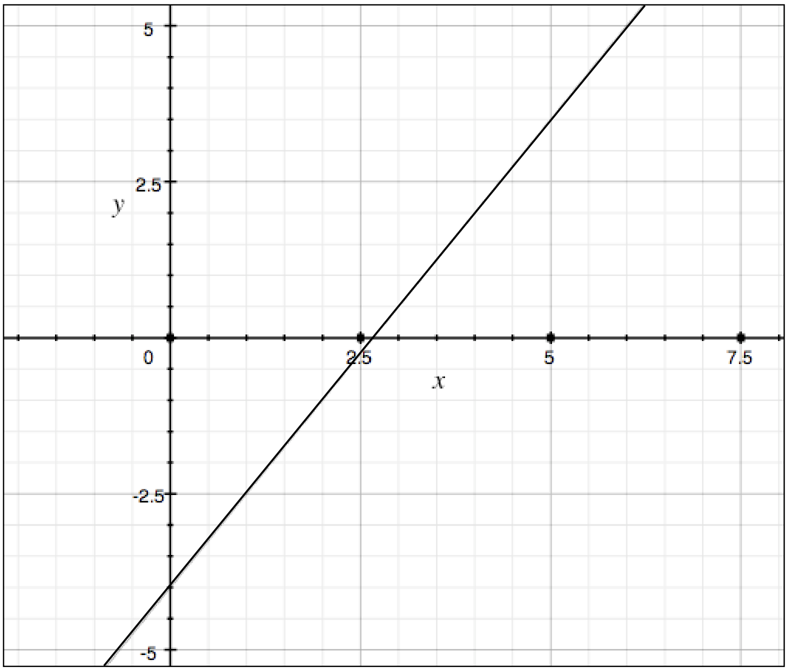
b) To get the related y-value for each x-value, multiply x by 3 and then subtract the result from 10. For every different x-value you will get a different y-value.

4. a) $f(y) = (180 - y) \div 2$ **b)** $f(x) = 180 - 2x$
c) You could decide on the x-value first, so the value of y would depend on x.
Or, you could decide on a y-value first so the value of x would depend on y.
d) $f(20) = 180 - 40 = 140$

5.a) $c = 0.3d + 3$ or $f(d) = 0.3d + 3$ **b)**

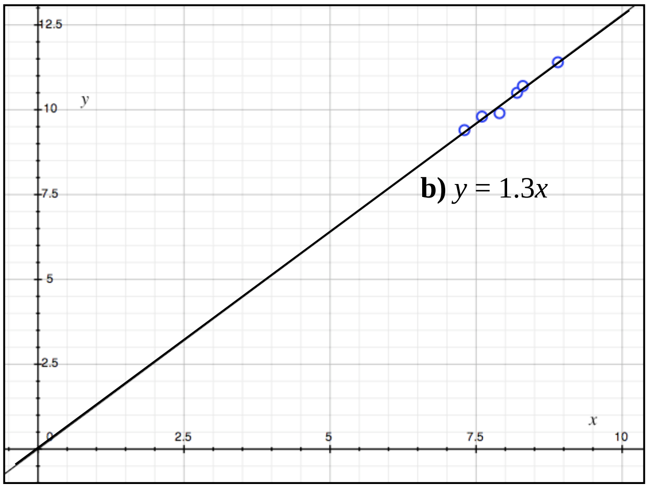


6. $y = 1.5x - 4$



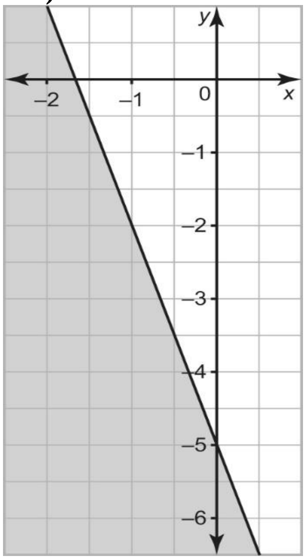
7. $f(x) = x + 1$

8. *Sample response*: **a)**

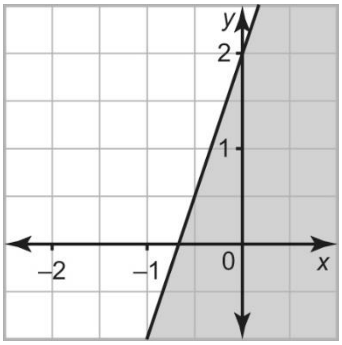


b) $y = 1.3x$

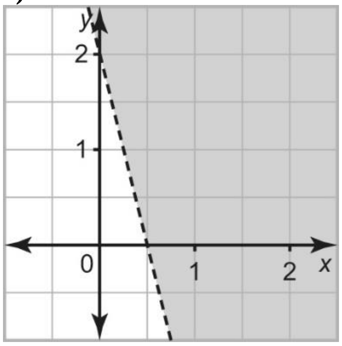
9. a)



b)



c)



10. a) $y > 4x - 8$ **b)** $y \leq 4x - 5$

11. a) $3y < 2x + 4$ will have a dashed line because < means values along the line are not included. $3y \leq 2x + 4$ will have a solid line because the ≤ sign means the values along the line are included. Otherwise they are the same.

b) The dashed line for
$4y + 2x < 10$ will be 2 units higher than the dashed line for $4y + 2x < 8$ because the
y-intercept is 10, not 8. Otherwise they are the same.

12. a) Shifted (translated) 4 units to the left.
b) Shifted (translated) 4 units to the right.
c) Slope is 4 times as steep and y-intercept is lower.
d) Slope is 4 times as steep but in the opposite direction and the y-intercept is higher.

13. a) (5, 22) **b)** (1, 2.5) **c)** (4, 6) **d)** (2, 17)

14. 21 items

15. 39 cm × 21 cm and
78 cm × 7 cm

16. a) $x = 2$ and $y = 4\frac{1}{3}$
b) $y = 4$ and $x = -2$
c) $x = 5$ and $y = 5$

17. a) $x = 5\frac{2}{3}$ and $y = 7$
b) $y = 4$ and $x = 2$
c) $x = 3$ and $y = 0.5$

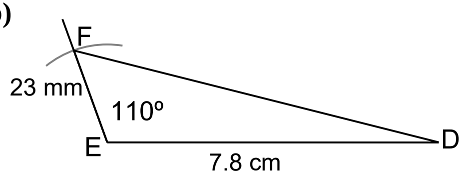
18. *Sample response:*
If you do not need an exact answer and you have grid paper available.

UNIT 4 MEASUREMENT pp. 103–126

Getting Started — Skills You Will Need pp. 103

1. a) 8.4 cm^2; 27.4 cm^2
b) 328 m^3; 295.2 m^2
c) 301.6 cm^3; 301.6 cm^2
d) 33.3 cm^3; 66.6 cm^2
e) 113.1 cm^3; 113.1 cm^2
f) 56.5 cm^3; 94.2 cm^2

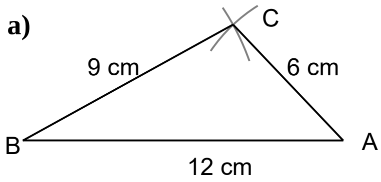
3. b)

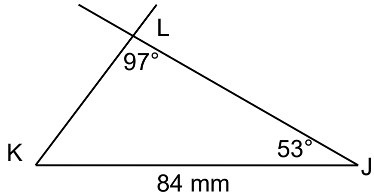


c)

2. a) 3.2 cm **b)** 1700 g
c) 0.270 L **d)** 20,000 cm^2
e) 7 cm^3 **f)** 4300 cm^3
g) 8200 mm

3. a)





4. a) 10π cm; 31.4 cm; 25π cm, 78.5 cm^2
b) 6.4π cm, 20.1 cm; 10.24π cm, 32.2 cm^2

5. a) 40° **b)** 66°

6. a) 3.01 cm **b)** 5.13 cm

4.1.1 Precision and Accuracy p. 111

1. a) 4 **b)** 4 **c)** 4 **d)** 3 **e)** 1
f) 1 **g)** 1 **h)** 3 **i)** 4 **j)** 1

2.a) 0.8 **b)** 4700 **c)** 3.2

3. *Sample response*:
a) 116 mm and 121 mm (anything between 115 mm and 125 mm is acceptable)
b) 7281 mL and 7312 mL (anything between 7250 mL and 7350 mL is acceptable)

4. a) *Sample response:* 0.200
b) Yes; *Sample response:* 50,300
c) *Sample response*: 0.000002
d) Greatest is 9,000,000; least is 0.000001

5. *Sample response*:
The calculator is about 15 cm long. This measurement is not very precise. Trying to be more precise by measuring to the nearest millimetre would be impossible because of the rounded corners and because the calculator is longer in the middle than along the sides, so one measurement to describe its length is not possible. The accuracy of any reported value will depend on the viewer's perspective in reading the ruler.

6. a) No; 65 km might have been reported to the nearest whole kilometre and 70 km to the nearest 10 km. That means the 65 km could have actually been 65.4 km and the 70 km could have been 65.1 km, making the distance from Paro to Thimphu longer than from Wangdi Phodrang to Thimphu.

b) Novin's measurement is given with more precision, but that does not mean Novin's measurement is accurate. It could be that one or both measurements come from an inaccurate source.

7. *Sample response:*
a) Drakpa tells you that her red bucket has a capacity of 10 L. Pema tells you that her blue bucket has a capacity of 9.6 L. Drapka's red bucket appears to hold more.

b) The red bucket may actually have a capacity of 9.51 L but Drakpa rounded to 10 L. The blue bucket could have had a capacity of 9.64 L but was rounded to 9.6 L. Thus the red bucket could be smaller than the blue bucket. A conclusion cannot be determined from the given measures.

CONNECTIONS: Precision Instruments p. 112

1. *Sample response:*
• A millimetre ruler is very precise. It measures length.
• The knobs on the stove that control the heat are less precise. They measure heat.
• The cups my mother uses to measure food are probably not very precise; they measure volume.

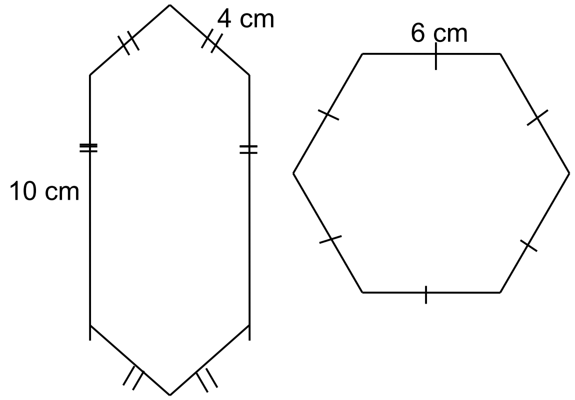
4.2.2 2-D Efficiency p. 120

1. The hexagon's area is greater.

2. *Sample response*:

5. a regular heptagon (7-sided polygon) with each side 3 m

The regular hexagon has a greater area because it is more circle-like.



3. *Sample response:*
a) 12 m × 9.0 m and 10 m × 10.8 m
b) I knew I needed a more square-like rectangle so I tried different lengths and widths that were close in size and had a product of 108.

4. *Sample response:*
a) A rectangle measuring 9 cm × 3 cm and a parallelogram of length 10 cm and slant height 2 cm
b) I knew I needed a less square-like shape so I tried different lengths and widths that were farther apart in size but added to 12.

6. a) 580 cm
b) About 66 cm

7. a) 29.3 cm, 29.3 cm, 41.4 cm
b) It makes the two leg dimensions as close to each other as possible in length so the shape is the most like a circle

8. No; If they have the same area, then the regular polygon with more sides has a shorter perimeter.
OR
No; If they have the same perimeter, then the regular polygon with more sides has a larger area.

9. *Sample response:*
a) You only have a certain amount of fencing, but you want to enclose a large area.
b) You do not want to require as much material to fence a garden, so you want to minimize the perimeter of the garden.

4.2.3 3-D Efficiency p. 124

1. a) 79.6 cm **b)** 62.9 cm
c) 50.9 cm **d)** 42.1 cm

2. I predict barrel **c)** because its dimensions (height and radius) are closest to being equal, making it the most sphere-like.
The surface areas are
a) 12,516.1 cm^2 **b)** 12,073.1 cm^2
c) 11,922.3 cm^2 **d)** 12,026.0 cm^2

3. Barrel **c)** because all have the same volume and **c)** has the least surface area.

4. A base of about 8.8 cm × 8.8 cm and a height of about 12.4 cm.

5. a) 6.5 cm **b)** 4.0 cm
c) 2.3 cm **d)** 1.0 cm

6. Prism **b).B**ecause its height is the same as its width, which makes it the most sphere-like. The volumes are
a) 59 cm^3 **b)** 64 cm^3 **c)** 58 cm^3 **d)** 36 cm^3

7. a) 2714 cm^2 **b)** 15 cm **c)** 22.6% greater

8. a) 21 cm
b) *Sample response*:
The volume of the cube will be smaller than the volume of the cylinder and sphere.

c) The cylinder volume is 10,857 cm^3 and the sphere volume is 13,306 cm^3. The cube volume is 9528 cm^3, which is less than the others.

9. Nu 237

10. a) 3217 cm^3; 1206 cm^2
b) 1053 cm^2
c) Sphere
d) *Sample response*:
The surface area of the sphere was smaller and the volumes were the same.

11. If the volume of two shapes is the same, the lower the value of the ratio, the more efficient the shape.
A lower ratio means that less surface area is required for the same volume.

CONNECTIONS: Animal Shapes and Sizes p. 125

1.

Radius (cm)	20.0	10.0	5.0	2.0	1.0	0.5
Surface area (cm^2)	5026.5	1256.6	314.2	50.3	12.6	3.1
Volume (cm^3)	33,510.3	4188.8	523.6	33.5	4.2	0.5
Surface area ÷ volume (SA : V ratio)	0.15	0.30	0.60	1.50	3.00	6.2

2.

Edge length (cm)	20.0	10.0	5.0	2.0	1.0	0.5
Surface area (cm^2)	2400	600	150	24	6.0	1.5
Volume (cm^3)	8000	1000	125	8.0	1.0	0.13
Surface area ÷ volume (SA : V ratio)	0.300	0.600	1.2	3.0	6.0	12

UNIT 4 Revision p. 126

1. a) 2 **b)** 4 **c)** 3

2. *Sample responses:*
a) 20.4 **b)** 12,000,000 **c)** 0.02134

3. No. Dawa's mass was given with only one significant figure. This means that even if Dawa measured his mass accurately, he could be as light as 65 kg (the margin of error for 70 kg is 65 kg to 75 kg), so Nima cannot be sure he is lighter.

4. a) 70 cm
4. b) *Sample response*:
727 mm (any value from 725 mm to 734 mm would be acceptable)

5. a) *Sample responses*:
Due to precision, the 20 cm measure could be actually as much as 5 cm off the actual measure. It is unlikely that the actual measure is precise.
b) 502.5 cm^2 **c)** 862.5 cm^2

6. The shape on the right since it is most like a circle.

7. The middle shape; it is most like a circle.

8. a) 28 cm^2 **b)** 36 cm^2
c) The square has more sides, so it is more like a circle and therefore has more area for the same perimeter.
d) *Sample response:*
Circle with circumference 24 cm

9. 3164.06 m^2

10. a) 9.0 cm **b)** 6.5 cm **c)** 4.7 cm **d)** 3.2 cm

11. Cylinder **b)** because the height is closest to the diameter, and so that cylinder is the most like a sphere. The volumes are
a) 175.9 cm^3 **b)** 183.8 cm^3
c) 180.5 cm^3 **d)** 158.8 cm^3

12. a) 1.15 m^3 **b)** 1.05 m sides
c) Cube; the less sphere-like, the greater surface area for the same volume
d) Cube is 6.62 m^2, sphere is 5.31 m^2

UNIT 5 NON-LINEAR FUNCTIONS AND EQUATIONS pp. 127–164

Getting Started — Skills You Will Need pp. 127–128

1. A; *Sample response:* It's a parabola.

2. a) –7 **b)** $-9x - 6$ **c)** $3x + 4$ **d)** $3x + 8$

3. a) $x = 4$ **b)** $x = 2$ **c)** $x = -\frac{2}{7}$

4. a) (2, 3) **b)** (4, –2) **c)** (4, –3) **d)** (1, –3)

5. a) $(3x + 1)(2x - 3)$ **b)** $6x^2 - 7x - 3$

6. a) $x^2 + 7x + 12$ **b)** $6x^2 + 13x + 6$
c) $x^2 - 9x + 20$ **d)** $-6x^2 + 13x + 5$

7. a) $y + 3$ if $y \neq 0$ **b)** $x + 5$ if $x \neq -2$

1. A, B, and E

2. a)

x	$f(x)$	$g(x)$
–2	1	1
–1	4	4
0	9	9
1	16	16
2	25	25

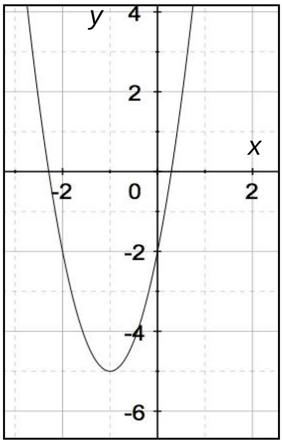
b) The functions produce the same output value for at least three values of x, so they are equivalent.

3. a) *Sample response*:
If $x = 0$, $g(0) = –12$, but $h(0) = –9$.
b) $f(x) = 3x^2 + 6x – 9$;
$h(x) = 3x^2 + 6x – 9$

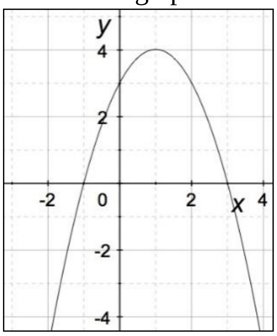
4. a) *See graph at right.*

b) *Sample response:*
(–1, –5); minimum

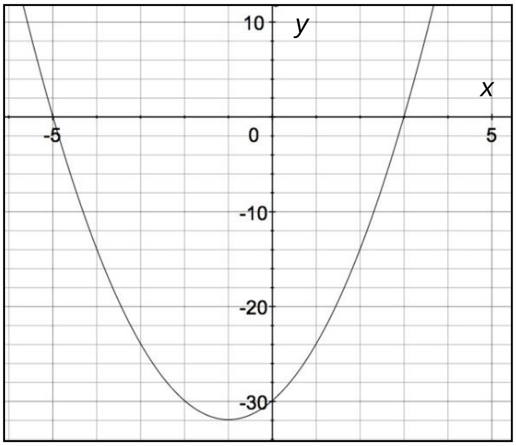
c) *Sample response*:
The x-intercepts are about 0.25 and –2.25 and the y-intercept is about –2.



5. All three graphs look like this:

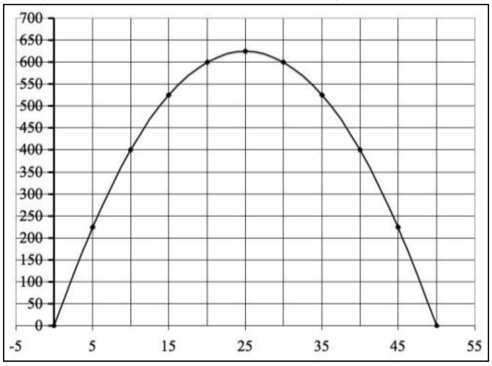


6. a)



6. b) *Sample response:* (–1, –32)

7. a) Vertical axis is Area (m^2)

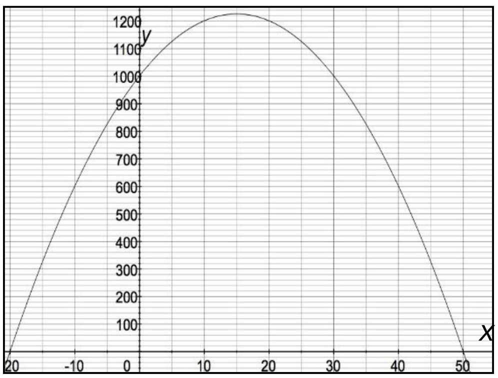


Width (m)

7. b) The coordinates of the vertex represent the greatest area and the width that relates to this area.

c) 25 m by 25 m; if the area is 625 m^2 at a width of 25 m, then the length has to be 25 m because $l = 50 – w$.

8. a) i) $20 + x$ **ii)** $50 – x$
b) $s(x) = (20 + x)(50 – x)$
c)



8. d) about Nu 35; about Nu 1225

9. I can create two different parabolas that go through two points, for example, the parabolas
$y = 2x^2 - x + 1$ and
$y = 3x^2 - 5x + 1$
both go through (0, 1) and (2, 3), but if I add the point (–1, 4), only $y = 2x^2 - x + 1$ works.

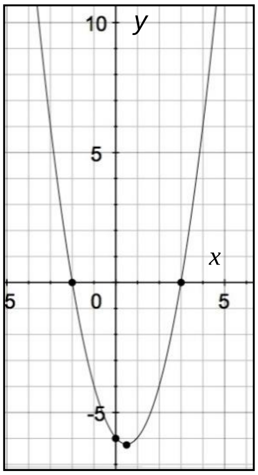
5.1.2 Graphs of Quadratic Functions in Factored Form p. 138

1. a)

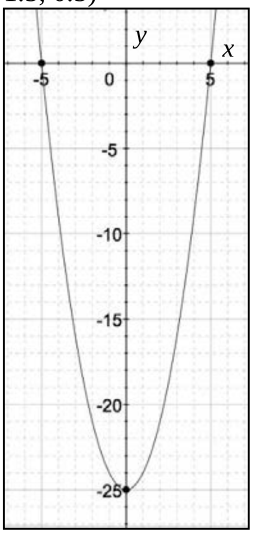
	Zeros	y-intercept
i)	–2, 3	–6
ii)	5, –5	–25
iii)	–1.5, –0.5	6
iv)	2, –2	–12
v)	–1, –2	–4

b) i) (0.5, –6.25) **ii)** (0, –25) **iii)** (–1, –2)
iv) (0, –12) **v)** (–1.5, 0.5)

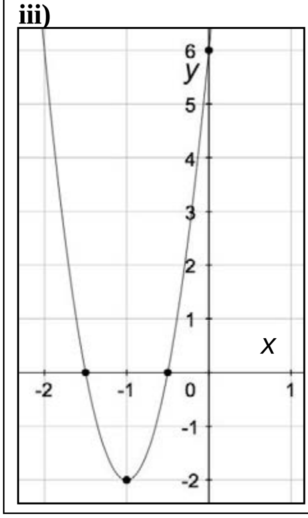
c) i)



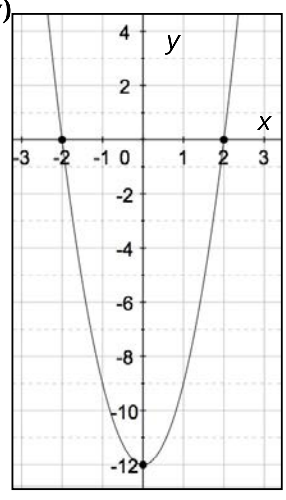
ii)



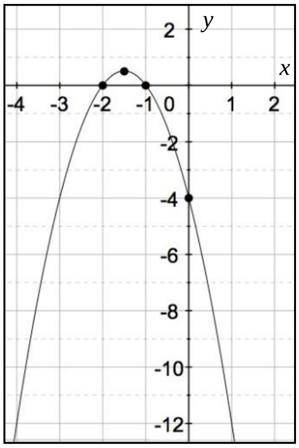
iii)



iv)



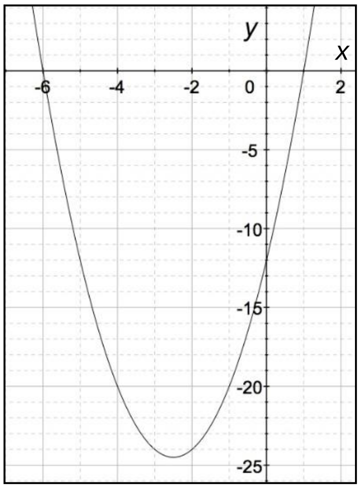
1. c) v)



2. a) $2(x - 1)(x + 6)$
$= 2(x^2 + 5x - 6)$
$= 2x^2 + 10x - 12$

b) When the function is in factored form, Mindu can determine the zeros easily in order to find the coordinates of the x-intercepts,
the vertex, and the y-intercept.

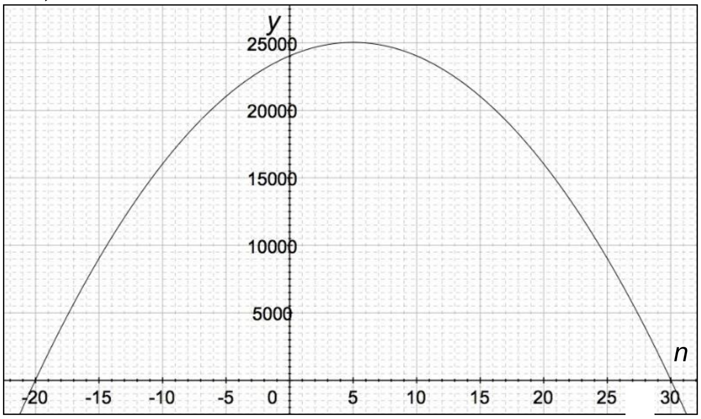
c)



3. a) i) $800 + 40n$
ii) $30 - n$
b) $f(n) = (800 + 40n)(30 - n)$
c) Nu 1000

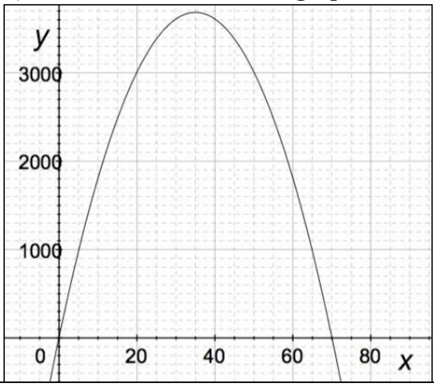
5.1.2 Graphs of Quadratic Functions in Factored Form [Cont'd] p. 138

3. d)



4. a) $a(w) = w(210 - 3w)$

b) 35 m; the vertex of the graph is at about (35, 3600)



The answer is reasonable.

5. Nu 150

6. The vertex is on the axis of symmetry. The axis of symmetry is halfway between the *x*-intercepts and is perpendicular to the *x*-axis. Any point on the axis of symmetry will have an *x*-coordinate that is halfway between the *x*-coordinates of the *x*-intercepts, which is the same as the mean of the zeros.

5.1.4 Relating Graphs of Quadratic Functions p. 147

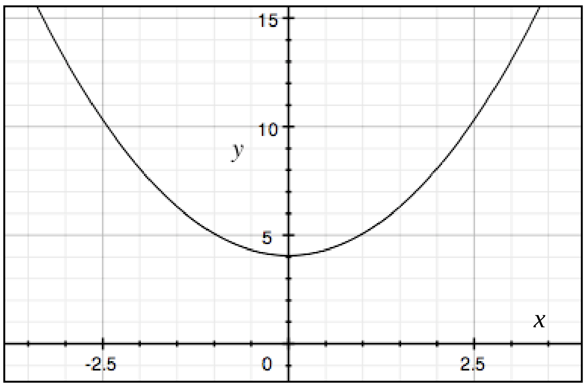
1. a) i) (0, 4) **ii)** (8, 0) **iii)** (3, –2) **iv)** (–1, –1) **v)** (1, 1) **vi)** (2, 2)

b) i) Up; not dilatated **ii)** Up; not dilatated

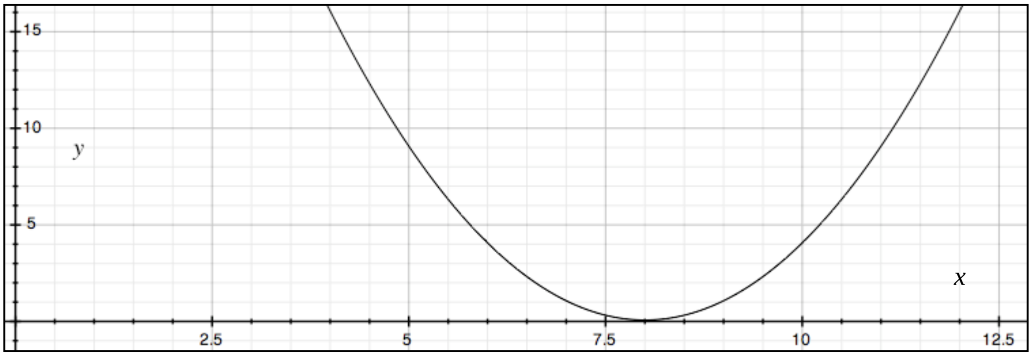
iii) Up; not dilatated **iv)** Down; dilatated by a factor of –2

v) Up; dilatated by a factor of $\frac{1}{5}$ **vi)** Down; dilatated by a factor of $-\frac{1}{5}$

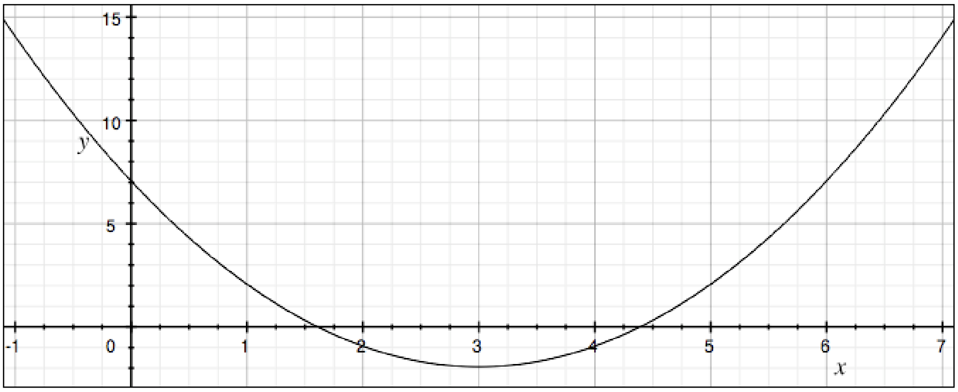
c) i)



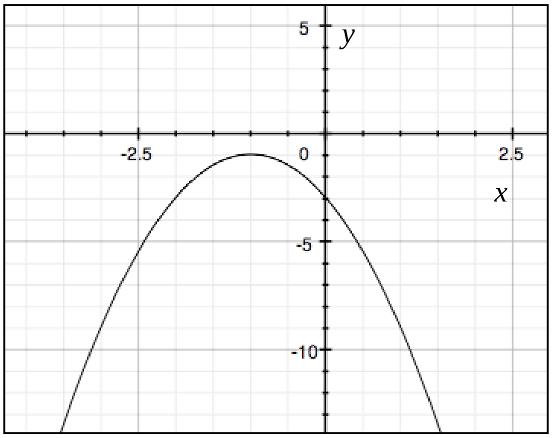
1. c) ii)


15
10
y
5
x
2.5
5
7.5
10
12.5

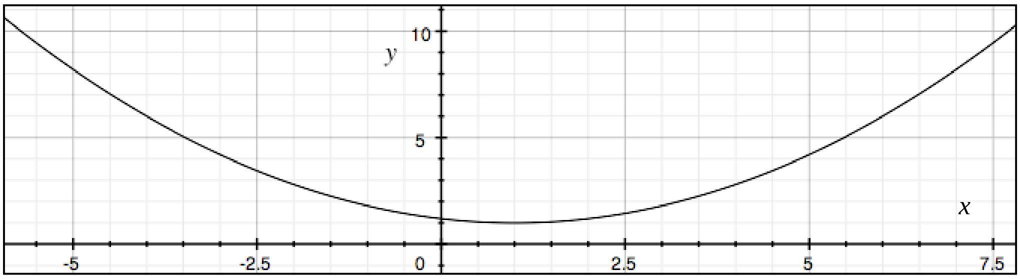
iii)


15
10
y
5
-1
0
1
2
3
4
5
6
7
x

iv)

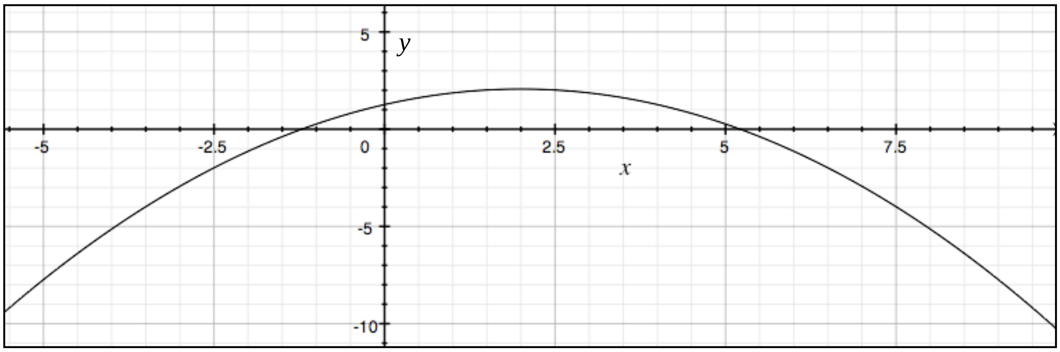

5
y
-2.5
0
2.5
x
-5
-10

v)


10
y
5
x
-5
-2.5
0
2.5
5
7.5

5.1.4 Relating Graphs of Quadratic Functions [Cont'd] p. 147

1. c) vi)



2. a) $y = (x - 4)^2$ **b)** $y = -\frac{1}{2}x^2$

c) $y = (x + 4)^2 - 3$ **d)** $y = \frac{1}{4}(x - 4)^2$

e) $y = -5(x - 4)^2$ **f)** $y = -3(x - 4)^2 + 6$

3. $y = -2(x + 2)^2 - 1$

4. a) When the dilatation factor is, for example, –1, the y-coordinate of every point on the parabola is multiplied by –1 because $(x, y) \rightarrow (x, -1 \times y)$. The x-coordinate stays the same but the y-coordinate becomes its opposite so every point is reflected across the x-axis, resulting in a parabola that is a refection of the original parabola.

4. b) The word vertical applies to vertical axis (the y-coordinate of each point),
so when the y-coordinate of each point on a parabola is multiplied by a factor of 2 while the x-coordinate stays the same, the parabola is stretched vertically.

5. It is a good name because the vertex is easy to figure out when it is in the form $f(x) = a(x - \mathbf{h})^2 + \mathbf{k}$; the vertex is $(\mathbf{h}, \mathbf{k})$. E.g., the vertex of $y = -3(x - \mathbf{4})^2 + \mathbf{6}$ is $(\mathbf{4}, \mathbf{6})$.

CONNECTIONS: Parabolas and Paper Folding p. 148

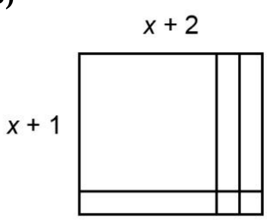
1. The vertex is halfway between the Focus and the line.

2.
- The distance between the Focus and the line influences the width of the parabola.
- If the focus point and the line are the same distance from the x-axis, the parabola's vertex is at the origin. Otherwise, the vertex is above or below the x-axis.
- If the Focus is not on the y-axis, the vertex will not have an x-coordinate of 0.

5.2.1 Factoring Quadratic Expressions p. 156

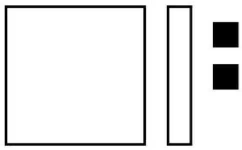
1. a) $x^2 + 3x + 2$

b)



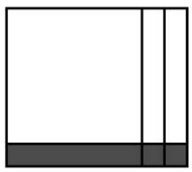
c) $(x + 1)(x + 2)$

2. a)



$x^2 + x - 2$

b) c) $(x - 1)(x + 2)$



3. a) $2x(2x + 3)$ **b)** $(x – 3)(x + 1)$
c) $(x + 3)(x + 1)$ **d)** $(x + 3)(x – 3)$
e) $(3x + 1)(3x – 1)$ **f)** $(x + 1)(3x + 2)$

4. a) $(x + 3)(x + \mathbf{4}) = x^2 + \mathbf{7}x + 12$
b) $(x + \mathbf{9})(x – \mathbf{9}) = x^2 – 81$
c) $(\mathbf{1}x + 2)(\mathbf{5}x – \mathbf{1}) = 5x^2 + 9x – 2$
d) $(2x + \mathbf{3})(\mathbf{3}x – 2) = 6x^2 + \mathbf{5}x – 6$

5. a) $(x + 11)(x – 11)$ **b)** $(x + 20)(x – 20)$
c) $(5x + 1)(5x – 1)$ **d)** $(6x + 5)(6x – 5)$

6. a) $6x(2x + 3)$
b) $5x(3x – 5)$
c) $ax(x + 1)$

7. a) $(x + 3)^2 = x^2 + \mathbf{6}x + \mathbf{9}$
b) $(x + \mathbf{6})^2 = x^2 + \mathbf{12}x + 36$
c) $x^2 – 12x + \mathbf{36} = (x – \mathbf{6})^2$
d) $x^2 + \mathbf{14}x + 49 = (x + \mathbf{7})^2$

8. a) $(x + 3)(x + 5)$ **b)** $(x – 6)(x – 1)$
c) $(x + 4)(x – 3)$ **d)** $(x + 2)(x – 8)$
e) $(x – 4)^2$ **f)** $(x + 8)(x – 8)$

9. a) $(2x – 3)^2$ **b)** $(2x + 3)(x – 2)$
c) $(3x + 1)(x – 4)$ **d)** $(5x – 1)(2x + 1)$
e) $(3x – 2)(x – 3)$ **f)** $(7x + 10)(7x – 10)$

10. *Sample response*:
Similar: If you think of factoring using the area model, you are still finding the side lengths of a rectangle with a particular area that you know.
Different: You have to juggle more pieces of information to figure out the factors of a quadratic compared to factors of a number.

5.2.3 Solving Quadratic Equations by Factoring p. 161

1. a) 4, 2 **b)** –3, 9 **c)** $\frac{-5}{2}, \frac{1}{3}$
d) $\frac{-4}{5}, \frac{9}{2}$ **e)** 0, 10 **f)** –2, 2

2. a) –3, –5 **b)** 11, –10 **c)** 0, –3
d) 6, –6 **e)** 10 **f)** –2

3. a) $5, \frac{-5}{2}$ **b)** $\frac{1}{4}, \frac{-1}{2}$ **c)** $-3, \frac{1}{2}$ **d)** 2, –1

4. a) –1, 7 **b)** 5, 2 **c)** 3, 4 **d)** 5, –5

5. a) $x^2 + (x + 1)^2 = 221$
b) $x = 10$ and $–11$
c) Two pairs are possible, 10 and 11 and –10 and –11. The roots are the lesser number in each pair.

6. 7 cm by 13 cm **7.** 6 cm

8. 17 m by 34 m

9. 8 m

10. 5 m, 12 m, and 13 m

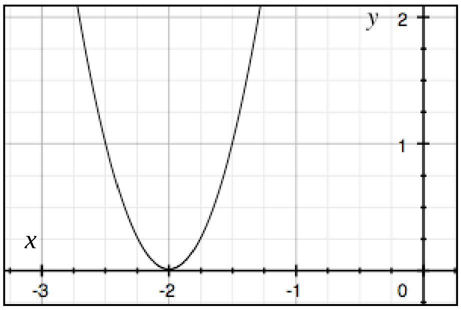
11. 11 and 13, or –11 and –13

12. 10 s and 30 s; at 10 s the rocket is on the way up and at 20 s it is on the way down.

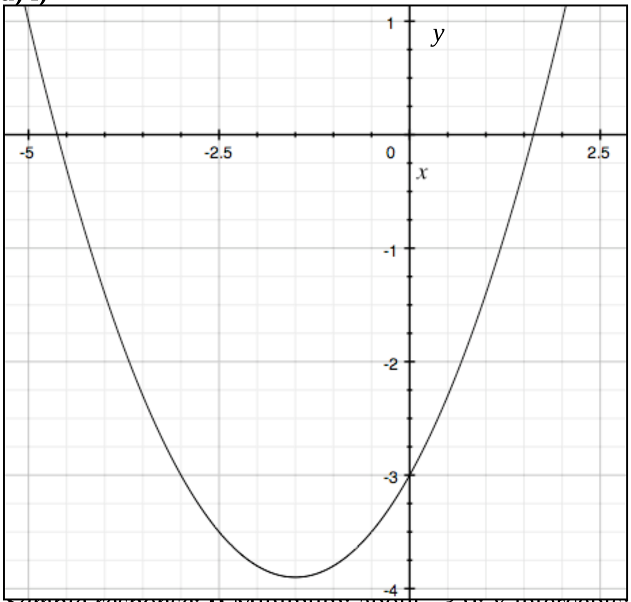
13. *Sample response:*
Some situations involve areas of rectangles, relationships of sides of a right triangle and relationships involving squares of numbers.

UNIT 5 Revision pp. 163–164

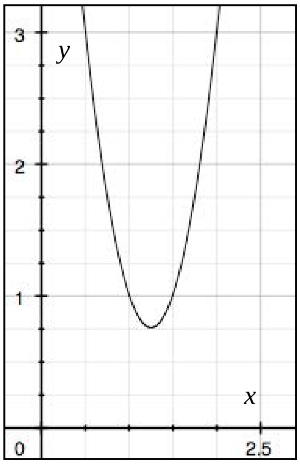
1. a) C and D
b) The graph is a parabola, so the functions are quadratic; The graphs are identical, so the functions are equivalent.
c) *Sample responses:*
I could substitute three values of x into each function and see if I get the same results each time (I only need three points on each to be the same to be sure they are equivalent since they're quadratic functions).



2. a) i)



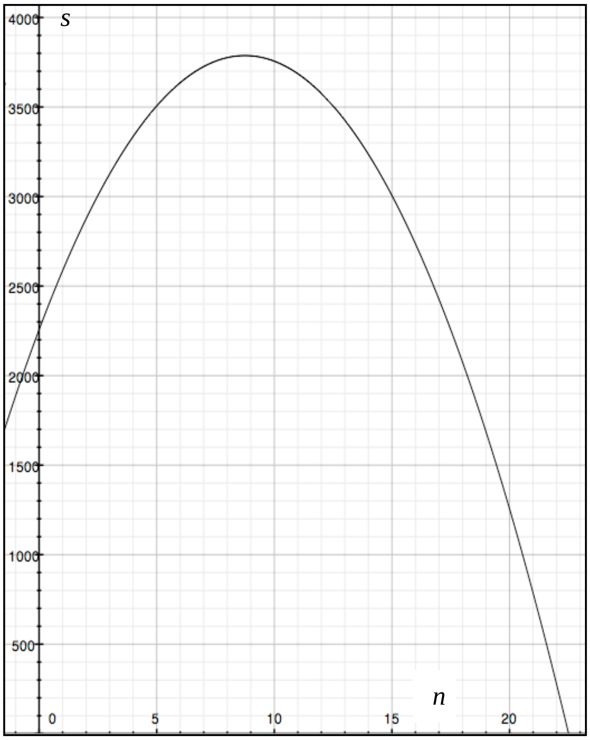
ii)



b) *Sample response*: **i)** Minimum: about –3.9; *x*-intercepts: –4.6 and 1.6
ii) Minimum: about 0.8; no *x*-intercepts

3. For negative values of x, each term is positive (because $-x$ would be positive and $4x^2$ is never negative) so the sum is positive and not 0. I think $4x^2$ will outweigh $-x$ for all positive values except maybe for very small values, so I only checked small positive values. I checked $x = 1$ and 2 and the values were positive. I am sure $f(x)$ will always be positive, because for small positive values, the +8 will outweigh anything else.

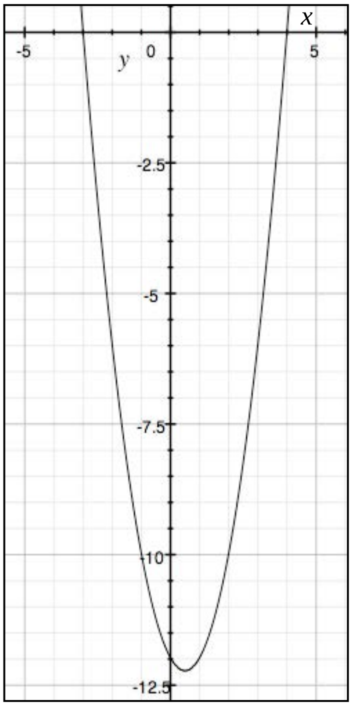
4. a) $s = (50 + 10n)(45 - 2n)$, where s is total sales and n is the number of Nu 10 price increases
b) *See graph at right.*
c) The best price to charge is Nu 140. The vertex shows you the maximum value (maximum sales) of the function (Nu 3780) and the number of Nu **10** increases (**9**) over the original price of Nu **50** at that level of sales. Using those values:
Nu **50** + **9** × Nu **10** = Nu 140



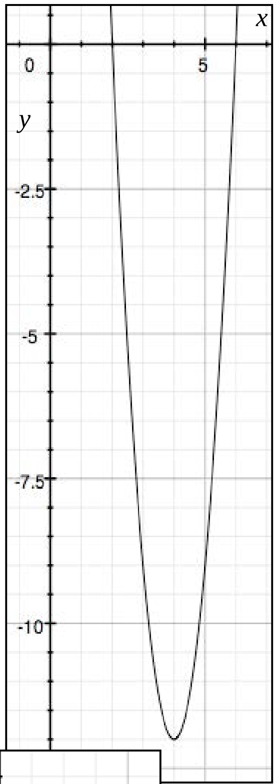
5. a) i) 4, – 3 **ii)** 6, 2
iii) 2.5, –1 **iv)** 0.5, –2.5
v) 0.6 , –5

b) i) (0.5, –12.25) **ii)** (4, –12)
iii) (0.75, –18.375) **iv)** (–1, –2.25)
v) (–2.2, –23.52)

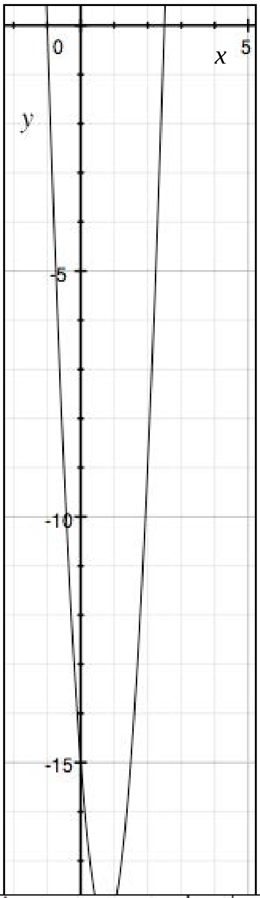
5. c. i)



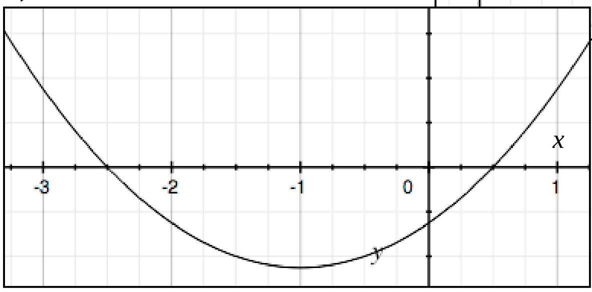
ii)



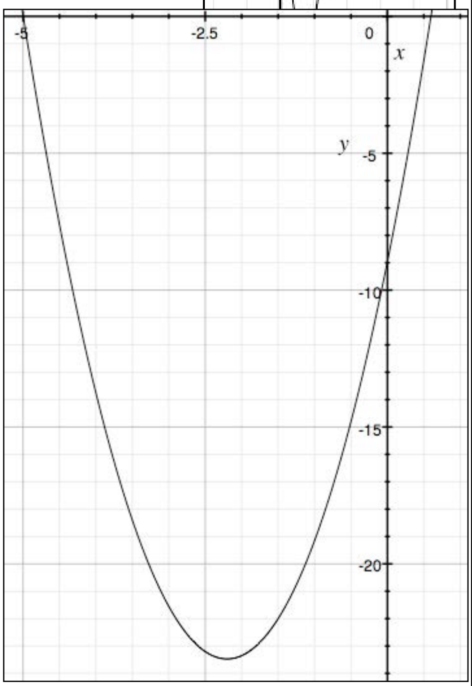
iii)



iv)



v)



6. h > –4; the *x*-intercepts are 4 and h; The *x*-coordinate of the vertex is at (4 + h) ÷ 2, which is only positive if 4 + h is positive, so h > –4.

7. a) i) (Vertical) translation down 30 units

ii) (Horizontal) translation left 30 unit

iii) Reflection in *x*-axis, (vertical) stretch by 2, and then (vertical) translation down 30 units

iv) (Horizontal) translation right 2 units and then (vertical) stretch by 3 (or reversed)

v) (Horizontal) translation left 30 units and then (vertical) translation down 7 units (or reversed)

vi) Reflection in the *x*-axis, (vertical) compression by 10, and then (vertical) translation up 8 units

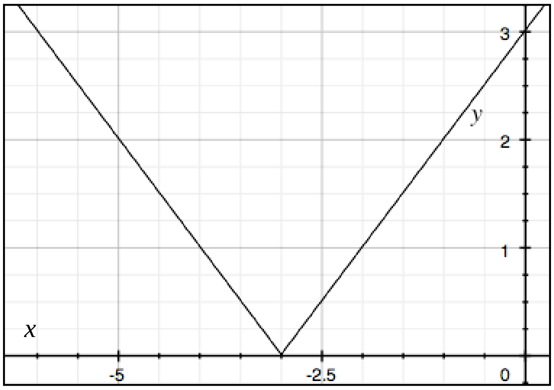
7. b) i) (0, –30) **ii)** (–30, 0)
iii) (0, –30) **iv)** (2, 0
v) (–30, –7) **vi)** (0, 8)

c) i) $(x, y) \to (x, y - 30)$
ii) $(x, y) \to (x - 30, y)$
iii) $(x, y) \to (x, -2y - 30)$
iv) $(x, y) \to (x + 2, 3y)$
v) $(x, y) \to (x - 30, y - 7)$
vi) $(x, y) \to (x, -0.1y + 8)$

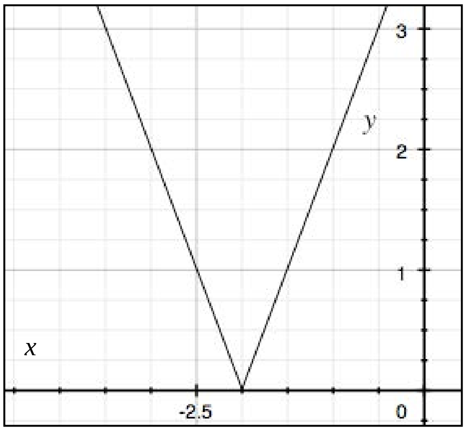
8. a) $f(x) = (x - 4)^2 + 3$
b) $f(x) = 0.5x^2 + 6$
c) $f(x) = -3(x + 3)^2$
d) $f(x) = (x + 8)^2 + 2$
e) $f(x) = 3(x + 8)^2 + 2$

9. a) $y = 3(x - 2)^2$
b) $y = 2(x + 3)^2$
c) $y = x^2 - x - 2$

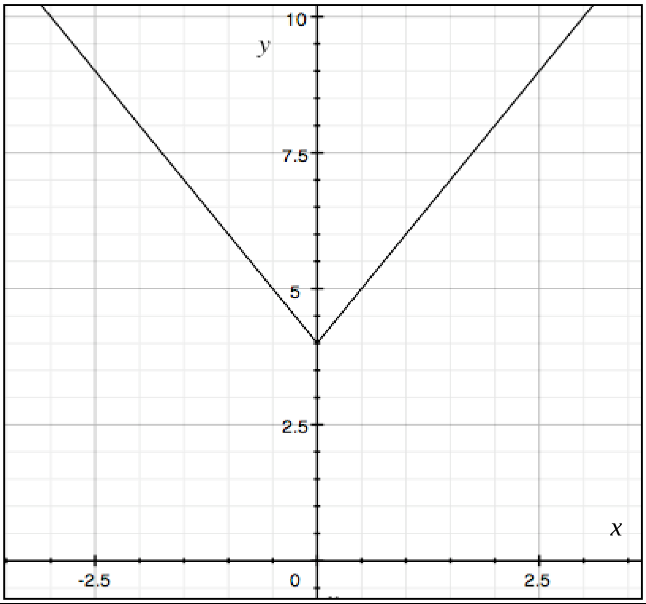
10. a)



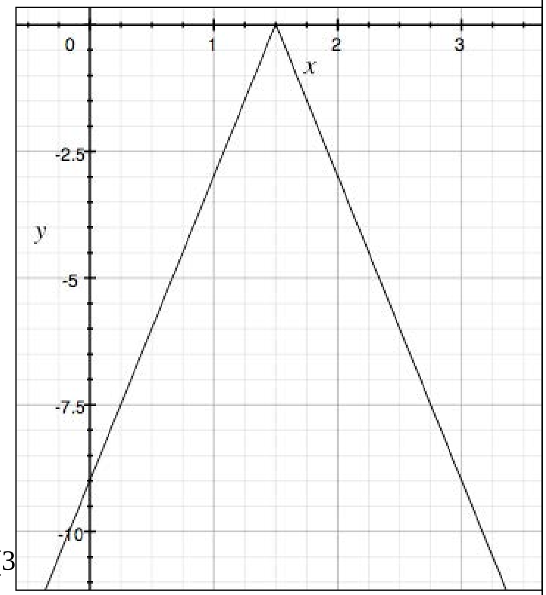
b)



10. c)



10. d)



11. a) $2x(3x - 4)$ **b)** (3
c) $(x + 4)(x + 2)$ **d)** $(x + 1)(x - 3)$
e) $(2x - 1)(x + 1)$ **f)** $(3x - 1)(2x + 1)$

12. a) $4(2x + 1)(2x - 1)$ **b)** $4x(4 - 3x)$
c) $(5x + 3)^2$ **d)** $(x + 11)(x - 8)$

13. a) $x^2 + \mathbf{10}x + 25 = (x + \mathbf{5})^2$
b) $(3x - \mathbf{4})^2 = \mathbf{9}x^2 - 24x + \mathbf{16}$

14. a) $x = 2, \frac{1}{2}$

b) $x = \frac{3}{2}, -\frac{1}{3}$

c) $x = -2, 5$

15. a) $x^2 - 11x + 24 = 0$
b) $x^2 + 9x + 14 = 0$

c) $20x^2 - 31x + 12 = 0$
d) $25x^2 - 45x + 18 = 0$

16. 3.39 m

17. 17 and 18 or –17 and –18

18. a) $x = 2, -2$

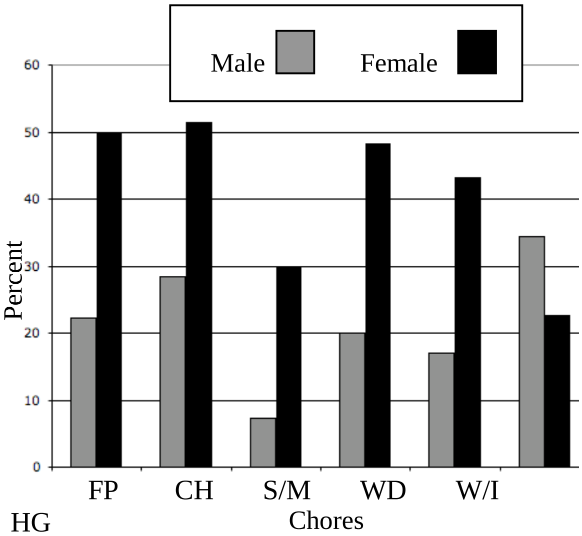
b) $x = \frac{8}{3}, -\frac{4}{3}$

c) $x = \frac{5}{3}, -\frac{7}{3}$

UNIT 6 DATA, STATISTICS, AND PROBABILITY pp. 165–226

Getting Started — Skills You Will Need pp. 165–166

1. a) Household Chores Performed by Thailand's Elderly



b) *Sample response:*
Females perform the majority of household chores, with the exception of gardening.

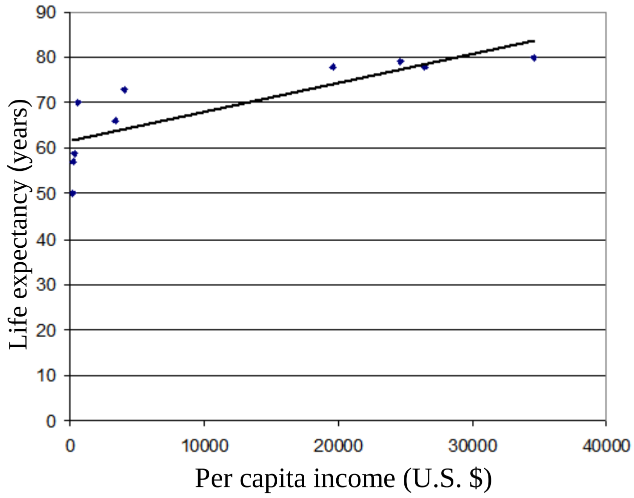
2. Mean 59.4, median 58.9, modes 66.0 and 53.5

3. Maximum Temperatures

Stems	Leaves
18	.7
19	.4 .6 .7
20	
21	.1 .2
22	.9
23	.1 .5 .5
24	.8 .9
25	.0 .1 .4
26	.6 .7 .8
27	.1 .2
28	.4
29	.1
30	.0 .3

Getting Started — Skills You Will Need [Cont'd] pp. 165–166

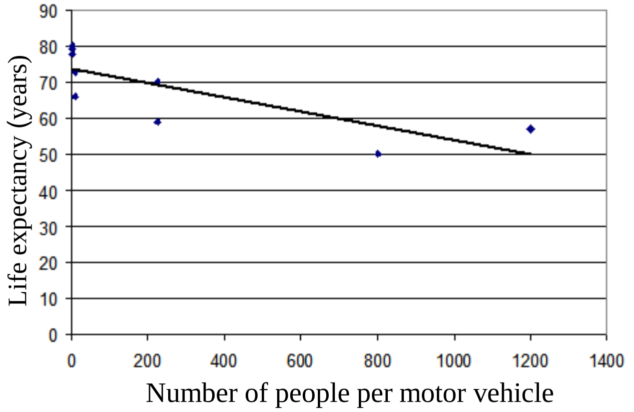
4. a) and c) Life Expectancy vs. Per Capita Income



c) The correlation is positive because it slopes up and weak because many of the points are far from the line.

d) *Sample response:*
A positive correlation make sense because if you have more money, it means you are better able to afford better health care and a healthy diet, so life expectancy increases. However, because there are other factors that affect life expectancy, the correlation is not very strong.

b) and c) Life Expectancy vs. Number of People Per Motor Vehicle



c) The correlation is negative because it slopes down. It is weak because many of the points are far from the line.

d) *Sample response:*
A negative correlation make sense because the more people per vehicle in a country, the less affluent are its people, and less money means they are less able to afford better health care and a healthy diet, so life expectancy decreases. However, because there are other factors that affect life expectancy, the correlation is not very strong.

5. First graph: discrete; one variable, countries
Second graph: continuous; two variables, independent variable is initial speed and dependent variable is skid length

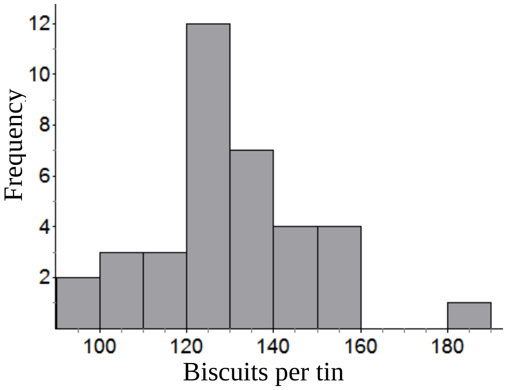
6.1.1 Histograms and Stem and Leaf Plots pp. 175–176

1. a) T **b)** F **c)** F **d)** T **e)** T **f)** F

2. a) Number of Biscuits Per Tin

Stems	Leaves
9	6 9
10	7 8 8
11	7 9 9
12	2 2 3 3 3 5 6 7 7 7 8 9
13	2 2 3 4 5 7 7
14	2 3 4 5
15	0 0 1 4
16	
17	
18	8

b) Number of Biscuits Per Tin



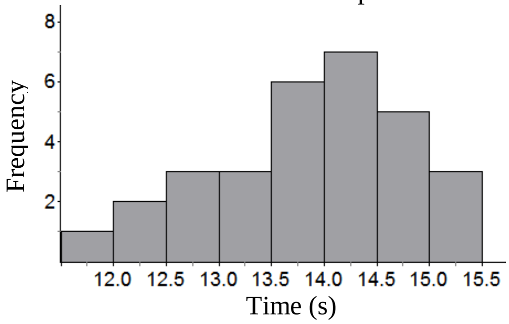
c) The manager would want to make sure each tin contained about the same number of biscuits. Both graphs show that only about half (19 out of 36) of the tins contained between 120 and 140 biscuits. There were 9 tins with more than 120 and 8 tins with fewer. The manager would use this data to support efforts to change the way the tins are filled.

3. a) *Sample response:*

Time (s)	Frequency
11.5–12.0	1
12.0–12.5	2
12.5–13.0	3
13.0–13.5	3
13.5–14.0	9
14.0–14.4	8
14.5–15.0	5
15.0–15.5	3

3. a) [Cont'd]

Times for the 100-m Sprint



b) *Sample response:*
The majority of students (18 out of 34) ran the race in a time between 13.5 s and 15.0 s. Nobody ran faster than 11.5 s or slower than 15.5 s.

4.

Group 2		Group 1
99 89	1	50 63 68 83
99 90 77 76 56 55	2	46 87 98
89 85 24 24 10	3	55 55 76
12	4	22 78
	5	05 99

5. a) 37 **b)** 11 **c)** 26
d) 29 km/h **e)** 96 km/h

6. Yes; Even though there are more data values on the male side (the plot shows data for 11 males and 7 females), the data values are clustered at the higher end for the females but not for the males. If you look at the total number of hours for each group — males at 240 h and females at 243 h — you notice that the females in this group watched more TV in a week.

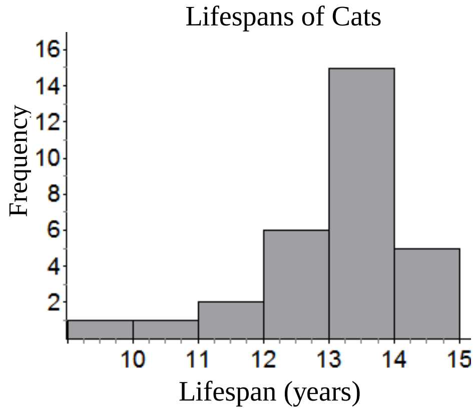
7. a) i) *Sample response:*
Most heights are between 155 and 165 cm. There are 30 students in the class. The range is less than 30 cm.
ii) *Sample response*:
There are usually between 100 and 130 passengers on the train in a six-week period. There are never more than 151 or fewer than 95. The mode is 129.

6.1.1 Histograms and Stem and Leaf Plots [Cont'd] pp. 175–176

7. b) i) Intervals of 5 are appropriate because they are easy to work with; six intervals are appropriate since there are data values in each interval but not too many. There are differences in the frequencies of the intervals.
ii) Place value intervals of 10 are appropriate because the data values range from 95 to 151. The only other reasonable choice is intervals of 100, and that would mean only 2 intervals, which would not be enough.

8. a) Lifespans of Cats (in years)

Stems	Leaves
9	.6
10	.4
11	.5 .8
12	.8 .8 .8 .9 .9 .9
13	.0 .1 .1 .2 .2 .3 .3 .4 .5 .5 .6 .6 .6 .8 .9
14	.1 .4 .5 .6 .8



b) The stem and leaf plot; It shows every piece of data and they are arranged in increasing size so it is fairly easy to count from the least or greatest number to find the one in the middle.

8. c) *Same*: They both use intervals of 1 so the frequencies for each interval are the same. This results in the same shape, even though the stem and leaf plot is turned sideways.
Different: The stem and leaf plot shows every piece of data, whereas the histogram does not.

9. a) Three age intervals occur most frequently: 1–10, 21–30, and 41–50
b) Two age intervals occur most frequently: 10–19 and 30–39
c) The conclusions are different because the graphs use different intervals:
- If there were a lot of 10-year-olds, Yuden's first bar would be tall, but Maya's second bar would be tall.
- If there were a lot of 30-year-olds, Yuden's third bar would be tall, but Maya's fourth bar would be tall.
- It looks like there were a number of 10-year-olds and 30-year-olds in the group and that is why the graphs look different.

10. Organizing the data into equal intervals, or bins, reduces a set of data from a number of single values to a smaller and more manageable number of intervals. Then all you have to do is compare the frequencies of the various intervals to see how the data values are distributed.

6.1.3 Histograms and Box and Whisker Plots pp. 184–185

1. a) minimum: 4 ppm

Q1: 29 ppm

Q2 (median): 48.5 ppm

Q3: 80 pp,

maximum: 141 ppm

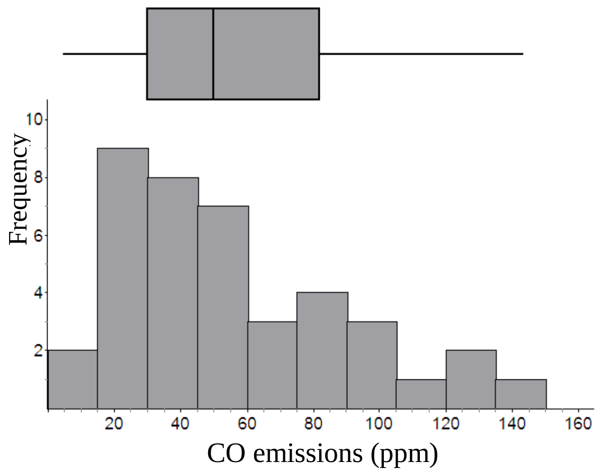
b) *See answer to **part d)**.*

1. c) *Sample response:*

Daily CO Measures

CO emissions (ppm)	Frequency
0 – 15	2
15 – 30	7
30 – 45	10
45 – 60	7
60 – 75	3
75 – 90	4
90 – 105	3
105 – 120	1
120 – 135	2
135 – 150	1

d) CO Emissions

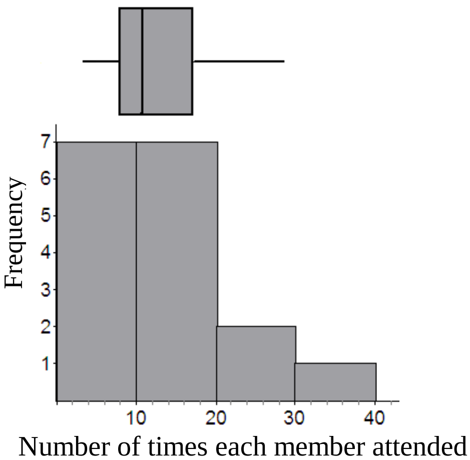


d) *Sample response:*

Box plot: median CO emission is about 48.5 ppm and 50% of the emission data values are between about 29 ppm and about 80 ppm

Histogram: the majority of CO emissions ($\frac{24}{40}$ or 60%) fall between 15 ppm and 59.9 ppm

2. a) and c) Number of Times Running Club Members Met in October



2. b) Minimum is 4, Q1 is 8, Q2 or Median is 11, Q3 is 18, Maximum is 30

d) *Sample response:*

- The box of the box plot is above the two highest bars in the histogram. The box plot shows that 50% of the data values are found between 8 and 18 but the histogram shows that 14 out of 17 (82%) of the data values are between 0 and 20.
- The box plot makes it easy to see that, even though the histogram shows that most of the data values are between 0 and 20, 50% of them are actually between 8 and 18.
- The box plot also shows the median, 11. Because it is above the histogram, it shows where the median occurs in the histogram.

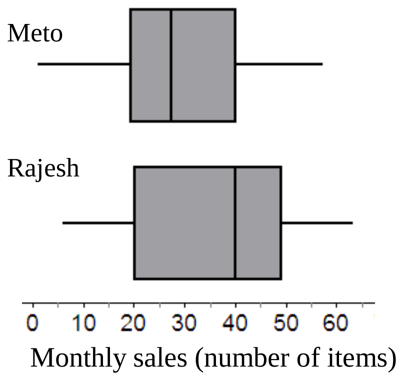
3. a) 34%

b) Between 10% and 43%

c) *Sample response*:

There are 2 types of breakfast cereal. One type has brands that are low in sugar and the other type has brands that are high in sugar. This explains the two tall bars, one at each end of the graph.

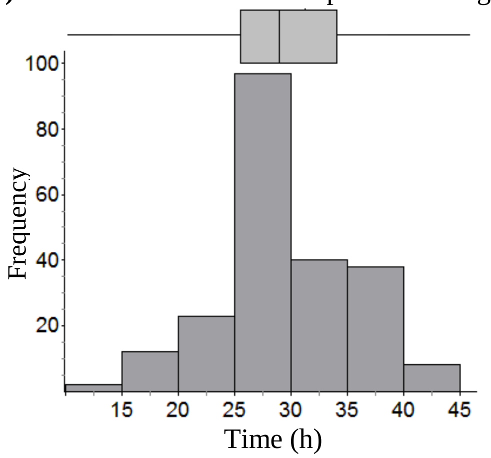
4. a) Monthly Sales Comparison



b) Rajesh seems to be the better salesman because of his significantly higher median. The range is similar for both though Rajesh's sales are consistently above the corresponding sales for Meto.

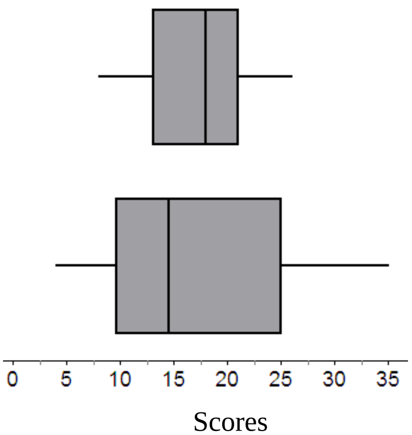
5. a) Lower quartile = 25.95
Median = 28.79
Upper quartile = 33.94
Minimum = 10
Maximum = 44

b) Time Canadian Students Spent Watching TV

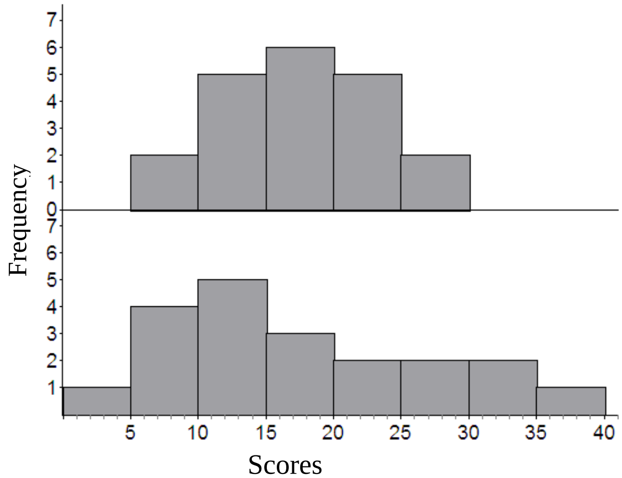


c) *Histogram*: the majority of the students ($\frac{175}{220}$ or about 80%) watch TV between 25 h and 40 h each week
Box plot: median time is about 29 h/week and half the students watch between 26 h and 34 h each week

6. a) Dema is top plot and Lemo is bottom plot



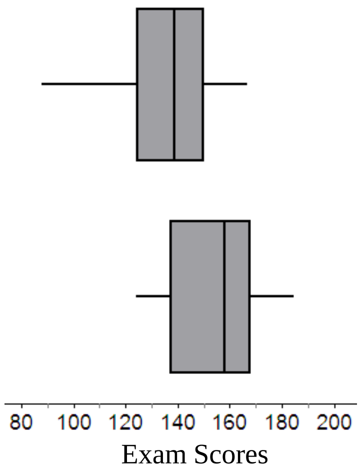
b) Dema is top histogram and Lemo is bottom histogram



c)
- From the box plot, you can tell that Dema has a higher median number of points.
- From both graphs, you can tell that Lemo is capable of scoring more points than Dema (above 28) but also fewer points because Lemo has a greater range with a minimum that is below Dema's and a maximum that is above Dema’s.

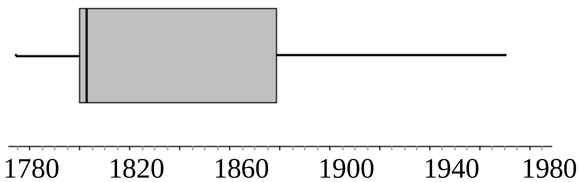
- *Conclusion*: Dema is the stronger player overall because of the higher median score and more consistent results.

7. a) Group 1 is top plot, Group 2 is bottom plot



7. b) Yes; Group 1 had a lower median score and lower maximum and minimum scores, indicating that they did not do as well as the Group 2 students. Group 1's wider range also indicates less consistent results.

8. a) Minimum is about 1775, maximum is about 1974, median is about 1831, lower quartile is about 1795, and upper quartile is about 1878.



8. b) Maybe, but only if the estimated values used for this box plot were the same as the actual values. For example, if no state joined in 1974, the whisker would be different.

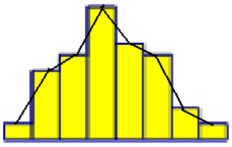
9. Count the total number of data values and divide by 4; call it *d*. Keep a running total of the frequencies starting at the lowest interval and stop once you have reached (or passed) *d*. Suppose this happened in the interval with the star (*see table to the right*). Calculate the total number of data values in the intervals before that interval and call it *x*.
Call the frequency of the interval with a star *y*. Subtract *x* from *d*. Divide *d* by *y*. Multiply that fraction by the size of the interval and add it to the lowest value in the interval.
For example, if there were 40 data values, you would look for the 10th value (so $d = 10$). If there were 7 data values in the first two intervals and 5 data values in the next interval, *x* would be 7 and you would calculate 10 – 7 = 3 to find *y*.
Multiply $\frac{3}{5}$ by the size of the interval and add it to the low value in the interval column for that row.

Interval	Frequency
*	*y*

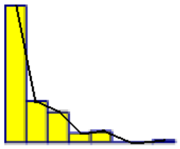
10. The box plot shows the median and how the data values are clustered around the median, where 50% of the data values lie. It also shows the location of the extreme values. You cannot directly determine these exact values from a histogram.

6.1.4 Data Distribution pp. 192–194

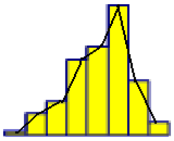
1. a) Normal



b) Positively or right skewed

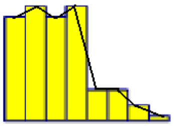


c) Negatively or left skewed

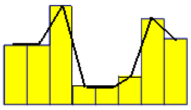


d) Uniform

e) Positively or right skewed

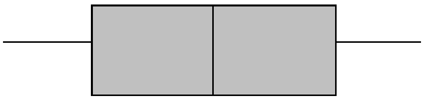


f) U-shaped

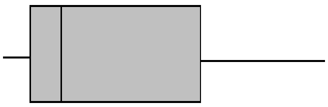


2. *Sample responses:*

a)



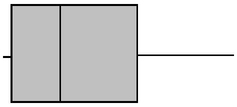
b)



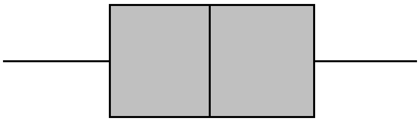
c)

d)

e)

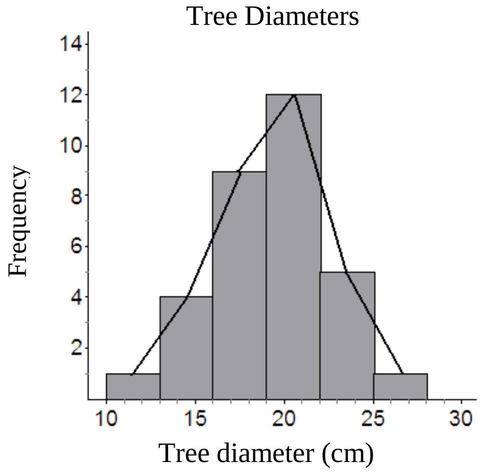


f)



3. a) Mean: 19.1 cm, median: 19.35 cm, modes: 18.7 cm and 19.5 cm

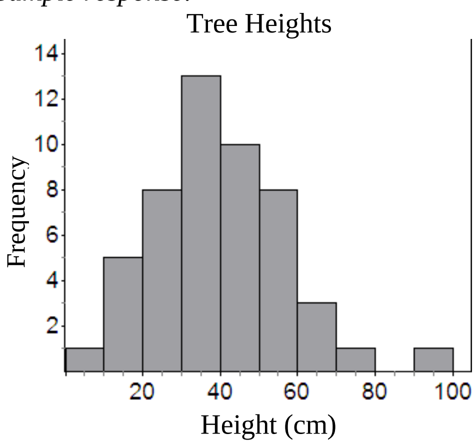
b) *Sample response:*



c) It is almost a normal distribution; it has an almost symmetrical mound shape.

d) Some trees may be smaller due to disease, or their poorer location with respect to sunlight, moisture, and soil.

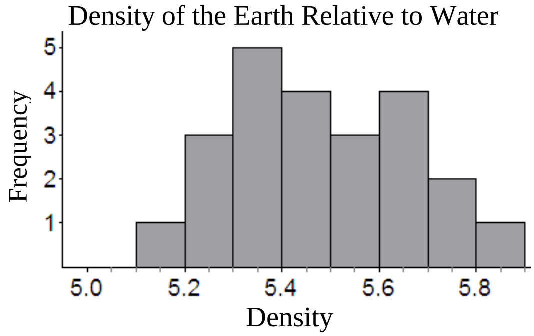
4. a) It is close to a normal distribution.
Sample response:



b) 38.8 cm; the median height will be close to the mean height since this is close to a normal distribution.

5. a) Mean: 5.48, median: 5.46, modes: 5.29 and 5.34
The mean and median are very close to each other but the modes differ. Both modes are lower than the mean and median.

5. b) *Sample response:*

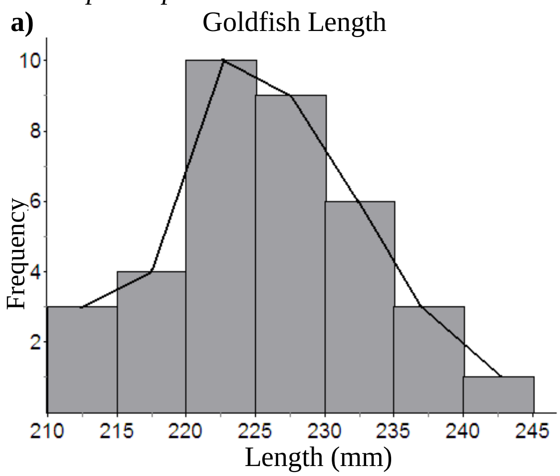


5. c) It is close to a normal distribution.

6. a) The Granny Smith sample is close to being a normal distribution because the distribution is symmetrical and the median is in the centre of the box.
b) The Red Delicious sample is left skewed because most of the values are large and the median is to the right of the centre of the box.
c) The McIntosh sample is right skewed because most of the values are small and the median is to the left of the centre of the box. The right whisker is longer.
d) Granny Smith: A lot of apples are close to the median mass while a few are heavier and a few more are light.
Red Delicious: There are a lot of heavier apples in the sample, with a few very heavy ones and no really light ones.
McIntosh: There a lot of lighter apples in the sample, with no very heavy ones and some very light ones.

7. a) The 323 has the most consistent ratings; shown by the short whiskers and narrow box
b) Mirage; The median is close to the centre of the box and the whiskers are about the same length.
c) Tracer and Festiva; There are long whiskers on the right with very small ones on the left.
d) The data set clearly shows that the car the consumers were most satisfied with was the Civic because it had the highest median, minimum, and maximum values.

8. *Sample response*:
a)



8. b) It is close to a normal distribution.
c) 210 to 220 mm: Unlikely; The data set shows that only 7 out of 36 (less than 20%) of the goldfish in the pond were within this range.
220 to 230 mm: Likely; The data set shows that 19 out of 36 (more than 50%) of the goldfish in the pond are within this range.

9. Table A is close to a normal distribution because the middle two intervals have the greatest frequencies and the frequencies of the intervals on either side decrease.
Table B is right skewed because the first three bins have the greatest frequencies.

CONNECTIONS: Normal Distribution and Sample Size p. 194

1. a) The data will vary from class to class.
b) The frequency table created will depend on the data collected and the bin width used to organize the data. The histogram and frequency polygons will vary from student to student unless the class uses a common table to represent the data set.
c) The shape depends on the results of **parts a) and b)**. It may be skewed, bi-modal, or some other shape.

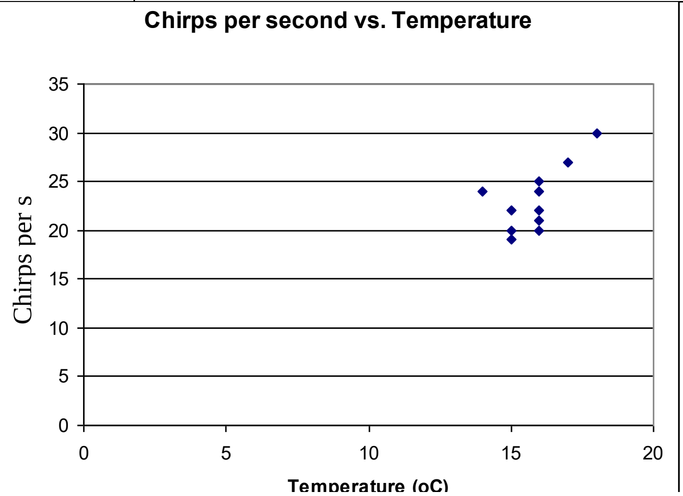
2. a) Data will vary from school to school
b) The frequency table created will depend on the data collected and the bin width used to organize the data.
c) Both histograms are similar because they show the heights of students. The scale on both axes and the units used will be the same. The range of data will be much greater for the data set of the entire school. This data set will also contain many more numbers. Its histogram should be close to a normal distribution. The class data set contains fewer numbers and its range will be much smaller. Its shape should look different than the data displayed for the entire school. This does not mean that the class data will not be symmetrical; it could be, but it's less likely.

6.2.1 Correlation and Lines of Best Fit pp. 203–204

1. a) I) Negative; data values are clustered around a line that goes from the upper left to the lower right.
II) Positive; data values are clustered around a line that goes from the lower left to the upper right.
III) No correlation; data values are not clustered around a line; they are scattered in all directions.
IV) Negative; data values are clustered around a line that goes from the upper left to the lower right.
b) I) r is about –1 II) r is about 1
III) r is about 0 IV) r is about –0.5

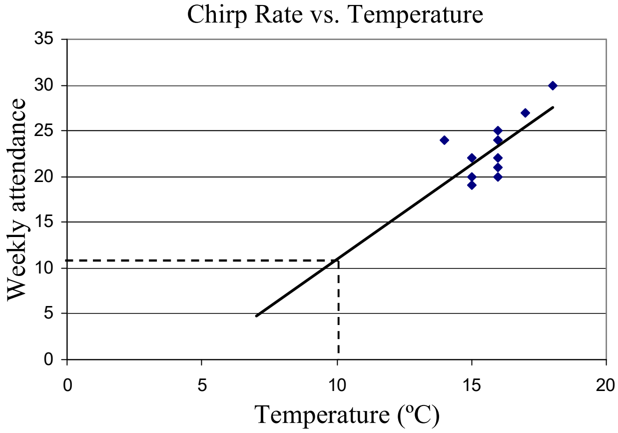
2. a) Negatively correlated; as you go up in altitude, the air temperature decreases.
b) Not correlated
c) Positively correlated; as the outside temperature increases and people get hot, there is a greater need for cold drinks so sales would increase.
d) Not correlated
e) Perfectly negatively correlated; as you read through a book, the number of pages you have read increases and the number of pages you have not yet read decreases.

3. a) The independent variable is temperature and the dependent variable is chirps per second. I suspect this because the table of values lists the independent variable in the left column and the dependent variable in the right column. Also, the number of chirps a cricket makes depends on the temperature. This means that the chirp rate must be the dependent variable.



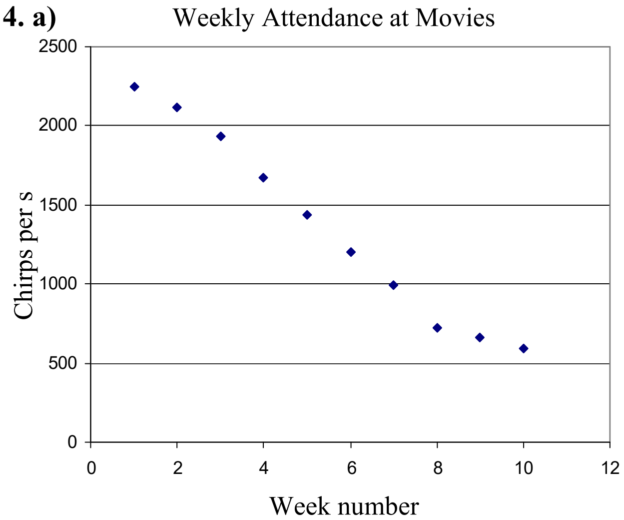
A line of best fit is appropriate since the data values are continuous and show a linear relationship — the points are in a reasonably linear pattern.
c) About 0.5; weak positive and linear

3. d) About 11 times/s; I drew a line through the dots and extended it beyond 10ºC. Then I used the line to predict the chirp rate for 10ºC.



3. e) I am not very confident in this prediction. Crickets might stop chirping altogether when the temperature drops below, for example, 14ºC. I do not have enough data to know for sure.

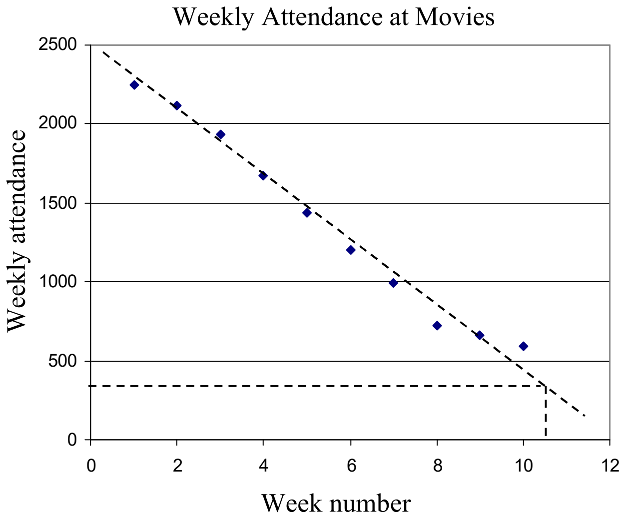
4. a)



A line of best fit is not appropriate since the data values are discrete, although they do follow a linear pattern. There are no data values for dates in between those plotted.

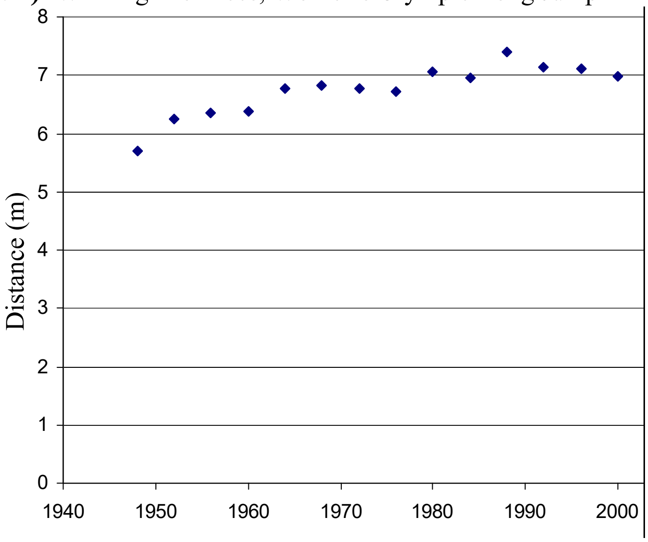
b) r is close to -1; the data values are closely scattered about a line that falls from left to right.

c) It will close around the 11th week



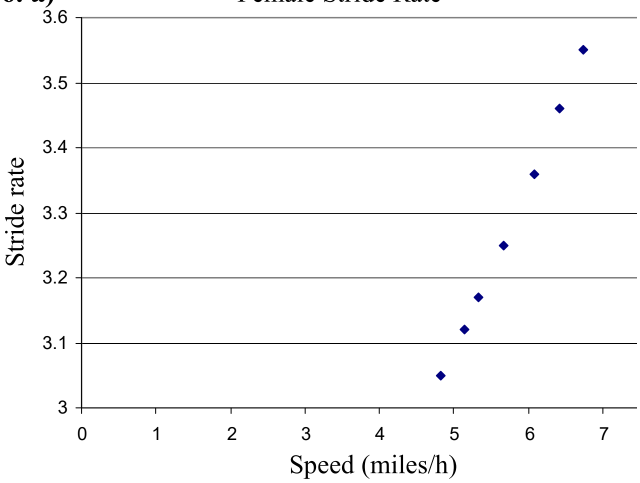
6.2.1 Correlation and Lines of Best Fit [Cont'd] pp. 203–204

5. a) Winning Distances, Women's Olympic Long Jump



A line of best fit is not appropriate as the Olympics are held every four years; the data values are discrete.
b) r is close to 1; the data values are closely clustered around a line that rises from left to right.
c) About 5.5 m
d) Since 1976, the distance has been up and down.

6. a) Female Stride Rate



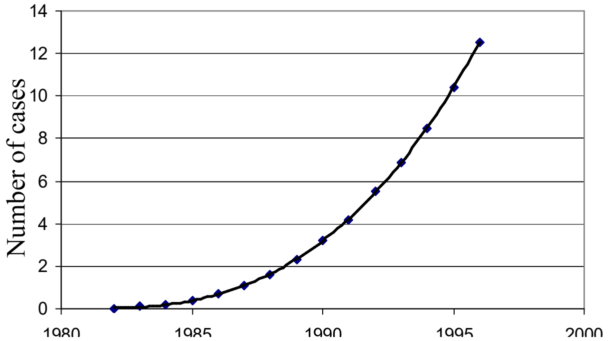
It is a linear relationship.
b) A line of best fit is appropriate since the data values are continuous and the points fall along a line.
c) No; Humans cannot run at a speed of 50 km/h as it is physically impossible.

7. a) Figure A: r is close to 1; Figure B: r is close to 0.5
b) Both show positive correlations but the correlation in A is stronger. Its points are closely scattered along a line rising from left to right. The points in B are more scattered about a line but still rising from left to right.

6.2.2 Non-Linear Data and Curves of Best Fit pp. 210–212

1. a) Exponential **b)** Linear **c)** Quadratic **d)** None of these

2. a) Cumulative AIDS Cases (millions)



2. b) It appears to be an exponential curve but could be a wide quadratic or cubic curve instead.

c) The number of AIDS cases is increasing rapidly so an exponential curve is a good model to use.

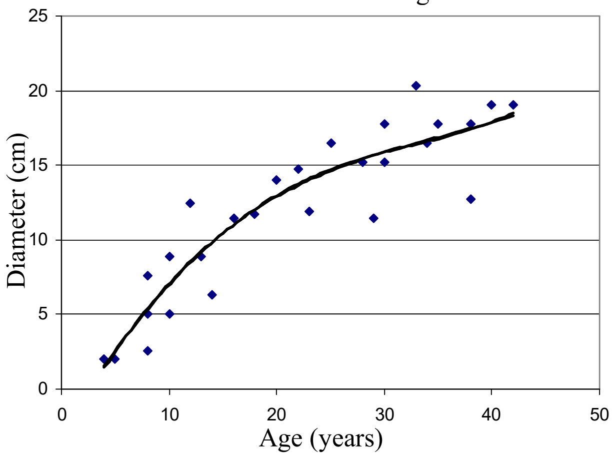
d) About 25 million; My prediction makes sense because the number of AIDS cases is increasing rapidly so a jump from 12.5 to 25 million from 1996 and 2000 is not surprising.

3. a) About 19.2 cm

b) Fairly confident based on the trend in the table.

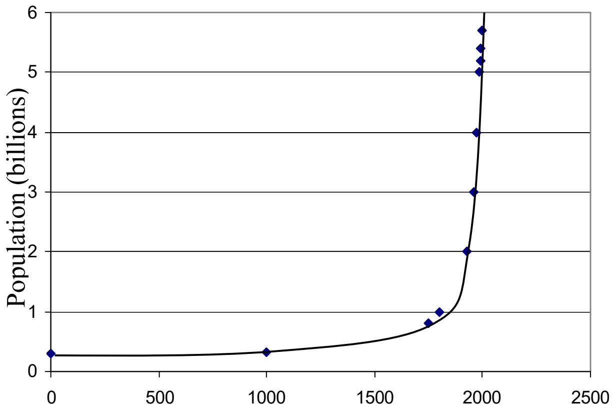
c) *Sample response:*

Tree Diameter vs. Age



3. d) From the curve of best fit, the diameter of a 32-year-old tree should be about 16 cm. I am more confident in this prediction than in my first prediction since this one is based on the graph that takes all the data values into account and not just the values that fall between 30 years and 33 years.

4. World's Population Over Time

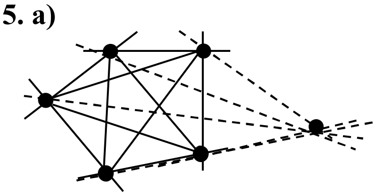


4. a) About 0.5 billion 0.3

b) About 0.8 billion 0.7

c) About 5.5 billion 7.0

5. a)

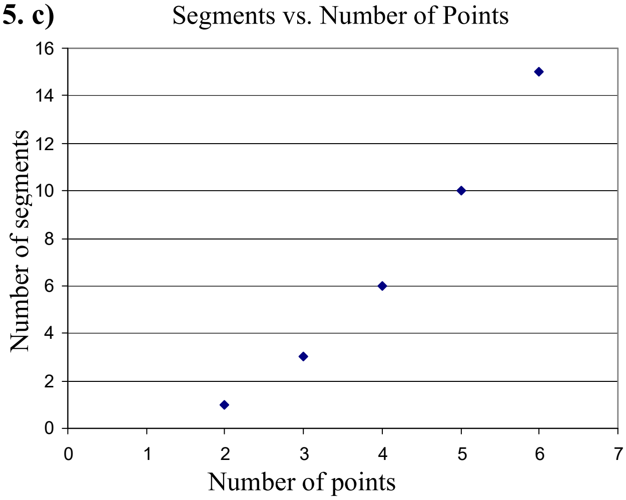


b)

Points	Segments
2	1
3	3
4	6
5	10
6	15

6.2.2 Non-Linear Data and Curves of Best Fit pp. 210–212

5. c)

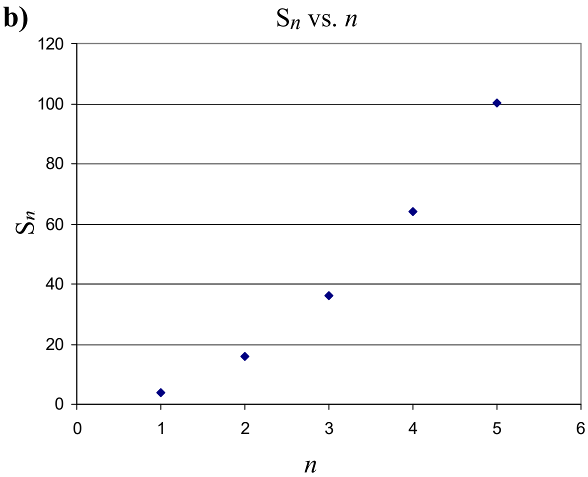


5. d) The data set is nonlinear; the points lie on a curve, not a line.
e) 21 segments; I saw that the pattern in the segments column was +2, +3, +4, +5, so the next number had to be 15 + 6 = 21.

6. a) 8 by 8 array of dots; 10 by 10 array of dots

n	1	2	3	4	5
S_n	4	16	36	64	100

b)



6. c) Nonlinear; the points lie on a curve, not on a line.
d) 3600;
S_n is the square of double n,
so if n is 30, then
S_n is $(2 \times 30)^2 = 60^2 = 3600$

7. a) Check to see if the data values are clustered closely around a line.
b) Check to see if the data values are closely clustered in a parabolic form or around a portion of a parabola that either increases from left to right or decreases from left to right.
c) Check to see if the dependent variable increases or decreases rapidly as the independent variable increases. Data values should be closely clustered around an exponential curve.

CONNECTIONS: Data Collection by Census p. 212

1. A census is the process of obtaining information about every member of a population (not necessarily a human population). It can be contrasted with sampling, in which information is obtained only from a subset of a population. It is a method used for accumulating statistical data, and it is vital to democracy (voting). Sets of census data are commonly used for research, business marketing, and planning purposes.

• Data sets were collected by Teachers, RNR Extension workers, Health workers, NFE instructors, and students of Sherbets College, NIE, NRTI, RBIT, RIM
• Data sets were collected about population characteristics, migration, health, education, labour and employment, and household and housing characteristics.
• The government wants to know this information so that it can adjust its programs. The records also form the basis for taxation.

2. One of the earliest documented censuses taken was in the year 500–499 BC by the military of the Persian Empire for the purpose of issuing land grants and taxation. Many countries conduct a census at either 5- or 10-year intervals

6.3.1 Dependent and Independent Events

p. 217

1. a) $\frac{2}{11}$ **b)** $\frac{1}{10}$

c) They are different. In **part a)**, you had 11 cards to draw from and two were B. In **part b)**, you had ten cards in the bag and only one was a B.

d) The events are dependent in **part b)** because the probability of drawing the letter B the second time was affected by what happened in the first draw.

2. a) $\frac{3}{7}$ **b)** $\frac{2}{6}$ or $\frac{1}{3}$

c) $\frac{3}{7}$ **d)** $\frac{3}{6}$ or $\frac{1}{2}$

3. a) The events are independent because the probability of a second spin of 4 is $\frac{1}{5}$ whether you get a 4 the first time or not.

b) The events are independent because the probability of spinning an odd number the second time is $\frac{3}{5}$ no matter what happens on the first spin.

c) The events are dependent because Event B can happen but it becomes impossible if Event A happens.

d) The events are dependent because the probability of spinning a number that results in a difference of 1 on the second spin depends on the result of the first spin.
For example, a first spin of 2 gives two possibilities with a difference of 1, whereas, a first spin of 5 would not.

4. Pair A are dependent since the probability of rolling a number that would result in a sum of 5 or more on the second roll, if a 2 is rolled first, is $\frac{4}{6}$ or $\frac{2}{3}$.
If a 2 is not rolled, the probability of rolling a number that would result in a total of 5 or more is $\frac{26}{30}$ or $\frac{13}{15}$.
Pair B is independent because the probability of rolling a number less than 5 on the second roll is $\frac{4}{6}$ or $\frac{2}{3}$, no matter what happens on the first roll.

5. *Sample response*:
a) Drawing two red cards in a row from a deck of playing cards if the first card is not replaced. These are dependent events because the probability of drawing a second red card will be affected by what happens in the first draw.
b) Drawing two red cards in a row from a deck of playing cards if the first card is replaced. These are independent events because the probability of drawing a second red card will not be affected by what is drawn the first time.

6.3.2 Calculating Probabilities — p. 222

1. a) i) $\frac{29}{40}$ **ii)** $\frac{27}{40}$ **iii)** $\frac{19}{40}$ **iv)** $\frac{3}{40}$

b) Dependent; $\frac{29}{40} \times \frac{27}{40} \neq \frac{19}{40}$

2. a) Count the number of pairs that have two odd numbers (in the left circle) and compare that number to the total number of pairs.
b) Count the number of pairs that have a total of 6 (in the right circle) and compare that number to the total number of pairs.
c) Multiply the probabilities from **parts a) and b)** and compare that product to the ratio of the number of pairs that have both two odd numbers and a total of 6 (in the intersection region of the two circles) to the total number of pairs.

3. a) i) $\frac{16}{50}$ **ii)** $\frac{10}{50}$ **iii)** $\frac{3}{50}$

b) Dependent; $\frac{16}{50} \times \frac{10}{50} \neq \frac{3}{50}$

4. a) i) $\frac{1}{2}$ **ii)** $\frac{1}{4}$ **iii)** $\frac{1}{4}$

b) Dependent; $\frac{1}{2} \times \frac{1}{4} \neq \frac{1}{4}$

4. c) i) $\frac{1}{4} = 1 \times \frac{1}{4}$ **ii)** $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$

5. About 0.42 (or $\frac{5}{12}$)

6. a) *By calculating*:

P(rolling a 4 first and a total of 10) = $\frac{1}{36}$; $\frac{1}{36} \neq \frac{1}{6} \times \frac{3}{12}$

Using reasoning:
The events are dependent because, to get a total of 10, the first roll must be at least 4. That means the result of the first roll affects the probability.
b) *By calculating*:

P(red, then blue) = $\frac{5}{10} \times \frac{5}{9}$; $\frac{25}{90} \neq \frac{5}{10} \times \frac{5}{10}$

Using reasoning:
The probability of selecting a blue marble second would be $\frac{5}{9}$ if a red marble was first drawn and not replaced and $\frac{5}{10}$ if it was replaced. Since the probability was affected, this means the events are dependent.

UNIT 6 Revision — pp. 223–226

1. *Sample responses:*
a) Examine the distribution of the masses of 100 dogs.
b) Compare the masses of 50 female and 50 male dogs.
c) Examine the distribution of the monthly income of all the residents in a community.
d) Compare the distribution relative to the median of the monthly income of all the residents in one community to that of residents in another community.

2. *Sample responses*:
a) *Advantages*: Shows how the data values are distributed. It is easy to group data into place value intervals and there is no scale to create. It is easy to find the extremes, range, median, and mode since it shows all the data and the data values are ordered.

2. a) [Cont'd] *Disadvantages*: Interval size is limited to place value intervals.

b) *Advantages*: Shows how the data values are distributed. You can select the interval size. It results in a simple image you can examine to determine the distribution.

Disadvantages: Does not show the actual data values. Changing the interval size can lead to different shapes and conclusions.

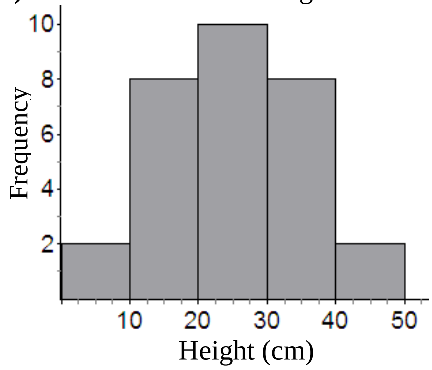
c) *Advantages*: It shows how the data values are distributed. It is good for examining the distribution of the data relative to the median and extremes. It can be easy to create depending on the data values.

Disadvantages: It does not show the actual data values. It can be difficult to create depending on the data values.

3. a)

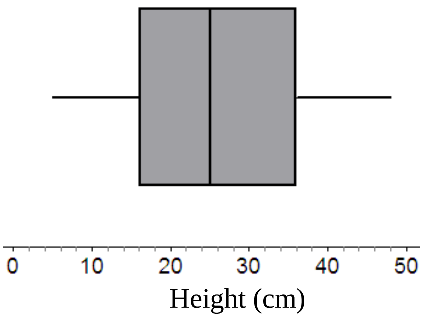
Stem	Leaves
0	5 7
1	0 0 1 3 5 6 7 8
2	1 2 2 4 5 5 5 6 7 9
3	2 3 6 6 7 7 8 9
4	5 8

b) Bean Plant Height



c) Normal distribution

d)

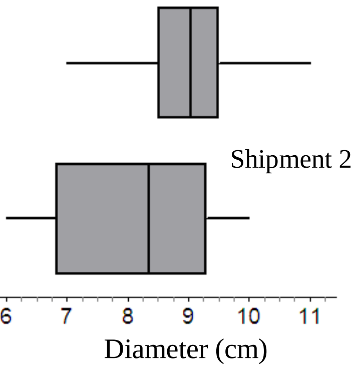


e) *Sample response:*

I prefer the stem and leaf plot because it is easy to construct and I do not have to choose an interval size or create a scale.

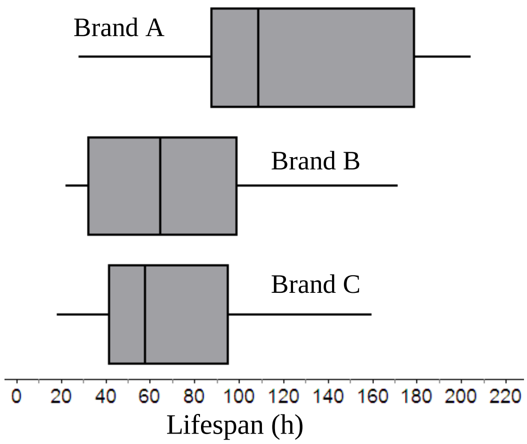
If I want to find values such as the median or extremes, I can use the plot without having to go back to the original data.

4. a) Shipment 1



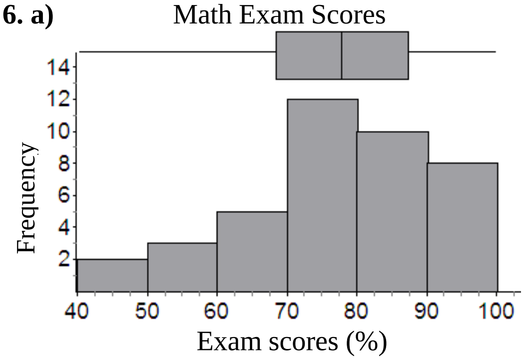
4. b) Shipment 2; The median and extremes in the box plot are lower than for Shipment 1, which means the oranges are generally smaller. The box in the box plot is wider than for Shipment 1, which means the oranges are less uniform in size.

5. a)



b) Brand A has the highest median lifespan. 50% of the time Brand A batteries last 90 h to 180 h whereas 50% of the time Brand B batteries only last about 30 h to 100 h, and 50% of the time Brand C batteries only last about 40 h to 95 h.

6. a) Math Exam Scores

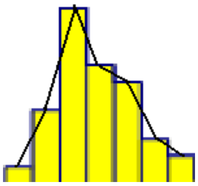


b) Both graphs show that the data set is skewed left, as the majority of the students scored above 70%.

The box plot shows that the median mark was about 79% and half the class scored between about 69% and 89%, but these are only estimates.

UNIT 6 Revision [Cont'd] pp. 223–226

7. SET I**a)**

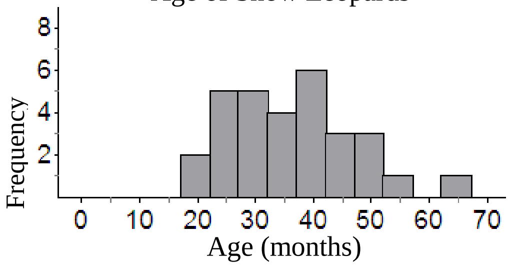


b) Rises quickly then tails off to the right; The distribution is right or positively skewed.
c) *Sample response*:
Marks on a test where most students did not perform well.

SET II **a)**

b) Relatively flat; It is close to a uniform distribution.
c) *Sample response*:
Heights of tomato plants grown inside a greenhouse under controlled conditions and planted at the same time.

8. a) *Sample response*:
Age of Snow Leopards



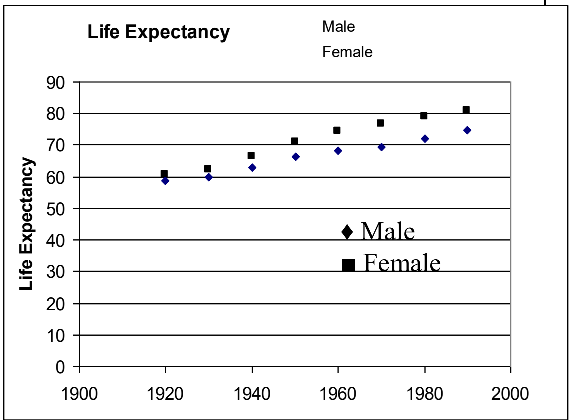
b) *Sample response*:
Close to mound-shaped with a normal distribution
c) Mean: about 35.37 months;
median: 35 months; mode: 37 months
d) Very likely, since 28 out of the 30 leopards were under 50 months old

9. a) IV **b)** I and IV **c)** III
d) I **e)** II

9. f) I: r is close to –0.5 II: r is close to 1
III: r is close to 0 IV: r is close to –1

10. a) Relationship A:
very strong negative correlation; close to –1
b) Relationship B:
weak positive correlation; close to 0.5

11. *Sample responses*:
a) The relationship between the age of a child up to age 18 and his/her height.
b) The number of sips you take from a drink and the volume of liquid remaining in the glass.
c) Shoe size and number of siblings.

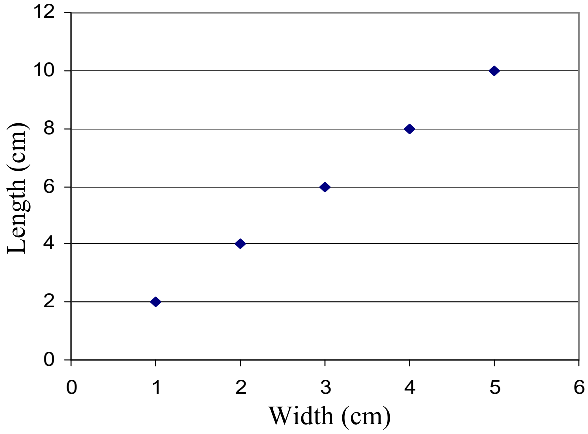


b) Strong positive correlation
c) Close to 1 for both data sets
d) Yes; Both sets of data are continuous and have a linear trend.
e) Males: about 81 years;
Females: about 91 years

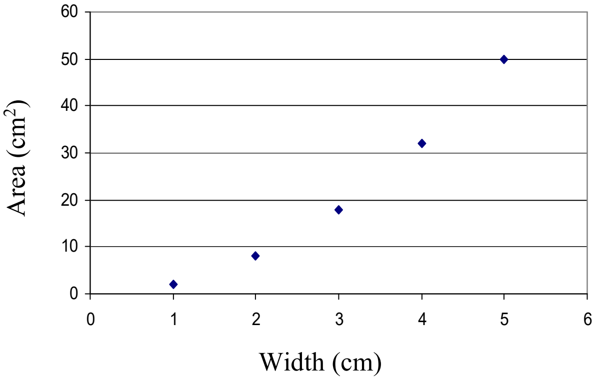
13. a) 4 row by 8 column array;
5 row by 10 column array

Width (cm)	Length (cm)	Area (cm^2)
1	2	2
2	**4**	**8**
3	**6**	**18**
4	**8**	**32**
5	**10**	**50**

13. b) Length vs. Width



Area vs. Width



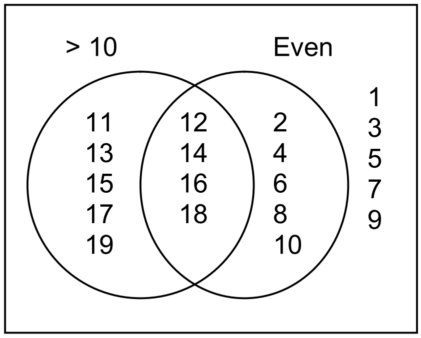
c) A line of best fit is appropriate for length vs. width because the data values are continuous and form a linear pattern. A curve of best fit is appropriate for area vs. width because the data values are continuous and form a non- linear pattern.

14. Not replacing the item changes the number of items in the bag and therefore changes the probability for the next draw.

15. The events are dependent because the outcome of the first roll affects the second roll. Novin has treated them as independent events and multiplied the probabilities of the two events together to find the probability of both events happening.

16. a) $\frac{5}{14}$ **b)** $\frac{4}{14}$ **c)** $\frac{5}{14}$ **d)** $\frac{4}{14}$

17. a) If the number selected has to be both greater than 10 and even, only 4 numbers in the sample space of 19 meet these criteria (12, 14, 16, and 18) so the probability is $\frac{4}{19}$.



b) P(even) = $\frac{9}{19}$, P(> 10) = $\frac{9}{19}$, so

P(even) × P(>10) = $\frac{9}{19} \times \frac{9}{19} = \frac{81}{361}$

The probability from **part a)** is $\frac{4}{19}$ or $\frac{76}{361}$, which is not equal to $\frac{81}{361}$, so the events must be dependent.

18. $\frac{1}{2}$;

You solve the equation

$0.3 = 0.6 \times P(\text{second blue}|\text{first blue})$

19. a) $\frac{1}{3}$

b) $\frac{5}{36}$

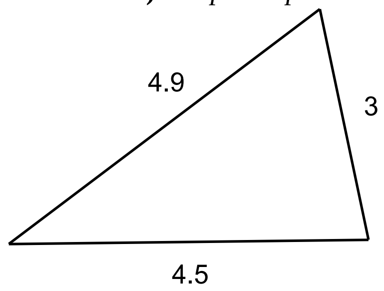
c) No, since $\frac{5}{36} \times \frac{1}{3} \neq \frac{1}{36}$ and $\frac{1}{36}$ is the probability of rolling 4, 2.

Getting Started — Skills You Will Need pp. 227–228

1. a) 7.5 **b)** 22.5 **c)** 2.333 **d)** 1.1

2. a) 60° **b)** 120°

3. a) 4.15 **b)** *Sample response*:



4. a) 10 cm
b) 4.36 cm

5. a) 1140 mm^2 or 11.4 cm^2
b) 5.6 cm^2

6. a and c, b and d; they are corresponding angles on parallel lines

7.1.1 Using Similarity Properties to Solve Problems pp. 233–234

1. a) 3.6 units **b)** 3.5 units **c)** 15 cm

2. 1.67 units **3.** 43.2 m

4. 5.35 units **5.** 79.69 m

6. 8.63 m

7. 24-32-40, 18-24-30, 10-24-26

8. 166.67 m

9. Since you know that the sides of similar triangles are proportional, if you know the dimensions of one triangle that might be easier to measure, it can give you information about the other triangle that is harder to measure.

CONNECTIONS: Using a Clinometer p. 237

1. Both are right triangles and they share a common angle at D, so they are similar (AAA).

2. 6.3 m

7.1.3 The Sine, Cosine, and Tangent Ratios p. 242

1. a) $\sin A = \frac{3}{5} = 0.60$; $\cos A = \frac{4}{5} = 0.80$;
$\tan A = \frac{3}{4} = 0.75$
b) $\sin A = \frac{8}{17} = 0.47$; $\cos A = \frac{15}{17} = 0.88$;
$\tan A = \frac{8}{15} = 0.53$

2. a) sin A = 0.34; cos A = 0.94; tan A = 0.36
b) sin A = 0.98; cos A = 0.21; tan A = 4.72

3. a) sin 35° = 0.574; cos 35° = 0.819;
tan 35° = 0.700
b) sin 55° = 0.819; cos 55° = 0.574;
tan 55° = 1.429
c) sin 12° = 0.208; cos 12° = 0.978;
tan 12° = 0.213
d) sin 80° = 0.984; cos 80° = 0.174;
tan 80° = 5.671

4. a) 58.2° **b)** 76.1° **c)** 11.3°

5. *Sample responses*:
a) a right triangle with height of 1.4 units and hypotenuse of 2 units
b) a right triangle with a 60° angle
c) a right triangle with a base of 2 units and a height of 4.8 units

6. a) $\angle A > \angle B$; *Sample response:*
This is because the values of the opposite sides increase as the angle increases.

6. b) No; *Sample response:*
sin 75° = 0.966 and cos 85° = 0.087, and 75° < 85°. But sin 75° = 0.966 and cos 60° = 0.5 and 75° > 60°.

7. The trig ratios work for all right triangles with a particular pair of acute angles because all right triangles with those angles are similar.

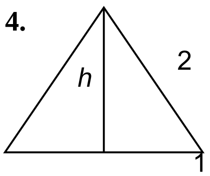
7.1.4 Trigonometric Identities p. 248

1. a) sec x = 1.152; csc x = 2.016; cot x = 1.75
b) sec x = 1.033; csc x = 4; cot x = 3.873

2. a) cos x = 0.6; tan x = 1.333; cot x = 0.75
sec x = 1.667; csc x = 1.25
b) sin x = 0.917; tan x = 2.29; cot x = 0.436
sec x = 2.5; csc x = 1.091
c) sin x = 0.707; cos x = 0.707; cot x = 1.0
sec x = 1.414; csc x = 1.414
d) sin x = 0.980; cos x = 0.2; tan x = 4.899
cot x = 0.204; csc x = 1.021

3. a) 30° **b)** 35.5° **c)** 72.9°

4.



$h^2 = 2^2 - 1^2 = 4 - 1 = 3$, so $h = \sqrt{3}$
$$\tan 60° = h \div 1 = \sqrt{3} \div 1 = \sqrt{3}$$

5. a) x is greater; if sec x > sec y, then cos x < cos y which means x is a bigger angle
b) y is greater; if csc x > csc y, then sin x < sin y and $x < y$
c) x is greater; if cot x < cot y, then tan x > tan y and since the numerator is the opposite side, x is the greater angle

6. a) 53° **b)** 48° **c)** 53° **d)** 37° **e)** 39°

7. Since sin 45° = 0.707, an angle with a sine of 0.7 must be close to 45°.

8. a) True; if x < 45°, its opposite side is shorter relative to its adjacent side so the cosine, $\frac{\text{adjacent}}{\text{hypotenuse}}$, will be greater than the sine, $\frac{\text{opposite}}{\text{hypotenuse}}$
b) False; if x < 45°, its opposite side is shorter relative to its adjacent side and the hypotenuse so the cosine, $\frac{\text{adjacent}}{\text{hypotenuse}}$, will be greater than the tangent.

c) True; sin x = (tan x)(cos x) because tan = $\frac{\text{opposite}}{\text{adjacent}}$ and cos = $\frac{\text{adjacent}}{\text{hypotenuse}}$, so
$\frac{\text{opposite}}{\text{adjacent}} \times \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}$,
which is sin.
d) False; I tried it with x = 30° and it did not work since cos 30° = 0.866, but sin 30° × tan 30° = 0.5 × 0.577 = 0.289.

9. *Sample responses*:
- The cotangent is the reciprocal of the tangent, so cot x = 2.
- If tan x is $\frac{1}{2}$, $\frac{\sin x}{\cos x} = \frac{1}{2}$, so cos x = 2 sin x.

Since $\sin^2 x + \cos^2 x = 1$, then $\sin^2 x + 4\sin^2 x = 1$.
Since $5\sin^2 x = 1$, sin x = $\sqrt{0.2}$ = 0.447.
- That means cos x = 2(0.447) = 0.894.
- If sin x = 0.447, then csc x = 2.236.
- If cos x = 0.894, then sec x = 1.118.

7.2.1 Calculating Side Lengths and Angles p. 254

1. a) $z = 18°$; $y = 3.25$; $x = 10.51$
b) $y = 5.7°$; $z = 84.3°$; $x = 9.95$

2. 3.09 m

3. 65.2°

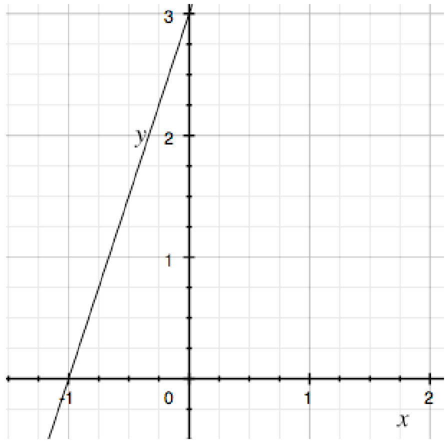
4. No. It would be closer to 8° when the base is 7. To find the base angle *a*, use

$\tan a = \frac{1}{7}$; $a = 8.13$

5. About 9.1°

6. 24.03 square units

7. a)



7. b) 71.6°
c) 71.6°, since the lines are parallel

8. 18.4°

9. a) Minimum 2.07 m, maximum 4 m
b) Closer at 75°

10. You only need one angle and one side length. If you know one acute angle, you can subtract it from 90° to find the other acute angle. If you know one acute angle and one side length, you can use sin or cos to find the other side length. Then you can use the Pythagorean theorem to find the third side length.

OR

You only need two side lengths. You can use the Pythagorean theorem to find the third side. You could use arc sin, arc cos, or arc tan to find the angles.

7.2.2 Angles of Elevation and Angles of Depression p. 251

1. 31.9°

2. 12.1°

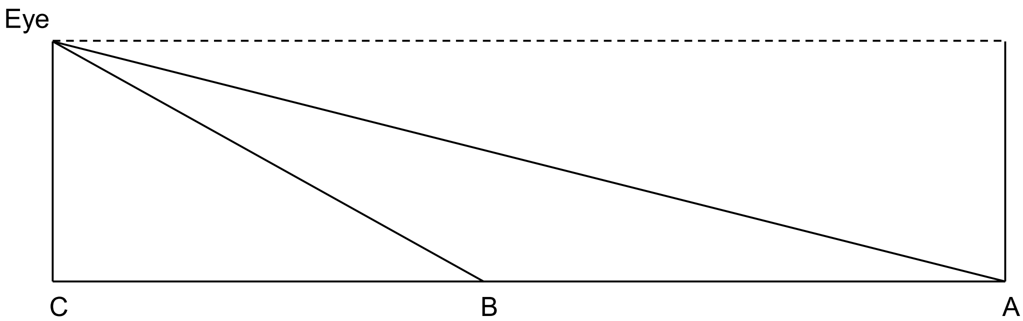
3. 45.1 m

4. 3.62 m

5. 20 m

6. 48.94 m

7. The angle of depression increases as you get closer to an object since the opposite side stays the same and the adjacent side gets smaller. If the object were directly below (C in the diagram), the angle of depression would be 90°.



7.2.3 Areas of Polygons — p. 258–259

1. 346.41 cm^2 **2.** 26.57 square units

3. 769.42 cm^2

4. a) A_{square} = 225 cm^2 $A_{pentagon}$ = 247.75 cm^2
$A_{hexagon}$ = 259.81 cm^2 $A_{decagon}$ = 276.99 cm^2

b) The area gets bigger as more sides are added even though the perimeter stays the same.

4. c) A circle is like a polygon with many, many sides.

5. $A = (b_1 + b_2)(b_1 \tan X) \div 2$ or $(b_1 + b_2)(b_2 \tan Z) \div 2$

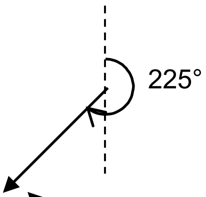
6. The lengths of two sides (*a* and *b*) and the angle between them, *x*, since the formula is $A = \frac{1}{2}ab \sin x$.

7.2.4 Vectors and Bearings — p. 264

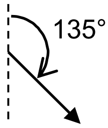
1. *Sample responses*
(values within 20° are reasonable):
a) 045° **b)** 330° **c)** 150°

2. *Sample responses* (distances may vary):

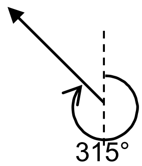
a)



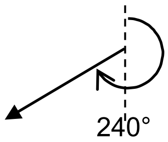
b)



c)



d)



3. a) i) 1 m at a bearing of 225°
ii) 0 m (no displacement)
iii) 1 m at a bearing of 050°
b) The directions were directly opposite each other.

4. a) 4.03 km at a bearing of about 060°
b) 2.2 km at a bearing of about 310°
c) 4.3 km at a bearing of about 145°
d) 3.12 km at a bearing of 045°

5. 5 km at a bearing of about 082°

6. *Sample response*:
5 km at a bearing of 120° and another 5 km at the same bearing;
12 km at a bearing of 120° and 2 km at a bearing of 300°;
8.7 km at a bearing of 90° and 5 km at a bearing of 180°

7. a) No; Both times the vector is the hypotenuse of a triangle with legs of 4 and 2 units and so the Pythagorean theorem would give the same value both times (about 4.5 km).
b) No; Both times the bearing is 153°

CONNECTIONS: Relating Trigonometric Ratios to Circles — p. 265

1 a) 0.64 cm **b)** 0.77 cm **c)** 0.84 cm

2. The values in **question 1** are the sine, cosine, and tangent of 40°.

3. Using ASA, ∠CAE = ∠FEA, AE = AE, and ∠AEC = ∠FAE

4. *Sample response*:
∠FEA = 40° because FE is parallel to AC
∠AEC = 50° since 180° – (90° – 40°)
∠CED = 40° since 90° – 50°
∠EFA = ∠ECD = 90°; ∠FAE = ∠CDE = 90°
∠AEF = ∠DEC = 40°
Using AAA, ΔEFA is similar to ΔECD.

5. $\frac{ED}{EA} = \frac{EC}{EF}$
AC = cos 40°(**question 2**)
EF = AC (**question 3**) so EF = cos 40°
EC = sin 40° (**question 2**)
So $\frac{ED}{1} = \frac{\sin 40^\circ}{\cos 40^\circ}$, but
$\frac{\sin 40^\circ}{\cos 40^\circ} = \tan 40^\circ$
so ED = tan 40°

UNIT 7 Revision — pp. 266–267

1. The pair in A are similar since the sides are in the same proportion (scale factor 1.4).

2. a) 1.8 units **b)** 6.67 units

3. 12 m **4.** 25 m

5. a) sin A = 0.467, cos A = 0.9,
tan A = 0.519; ∠A = 27°

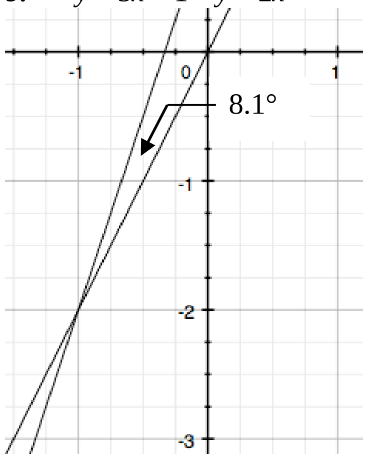
b) sin A = 0.514, cos A = 0.857,
tan A = 0.60, ∠A = 31°

6. For **part a**, sec A < cot A < csc A
For **part b**, sec A < cot A < csc A
In both cases, the order is the same; because the sizes are in reverse order to the sizes of the reciprocal primary ratios.

7. a) 78.5° **b)** 11.5° **c)** 56.3° **d)** 5°

8. 4.97 m

9. $y = 3x + 1$ $y = 2x$



10. 29° **11.** 24.4° **12.** 103.92 m

13. 12.05 m **14.** 2472.73 cm^2

15. *Sample responses*
(values within 20° are reasonable):
a) 150° **b)** 65°

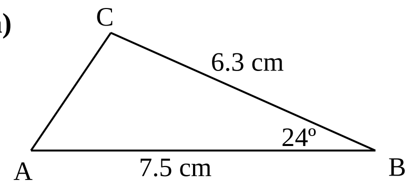
16. 7.8 km at a bearing of 140°

UNIT 8 GEOMETRY — pp. 269–296

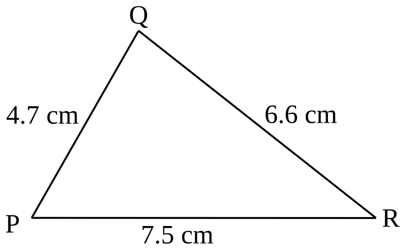
Getting Started — Skills You Will Need — p. 269

1. a) Pentagon-based prism
b) Octagon-based pyramid
c) Cone
d) Cylinder
e) Sphere

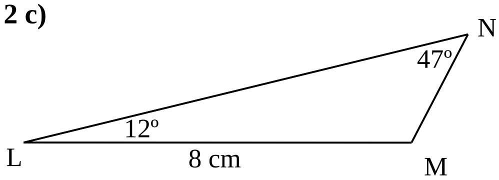
2. a)



b)



2 c)

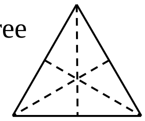


3. a) Three corresponding sides the same in both triangles (SSS), two corresponding sides and the angle between them the same in both triangles (SAS), or two corresponding angles and one corresponding side the same in both triangles (ASA).
b) *Sample response:*
knowing VZ = VX would show congruence using SAS; or
knowing ∠Y = ∠W would show congruence using ASA; or
knowing ∠Z = ∠X would show congruence using AAS.

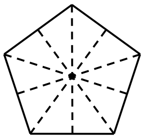
8.1.1 2-D and 3-D Reflectional Symmetry p. 274

1. Four

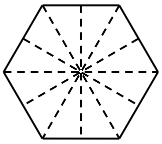
2. a) Three



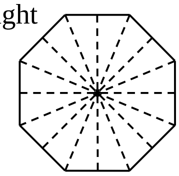
b) Five



c) Six



d) Eight



3. a) Four **b)** Six **c)** Seven **d)** Nine

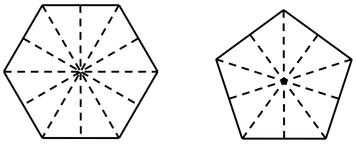
4. a) The number of lines of symmetry is same as number of sides. If the polygon has an even number of sides, *n*, there are

$\frac{n}{2}$ lines connecting pairs of vertices and

$\frac{n}{2}$ lines connecting midpoints of sides to

midpoints of opposite sides, and $\frac{n}{2} + \frac{n}{2} = n$.
If the polygon has an odd number of sides, *n*, there are *n* lines connecting vertices to the midpoints of the opposite sides.

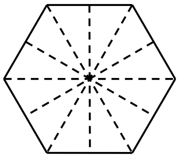
Number of lines of symmetry = Number of sides

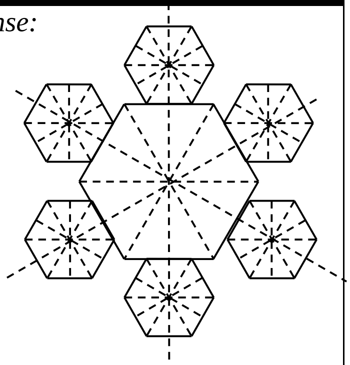


Even Odd

b) Each of these lines of symmetry results in a plane of symmetry if it is extended. There is also one more plane of symmetry that extends from the lines of symmetry that cut the lateral rectangular faces in half.
This means there is one more plane of symmetry for a prism than the number of lines of symmetry in the regular polygon base.
c) Each of these lines of symmetry results in a plane of symmetry if it is extended so the number of planes of symmetry for a pyramid is the same as the number of lines of symmetry in the regular polygon base.

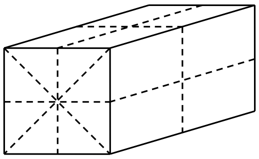
5. *Sample response:*

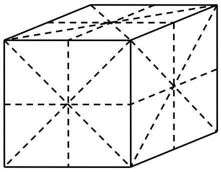




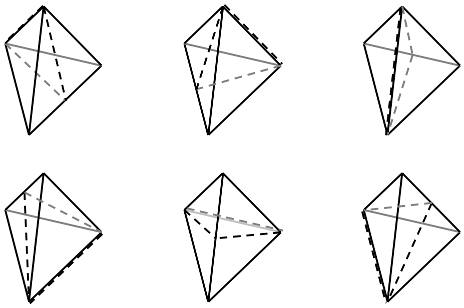
6. No. Scalene triangles do not have any, isosceles triangles have only one, and equilateral triangles have three. There are no other kinds of triangles.

7. a) Five **b)** Nine





8. Six planes of symmetry; Each plane passes through an edge of the tetrahedron and the midpoint of of the opposite edge.



9. a) A circle has an infinite number of lines of symmetry.
b) A cylinder has an infinite number of planes of symmetry on the base corresponding to the infinite number of lines of symmetry in the circle base. It also has one more plane that cuts across the lateral curved surface
c) A cone has an infinite number of planes of symmetry on the base corresponding to the infinite number of lines of symmetry in the circle base.

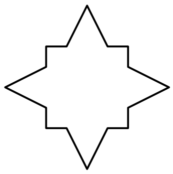
8.1.2 2-D and 3-D Rotational Symmetry p. 278

1. a) 3 **b)** 5 **c)** 6 **d)** 8

2. The order of turn symmetry is the same as the number of sides. You can divide any regular polygon with *n* sides into *n* congruent triangles. If you were to focus on one of the triangles while turning the polygon, that triangle would match each other triangle *n* times.

3. No. Isosceles triangles and scalene triangles have order 1 and equilateral triangles have order 3. There are no other kinds of triangles.

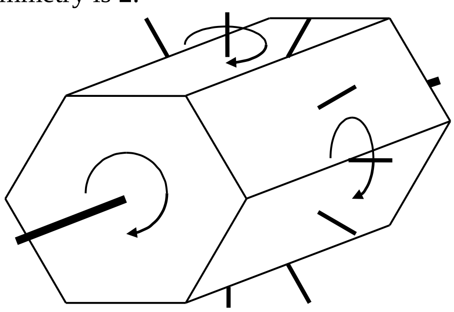
4. *Sample response:*



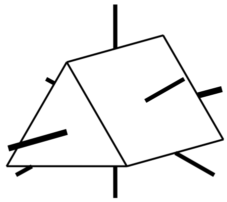
5. a) Opposite vertices, midpoints of edges to midpoints of diagonally opposite edges, and centres of faces to centres of opposite faces.
b) Thirteen

6. Order of 8 around an axis passing through the centre of the base and the apex of the pyramid

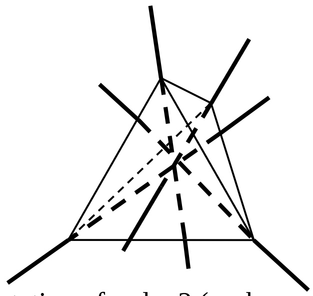
7. a) Seven
b) Using the axis that passes through the centres of the bases, the order of turn symmetry is 6.
Using any of the other axes (one passes through the centres of opposite lateral faces and the others pass through the midpoints of opposite lateral edges), the order of turn symmetry is 2.



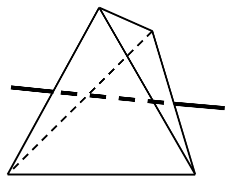
8. a) Four
b) Using the axis that passes through the centres of the bases, the order of turn symmetry is 3.
Using any of the other axes (each other axis passes through the centre of a lateral face and the midpoint of the opposite lateral edge), the order of turn symmetry is 2.



9. a) Seven
b) Four axes of rotation (each passes through a vertex and the centre of the opposite face) of order 3



Three axes of rotation of order 2 (each one passes through the midpoint of an edge and the midpoint of the opposite edge) One is shown below, but there are two others since there are three pairs of opposite edges:



10. Both have an infinite number of axes of rotational symmetry:
- For a cylinder, visualize an axis passing through the centres of the bases, and visualize an infinite number of axes passing from any point on the lateral surface at the "equator" to a point on the opposite side.
- For a sphere, visualize an infinite number of axes passing through the centre of the sphere from any point on the surface to a point on the opposite surface.

8.1.3 Reasoning p. 283

1. a) Deductive **b)** Inductive
c) Inductive **d)** Deductive

2. a)

Number of sides in the polygon base	Number of planes of symmetry
3	4
4	5
5	6
6	7

b) *Sample response*:
The number of planes of symmetry of a regular polygon-based prism is always one more than the number of sides in the base.
c) I used the examples in the chart to come up with the conjecture.
d) If a prism has an *n*-sided base, there are *n* lines of symmetry in the base. If each line is extended into a plane of symmetry, there are *n* planes of symmetry. In addition, there is one more plane that cuts through the "equator" of the prism.
d) So, if there are *n* sides on the base of a regular polygon-based prism, there are $n + 1$ planes of symmetry.

3. a) I checked some prisms and pyramids and found that a pyramid always has one face fewer than a prism with the same base.

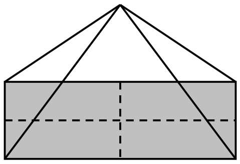
Number of sides in the base	Number of faces of prism	Number of faces of pyramid
3	5	4
4	6	5
5	7	6
6	8	7

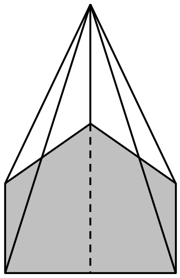
b) For a prism, if the base has *n* sides, there are *n* edges joining the vertices of the top base with the vertices of the bottom base. This results in *n* lateral rectangle faces along with the two faces that are bases. So, the total number of faces for a prism is $n + 2$.

For a pyramid, if the base has *n* edges, there are *n* edges joining the vertices of the base to the apex. This results in *n* lateral triangle faces along with the one face that is the base. So, the total number of faces for a pyramid is $n + 1$.

Since $(n + 2) – (n + 1) = 1$, a pyramid always has one face fewer than a prism with the same base.

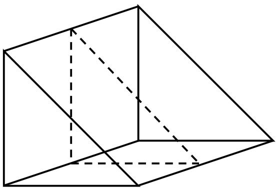
4. a) I used bases that were a rectangle (but not a square) and a pentagon. The pyramid with the rectangle base has 2 planes of symmetry that come up from the two lines of symmetry on the base. The same is true for the pentagonal base — there is one line of symmetry and one plane that goes up from that line.





b) I realized that if I cut horizontally across a pyramid anywhere along the height, I would not get a plane of symmetry since the bottom would be much wider than the top. So, I knew each of the planes had to go through the top vertex and the base, cutting the base into congruent mirror halves.
That means each plane cuts the base through one of the lines of symmetry.
So, a pyramid always has the same number of planes of symmetry as lines of symmetry on the base.
5) and 6. c) To be guided by teacher with reference to Teachers' Guide Book.(p. 290)
6. a) The sum is 360° each time.
b) My conjecture is that the sum of the exterior angles of any polygon is 360°.

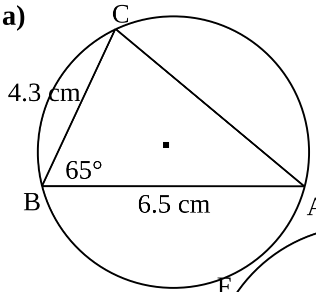
7. There may not be any planes of symmetry travelling through the bases because the bases are not symmetrical, but any prism always has a plane of symmetry through its "equator."



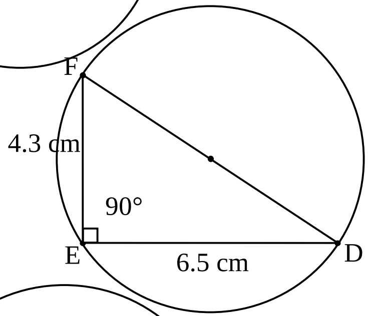
8. With deductive reasoning, you can be sure that it is true. With inductive reasoning, you must depend on examples, and there may always be a counterexample that you have not yet discovered.

1. *NOTE: These are not actual size.*

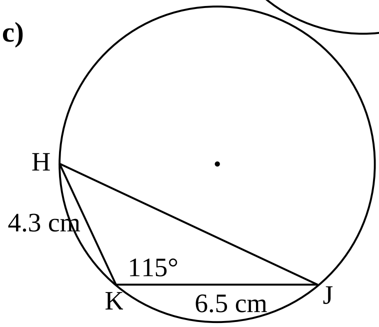
a)



b)



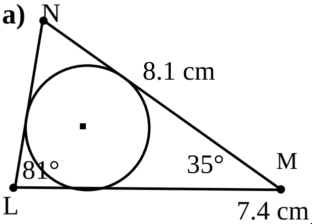
c)



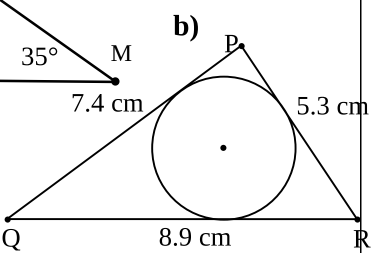
2. a) In the acute triangle, the circumcentre is inside the triangle. In the right triangle, the circumcentre is on the hypotenuse. In the obtuse triangle, the circumcentre is outside the triangle.
b) Obtuse triangles
c) *Sample response:*
I could verify it by induction, using many examples of different obtuse triangles.

3. NOTE: *These are not actual size.*

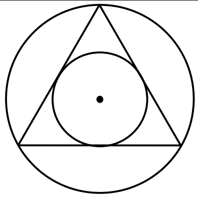
a)



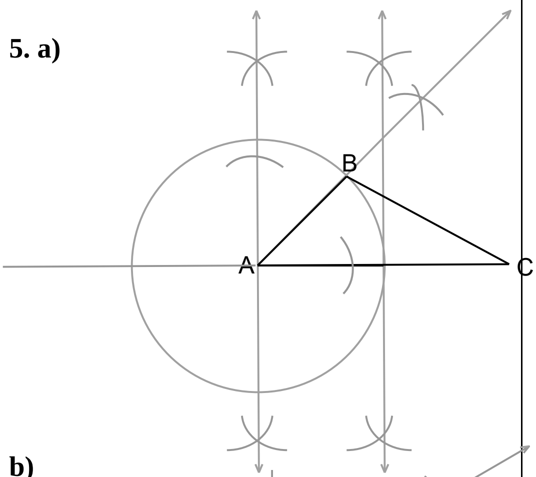
b)



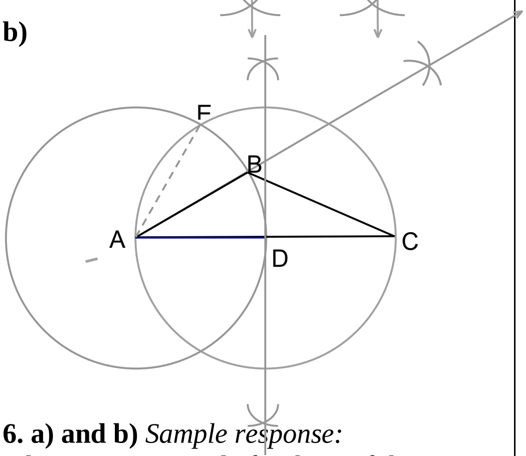
4. a) and b)
The circumcentre and incentre are the same point.



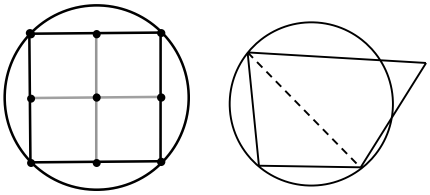
5. a)



b)

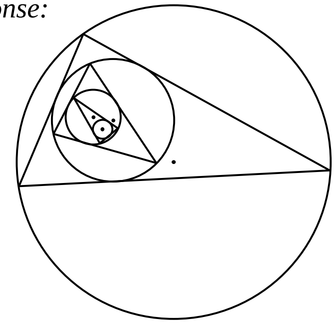


6. a) and b) *Sample response:*
I drew a circumcircle for three of the vertices and it went through the fourth one for a square. When I tried with a non-isosceles trapezoid, it did not work.

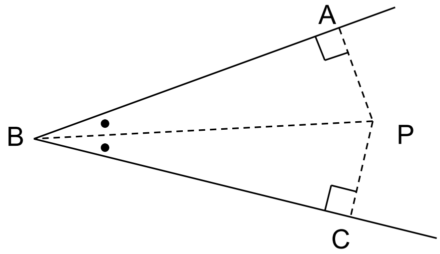


I conclude that not all quadrilaterals have circumcircles. One counterexample was enough to prove that this is true.

7. There is no pattern.
Sample response:



8. a) Construct perpendiculars from P to the arms of the angle to make points A and B. The angles at B are equal.

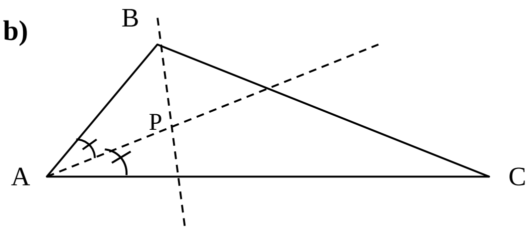


To prove that AP = CP:

Since, sin $\angle$B= $\frac{AP}{BP}$ and sin $\angle$B = $\frac{CP}{BP}$

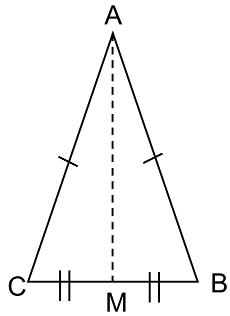
AP = BP × sin $\angle$B and CP = BP × sin $\angle$B.
Therefore AP = CP.

b)

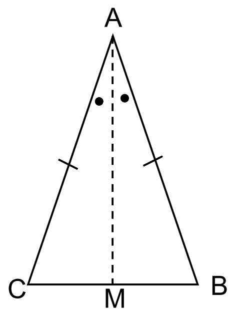


Since P is on the bisector of $\angle$A, it is as far from AB as from AC. Since it is also on the bisector of $\angle$B, it is as far from AB as from BC. That means P is equally far from all three sides, which makes it the incentre.

9. a)
AM is the perpendicular bisector of BC.
ΔABM ≅ ΔACM because of SSS:
• CM = BM (M is the midpoint of CB because AM bisects CB)
• CA = BA (triangle is isosceles)
• MA = MA (shared side length)



AM is the angle bisector of $\angle$A.
ΔABM ≅ ΔACM because of SAS:
• MA = MA (shared side length)
• CA = BA (triangle is isosceles)
• $\angle$MAC = $\angle$MAB (AM bisects $\angle$A)

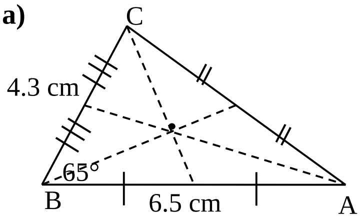


This proves that the line that is the perpendicular bisector must also be the angle bisector.

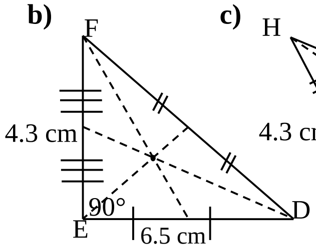
b) Equilateral triangles have this property for all three angles and opposite side lengths because the arms of the angle from each vertex are of equal length.

1. *Note: these shapes are not actual size and the third median is optional.*

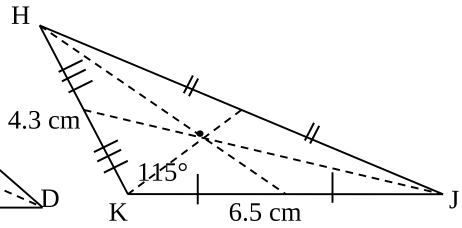
a)



b)

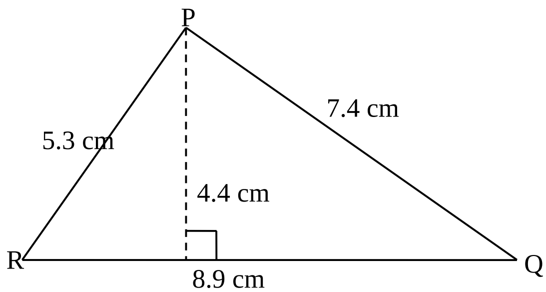


c)

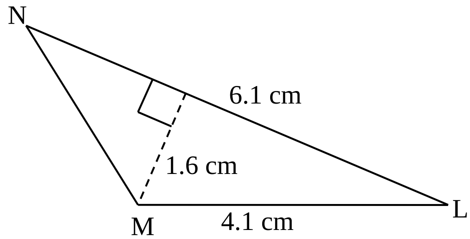


2. No, it's impossible; the centroid is the intersection of the medians and medians are always inside the triangle because they go from a vertex to the midpoint of its opposite side.

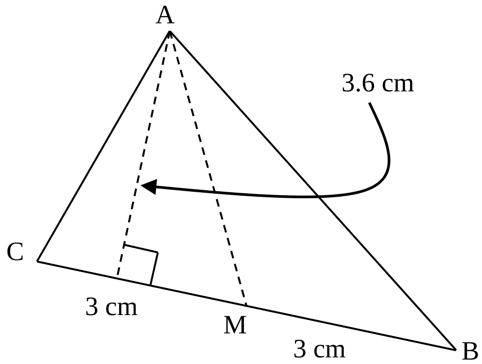
3. a) *Sample response*:
Using the altitude from P,
$A = bh \div 2 = (8.9)(4.4) \div 2$
$= 19.58 \text{ cm}^2$
$\approx 19.6 \text{ cm}^2$



3. b) *Sample response*:
Using the altitude from M,
$A = bh \div 2 = (6.1)(1.6) \div 2$
$= 4.88 \text{ cm}^2$
$\approx 4.9 \text{ cm}^2$

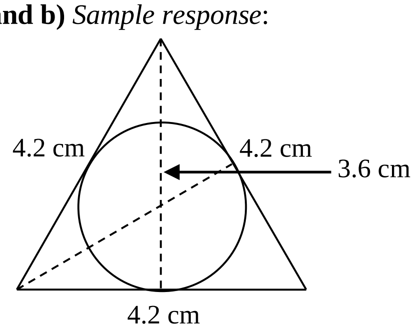


4. a) *Sample response*:



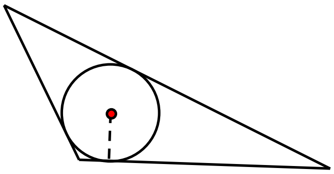
b) ΔABC's altitude is also the altitude of ΔAMB and ΔAMC.
c) ΔAMB: $A = bh \div 2 = (3)(3.6) \div 2 = 5.4 \text{ cm}^2$
ΔAMC: $A = bh \div 2 = (3)(3.6) \div 2 = 5.4 \text{ cm}^2$
The median divides the area of ΔABC in half.
d) The median of a triangle divided the area of the triangle in half regardless of triangle's dimensions.
e) The median of a triangle divides the area of any triangle in half. I used inductive reasoning (through examples).

5. a) and b) *Sample response*:



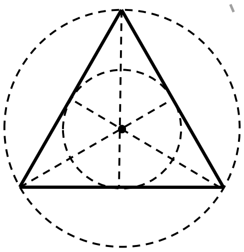
I estimate that the incircle covers more than half the triangle.
c) The area of the triangle is 7.6 cm^2.
The area of the circle is 4.5 cm^2.
Therefore, the circle covers 59% of the triangle ($4.5 \div 7.6 \approx 0.6$), so my estimate was reasonable.

d) *Sample response*:
Area of triangle is 7.85 cm^2 and area of incircle is 3.04 cm^2. The circle covers approximately 39% of the area of the triangle.



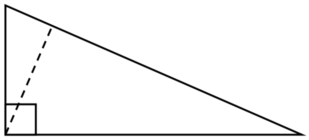
e) The percentage of the triangle's area covered by the incircle depends on the type of triangle. It can be more than half or less than half.

6. a), b), c)
All four centres are in the same location in any equilateral triangle.



7. d) The ratio for PQ to PS and for PR to PT is the same, approximately 3:1.
e) No matter what the dimensions of the right triangle were, the ratio for PQ to PS and for PR to PT was always the same, approximately 3:1.

8. The two legs of the right triangle are also two of the altitudes.

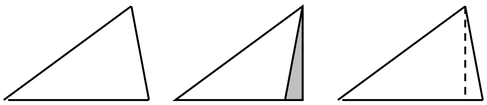


9. Each median divides the area of the triangle in half. Therefore, the location where all the medians intersect has half the area on each side, no matter which way you are dividing the triangle in half.

CONNECTIONS: Paper Folding Constructions p. 293

1. a) The circumcentre

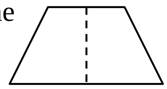
2. a) Fold the angle in half so the arms match and the fold goes through the vertex. The fold line is the angle bisector.
b) Fold to find the angle bisectors of two of the triangle's angles (using the method described in **question 2a**). The incentre will be the intersection of the two fold lines.
c) Fold one side so the sides match and the fold also goes through the vertex opposite to it. The fold line is an altitude of the triangle.



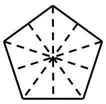
2. d) Fold to find two altitudes of the triangle (using the method described in **part 2c)**). The orthocentre will be the intersection of the two fold lines.
e) Fold the triangle so that two vertices match. The fold line is the midpoint of the side that connects those vertices. Unfold and fold again so that the new fold line travels through the midpoint of the side and the vertex opposite to it. The new fold line is a median of the triangle.
f) Fold to find two medians of the triangle (using the method described in **part 2e**). The centroid will be the intersection of the two fold lines.

UNIT 8 Revision p. 294

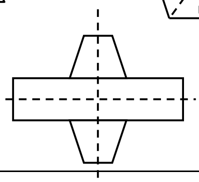
1. a) One



b) Five



2. *Sample response:*



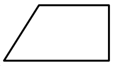
3. *Sample response:*
A square based prism that is not a cube.

4. a) No turn symmetry, so the order of turn symmetry is 1.
b) Turn symmetry of order 5 using the turn centre that is at the intersection of the five lines of symmetry.

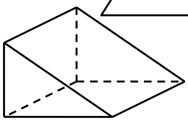
UNIT 8 Revision [Cont'd] p. 294

5. a) The order of turn symmetry is 5 using the axis of turn symmetry that passes through the centre of the base and the apex of the pyramid.
b) The order of turn symmetry is 5 using the axis of turn symmetry that passes through the centres of the two bases of the prism.
The order of turn symmetry is 2 using any of the five axes of turn symmetry that pass through the centre of one of the lateral faces and the midpoint of the edge opposite to it.

6. a) Non-isosceles trapezoid



b) Scalene right triangle-based prism



7. a) A cone has an infinite number of planes of symmetry, each of which passes through the centre of the base circle and the apex of the cone.
b) A cone has an order of turn symmetry that is infinite, around the only axis of symmetry, which passes through the centre of the base circle and the apex of the cone.

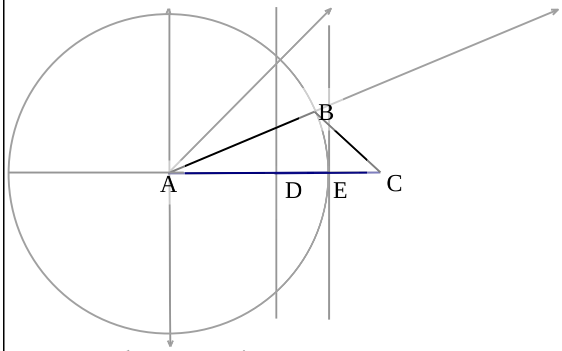
8. The number of edges of a pyramid is double the number of sides on the base.

Number of sides on the base	Number of edges on the pyramid
3	6
4	8
5	10
6	12

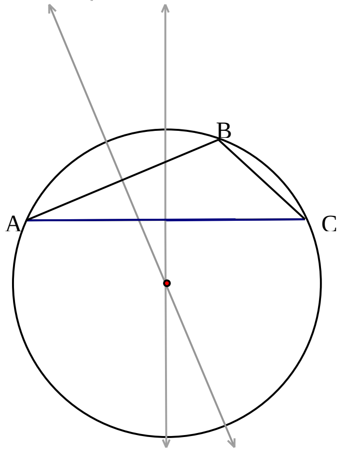
9. The order of turn symmetry is the same as the number of sides in the equilateral triangle. If you divide the triangle into three congruent triangles, by drawing lines from each vertex to the centre, and then turn the equilateral triangle around the centre, each small congruent triangles matches with the original triangle 3 times in a full rotation. This means the order of turn symmetry is 3, which is the same as the number of sides.

10. If she put a diagonal on every face, the prism would be rigid because each face would be made up of triangles.

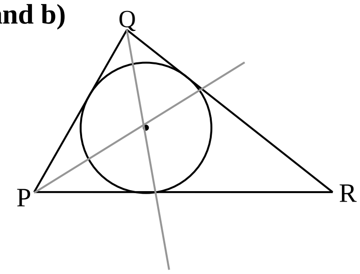
11. a)



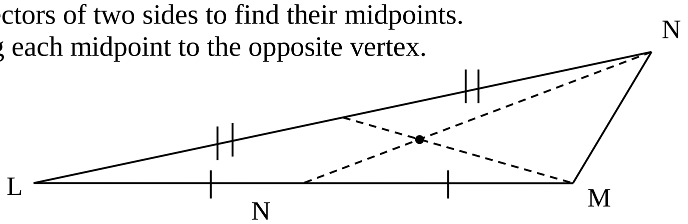
b)



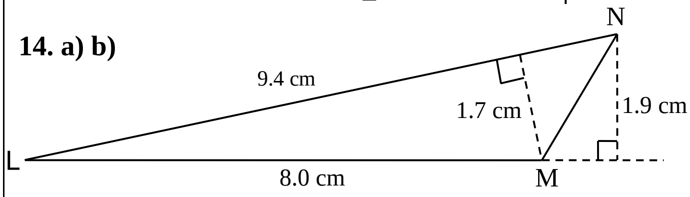
12. a) and b)



13. a) b) c)
- Construct two medians. The intersection is the centroid:
- Construct perpendicular bisectors of two sides to find their midpoints.
- Draw the medians by joining each midpoint to the opposite vertex.



14. a) b)



Using the altitude from M, $A = bh \div 2 = (9.4)(1.7) \div 2 = 7.99 \approx 8.0$ cm^2
Using the altitude from N, $A = bh \div 2 = (8.0)(1.9) \div 2 = 7.6$ cm^2
b) The answers can be slightly different because of construction and measuring inaccuracies. If you could construct and measure perfectly, the answers would be the same.

15. A median divides a triangle into two triangles. Each of these has the same height. Each has a base that is half the base of the original triangle, so their bases are also equal. Base and height are the only two measurements needed to find the area of a triangle. Therefore, the two triangles have equal area.

16. *Sample response:*
I could construct various kinds of isosceles triangles, including acute, right, and obtuse triangles, and then construct the three centres of each to see if they are collinear. This is inductive reasoning.

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MEASUREMENT REFERENCES

Measurement Abbreviations and Symbols

Time		**Capacity**	
second	s	millilitre	mL
minute	min	litre	L
hour	h	kilolitre	kL
Length		**Volume**	
millimetre	mm	cubic centimetre	cm^3
centimetre	cm	cubic metre	m^3
metre	m		
kilometre	km	$1\ cm^3 = 1\ mL$	
Mass		**Area**	
milligram	mg	square centimetre	cm^2
gram	g	square metre	m^2
kilogram	kg	hectare (10,000 m^2)	ha
ton(ne)	t	square kilometre	km^2

Metric Prefixes

Prefix	milli × 0.001	centi × 0.01	deci × 0.1	unit 1	deka × 10	hecto × 100	kilo × 1000
Example	*milli*metre mm	*centi*metre cm	*deci*metre dm	metre m	*deka*metre dam	*hecto*metre hm	*kilo*metre km

Measurement Formulas

Perimeter (Polygons)

rectangle	$P = 2(l + w)$
square	$P = 4s$
regular polygon	$P = ns$ (n is number of sides)

Circumference

circle	$C = \pi d$ or $C = 2\pi r$

Area (Polygons)

rectangle	$A = lw$
square	$A = s^2$
parallelogram	$A = bh$
triangle	$A = \frac{1}{2}bh$

Area (Non-Polygon)

circle	$A = \pi r^2$

Volume (Polyhedra)

rectangular prism	$V = lwh$
cube	$V = e^3$ (e is edge length of cube)
any prism	$V = Ah$ (A is area of base)
pyramid	$V = \frac{1}{3}Ah$ (A is area of base)

Volume (Non-Polyhedra)

cylinder	$V = \pi r^2 h$
cone	$V = \frac{1}{3}\pi r^2 h$
sphere	$V = \frac{4}{3}\pi r^3$

Surface Area (Polyhedra)

rectangular prism	$SA = 2(lw + wh + lh)$
cube	$SA = 6s^2$ (s is side length of face)
any prism	$SA = 2A + hP$ (A is area of base, P is perimeter of base)
pyramid	$SA = A$ + Area of lateral faces (A is area of base)

Surface Area (Non-Polyhedra)

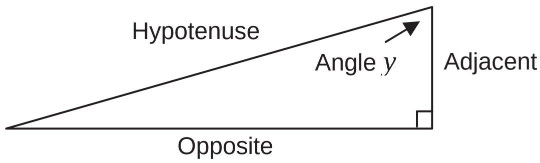
cylinder	$SA = 2\pi r^2 + 2\pi rh$
cone	$SA = \pi r^2 + \pi rs$ (s is slant height of cone)
sphere	$SA = 4\pi r^2$

TRIG TABLE AND REFERENCES

You can use this table instead of your calculator to find the trigonometric ratios for angles that are multiples of 5°. It can also be used to estimate the ratios for angles between those listed.

Angle	Sine	Cosine	Tangent
0°	0.000	1.000	0.000
5°	0.087	0.996	0.087
10°	0.174	0.985	0.176
15°	0.259	0.966	0.268
20°	0.342	0.940	0.364
25°	0.423	0.906	0.466
30°	0.500	0.866	0.577
35°	0.574	0.819	0.700
40°	0.643	0.766	0.839
45°	0.707	0.707	1.000
50°	0.766	0.643	1.192
55°	0.819	0.574	1.428
60°	0.866	0.500	1.732
65°	0.906	0.423	2.145
70°	0.940	0.342	2.747
75°	0.966	0.259	3.732
80°	0.985	0.174	5.671
85°	0.996	0.087	11.430
90°	1.000	0.000	---

Trig Ratios and Identities



$$\sin y = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc y = \frac{\text{hypotenuse}}{\text{opposite}} \text{ or } \frac{1}{\sin y}$$

$$\cos y = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec y = \frac{\text{hypotenuse}}{\text{adjacent}} \text{ or } \frac{1}{\cos y}$$

$$\tan y = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot y = \frac{\text{adjacent}}{\text{opposite}} \text{ or } \frac{1}{\tan y}$$

$$\cos(90^\circ - y) = \sin y \qquad \cos^2 y + \sin^2 y = 1$$

$$\sin(90^\circ - y) = \cos y$$

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